Problem Set #5

Problem 1

The magnetic field in the conductor layer has two configurations, one where the fields from the two layers add up (parallel), one where the fields (approximately) cancels with each other (antiparallel). Hence, in this problem we are effectively distinguishing between two resistor R_l and R_h, where R_l is the (lower) baseline resistance, and R_h is the (higher) resistance under magnetic field. The magnetoresistance is defined as:

$$\frac{R_h - R_l}{R_l} = \delta_H \tag{1.1}$$

1 point for the correct expression for magnetoresistance.
The question asks you to:

1. Find a mathematical relationship between MR, readout speed ($f = \tau^{-1}$), and R_l.
2. Show that higher MR leads to faster readout.
3. State whether you want higher or lower R_l for faster readout (explain why).

To approach this problem, define measurement as distinguishing I_l and I_h relative to the noise current. For instance, you may define a successful measurement as when the noise current is 1/10th of the current difference between the two resistors:

$$\frac{I_n}{I_l - I_h} \simeq 0.1 \tag{1.2}$$

1 point for writing down a reasonable noise criterion.
Note that if we set a single cutoff to be such that the type I/II error rates are equal, the error probability is

$$p = \Phi\left(-\frac{I_l - I_h}{I_{nl} + I_{nh}}\right) \tag{1.3}$$

where Φ is the cumulative of standard normal distribution. Now we can combine this with equations for the electrical noise current for shot noise and Johnson noise, and derive equations relating δ_H, R_l, f, and V (voltage). The equation for shot noise current is:

$$I_s = (\frac{eI}{t})^{1/2} = \left(\frac{eVf}{R}\right)^{1/2} \tag{1.4}$$

1 point for the shot noise expression.
The equation for Johnson noise current is:

$$I_J = \left(\frac{2k_BT}{Rt}\right)^{1/2} = \left(\frac{2k_BTf}{R}\right)^{1/2} \tag{1.5}$$

1 point for the Johnson noise expression.
$I_l - I_h$ is given by:

$$I_l - I_h = \frac{V}{R_l} - \frac{V}{R_h} = \frac{V}{R_l R_h}(R_h - R_l) = \frac{V}{R_l\left(1+\delta_H^{-1}\right)} \tag{1.6}$$

1 point for combining the magnetoresistance with the noise criterion by writing current in terms of voltage and resistance.
1 point for relating shot noise current to voltage and resistance.
1 point for substituting shot noise and Johnson noise into the noise criterion (2 points).
From these, we can calculate the effective SNR, assuming the two noise sources are independent (note that in reality the actual variance is less than the sum):

$$\begin{aligned}\frac{I_l - I_h}{I_{nl} + I_{nh}} &= \frac{V}{R_l\left(1+\delta_H^{-1}\right)\sqrt{(eV+2k_BT)f}\left(R_l^{-1/2} + R_l^{-1/2}\left(1+\delta_H\right)^{-1/2}\right)} \\ &= \frac{V\left(1-\left(1+\delta_H\right)^{-1/2}\right)}{\sqrt{fR_l\left(eV+2k_BT\right)}}\end{aligned} \tag{1.7}$$

To answer the questions:

1. **Show higher MR leads to faster readout.**
   Clearly as δ_H increases $\delta_H^{-1/2}$ decreases, so f can increase to make the measurement at the same noise level, but even at the limit of $\delta_H \to \infty$, if R_l is constant, the SNR does not go to ∞. (However, if we use 2 cutoffs instead of 1, the speed can still approach ∞.) Similarly, if increasing δ_H will increase R_l simultaneously, it can possibly worsen the measurement.
   2 points for showing that higher MR leads to faster readout.

2. **Do you want higher resistance or lower resistance for R_l?**
   If R_l is decreased then f can increase to achieve a measurement with the same noise level, but similarly if R_h decreases faster than R_l in scale, it can possibly worsen the measurement.
   2 points for correct conclusion about the relationship between R_l and readout time.

Note that if other parameters can also be changed and the actual total variance is used, the effective SNR expression can be written as

$$\frac{I_l - I_h}{I_{nl} + I_{nh}} = \sqrt{\frac{P_l}{feV\coth\frac{eV}{2k_BT}}}\left(1-\left(1+\delta_H\right)^{-1/2}\right) \tag{1.8}$$

so for a given noise level, the speed can be high when power is high, voltage is low, temperature is low, and/or MR is high.

