Chapter 4

Quantum noise and measurement

The **Poisson distribution** is a **discrete** probability distribution that tells us the probability that a given number of events will occur in a fixed interval of time, if these events occur with a known constant average rate and independently of the time since the last event. In quantum mechanics, since we treat particles and excitations as discrete energy levels, it is only natural that Poissonian statistics will play a role in quantum engineering.

I Poissonian Statistics

I.1 The Poisson Distribution Function

Discrete events often obey **Poissonian statistics**, which instead of a probability distribution function, we evaluate the integer frequency of events in the Poissonian distribution to find what is the probability for this frequency to occur.

Theorem 4.1: the Poissonian Distribution

The probability that k discrete events occurs in an interval of time, where these events occur with a known **constant mean occurrence number** (λ) and **occur independently** of the time since the last event, would be,

$$P(k \text{ events occurring}) = \frac{\lambda^k e^{-\lambda}}{k!}$$

where λ is the **mean** amount of time for the events to occur.

For instance, consider the following scenario.

Problem 4.1: Floods in a River

On a particular river, overflow floods occur once every 100 years on average. Calculate the probability of $k = 0, 1, 2, 3, 4, 5,$ or 6 overflow floods in a 100-year interval, assuming the

Poisson model is appropriate.

Solution. Because the average event rate is one overflow flood per 100 years, $\lambda = 1$. Thus, the Poissonian distribution suggests that the occurrence frequencies obey the expression,

$$P(k \text{ overflow floods in 100 years}) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{1^k e^{-1}}{k!}$$

Therefore, substituting the different frequencies k, we yield,

$$\begin{cases} P(k=0 \text{ overflow floods in 100 years}) = \dfrac{1^0 e^{-1}}{0!} = \dfrac{e^{-1}}{1} \approx 0.368 \\ P(k=1 \text{ overflow flood in 100 years}) = \dfrac{1^1 e^{-1}}{1!} = \dfrac{e^{-1}}{1} \approx 0.368 \\ P(k=2 \text{ overflow floods in 100 years}) = \dfrac{1^2 e^{-1}}{2!} = \dfrac{e^{-1}}{2} \approx 0.184 \end{cases}$$

which means that there is 36% chance that no floods occur at all.

I.2 Poissonian Photon Statistics

[Note: this presentation largely follows Fox's *Quantum Optics*, Section 5.3, *Coherent light: Poissonian photon statistics*.] Consider a beam of light with constant power P. The **average number** of photons within a beam segment of length L would therefore be given by,

$$\langle n \rangle = \Psi_\gamma L/c,$$

where Ψ_γ is the **photon flux**, or the average number of photons passing through a cross-section of our beam per unit time. Assume now we split the beam into N subsegments. Here,N is very large, such that there is only a very small **probability** $p \equiv \bar{n}/N$ of finding a photon within a subsegment.

We now ask ourselves, what is the probability $P(n)$ of finding n photons within a beam of length L containing N sub-segments? This would be equivalent to calculating the probability of finding n subsegments containing one photon and $N-n$ subsegments containing no photon. Naturally, the probability for n segments to be *occupied* would be,

$$\begin{aligned} P(n \text{ segments occupied}) &= \binom{N}{n} (p)^n (1-p)^{N-n} \xrightarrow[p=\langle n \rangle/N]{} \frac{N!}{n!(N-n)!} \left(\frac{\langle n \rangle}{N}\right)^n \left(1 - \frac{\langle n \rangle}{N}\right)^{N-n} \\ &= \frac{1}{n!} \left(\frac{N!}{(N-n)! N^n}\right) \langle n \rangle^n \left(1 - \frac{\langle n \rangle}{N}\right)^{N-n} \end{aligned}$$

Now, let's take this one term at a time. The first step is to find the limit of Taking the limit as $N \to \infty$, we can use Stirling's formula,

$$\lim_{N \to \infty} \ln N! = N \ln N - N.$$

In this limit, we yield,

$$\lim_{N \to \infty} P(n) = \lim_{N \to \infty} \frac{1}{n!} \left(\frac{N!}{(N-n)! N^n}\right) \langle n \rangle^n \left(1 - \frac{\langle n \rangle}{N}\right)^{N-n}.$$

First, recognise that, assuming $N \gg n$ such that $N - n \approx N$, we yield,

$$\begin{aligned}\lim_{N\to\infty}\left[\ln\left(\frac{N!}{(N-n)!N^n}\right)\right] &= \lim_{N\to\infty}\left[\ln N! - \ln(N-n)! - \ln N^n\right] \\ &= \lim_{N\to\infty}\left[(N\ln N - N) - (N-n)\ln(N-n) + (N-n) - n\ln N\right] = 0,\end{aligned}$$

