Problem 1

This problem identifies some of the choices one might want to make in a basis. Consider the free particle in one dimension, $\hat{H} = \hat{p}^2/2m$. This Hamiltonian has both translational symmetry and inversion symmetry.

(a) Show that the translation operator commutes with Hamiltonian, and that the inversion operator commutes with the Hamiltonian.

(b) Show that translations and inversions do not commute.

(c) Because of the translational symmetry we know that the eigenstates of $\hat{H}$ can be chosen to be simultaneous eigenstates of momentum, namely $\psi_p(x) = V^{-1/2}\exp(ipx/\hbar)$ with energy $p^2/2m$. Show that the inversion operator turns $\psi_p(x)$ into $\psi_{-p}(x)$; these two states must therefore have the same energy.

(d) Alternatively, because of the inversion symmetry we know that the eigenstates of $\hat{H}$ can be chosen to be simultaneous eigenstates of inversion namely $(\pi\hbar)^{-1/2}\cos(px/\hbar)$ and $(\pi\hbar)^{-1/2}\sin(px/\hbar)$. Show that the translation operator mixes these two states together; they therefore must be degenerate.

(a). translation operator $\hat{T}(a)\psi(x) = \psi(x-a)$

inversion operator $\hat{P}\psi(x) = \psi(-x)$

so we have:

$$[\hat{T}(a), \hat{H}]\psi(x) = (\hat{T}(a)\hat{H} - \hat{H}\hat{T}(a))\psi(x)$$

$$= \hat{T}(a)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x)\right) + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\hat{T}(a)\psi(x)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial (x-a)^2}\psi(x-a) + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x-a)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x)$$

$$= 0$$

hence. $[\hat{T}(a), \hat{H}] = 0$

$$[\hat{P}, \hat{H}]\psi(x) = (\hat{P}\hat{H} - \hat{H}\hat{P})\psi(x) = \hat{P}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x)\right) + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(-x)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial (-x)^2}\psi(-x) + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(-x)$$

$$= 0$$

hence. $[\hat{P}, \hat{H}] = 0$

(b) $[\hat{T}(a), \hat{P}]\psi(x) = (\hat{T}(a)\hat{P} - \hat{P}\hat{T}(a))\psi(x) = \hat{T}(a)\psi(-x) - \hat{P}\psi(x-a)$

$$= \psi(-(x-a)) - \psi(-x-a)$$

$$= \psi(-x+a) - \psi(-x-a) \neq 0$$

(c). $\hat{P}\psi_p(x) = \psi_p(-x) = V^{-\frac{1}{2}}\exp[ip(-x)/\hbar] = V^{-\frac{1}{2}}\exp(-ipx/\hbar)$

which is also $\psi_{-p}(x) = V^{-\frac{1}{2}}\exp[i(-p)x/\hbar] = V^{-\frac{1}{2}}\exp(-ipx/\hbar)$.

so $\hat{P}\psi_p(x) = \psi_{-p}(x)$, $\psi_p(x)$ and $\psi_{-p}(x)$ have the same energy, which is $\frac{p^2}{2m}$.

(d). let's set $\psi_1(x) = (\pi\hbar)^{-\frac{1}{2}}\cos(px/\hbar)$

$\psi_2(x) = (\pi\hbar)^{-\frac{1}{2}}\sin(px/\hbar)$

so $\hat{T}(a)\psi_1(x) = \psi_1(x-a) = (\pi\hbar)^{-\frac{1}{2}}\cos\left(\frac{px}{\hbar} - \frac{pa}{\hbar}\right)$

$$= (\pi\hbar)^{-\frac{1}{2}}\cos\frac{px}{\hbar}\cos\frac{pa}{\hbar} + (\pi\hbar)^{-\frac{1}{2}}\sin\left(\frac{px}{\hbar}\right)\sin\left(\frac{pa}{\hbar}\right)$$