Problem 2

Due to properties of Poisson distribution, this problem is equivalent to a simpler version, where every photon from the satellite is always detected, but with effective pair generation rate of

$$r_e = rp^2 \tag{2.1}$$

and effective dark count rate of

$$R_e = R + rp(1-p) \tag{2.2}$$

where $p = 0.7 \times 10^{-3.5} = 2.2 \times 10^{-4}$ is the original probability of a satellite photon being detected by a detector. The total coincidence rate is proportional to the probability of at least 1 pair generated or both detector have at least 1 dark count, i.e.

$$r_t = \frac{1}{\tau}\left(1 - e^{-r_e\tau} + e^{-r_e\tau}\left(1 - e^{-R_e\tau}\right)^2\right) = \frac{1 - 2e^{-(r_e+R_e)\tau} + e^{-(r_e+2R_e)\tau}}{\tau} \tag{2.3}$$

The success rate depends on the architecture, but can be bounded with sufficient (1 generated pair and no dark counts) and necessary (at least 1 generated pair) conditions, i.e.

$$r_e e^{-(r_e+2R_e)\tau} \le r_s \le \frac{1 - e^{-r_e\tau}}{\tau} \tag{2.4}$$

Because $r_e\tau = 2.45 \times 10^{-9}$ and $R_e\tau = 1.16 \times 10^{-5}$, $r_e > R_e^2\tau \ggg R_e r_e \tau + R_e^3\tau^2 + r_e^2\tau$, so we only need to consider events with 2 photons detected, i.e.

$$r_s = r_e + O(r_e R_e \tau) = 0.49s^{-1} = 1764h^{-1} \tag{2.5}$$

$$r_f = r_t - r_s = R_e^2\tau + O(r_e R_e \tau) = 0.0268s^{-1} = 96.3h^{-1} \tag{2.6}$$

Problem 3

(a) Using Einstein notation, a generic pure state could be written as

$$|\psi\rangle = \psi_{ij}\,|ij\rangle \tag{3.1}$$

where $\psi_{ij}\psi_{ij}^* = 1$. Trace over particle #1, we have

$$\mathrm{Tr}_1\left(|\psi\rangle\!\langle\psi|\right) = |j\rangle\,\psi_{ij}\psi_{ik}^*\,\langle k| \tag{3.2}$$

$$\mathrm{Tr}\left(\left(\mathrm{Tr}_1\,|\psi\rangle\!\langle\psi|\right)^2\right) = \psi_{ij}\psi_{ik}^*\psi_{lk}\psi_{lj}^* \tag{3.3}$$

Trace over particle #2, we have

$$\mathrm{Tr}_2\left(|\psi\rangle\!\langle\psi|\right) = |i\rangle\,\psi_{ij}\psi_{lj}^*\,\langle l| \tag{3.4}$$

$$\mathrm{Tr}\left(\left(\mathrm{Tr}_2\,|\psi\rangle\!\langle\psi|\right)^2\right) = \psi_{ij}\psi_{lj}^*\psi_{lk}\psi_{ik}^* = \mathrm{Tr}\left(\left(\mathrm{Tr}_1\,|\psi\rangle\!\langle\psi|\right)^2\right) \tag{3.5}$$

(b) For a spin-1/2 system, the purity can be evaluated using the Bloch vector

$$\mathrm{Tr}\,\rho^2 = \mathrm{Tr}\left(\left(\frac{1}{2}\left(I + a_i\sigma_i\right)\right)^2\right) = \frac{1}{4}\,\mathrm{Tr}\left(I + 2a_i\sigma_i + a_i a_j\left(\delta_{ij} I + i\epsilon_{ijk}\sigma_k\right)\right) = \frac{1 + a_i a_i}{2} = \frac{1 + |a|^2}{2} \tag{3.6}$$

The total density matrix is

$$\rho = |{\uparrow\downarrow}\rangle\!\langle{\uparrow\downarrow}| \tag{3.7}$$

Trace over particle #1, we have

$$\mathrm{Tr}_1\,\rho = |{\downarrow}\rangle\!\langle{\downarrow}| = \frac{1}{2}\left(I - \sigma_z\right) \tag{3.8}$$

which implies $|a| = 1$, so the purity is 1 and there is no entanglement.

(c) The total density matrix is

$$\rho = |T_0\rangle\langle T_0| = \frac{1}{2}\left(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|\right) \tag{3.9}$$

Trace over particle #1, we have

$$\mathrm{Tr}_1\,\rho = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) = \frac{I}{2} \tag{3.10}$$

which implies $|a| = 0$, so the purity is $\frac{1}{2}$ and the two particles are maximally entangled.

(d) The total density matrix is

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{3.11}$$

Trace over particle #1, we have

$$\mathrm{Tr}_1\,\rho = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) = \frac{1}{2}\left(I - \sigma_x\right) \tag{3.12}$$

which implies $|a| = 1$, so the purity is 1 and there is no entanglement between these two particles.

(e) The two particle state could be rewritten in x basis as

$$|\psi\rangle = -\frac{1}{2}\left(|\uparrow\rangle - |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle - |\downarrow\rangle\right) = -\,|\downarrow_x\downarrow_x\rangle \tag{3.13}$$

From this, we could easily see that the state is a product state, hence cannot be a entangled state.