Hence,

$$\lim_{N\to\infty}\frac{N!}{(N-n)!N^n} = 1.$$

Furthermore, we see,

$$\begin{aligned}\left(1 - \frac{\bar{n}}{N}\right)^{N-n} &= 1 - (N-n)\frac{\langle n\rangle}{N} + \frac{1}{2!}(N-n)(N-n-1)\left(\frac{\langle n\rangle}{N}\right)^2 - \cdots \\ &\xrightarrow[N-n\approx n]{} 1 - \langle n\rangle + \frac{\langle n\rangle^2}{2!} - \cdots = \exp\left(-\langle n\rangle\right).\end{aligned}$$

Using the two limits presented here, we see,

$$\lim_{N\to\infty} P_n = \frac{1}{n!}\times 1\times\langle n\rangle^n\times\exp\left(-\langle n\rangle\right).$$

We thus can conclude that the probability of finding n photons in a beam of length L would be,

$$\boxed{P(n) = \frac{\langle n\rangle^n}{n!}\exp(-\langle n\rangle), \qquad n = 0, 1, 2, \cdots},$$

which is precisely the form of a Poisson distribution. Now, using the Poissonian, we can solve the following difficult problem.

Problem 4.2: the Micius Satellite

Quantum satellite is a state-of-the-art technology to distribute quantum entanglement over large (¿1000 km) geographical distance[a] for secure quantum communication and distributed quantum computation. Here we consider a quantum satellite that is equipped with an onboard entangled photon pair source. The source generates pairs of polarization entangled photons in the state,

$$\frac{|HV\rangle + |VH\rangle}{\sqrt{2}}$$

at a rate of 1×10^7 pairs per second. Then each of the entangled photon is beamed down from the satellite to two distant ground stations. We assume the distances from the satellite to each ground station are identical. The optical attenuation from the satelliate to either ground station is 35 dB, due to absorption by the atmosphere and air turbulence. At each ground state, a single photon detector with 70% detection effciency and 100 counts per second dark count is used to detect photons from the satellite. The time of arrival of photons is recorded at both stations. Two detection events, one by each detector, occurred within a 5 ns time window is considered a coincidence event.

- **a.** Given these conditions, what is the coincidence rate (in units of events per hour) that results from **one** pair of entangled photon pairs?
- **b.** What is the "false" coincidence rate (also called "accidental" coincidence) which are coincidence events not due to entangled photons?

To correctly solve this problem, you should understand that the photon pairs are generated with a Poissonian statistics. The mean generation rate is 1×10^{-7} per second. But there are infinite probabilities that zero or more than one pair of photons being generated as well. Due to significant loss during transmission and detection, there are false coincidences due to photons from the satellite but those photons are actually not entangled. Finally, you can assume there is no background photons from the ambient impinging on both detectors.

[a]From Beijing to Xinjiang!

[2]Each pair is not entangled. Think of the classical analogy of follows. We have n jars which are distinguishable (entanglement), but the balls inside the jars are indistinguishable (photons).

[3]bad code

II the Semiclassical Noise

Noise is defined as random or unwanted signals, and is an intrinsic property of a wave[4] phenomena. In an ideal world, the equation that describes a wave would look like,

$$f(t) = A\cos(\omega_0 t).$$

However, due to the presence of noise, the *actual* function of a wave would become,

$$\boxed{f(t) = (A + \underbrace{\delta A(t)}_{\text{amplitude uncertainty}})\cos(\omega_0 t + \underbrace{\phi(t)}_{\text{phase uncertainty}})}$$

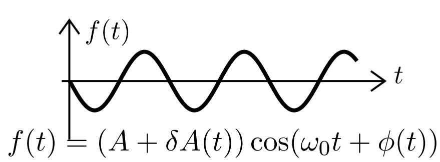
where $\delta A(t)$ is noise (uncertainty) associated with the amplitude, known as **amplitude noise**, and $\phi(t)$ is analogously known as the **phase noise**.

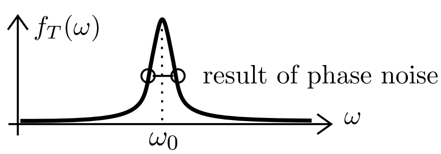
II.1 General Properties of Noise

Noise can also be well expressed in the spectral (frequency) domain through a transformation of the wavefunction itself[5],

$$\boxed{f_T(\omega) = \frac{1}{T}\int_0^T f(t)\exp(i\omega t)dt}.$$

If $f(t)$ iis a noisy sinusoidal function, the transformed $f_T(\omega)$ in the frequency domain would look as follows.





In the ideal system where we have no noise, the width of the frequency domain peak would be zero,

$$\boxed{\delta(\omega_0) \text{ with no width } \leftrightarrow \text{ no noise}};$$

for a sinusoidal wave, the spread of the peak is almost always associated with the noise.

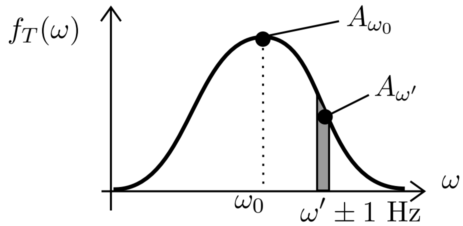
There are three vocabulary terms that we should familiarise ourselves with, as it is often encountered in various fields of engineering.