so. $\hat{T}(a)\psi_1(x) = \cos\frac{pa}{\hbar}\psi_1(x) + \sin\frac{pa}{\hbar}\psi_2(x)$

$$\hat{T}(a)\psi_2(x) = \psi_2(x-a) = (\pi\hbar)^{-\frac{1}{2}}\sin\left(\frac{px}{\hbar} - \frac{pa}{\hbar}\right)$$

$$= (\pi\hbar)^{-\frac{1}{2}}\sin\left(\frac{px}{\hbar}\right)\cos\left(\frac{pa}{\hbar}\right) - (\pi\hbar)^{-\frac{1}{2}}\cos\left(\frac{px}{\hbar}\right)\sin\left(\frac{pa}{\hbar}\right)$$

so $\hat{T}(a)\psi_2(x) = \cos\frac{pa}{\hbar}\psi_2(x) - \sin\frac{pa}{\hbar}\psi_1(x)$

hence. the $\hat{T}(a)$ mixes these two states $\psi_1(x)$. $\psi_2(x)$ together.

suppose that: $\hat{H}\psi_1(x) = E_1\psi_1(x)$

$\hat{H}\psi_2(x) = E_2\psi_2(x)$.

considering $[\hat{T}(a), \hat{H}] = 0$, so $\hat{T}(a)\hat{H} = \hat{H}\hat{T}(a)$

$$\hat{T}(a)\psi_1(x) = \cos\frac{pa}{\hbar}\psi_1(x) + \sin\frac{pa}{\hbar}\psi_2(x)$$

$$\Rightarrow \hat{H}\hat{T}(a)\psi_1(x) = \cos\frac{pa}{\hbar}E_1\psi_1(x) + \sin\frac{pa}{\hbar}E_2\psi_2(x)$$

and $\hat{H}\hat{T}(a)\psi_1(x) = \hat{T}(a)\hat{H}\psi_1(x) = \hat{T}(a)E_1\psi_1(x) = \cos\frac{pa}{\hbar}E_1\psi_1(x) + \sin\frac{pa}{\hbar}E_1\psi_2(x)$

so $\sin\frac{pa}{\hbar}E_2\psi_2(x) = \sin\frac{pa}{\hbar}E_1\psi_2(x)$

$\Rightarrow E_1 = E_2 = E$.

these two states $\psi_1(x)$. $\psi_2(x)$ must be degenerate, they have the same eigenvalue of E.

Problem 2

For two spin-1/2 particles you can construct symmetric and antisymmetric states (the triplet and single combinations). For three spin-1/2 particles you can construct symmetric combinations, but no completely antisymmetric configuration is possible.

(a) Prove that statement. (Hint: one approach is to write down the most general linear combination. Require antisymmetry for exchanging particle 1 with particle 2. Then require antisymmetry for exchanging particle 2 with particle 3)

(b) Suppose you put three identical noninteracting spin-1/2 fermions in the square with infinite potential hard walls. What is the ground state for this system, its energy, and degeneracy?

(c) Show that your answer to (b), properly nomalized, can be written in the form

$$\Phi(1,2,3) = 3^{-1/2}[\Phi(1,2)\phi(3) - \Phi(1,3)\phi(2) + \Phi(2,3)\phi(1)] \tag{2.1}$$

where $\Phi(i,j)$ is the wave function of two particles in the $n=1$ spatial state and the singlet spin, and $\phi(i)$ is the wavefunction of the ith particle in the $n=2$ spin up state. Check that $\Phi(1,2,3)$ is antisymmetric under all three exchanges of two particles.

