Problem 1

When performing spin measurements in a traditional spin resonance experiment, there are several steps, which you can now follow with the tools from class. The initial situation is that you have a large number N of spins (each spin 1/2), which are placed in a large magnetic field B_0 (which, for convenience, we will assume is parallel to $\hat{z}$). Let's measure that spin polarization.

(a) We wish to write the density matrix for the initial configuration. This is a distribution in thermal equilibrium, so the relative probability of occupation of states with different energies (E_1 and E_2) is $P(E_1)/P(E_2) = \exp(-(E_1 - E_2)/k_B T)$, where k_B is the Boltzmann constant and T is the temperature. Let's choose the normalization for the density matrix such that $\mathrm{Tr}\rho = N$. What is the density matrix for this initial configuration?

(b) Now a $\pi/2$ pulse is applied (along an axis perpendicular to the static field, so for concreteness choose along the $\hat{x}$ direction) using a field on resonance with the spin-down to spin-up transition. What is the new density matrix?

(c) Calculate the time-dependent magnetization $\boldsymbol{M}(t)$ from the sample. This time-dependent magnetization is traditionally measured using a microwave pickup coil. Note: $\boldsymbol{M}(t) = N\boldsymbol{a}(t)$

(d) Let's put in some numbers. Assume the spin-1/2 object is the nuclear spin of a hydrogen nucleus (proton), and the density of hydrogen is that corresponding to liquid water at its maximum density (1g/cc). What is the time-dependent magnetization in a field of 1T at a temperature of 300K?

(e) Now hyperpolarize the hydrogen nuclei to 50%. What is the time-dependent magnetization now (after following steps (b) and (c))

(a). the magnetic field // $\vec{z}$: $\vec{B} = B_0 \vec{z}$, $\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$

the hamiltonian of spin ensemble: $\hat{H} = -g\vec{\mu}\cdot\vec{B} = g\frac{\mu_B}{\hbar}\hat{S}_z\cdot\vec{B} = \frac{\mu_B g B_0}{2}\hat{\sigma}_z$

for thermal equilibrium state: $\rho = a\exp(-\frac{H}{k_B T})$. a is normalization constant.

spin up along z: $|\uparrow_z\rangle$. spin down along z: $|\downarrow_z\rangle$.

so the pure state: $|\uparrow_z\rangle\langle\uparrow_z|$. $|\downarrow_z\rangle\langle\downarrow_z|$

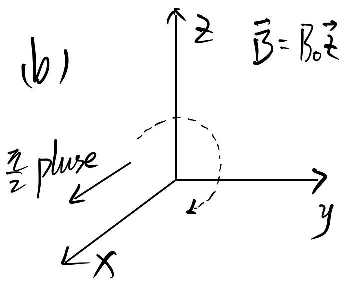
so the density operator:

$$\rho = \sum_i W_i|\psi_i\rangle\langle\psi_i| = P_\uparrow|\uparrow\rangle\langle\uparrow| + P_\downarrow|\downarrow\rangle\langle\downarrow| = a\exp\left(\frac{\mu_B g B_0}{2k_B T}\right)|\uparrow_z\rangle\langle\uparrow_z| + a\exp\left(\frac{-\mu_B g B_0}{2k_B T}\right)|\downarrow_z\rangle\langle\downarrow_z|$$

considering $\mathrm{Tr}(\hat{\rho}) = N$:

$$a\exp\left(\frac{\mu_B g B_0}{2k_B T}\right) + a\exp\left(\frac{-\mu_B g B_0}{2k_B T}\right) = N \Rightarrow a = \frac{N}{2\cosh\left(\frac{\mu_B g B_0}{2k_B T}\right)}$$

(b)



because a $\frac{\pi}{2}$ pulse is along x-axis. so the spin state $|\uparrow_z\rangle$ will have a clockwise from z-axis to y-axis. hence the pure states are $|\uparrow_y\rangle\langle\uparrow_y|$ and $|\downarrow_y\rangle\langle\downarrow_y|$.

so the density operator:

$$\rho=\sum_i w_i|\psi_i\rangle\langle\psi_i| = a\exp\left(\frac{\mu_B g B_0}{2k_BT}\right)|\uparrow_y\rangle\langle\uparrow_y| + a\exp\left(\frac{-\mu_B g B_0}{2k_BT}\right)|\downarrow_y\rangle\langle\downarrow_y|$$

where $a=\dfrac{N}{2\cosh\left(\frac{\mu_B g B_0}{2k_BT}\right)}$

(c). Spin $\frac{1}{2}$ particle in a $\vec{B}$ field.