Term	**Definition**
Drift	Growing difference of a phase from a reference with time, $\phi(t) > \phi(0)$
Jitter	Phase error per cycle (units of seconds)
Phase Noise	Relative amplitude of noise w.r.t to the carrier over a bandwidth of 1 Hz (units of dBc/Hz)

[4] of all forms, including EM waves, sound, quantum, etc.

[5] Note that this differs from a Fourier series since it is not properly normalised over an entire period, but instead over a measurement period T.

Consider the following diagram that indicates geometrically what is meant by phase noise.



$$\boxed{\text{Phase noise } \equiv \frac{A_{\omega'}}{A_{\omega_0}} \text{ in dBc/Hz}}$$

As a reminder, the unit dB is known as **decibels**. The decibel is the relative ratio of power or amplitude, and have different definitions.

$$\text{(power decibel)} \equiv 10\log_{10}\left(\frac{P}{P_0}\right) \text{ dB}, \quad \text{(amplitude decibel)} \equiv 20\log_{10}\left(\frac{A}{A_0}\right) \text{ dB}$$

For instance, if the power ratio between P and P_0 is 20 decibels, then the actual ratio P/P_0 would be 100. On the contrary, if the amplitude ratio between A and A_0 is 20 decibels, then the actual ratio P/P_0 would only be 10.

Example: Noise in Quantum Waves

As an example, we can see that noise also exists in quantum waves simply due to its wave nature. Let's consider the wavefunction of a free particle in a 1D "box" with dimension V,

$$\Psi(x,t) = \frac{1}{\sqrt{V}}\exp(ikx - \omega t).$$

However, recall that the **uncertainty principle** suggests that there exists uncertainty relations between the measurement of pairs of conjugate variables - for instance, position and momentum,

$$[\hat{x}, \hat{p}_x] = \left[x, -i\hbar\frac{d}{dx}\right] = i\hbar.$$

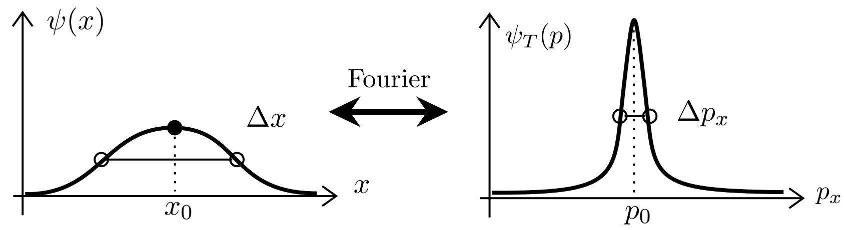
Physically, there will always be **uncertainty**, or deviations of the measurement results from the mean, if we try to measure the position and the momentum simultaneously. In this sense, we will *never* know where the particle is with absolute certainty. **Heisenberg's uncertainty principle** states that, for a conjugate pair of position and momentum, the uncertainty of measurements will obey,

$$\boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}}.$$

Uncertainty in this sense comes fully from the inability to fully localise a wave in both the position and momentum domains. In order to fully localise a quantum wave at a given position, we need an infinite series of sinusoidal waves,

$$\psi_{k,\Delta k,x_0}(x) = A\int \exp(ik'(x-x_0))\exp\underbrace{\left(-\frac{1}{2}\frac{k'-k}{\Delta k}\right)^2}_{\text{assume wavepackets are Gaussians}} \quad dk',$$

where k is the average momentum of the wavepacket, and Δk is the spread of the momentum of the system. In essence, the same wavefunction can be expressed in the position and momentum domains as follows.



From the above figure, we see that in order to localise a wave in one domain, we have to spread in the conjugate domain. In other words, the more localised the momentum is, the less localised the position is.

This is characteristic of the Fourier transform itself. For instance, another similar example would be that from electrical engineering - there also exists an uncertainty relation known as **time-bandwidth product**,

$$\Delta t \Delta \omega \geq \frac{1}{2}.$$

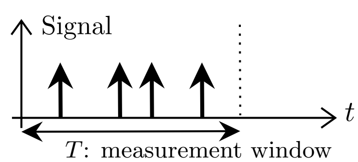
II.2 the Shot Noise

A very important source of noise in quantum engineering would be the **shot noise**, due to the fact that elementary particles are quantised in integer amount of particles.

For instance, in an electrical current it is an electron that carries the charge in the current: it does make sense for there to be 1.1 electrons, since electrons are discrete particles. Therefore, when the current is small enough, the "edge effect" due to the discrete electrons will start to play a larger role, known as the shot noise. Similarly, in optics, light comes in wavepackets of discrete photons.