(a). For three $-\frac{1}{2}$ spin particles.

spin up: 1 , spin down : 0

so the wave function can be expanded as the follows:

$$|\psi\rangle_{1\cdot2\cdot3} = a_1|000\rangle + a_2|100\rangle + a_3|010\rangle + a_4|001\rangle + a_5|110\rangle + a_6|101\rangle + a_7|011\rangle + a_8|111\rangle$$

exchang the 1st and 2nd particles: $|\psi\rangle_{1\cdot2\cdot3} = -|\psi\rangle_{2\cdot1\cdot3}$

$$|\psi\rangle_{2\cdot1\cdot3} = a_1|000\rangle + a_2|010\rangle + a_3|100\rangle + a_4|001\rangle + a_5|110\rangle + a_6|011\rangle + a_7|101\rangle + a_8|111\rangle$$

so $a_1 = -a_1$, $a_3 = -a_2$, $a_4 = -a_4$. $a_5 = -a_5$, $a_7 = -a_6$, $a_8 = -a_8$

$\Rightarrow a_1 = a_4 = a_5 = a_8 = 0$.

hence. $|\psi\rangle_{1\cdot2\cdot3} = a_2|010\rangle - a_2|100\rangle + a_6|011\rangle - a_6|101\rangle$

exchang the 2nd and 3rd particles: $|\psi\rangle_{1.2.3} = -|\psi\rangle_{1.3.2}$

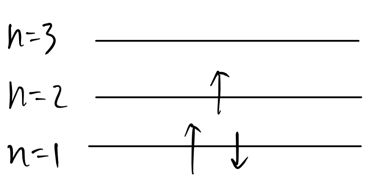
$$|\psi\rangle_{1.3.2} = a_2|001\rangle - a_2|100\rangle + a_6|011\rangle - a_6|110\rangle$$

so $a_2 = -a_2$. $a_6 = -a_6$

so it's no possible to construct an antisymmetric state for three $-\frac{1}{2}$ spin particles.

(b). the eigenvalue and eigenfunction of a particle in an infinite well:

$$\begin{cases} E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2} & (n=1.2,\cdots) \\ \psi_n(x) = \sqrt{\dfrac{2}{a}}\sin\dfrac{n\pi}{a}x & (n=1.2.\cdots) \end{cases}$$



there are totally three fermions in a square well, so there are two particles in ground states, one is spin up. and the other is spin down. the 3rd particle is in the first excited state, spin up or spin down.

The total energy for the system: $E_{tot} = E_1 + E_1 + E_2 = \frac{3\pi^2\hbar^2}{ma^2}$

and the dengeneracy of ground state is 2.

When the spin in the first excited state is $\uparrow$:

$$|\psi\rangle_{1,2,3} = \frac{1}{\sqrt{6}}\Big\{\left[\psi_1(x_1\uparrow)\psi_1(x_2\downarrow)\psi_2(x_3\uparrow) - \psi_1(x_1\downarrow)\psi_1(x_2\uparrow)\psi_2(x_3\uparrow)\right]$$
$$+\left[\psi_1(x_2\uparrow)\psi_1(x_3\downarrow)\psi_2(x_1\uparrow) - \psi_1(x_2\downarrow)\psi_1(x_3\uparrow)\psi_2(x_1\uparrow)\right]$$
$$-\left[\psi_1(x_1\uparrow)\psi_1(x_3\downarrow)\psi_2(x_2\uparrow) - \psi_1(x_1\downarrow)\psi_1(x_3\uparrow)\psi_2(x_2\uparrow)\right]\Big\}$$

When the spin in the first excited state is $\downarrow$:

$$|\psi\rangle_{1,2,3} = \frac{1}{\sqrt{6}}\Big\{\left[\psi_1(x_1\uparrow)\psi_1(x_2\downarrow)\psi_2(x_3\downarrow) - \psi_1(x_1\downarrow)\psi_1(x_2\uparrow)\psi_2(x_3\downarrow)\right]$$
$$+\left[\psi_1(x_2\uparrow)\psi_1(x_3\downarrow)\psi_2(x_1\downarrow) - \psi_1(x_2\downarrow)\psi_1(x_3\uparrow)\psi_2(x_1\downarrow)\right]$$
$$-\left[\psi_1(x_1\uparrow)\psi_1(x_3\downarrow)\psi_2(x_2\downarrow) - \psi_1(x_1\downarrow)\psi_1(x_3\uparrow)\psi_2(x_2\downarrow)\right]\Big\}$$