from the dynamics of ρ:

$$\begin{cases} \dot{a}_x = -\dfrac{g\mu_B B_0}{\hbar}a_y \\ \dot{a}_y = \dfrac{g\mu_B B_0}{\hbar}a_x \\ \dot{a}_z = 0 \end{cases} \Rightarrow \begin{cases} a_x(t) = a_x(0)\sin\left(\dfrac{g\mu_B B_0}{\hbar}t\right) \\ a_y(t) = -a_y(0)\cos\left(\dfrac{g\mu_B B_0}{\hbar}t\right) \\ a_z(t) = a_z(0) \end{cases}$$

according to problem (b) at $t=0$. the Bloch vector is:

$$\vec{a}(t)=\vec{a}(0)=\left(0,\tanh\frac{g\mu_B B_0}{2k_BT},0\right)$$

so we have $\vec{a}(t)=\vec{a}_x(t)+\vec{a}_y(t)+\vec{a}_z(t)$

$$=\tanh\frac{g\mu_B B_0}{2k_BT}\sin\left(\frac{g\mu_B B_0}{\hbar}t\right)-\tanh\frac{g\mu_B B_0}{2k_BT}\cos\left(\frac{g\mu_B B_0}{\hbar}t\right)$$

so $\vec{M}(t) = Na(t) = N\tanh\frac{g\mu_B B_0}{2k_BT}\sin\left(\frac{g\mu_B B_0}{\hbar}t\right) - N\tanh\frac{g\mu_B B_0}{2k_BT}\cos\left(\frac{g\mu_B B_0}{\hbar}t\right)$

(d). $N_A = 6.02\times10^{23}$ atoms/mol

the # Hydrogen per cm^3:

$$\frac{1g}{18g/mol}\times\frac{6.02\times10^{23}\text{ atoms}}{mol}\times 2 \simeq 6.69\times10^{22} \text{ Hydrogen atoms}$$

$$\frac{g\mu_B B_0}{\hbar} = 2.7\times10^8\ Hz$$

$$\tanh\frac{g\mu_B B_0}{2k_BT} \simeq 3.4\times10^{-6}$$

so the time-dependent magnetization:

$$\vec{M}(t) = Na(t) = 6.69\times10^{22}/cm^3 \times 3.4\times10^{-6}\left\{\sin[(2.7\times10^8 H_z)t] - \cos[(2.7\times10^8 H_z)t]\right\}$$

$$= 2.27\times10^{17}/cm^3 \cdot \left\{\sin[(2.7\times10^8 H_z)t] - \cos[(2.7\times10^8 H_z)t]\right\}$$

(e). with hyperpolarize the hydrogen nuclei to 50%.

the density operator: $\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

after the first $\frac{\pi}{2}$ rotation. $\rho = \begin{pmatrix} \frac{1}{2} & \frac{i}{4} \\ -\frac{i}{4} & \frac{1}{2} \end{pmatrix}$

hence at $t=0$. $\vec{a}(0)=(0,-\frac{1}{2},0)$. from $\begin{cases} \dot{a}_x = -\frac{g\mu_B B_0}{\hbar} a_y \\ \dot{a}_y = \frac{g\mu_B B_0}{\hbar} a_x \\ \dot{a}_z = 0 \end{cases}$

we have $\vec{a}(t)=(\frac{1}{2}\sin g\mu_B B_0 t, -\frac{1}{2}\cos g\mu_B B_0 t, 0)$

$\vec{M}(t)=N\vec{a}(t)=N(\frac{1}{2}\sin g\mu_B B_0 t, -\frac{1}{2}\cos g\mu_B B_0 t, 0)$

Problem 2

There are various tricks to implement resonance conditions that avoid using a time-dependent magnetic field at microwave frequencies. A popular approach is to use electric fields, which can be localized far more easily. An approach is to use g-tensor modulation resonance, in which the g-tensor depends on an applied electric field. Consider a case in which $g_z(\mathcal{E}) = g_z(0) - \alpha_z \mathcal{E}$ and $g_x(\mathcal{E}) = g_x(0) + \alpha_x \mathcal{E}$. Both α_x and α_z are positive, as are $g_x(0)$ and $g_z(0)$. What is the directon a magnetic field should be applied (in the $x - z$ plane) in order to generate the largest Rabi oscillation frequency for a given magnetic field strength and a given oscillation electric field strength? (The average electric field is zero.) **Useful reading: Science 299, 1201 (2003). Hint: first construct the unperturbed Hamiltonian H_0 without modulation (electric field is off). Find the spin quantization axis by diagonalize H_0. Then the maximum Rabi frequency is obtained by maximizing the off-diagonal matrix elements.**