Poissonian Distribution and the Shot Noise

Now, assume we have a signal[6] that consists of N discrete objects over a window of time T, as seen in the figure below.



Since the ensemble of discrete objects will obey the Poissonian distribution, the **noise**, or the **standard deviation**, N-discrete events occurring would be as follows.

> **Theorem 4.2: the Standard Deviation in Poissonian Statistics**
>
> $$\text{standard deviation for } N \text{ events occurring} = \sqrt{N}.$$

The variance of the number of events measured would naturally be N, since the variance is defined as the standard deviation squared. In essence, at a given time t, the number of particles that I

[6] For instance, a flux of electrons at a rate of N electrons per second

would measure would actually be,

$$\text{number of particles measured} = N \pm \sqrt{N}.$$

We can also calculate another quantity, known as the **signal-noise ratio**, by performing a ratio between the amplitude of the signal and the noise to be,

$$\text{SNR} \equiv \frac{\text{signal}}{\text{noise}} \xrightarrow{\text{shot noise}} \frac{N}{\sqrt{N}} = \sqrt{N_{\text{total}}}.$$

Example: Electronic Shot Noise in a Current

As an example, we can consider the shot noise that a current[7] I experiences. First, we count the number of electrons per second to be,

$$N_{\text{per second}} = \frac{I}{e}.$$

Therefore, according to the Poissonian distribution, the "noise" (or the standard deviation) of the electron count over a time T would be,

$$N_{\text{noise}} = \sqrt{N_{\text{per second}} T} = \sqrt{\frac{IT}{e}}, \quad \xrightarrow[\text{convert back to current}]{} I_{\text{noise}} = \sqrt{\frac{IT}{e}} \underbrace{\frac{e}{T}}_{I \propto \text{charge/sec} = Ne/T} = \sqrt{\frac{eI}{T}},$$

where T is the measurement parameter, or how long we are measuring the observing the system to get this mean. We can define the **single-sided bandwidth**, which maps T into the frequency domain, to be,

$$\Delta f \equiv \frac{1}{2T},$$

which then generates I_{noise} to be,

$$I_{\text{noise}} = \sqrt{2eI\Delta f},$$

and an associated signal-noise ratio of,

$$\text{SNR} = \frac{I}{I_{\text{noise}}} = \left(\frac{2e\Delta f}{I}\right)^{-1/2} = \sqrt{\frac{I}{2e\Delta f}}.$$

In fact, we can actually show that the amplitude signal-noise ratio of all "flows" of discrete particles, with N particles per unit time, to be,

$$\boxed{SNR = \sqrt{\frac{N_{\text{unit time}}}{2\Delta f}}},$$

which is equivalent to the signal noise ratio we have derived above for a Poisson distribution.

[7] units of charge per seconds.

Photon Shot Noise

Similarly, we can calculate the shot noise due to the quantisation of photons in a beam of light. First, we know that the power[8] of beam of N photons can be expressed as,

$$P = \text{energy per photon} \times \frac{\text{number of photons}}{\text{time}} = h\nu \left(\frac{N}{T}\right) = \frac{Nh\nu}{T}.$$

We can calculate the amount of photons measured after a total time T to be,

$$N_{\text{total}} = \frac{PT}{\hbar\nu}.$$

According to the boxed formula given above, the photon shot noise would naturally be,

$$SNR = \sqrt{N_{\text{total}}} = \sqrt{\frac{N_{\text{unit time}}}{2\Delta f}} \xrightarrow[\text{photon noise}]{\text{evaluating}} \sqrt{\frac{PT}{h\nu}} = \sqrt{\frac{P}{2h\nu\Delta f}}$$

If we define the **photon flux** Φ_γ as the number of incident photons per unit area per unit time, such that the total power of the photon at a given area A can be expressed as,

$$P = (h\nu)(\Phi_\gamma)(A),$$

we can re-express the photon signal-noise ratio as,

$$SNR = \frac{h\nu\Phi_\gamma A}{2\hbar\nu\Delta f} = \sqrt{\frac{\Phi_\gamma A}{2\delta f}}.$$

Noise in Photodetectors

The photon shot noise has a very useful application when it comes to **photon detection**. Usually, a photon measurement device - known as **photodetector**, works as follows: as the photon collides with the device, a current is created by the device due to the photoelectric device. The device then measures the current to compute the total power of the beam of light.

$$\text{optical power} \;\to\; \text{current} \;\to\; \text{counting the current}.$$

The aforementioned current associated with the device would take the value,

$$\boxed{I_{\text{signal}} = \frac{\eta P_S e}{h\nu} \equiv RP_S} \xrightarrow[\text{photon flux}]{\text{in terms of}} e\eta\Phi_\gamma A,$$

where η, the **efficiency**, is a intrinsic parameter of the given device that measures the efficiency of power conversion between optical power and electric current. The factor associated with I_S can be combined as a parameter known as the **responsivity**, in units of amperes per watt. This parameter essentially tells us the amount of current is generated in the photodetector per unit watt of light.