(c). $\phi(i)$ is the ith particle in the n=2 spin up state.

$$\Phi(1.2.3)=3^{-\frac{1}{2}}[\Phi(1.2)\phi(3)-\Phi(1.3)\phi(2)+\Phi(2.3)\phi(1)].$$

for two spin-$\frac{1}{2}$ particles there are antisymmetric states.

which means $\Phi(1.2)=-\Phi(2.1)$. $\Phi(1.3)=-\Phi(3.1)$

$$\Phi(2.3)=-\Phi(3.2).$$

$$\Phi(2.1.3)=3^{-\frac{1}{2}}[\Phi(2.1)\phi(3)-\Phi(2.3)\phi(1)+\Phi(1.3)\phi(2)].$$
$$=3^{-\frac{1}{2}}[-\Phi(1.2)\phi(3)+\Phi(1.3)\phi(2)-\Phi(2.3)\phi(1)]$$
$$=-\Phi(1.2.3).$$

$$\Phi(3.2.1)=3^{-\frac{1}{2}}[\Phi(3.2)\phi(1)-\Phi(3.1)\phi(2)+\Phi(2.1)\phi(3)].$$
$$=3^{-\frac{1}{2}}[-\Phi(1.2)\phi(3)+\Phi(1.3)\phi(2)-\Phi(2.3)\phi(1)]$$
$$=-\Phi(1.2.3)$$

$$\Phi(1.3.2)=3^{-\frac{1}{2}}[\Phi(1.3)\phi(2)-\Phi(1.2)\phi(3)+\Phi(3.2)\phi(1)].$$
$$=3^{-\frac{1}{2}}[-\Phi(1.2)\phi(3)+\Phi(1.3)\phi(2)-\Phi(2.3)\phi(1)]$$
$$=-\Phi(1.2.3)$$

so $\Phi(1.2.3)$ is antisymmetric

Problem 3

Two-level-systems (TLS) are common quantum mechanical objects in amorphous solids that arises from random disorders of the atomic lattices. It can be modeled as two minima in a double well potential which are separated by a barrier. At low temperatures, the dynamics of the TLS are governed by quantum tunneling through the barrier.

(a) Suppose a typical potential wells of a TLS as show in Fig. 3.1. The difference between the two ground-state eigen energies of the two wells is ϵ. The matrix element for the tunneling through the barrier is $\langle L|H_T|R\rangle = \Delta/2$. Find the two lowest energy eigenstates and eigenenergies. Treating the TLS as a qubit, what is its transition frequency?

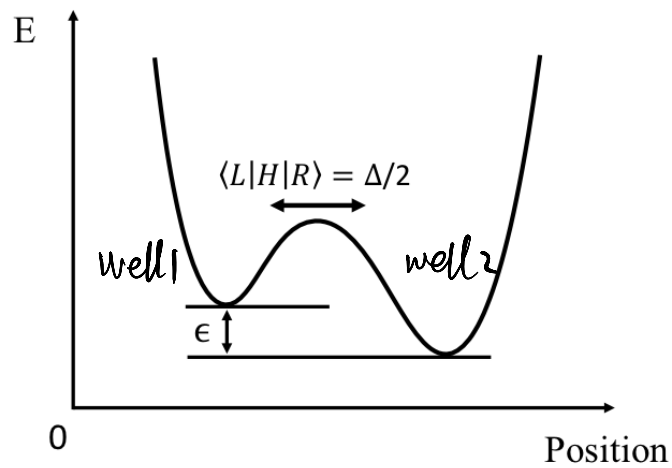
(b) The tunneling energy Δ is a function of the barrier height V, barrier width d and the effective mass of the particle m:

$$\Delta = \hbar\omega_o e^{-\lambda}, \tag{3.1}$$

Where $\lambda = \sqrt{2mV/\hbar^2}d$

Due to its random nature, ensembles of TLS exhibit a range of parameters in λ and ϵ with uniform probability. That is $P(\epsilon, \Delta)d\epsilon d\lambda = P_0 d\epsilon d\lambda$. Assuming there is a minimum tunneling energy Δ_0, find the density of states of TLS, $D(E)$ by integrating over all possible Δ values. Here E is the transition energy of the TLS.