2. considering $\hat{H}(t)=\frac{\mu_B}{\hbar}\cdot\vec{S}\cdot\overleftrightarrow{g}\cdot\vec{B}=\vec{S}\cdot\vec{\Omega}(t)$

where $\Omega(t)$ is spin precession vector.

$$\hat{H} = \mu \vec{S}\cdot\overleftrightarrow{g}\cdot\vec{B} = \frac{\mu_B}{\hbar}(S_x\ S_y\ S_z)\cdot\begin{pmatrix} g_x(\varepsilon) & 0 & 0 \\ 0 & g_y(\varepsilon) & 0 \\ 0 & 0 & g_z(\varepsilon)\end{pmatrix}\begin{pmatrix} B_x \\ 0 \\ B_z\end{pmatrix}$$

$$= \frac{\mu_B}{\hbar} S_x g_x(\varepsilon)\vec{B}_x + \frac{\mu_B}{\hbar} S_z g_z(\varepsilon)\vec{B}_z$$

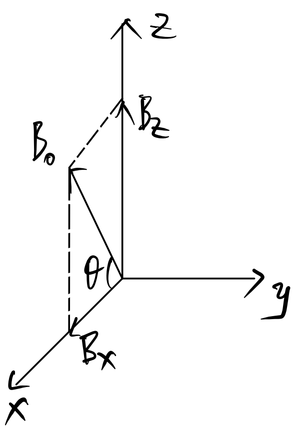
$$= \frac{\mu_B}{\hbar}\left[S_x(g_x(0)+\alpha_x\varepsilon)\vec{B}_x + S_z(g_z(0)-\alpha_z\varepsilon)\vec{B}_z\right]$$

$$= \frac{\mu_B}{\hbar}\left[S_x g_x(0)\vec{B}_x + S_z g_z(0)\vec{B}_z + S_x\alpha_x\varepsilon\vec{B}_x - S_z\alpha_z\varepsilon\vec{B}_z\right]$$

$$= \hat{H}_0 + \hat{H}_1(t)$$

so $\hat{H}_0 = \frac{\mu_B}{\hbar}\vec{S}\cdot\overleftrightarrow{g}\cdot\vec{B}_0 = \vec{S}\cdot\vec{\Omega}_0 \Rightarrow \vec{\Omega}_0 = \frac{\mu_B}{\hbar}(g_x(0)\vec{B}_x + g_z(0)\vec{B}_z)$

$\hat{H}_1(t) = \vec{S}\cdot\vec{\Omega}_1(t) \Rightarrow \vec{\Omega}_1(t) = \frac{\mu_B}{\hbar}(\alpha_x\varepsilon\vec{B}_x - \alpha_z\varepsilon\vec{B}_z)$



To generate the largest Rabi frequency, the $\vec{\Omega}_1(t)$ along axis perpendicular to $\vec{\Omega}_0$ should be the maximum: $\vec{\Omega}_0 = \frac{\mu_B}{\hbar}(g_x(0)\vec{B}_x + g_z(0)\vec{B}_z)$

$$= \frac{\mu_B}{\hbar} g_x(0) B_x\vec{x} + \frac{\mu_B}{\hbar} g_z(0) B_z\vec{z}$$

so we may choose $\vec{\Omega}_\perp$ that $\vec{\Omega}_\perp\cdot\vec{\Omega}_0 = 0$

$\vec{\Omega}_\perp = c\,(g_z(0)B_z\hat{x} - g_x(0)B_x\hat{z})$, c is normalization constant

so $c = \frac{1}{\sqrt{g_z^2(0)B_z^2 + g_x^2(0)B_x^2}}$

so $\vec{\Omega}_\perp \cdot \vec{\Omega}_0 = C \cdot \frac{\mu_B}{\hbar}\left[g_z(0) g_x(0) B_z B_x - g_x(0) g_z(0) B_x B_z \right] = 0$

hence. $\vec{\Omega}_1(t) \cdot \vec{\Omega}_\perp = \frac{\mu_B}{\hbar}(\alpha_x \varepsilon B_x \vec{x} - \alpha_z \varepsilon B_z \vec{z}) \cdot C\left(g_z(0) B_z \hat{x} - g_x(0) B_x \hat{z}\right)$

$$= \frac{\mu_B \varepsilon (\alpha_x B_x g_z(0) B_z + \alpha_z B_z g_x(0) B_x)}{\hbar \sqrt{g_z^2(0) B_z^2 + g_x^2(0) B_x^2}}$$