However, due to the "quantised" nature of the electron, there will exist an uncertainty of the current measured by the photodetector. As computed previously, the signal-noise ratio of our photodetector would therefore be,

$$SNR = \frac{I_{\text{noise}}}{I_S} = \frac{\sqrt{ReIS\Delta f}}{I_S} = \frac{R\phi A h\nu}{\sqrt{2e^2\eta\phi_\gamma \Delta f A}}.$$

[8] energy per time

II.3 Dark Currents

Quite strangely, if we perform current measurements on objects without any external excitation, we would still measure a non-zero current. This is simply due to the fact the **environment** acts as a thermal reservoir, which in turn excites[9] charge carriers (mostly electrons) to move around. If our environment is at temperature T, the thermal energy of one charged particle would be k_BT. If all of this energy goes to electric energy, we will yield a potential difference due to thermal effects as,

$$k_BT = e\Delta V, \quad \to \quad \Delta V_{\text{thermal}} = \frac{k_BT}{e}.$$

The current due to this newly generated potential difference is called the **Johnson current**, defined as,

$$\Delta V = IR_0, \quad \boxed{I_{\text{Johnson}} = \frac{\Delta V}{R_0} = \frac{k_BT}{eR_0}},$$

where R_0 is the electric resistance of our system. Of course, when we measure Johnson currents, we would also pick up a noise called the **Johnson noise**. The Johnson noise can be computed using the same formula a derived above. In the language of AC circuitry[10], we add a factor of $\sqrt{2}$, which yields the Johnson noise to be,

$$I_{\text{noise}} = \sqrt{4\left(\frac{k_BT\Delta f}{R_0}\right)}.$$

Now, we will need to perform an example to more thoroughly understand the above concepts.

Problem 4.3: Hard Disk Read Head

The noise that occurs in a resistor has important implications for using the difference between two resistance states as the indication of the readout of a sensor. One situation where this is used is in the hard disk read head shown below, in which a magnetic region responds to the local magnetic field of the hard disk media and orients one way or the other (this is the so-called free region). Current flows through this magnetic region and also a magnetic region with a fixed magnetic orientation (so-called fixed region). The difference in resistance between the two magnetic configurations (parallel or antiparallel) is called the magnetoresistance (MR), and it is usually reported as a percentage of the resistance of the low-resistance (baseline) configuration (thus yielding percentages greater than 100% for some systems).

For a period of time the community focused on these MR percentages without concerning themselves with the actual baseline resistance. Analyse the trade-off between MR, baseline resistance, and speed of reading out information from media. From this analysis you should find you want higher MR percentages for faster readout. Do you want higher resistance or lower resistance for the low-resistance configuration for faster readout?

Solution.

[9] albeit, quite minimally

[10] The Δf defined here and Δf defined in electronics is defined somewhat ambiguously.

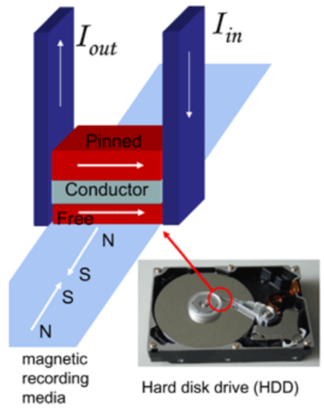


Figure 4.1: Hard disk read head

11

Commentary

Up until now, although we have considered the "discrete-ness" of photons and electrons, we have not touched upon the *wave* properties of particles[12]. As seen in the previous discussion, the source of noises in these semi-classical systems would be the shot noise. Therefore, as a corollary, we can deduce the following.

> If we measure a system and find that its noise is less than the shot noise, then the system must be completely quantum.

The distinguishing between classical and quantum noise can serve as one of the "metrics" in determining whether a system is in a classical regime or a quantum regime.

III Quantum Noise

In this section, we will discuss the effects of wave-like properties on the measurement of noise. In this entirely quantum regime, there will be two peculiar *types* of quantum states that would be of great interest to us.

[12] Such as superposition, etc.

III.1 Fock Noise of a Single Particle

The two types of eigenstates we are interested in are as follows:

1. **Fock states.** Eigenstates of "particle number",
2. **Coherent (Glarber) states.** Eigenstates of phase / amplitude.