(c) In superconducting circuits, TLS are commonly present in the material interfaces within the Josephson junction. This causes a coupling of superconducting transmon qubit to a bath of TLSs. Given a SC qubit (of frequency ω_o) to TLS coupling matrix element of J_{TLS}, use Fermi's golden rule to calculate the decay rate of a SC qubit initially in the excited state due to resonant interaction with the TLSs.



(a). Hamiltonian:

$$\hat{H} = \begin{pmatrix} \frac{\varepsilon}{2} & \langle L|H_T|R\rangle \\ \langle L|H_T|R\rangle & -\frac{\varepsilon}{2} \end{pmatrix}$$

$$\hat{H}|\psi\rangle = \hat{E}I|\psi\rangle$$

$$(H - EI)|\psi\rangle = 0 \Rightarrow \begin{vmatrix} \frac{\varepsilon}{2} - E & \frac{\Delta}{2} \\ \frac{\Delta}{2} & -\frac{\varepsilon}{2} - E \end{vmatrix} = 0 \Rightarrow E = \frac{\pm\sqrt{\varepsilon^2 + \Delta^2}}{2}$$

accordingly. eigenstates: from HW-1.

$$|\psi\rangle = \frac{\left[\left(-\frac{\varepsilon}{\lambda} \pm \sqrt{\varepsilon^2/\lambda^2+1}\right)|L\rangle + |R\rangle\right]}{\left(-\frac{\varepsilon}{\lambda} \pm \sqrt{\frac{\varepsilon^2}{\lambda^2}+1}\right)^2 + 1}$$

the transition frequency of TLS is the energy difference: $\sqrt{\varepsilon^2+\Delta^2}$

(b). the number of states in $(E, E+dE)$ is $D(E)dE$. the uniform probability

$P(\varepsilon, \Delta)\, d\varepsilon\, d\lambda = P_0\, d\varepsilon\, d\lambda$. so

$$\begin{cases} \Delta = \hbar w_0 e^{-\lambda} \\ \lambda = \sqrt{\frac{2mV}{\hbar^2}}\, d \end{cases} \Rightarrow d\Delta = -\hbar w_0 e^{-\lambda} d\lambda$$

$$E = \sqrt{\varepsilon^2+\Delta^2} \Rightarrow dE = \frac{1}{2}\frac{2\varepsilon}{\sqrt{\varepsilon^2+\Delta^2}} d\varepsilon = \frac{\varepsilon\, d\varepsilon}{E}$$

$$\text{so } d\varepsilon = \frac{E\,dE}{\varepsilon} = \frac{E\,dE}{\sqrt{E^2-\Delta^2}}$$

integrating over all possible Δ: (minimum tunneling energy $=\Delta_0$)

$$D(E) = \int D(E)dE = \int P(\varepsilon, \Delta)\, d\varepsilon\, d\lambda = \int P_0 \frac{E\,dE}{\sqrt{E^2-\Delta^2}} \frac{-d\Delta}{\Delta}$$

$$= P_0 \int dE \int_{\Delta_0}^{\Delta_t} \frac{E}{\Delta\sqrt{E^2-\Delta^2}} d\Delta$$

c). Considering Fermi's Golden Rule. the decay rate of the SC qubit at initial time is:

$$R = \frac{2\pi}{\hbar} |\langle \phi_f | \hat{H} | \phi_i \rangle|^2 \delta(\omega_{fi} - \omega) = \frac{2\pi}{\hbar} J_{TLS}^2 \delta(\omega_0 - \omega)$$

J_{TLS} is TLS coupling matrix elements