$B_x = B_0 \cos\theta$. $B_z = B_0 \sin\theta$

$$\vec{\Omega}_1(t) \cdot \vec{\Omega}_\perp = \frac{\mu_B \varepsilon}{\hbar} \frac{[\alpha_x g_z(0) + \alpha_z g_x(0)] B_0^2 \sin\theta \cos\theta}{B_0 \sqrt{g_z^2(0) \sin^2\theta + g_x^2(0) \cos^2\theta}}$$

$$= \mu_B \varepsilon [\alpha_x g_z(0) + \alpha_z g_x(0)] \frac{B_0 \sin\theta \cos\theta}{\sqrt{g_z^2(0) \sin^2\theta + g_x^2(0) \cos^2\theta}}$$

the denominator: $g_z^2(0) \sin^2\theta + g_x^2(0) \cos^2\theta \geq 2\sqrt{g_z^2(0) \sin^2\theta \, g_x^2(0) \cos^2\theta}$

$$= 2 g_z(0) g_x(0) \sin\theta \cos\theta.$$

So the maximum of $\vec{\Omega}_1(t) \cdot \vec{\Omega}_\perp$ is

$$\mu_B \varepsilon [\alpha_x g_z(0) + \alpha_z g_x(0)] \frac{B_0 \sin\theta \cos\theta}{2 g_z(0) g_x(0) \sin\theta \cos\theta} = \frac{\mu_B \varepsilon B_0 [\alpha_x g_z(0) + \alpha_z g_x(0)]}{2 g_z(0) g_x(0)}$$

if and only if $g_z^2(0) \sin^2\theta = g_x^2(0) \cos^2\theta \Rightarrow \tan^2\theta = \frac{g_x^2(0)}{g_z^2(0)} \Rightarrow \tan\theta = \pm \frac{g_x(0)}{g_z(0)}$

Problem 3

Let's look at another case of an "unusual" two-state system. Consider a system with one spatial dimension ($\hat{y}$) but three directions for spin ($\hat{x}$, $\hat{y}$, $\hat{z}$). The confining potential in the $\hat{y}$ direction is that of a quantum well with width a. The spin-orbit interaction for this system is $\alpha \mathcal{E} p_y \sigma_x$, where α is a coefficient and $\mathcal{E}$ is the electric field strength. Note that this Hamiltonian does not depend on the electric field direction, so the electric field direction is assumed in constructing this Hamiltonian. The two-state system has as one state an electron in the lowest-energy spatial state of the quantum well and spin down, and the other state is an electron in the first excited spatial state of the quantum well and spin up. Use standard variables for the mass, g factor, magnetic moment of the electron. The spin directions here are in the $\hat{z}$ direction and a static magnetic field is applied parallel to $\hat{z}$ with strength B_0, with spin-down the lowest energy state.

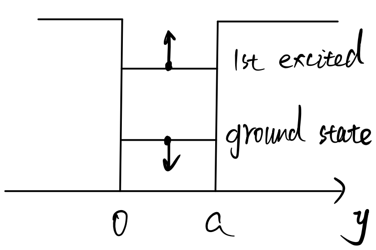
(a) If the electric field is oscillated at a specific frequency ω the system can be in resonance. What is that resonant frequency ω?

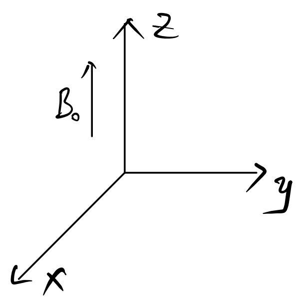
(b) In terms of α and the maximum electric field strength $\mathcal{E}$, what is the Rabi frequency?

(c) Starting with the system in the state of the lowest-energy spatial state and spin down, as a function of time, what is $\langle S_z(t) \rangle$?

(d) For the case of (c)), what is $\langle S_x(t) \rangle$ and $\langle S_y(t) \rangle$?