The most simple states that demonstrates the above two types of wavefunctions originate from the Hamiltonian of the harmonic oscillator,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

This form of expressing the harmonic oscillator potential is known as the **canonical quantisation**. However, we can rewrite the position and momentum operators in terms of ladder operators,

$$\hat{a} = \hat{x}\sqrt{\frac{m\omega}{2\hbar}} + i\hat{p}\left(\frac{1}{2m\hbar\omega}\right), \quad \hat{a}^\dagger = \hat{x}\sqrt{\frac{m\omega}{2\hbar}} - i\hat{p}\left(\frac{1}{2m\hbar\omega}\right),$$

where we have taken the Hermiticity of $\hat{x}$ and $\hat{p}$ for granted. The process of writing the Hamiltonian in terms of ladder operators is known as **second quantisation**. If we multiply $\hat{a}$ and $\hat{a}^\dagger$, we see that,

$$\hat{a}^\dagger\hat{a} = \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2m\hbar\omega}\hat{p}^2 + \frac{i}{2\hbar}[\hat{x},\hat{p}].$$

Notice that the first two terms are directly proportional to the original Hamiltonian $\hat{H}$, while the second term can be computed using the canonical commutator relationship, $[\hat{x},\hat{p}] = -i\hbar$. Therefore, we see,

$$\boxed{\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hbar\omega = \hat{H}}.$$

Properties of the Ladder Operators; Fock States

We can compute the commutator relationships of the ladder operators using the canonical commutation relationships as follows,

$$[\hat{a},\hat{a}^\dagger] = 1, \quad [\hat{a},\hat{H}] = \hat{a}\hbar\omega, \quad [\hat{a}^\dagger,\hat{H}] = -\hat{a}^\dagger\hbar\omega.$$

Let's now attempt to construct eigenstates for the harmonic oscillator Hamiltonian. We label the eigenstates of the Hamiltonian as $|n\rangle$, where n, known as the **particle number**, represents the number of excitations (or particles) of the system in the given state. Demanding that,

$$\hat{H}\,|n\rangle = E_n\,|n\rangle\,,$$

we can see that,

$$\hat{H}\hat{a}\,|n\rangle = \hat{a}\hat{H}\,|n\rangle - \hbar\omega\hat{a}\,|n\rangle = \hat{a}E_n\,|n\rangle - \hbar\omega\hat{a}\,|n\rangle = (E_n - \hbar\omega)\hat{a}\,|n\rangle\,.$$

Therefore, if $|n\rangle$ is an eigenstate of $\hat{H}$, then $\hat{a}\,|n\rangle$ would also be an eigenstate.

However, if we are in the ground state where $n = 0$, we cannot subtract anymore energy from the ground state. Therefore, we demand that, in the ground state,

$$\boxed{\hat{a}\,|0\rangle = 0}.$$

Moreover, if we apply $\hat{a}^\dagger$ on top of $\hat{a}\,|0\rangle$, we see,

$$\hat{a}^\dagger \hat{a}\,|0\rangle = \left(\hat{H} - \frac{\hbar\omega}{2}\right)|0\rangle = E_0\,|0\rangle - \frac{\hbar\omega}{2}\,|0\rangle\,.$$

Therefore, the energy value of the lowest eigenenergy would be,

$$\hat{H}\,|0\rangle = \underbrace{\frac{\hbar\omega}{2}}_{E_0}\,|0\rangle\,.$$

Applying $\hat{a}^\dagger$ recursively to the ground state, we see,

$$\hat{H}\hat{a}^\dagger\,|n\rangle = (E_n + \hbar\omega)\hat{a}^\dagger\,|n\rangle\,, \quad \rightarrow \quad \boxed{\hat{H}\,|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega\,|n\rangle}.$$

From here, we have calculated the energy of a harmonic oscillator. By matching coefficients betwen $\hat{H}$ and $\hat{a}^\dagger\hat{a}$, we would find that,

$$\boxed{\hat{a}^\dagger\hat{a} = n},$$

where $\hat{a}^\dagger\hat{a}$ is appropriately known as the **number operators**. As we have defined $|n\rangle$ to be eigenstates of the particle number $\hat{a}^\dagger\hat{a}$, $|n\rangle$ are **Fock states** of the harmonic oscillator.

Now that we have introduced the Fock state for one particle, we want to ask ourselves three conceptual questions.

Where are Fock States seen?

The Hamiltonian of a harmonic oscillator did not simply spawn from the vaccum: there are many physical realisations of the harmonic oscillator Hamiltonian. For instance, if $\hat{x}$ and $\hat{r}$ represents the position and momentum of oscillators in a mechanical oscillation, then the resultant excitation is known as a **phonon**. Similarly, if $\hat{x}$ and $\hat{p}$ represents *quadratures* of an electromagnetic wave, then the excitation is a **photon**.

The term **quadrature** is defined as the cosine and sine components of an oscillator - for instance, since in phase space the position and momentum of a classical (non-dissipative) harmonic oscillator traces out a circle, the position and momentum are said to be quadratures of the harmonic oscillator Hamiltonian.

How Empty is the Vaccum?

Notice that the ground state of a harmonic oscillator will still have a nonzero finite energy of half a quanta, $E_0 = \hbar\omega/2$. Quite intriguingly, the Hamiltonian of an electromagnetic field is in that of a form of a harmonic oscillator; thus, the ground state of an electromagnetic field is nonzero.

This then connects to the question we have asked ourselves previously: the vacuum is **not empty** at all; there will always be at least a half-photon energy lying somewhere in the of the electromagnetic field[13].

[13] which propagates through the entirety of the Universe.

Does the Fock states have any noise during a measurement of the degree of excitation?

Let's say that we want to measure the degree of excitation of an ensemble of Fock states, $|n\rangle$. Since Fock states are eigenstates of the number operator, there is no wavefunction that our measurement can collapse into, other than the eigenstate $|n\rangle$. When the Fock state describes the number of particles created due to an excitation by a Hamiltonian, *any* measurement of the particle number will give us the same result n without any standard deviation, which is in stark contrast to the semi-classical shot noise that we would otherwise expect.

III.2 Coherent States

Another type of quantum states we are particularly interested in would be eigenstates of the ladder operator $\hat{a}$, which we denote to be $|\alpha\rangle$. If we construct $|\alpha\rangle$ such that it is an eigenstate to the lowering operator,

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,,$$

where α represents an **amplitude** and **phase** parameter,

$$\alpha = A\exp(i\phi).$$

The state $|\alpha\rangle$ is called the **coherent state**. We can therefore express the coherent state in terms of a superposition of Fock states,

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_{\alpha_n}\,|n\rangle\,.$$

Applying the lowering operator, we yield,

$$\hat{a}\,|\alpha\rangle = \sum_{n=0}^{\infty} c_{\alpha,n}\,[\hat{a}\,|n\rangle] = \sum_{n=1}^{\infty} c_{\alpha,n}\left[\sqrt{n}\exp(i\phi)\,|n-1\rangle\right] \xrightarrow[\text{change index}]{n\to n+1} \sum_{n=0}^{\infty} c_{\alpha,n+1}\sqrt{n+1}\exp(i\phi)\,|n\rangle\,.$$

Comparing indices where,

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,,$$

we will yield,

$$c_{\alpha,n+1}\sqrt{n+1}\exp(i\phi)\,|n\rangle = \alpha c_{\alpha,n}.$$

Since this relationship is a recursion, we should define the base term as,

$$\alpha c_{\alpha,0} = c_{\alpha,1}\exp(i\phi).$$

By induction, we can therefore see that,

$$c_{\alpha,} = \frac{c_{\alpha,0}\alpha^n}{\sqrt{n!}}\exp(-in\phi),$$

which in terms mean that the full coherent state would be,

$$\boxed{|\alpha\rangle = c_{\alpha,0}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp(-in\phi)\,|n\rangle},$$

where, fortunately, the Fock states are orthogonal to each other. We can get $c_{\alpha,0}$ through normalisation,

$$\langle\alpha|\,\alpha\rangle = 1 \qquad \rightarrow \qquad |c_{\alpha,0}|^2 \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} = 1.$$

However, mathematically, we know that,

$$\sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} = \frac{|\alpha^2|^n}{n!} = \exp(|\alpha|^2).$$

Therefore, we see that,

$$c_{\alpha,0} = \exp\left(-\frac{|\alpha|^2}{2}\right),$$

implying that the full coherent state would be,

$$\boxed{|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle},$$

where α is again a complex quantity denoting the amplitude and phase of the coherent state.

Measurement of the Particle Number

We now what to find out the probability distribution of particle numbers if we are in a coherent state[14]. The observable, in this case, would be

$$\hat{n} = \hat{a}^\dagger \hat{a},$$

or the particle number of the system. The probability of measuring a particle number of n would be the coefficient in front of $|n\rangle$, which would be,

$$\boxed{P(\text{particle number } = n) = |\langle\alpha|\,n\rangle|^2 = \frac{|\alpha|^{2n} \exp(-|\alpha|^2)}{n!}},$$

which, very interestingly, recovers Poissonian statistics!

In other words, if we perform a measurement of the particle number n, the distribution function we get for n would be a Poisson distribution. We can now calculate various parameters of the coherent state.

First, the mean particle number of a coherent state $|\alpha\rangle$ would be,

$$\langle\hat{n}\rangle = \langle\alpha|\,\hat{a}^\dagger \hat{a}\,|\alpha\rangle = |\alpha|^2.$$

The noise, or the standard deviation, of this measurement, would therefore be,

$$\Delta n^2 = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2.$$

We can calculate $\langle\hat{n}^2\rangle$ of a coherent state to be,

$$\langle\hat{n}^2\rangle = \langle\alpha|\,(\hat{a}^\dagger\hat{a})(\hat{a}^\dagger\hat{a})\,|\alpha\rangle = \alpha^* \langle\alpha|\,\hat{a}\hat{a}^\dagger\,|\alpha\rangle\,\alpha = |\alpha|^2 \langle\alpha|\left(1 + \hat{a}^\dagger\hat{a}\right)|\alpha\rangle = |\alpha|^2\left(1 + |\alpha|^2\right).$$

[14] As in, what is the probability that there exists one particle ($n = 1$)? How about $n = 2$?