(e) Introduce a new operator $\hat{O} = y\sigma_y$. What is $\langle \hat{O}(t) \rangle$





(a). the eigenvalue and eigenfunction of quantum well:

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad (n=1,2,\cdots), \qquad \psi_n(y) = \sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}y \quad (n=1,2,\cdots)$$

$$n=1: E_1 = \frac{\pi^2\hbar^2}{2ma^2}, \quad n=2: E_2 = \frac{2\pi^2\hbar^2}{ma^2}$$

hence with magnetic field: $\vec{B} = B_0\hat{z}$. $\hat{H}' = \frac{\mu_B}{\hbar} g B_0 \hat{S}_z = \frac{\mu_B g B_0}{2}\hat{\sigma}_z$

so $E_1' = \frac{\pi^2\hbar^2}{2ma^2} - \frac{\mu_B g B_0}{2}$, $E_2' = \frac{2\pi^2\hbar^2}{ma^2} + \frac{\mu_B g B_0}{2}$

$$\Delta E = E_2' - E_1' = \frac{3\pi^2\hbar^2}{2ma^2} + \mu_B g B_0$$

(b). $\psi_1(y,\downarrow) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}y\right)\chi(\downarrow)$, $\psi_2(y,\uparrow) = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi}{a}y\right)\chi(\uparrow)$.

$$\langle \psi_1(y.\downarrow) | \alpha\varepsilon P_y \sigma_x | \psi_2(y.\uparrow) \rangle$$

$$= \left(\sqrt{\frac{2}{a}}\sin\frac{\pi}{a}y, \alpha\varepsilon\left(-i\hbar\frac{\partial}{\partial y}\right)\sqrt{\frac{2}{a}}\sin\frac{2\pi}{a}y\right)\cdot\langle \downarrow_z | \sigma_x | \uparrow_z \rangle$$

$$= \int_0^a \left(\sqrt{\frac{2}{a}}\sin\frac{\pi}{a}y\right)^* \cdot \alpha\varepsilon\left(-i\hbar\frac{\partial}{\partial y}\right)\sqrt{\frac{2}{a}}\sin\frac{2\pi}{a}y\, dy \cdot (0\ 1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{-2i\hbar\alpha\varepsilon}{a}\int_0^a \sin\frac{\pi}{a}y\left(\frac{2\pi}{a}\right)\cos\frac{2\pi}{a}y\, dy$$

$$= \frac{-4\pi i\hbar\alpha\varepsilon}{a^2}\int_0^a \sin\frac{\pi}{a}y\left(1-2\sin^2\frac{\pi}{a}y\right)dy$$

$$= \frac{8i\hbar\alpha\varepsilon}{3a}$$

so the Rabi frequency is $\Omega = \frac{8\alpha\varepsilon}{3a}$

(c). $\langle \hat{S}_z(t) \rangle$

We use rotating wave approximation to transform hamiltonian:

$$\hat{H} = \frac{\hat{P}_y^2}{2m} + \frac{\mu_B g B_0}{\hbar} \hat{S}_z + \alpha \varepsilon \hat{P}_y \sigma_x \cos W_d t$$

after transformation:

$$\hat{H}_{rot} = -(W - W_d)\hat{S}_z - \Omega \hat{S}_y = -\Delta \hat{S}_z - \Omega \hat{S}_y . \qquad \Omega \text{ is Rabi frequency.}$$

So the evolution:

$$T = \exp\left\{ \frac{1}{i\hbar} [-\Delta \hat{S}_z - \Omega \hat{S}_y] t \right\}$$

$$= \cos \frac{\sqrt{\Delta^2 + \Omega^2}}{2} t + i \sin \frac{\sqrt{\Delta^2 + \Omega^2}}{2} t \ \frac{1}{\sqrt{\Delta^2 + \Omega^2}} \begin{pmatrix} \Delta & -i\Omega \\ i\Omega & -\Delta \end{pmatrix}$$

in rotating frame.

$$|\psi(t)\rangle_{rot} = T(t) |\psi(0)\rangle$$

$$= \left[\cos \frac{\sqrt{\Delta^2 + \Omega^2}}{2} t + i \sin \frac{\sqrt{\Delta^2 + \Omega^2}}{2} t \ \frac{1}{\sqrt{\Delta^2 + \Omega^2}} \begin{pmatrix} \Delta & -i\Omega \\ i\Omega & -\Delta \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \cos \frac{\sqrt{\Delta^2 + \Omega^2}}{2} t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{\Delta^2 + \Omega^2}} \sin \frac{\sqrt{\Delta^2 + \Omega^2}}{2} t \begin{pmatrix} \Delta \\ i\Omega \end{pmatrix}$$