Therefore, we see,

$$\Delta n^2 = \underbrace{|\alpha|^2 \left(1 + |\alpha|^2\right)}_{\langle \hat{n}^2 \rangle} - \underbrace{|\alpha|^4}_{\langle n \rangle^2} = |\alpha|^2 .$$

Therefore, the standard deviation (noise) of a coherent state would be,

$$\boxed{\Delta n = |\alpha| = \sqrt{n}},$$

which again forms a strong parallel with the shot noise coming from the Poissonian distribution. In practice, consider the follows.

> Although number states are solely a quantum effect, one cannot distinguish between quantum and classical noise[a] in the coherent state. Anything that yields measurement statistics different than the coherent state[b] must be entirely quantum mechanical.
>
> [a]i.e. shot noise
> [b]e.g. noise less than the shot noise

An ensemble of coherent states $|\alpha\rangle$ will demonstrate a classical noise distribution: since the coherent state behaves so much so as a classical object, it is often appropriately named as the **semiclassical state**.

Coherent States are not orthogonal.

Unlike the Fock states, the coherent states are not orthogonal to each other. We can calculate the inner product between two coherent states to be,

$$\begin{aligned} \langle \alpha |\, \beta \rangle &= \exp\left(-\frac{|\beta|^2}{2} - \frac{|\alpha|^2}{2}\right) \sum_n \sum_{n,m} \frac{\alpha^{*m} \beta^n}{\sqrt{n!m!}} \underbrace{\langle n |\rangle m}_{\delta_{nm}} = \exp\left(-\frac{|\beta|^2}{2} - \frac{|\alpha|^2}{2}\right) \sum_n \frac{(\alpha^* \beta)^n}{n} \\ &= \exp\left(-\frac{|\beta|^2}{2} - \frac{|\alpha|^2}{2} + \alpha^* b\right) \xrightarrow[\text{if } \alpha, \beta \in \mathbb{R}]{} \exp\left(-\frac{|\alpha - \beta|^2|}{2}\right) \end{aligned}$$

In general, we see that,

$$\boxed{\langle \alpha |\, \beta \rangle \neq 0}.$$

They form what is called an **overcomplete basis**, where instead of resolving to the identity, we yield,

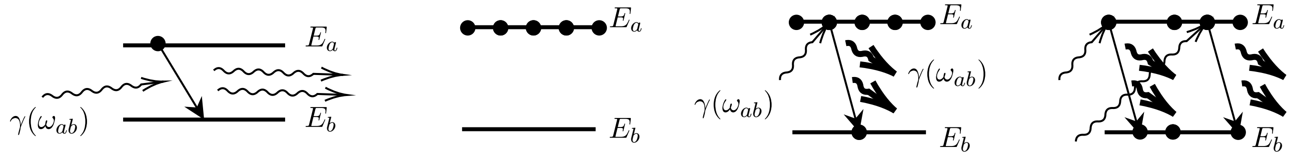
$$\int |\alpha\rangle \langle \alpha|\, d\alpha = \pi \mathbf{1}.$$

III.3 Applications of the Coherent State

There are many instances where the coherent state are represented in quantum engineered objects.

the LASER

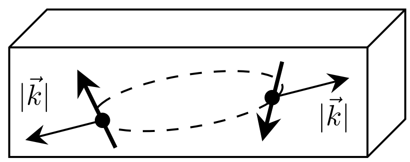
For instance, photons created by LASERs (Light Amplification of Stimulated Emission of Radiations) are represented by coherent states. All the photons created by the stimulated emissions are described by the coherent state $|\alpha\rangle$ of the electromagnetic field. All photons in the states of $|\alpha\rangle$ will have amplitude $|\alpha|$ and a phase $\alpha/|\alpha|$.



stimulated emission　original population　$1 \to 2$ photons　$2 \to 4$ photons

Superconductors

Recall that superconductors form by Cooper pairs created from a pair of correlated particles that share opposite $\vec{k}$ wavevectors.



Cooper pair

The **BCS wavefunction**, which describes the creation of Cooper pairs in a superconductor, can be expressed as,

$$|\Psi\rangle_{BCS,\vec{k}} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} |n\rangle ,$$

where $|n\rangle$ is the number state that corresponds to the number of Cooper pairs in our system. However, notice that we can only have one Cooper pair per pair of opposite $\vec{k}$-vector.

In an entire superconductor, there would be multiple possible $\vec{k}$-vectors; each pair of $\vec{k}$ vectors can either have zero or one Cooper pairs. Therefore, the total BCS wavefunction would be,

$$|\phi\rangle_{\mathrm{BCS}} = \prod_{\text{all } \vec{k}} \sum_{n=0,1} \frac{\alpha^n}{n!} |n\rangle ,$$

which is a coherent state of the semiconductor with amplitude α and phase $\alpha/|\alpha|$. The coherent state is very important in terms of superconductors, as it guarantees that *every* number state has the same phase, which is integral in the development on transmon qubits.