Let's transform back to lab frame by operator $\exp\left(\frac{1}{i\hbar}(-W_d \hat{S}_z) t\right)$

$$|\psi(t)\rangle_{lab} = \exp\left(\frac{1}{i\hbar}(-\omega_d \hat{S}_z)t\right)|\psi(t)\rangle_{rot}$$

$$= \begin{pmatrix} e^{\frac{i}{2}\omega_d t} & 0 \\ 0 & e^{-\frac{i}{2}\omega_d t} \end{pmatrix}\left[\cos\frac{\sqrt{\Delta^2+\Omega^2}}{2}t\begin{pmatrix}1\\0\end{pmatrix} + \frac{i}{\sqrt{\Delta^2+\Omega^2}}\sin\frac{\sqrt{\Delta^2+\Omega^2}}{2}t\begin{pmatrix}\Delta\\ i\Omega\end{pmatrix}\right]$$

$$= \cos\frac{\sqrt{\Delta^2+\Omega^2}}{2}t\begin{pmatrix} e^{\frac{i}{2}\omega_d t} \\ 0\end{pmatrix} + \frac{i}{\sqrt{\Delta^2+\Omega^2}}\sin\frac{\sqrt{\Delta^2+\Omega^2}}{2}t\begin{pmatrix}\Delta e^{\frac{i}{2}\omega_d t}\\ i\Omega e^{-\frac{i}{2}\omega_d t}\end{pmatrix}$$

$\hat{S}_z = -\frac{\hbar}{2}\hat{\sigma}_z$.

so $\langle S_z(t)\rangle = \langle\psi(t)|\hat{S}_z|\psi(t)\rangle_{lab} = -\frac{\hbar}{2}\cos^2\frac{\sqrt{\Delta^2+\Omega^2}}{2}t$

(d) Similar to (c) above.

$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x$. $\hat{S}_y = -\frac{\hbar}{2}\hat{\sigma}_y$

$$\langle S_x(t)\rangle = \frac{\hbar}{2}\left\{-\frac{\Omega}{\sqrt{\Delta^2+\Omega^2}}\cos\omega_d t\sin\sqrt{\Delta^2+\Omega^2}\,t + \frac{\Omega\Delta}{\Delta^2+\Omega^2}\sin\omega_d t\left[1-\cos\sqrt{\Delta^2+\Omega^2}\,t\right]\right\}$$

$$\langle S_y(t)\rangle = \frac{-\hbar}{2}\left\{\frac{\Omega}{\sqrt{\Delta^2+\Omega^2}}\sin\omega_d t\sin\sqrt{\Delta^2+\Omega^2}\,t + \frac{\Omega\Delta}{\Delta^2+\Omega^2}\cos\omega_d t\left[1-\cos\sqrt{\Delta^2+\Omega^2}\,t\right]\right\}$$

(e). the matrix of $\hat{O} = y\hat{\sigma}_y$ in the unusual two level system.

$$\hat{O} = y\hat{\sigma}_y = \begin{pmatrix} \langle 1\downarrow| y\sigma_y |1\downarrow\rangle & \langle 1\downarrow| y\sigma_y |2\uparrow\rangle \\ \langle 2\uparrow| y\sigma_y |1\downarrow\rangle & \langle 2\uparrow| y\sigma_y |2\uparrow\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i\langle 1|y|2\rangle \\ -i\langle 2|y|1\rangle & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{16ai}{9\pi^2} \\ \frac{16ai}{9\pi^2} & 0 \end{pmatrix} = \frac{16a}{9\pi^2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{16a}{9\pi^2}\hat{\sigma}_y$$

hence.

$$\langle \hat{O} \rangle = \frac{16a}{9\pi^2}\langle \hat{\sigma}_y \rangle = \frac{16a}{9\pi^2}\frac{\langle S_y(t)\rangle}{-\frac{\hbar}{2}}$$

$$= \frac{16a}{9\pi^2}\left\{ \frac{\Omega}{\sqrt{\Delta^2+\Omega^2}} \sin\omega_0 t \sin\sqrt{\Delta^2+\Omega^2}\, t + \frac{\Omega\Delta}{\Delta^2+\Omega^2}\cos\omega_0 t\left[1-\cos\sqrt{\Delta^2+\Omega^2}\, t\right]\right\}$$