Problem 1

The noise that occurs in a resistor has important implications for using the difference between two resistance states as the indication of the readout of a sensor. One situation where this is used is in the hard disk read head shown below, in which a magnetic region responds to the local magnetic field of the hard disk media and orients one way or the other (this is the so-called free region). Current flows through this magnetic region and also a magnetic region with a fixed magnetic orientation (so-called fixed region). The difference in resistance between the two magnetic configurations (parallel or antiparallel) is called the magnetoresistance (MR), and it is usually reported as a percentage of the resistance of the low-resistance (baseline) configuration (thus yielding percentages greater than 100% for some systems). For a period of time the community focused on these MR percentages without concerning themselves with the actual baseline resistance. For a sensor which is about 100 nanometers in linear dimension, analyze the tradeoff between MR, baseline resistance, and speed of reading out information from media. From this analysis you should find you want higher MR percentages for faster readout. Do you want higher resistance or lower resistance for the low-resistance configuration for faster readout?

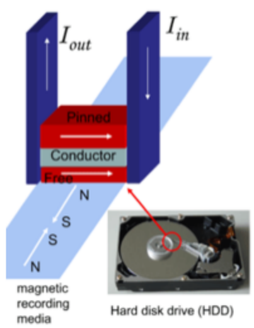


Figure 1.1: Hard disk read head

tunable current: $I_t = \frac{V}{R_t}$. where R_t is the tunable resistance.

fixed current: $I_f = \frac{V}{R_f}$, where R_f is the fixed resistance

hence. the signal of the readout of a sensor is the difference of these two

resistance: $S = \frac{V}{R_f} - \frac{V}{R_t} = V\left(\frac{R_t - R_f}{R_t R_f}\right)$

from the definition. $MR = \frac{R_t - R_f}{R_f}$, so we have $MR + 1 = \frac{R_t}{R_f}$

$$S = V\left(\frac{1-\frac{R_f}{R_t}}{R_f}\right) = V\left(\frac{1-\frac{1}{MR+1}}{R_f}\right) = \frac{V}{R_f}\frac{MR}{MR+1}$$

the Johoson noise $N_J = \sqrt{4\frac{k_B T \Delta f}{R_f}}$

signal to noise ratio $SNR = \frac{S}{N_J} = \frac{V}{\sqrt{4k_B T \Delta f R_f}}\frac{MR}{MR+1}$

Hence, to achieve faster readout, we need smaller baseline resistance R_f.

considering shot noise: $N_s = \sqrt{2eI_f\Delta f} = \sqrt{2e\Delta f\frac{V}{R_f}}$

$$SNR = \frac{S}{N_s} = \frac{\frac{V}{R_f}\frac{MR}{MR+1}}{\sqrt{2e\Delta f\frac{V}{R_f}}} = \sqrt{\frac{V}{2e\Delta f R_f}}\frac{MR}{MR+1}$$

we also need smaller baseline resistance R_f for faster readout.

Problem 2

Quantum satellite is a state-of-the-art technology to distribute quantum entanglement over large (>1000 km) geographical distance for secure quantum communication and distributed quantum computation. Here we consider a quantum satellite that is equipped with an onboard entangled photon pair source. The source generates pairs of polarization entangled photons in the state $(|HV\rangle + |VH\rangle)/\sqrt{2}$ at a rate of 1×10^7 λ_1 pairs per second. Then each of the entangled photon is beamed down from the satellite to two distant ground stations. We assume the distances from the satellite to each ground station are identical. The optical attenuation from the satellite to either ground station is 35 dB $10^{-3.5}$ due to absorption by the atmosphere and air turbulence. At each ground state, a single photon detector with 70% detection efficiency and 100 counts per second dark count is used to detect photons from the satellite. The time of arrival of photons is recorded

at both stations. Two detection events, one by each detector, occurred within a 5 ns time window is considered a coincidence event. Given these conditions, what is the coincidence rate (in units of # events per hour) that results from a pair of entangled photon pairs? What is the "false" coincidence rate (also called "accidental" coincidence) which are coincidence events not due to entangled photons? To correctly solve $\Sigma P = 1$ this problem, you should understand that the photon pairs are generated with a Poissonian statistics. The mean generation rate is 1×10^7 per second. But there are finite probabilities that zero or more than one pair of photons being generated as well. Due to significant loss during transmission and detection, there are false coincidences due to photons from the satellite but those photons are actually not entangled. Finally, you can assume there is no background photons from the ambient impinging on both detectors.

the photon pairs are generated with a Poissonian statistics.

the mean generation rate is 1×10^7 per second

within 5 ns time window: $1\times10^7 \times 5\times10^{-9} = 5\times10^{-2}$

so the probability of n_1 photon pairs generated within the time window:

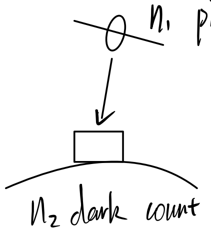
$$P(X=n_1) = \frac{e^{-\lambda_1}\lambda_1^{n_1}}{n_1!}, \quad \text{where } \lambda_1 = 0.05$$

Similarly, the dark count is also a Poissonian statistics.

100 counts per second. within the time window: $100\times5\times10^{-9} = 5\times10^{-7}$

so the probability of n_2 dark count within the time window:

$$P(Y=n_2)=\frac{e^{-\lambda_2}\lambda_1^{n_2}}{n_2!}\ , \text{ where } \lambda_2=5\times10^{-7}$$



the probability of n_1 photon pairs generated within the time window:

$P(n_1)=\dfrac{e^{-\lambda_1}\lambda_1^{n_1}}{n_1!}$, on the condition of n_1 photon pairs generated,

if the detector counts 1 photon, so there exactly 1 pair of entangled photon pairs detected.

from the conditional probability. $P(B|A)=\dfrac{P(AB)}{P(A)} \Rightarrow P(AB)=P(A)P(B|A)$

so within the time window: Δt.

the real count rate: $N_{\Delta t}=\sum_{n_1=0}^{\infty}P(n_1)P(\geq1|n_1)=\sum_{n_1=0}^{\infty}\dfrac{e^{-\lambda_1}\lambda_1^{n_1}}{n_1!}\left\{1-[1-(\alpha\eta)^2]^{n_1}\right\}$

$$N_{\Delta t}=\sum_{n_1=0}^{\infty}\frac{e^{-\lambda_1}\lambda_1^{n_1}}{n_1!}-\sum_{n_1=0}^{\infty}\frac{e^{-\lambda_1}\lambda_1^{n_1}}{n_1!}[1-(\alpha\eta)^2]^{n_1}$$

$$=1-\sum_{n_1=0}^{\infty}\frac{e^{-\lambda_1}}{n_1!}\left\{\lambda_1[1-(\alpha\eta)^2]\right\}^{n_1}$$

$$=1-e^{-\lambda_1(\alpha\eta)^2},$$

so the real count rate: $R_1=\dfrac{N_{\Delta t}}{\Delta t}$.

$\lambda_1 = 0.05$. $\alpha = 10^{-3.5}$. $\eta = 70\%$. $\Delta t = 5\,ns$. so $R_1 \simeq 1764/h$.

if we consider the photons detected on two ground stations

$$N_{\Delta t}^{tot} = \sum_{n_1=0}^{\infty} P(n_1) P_{tot}(\geq 1 | n_1) = \sum_{n_1=0}^{\infty} P(n_1) P^2(\geq 1 | n_1)$$

$$= \sum_{n_1=0}^{\infty} P(n_1) \left\{ 1 - P(n_1=0) + P(n_1=0)\left[1 - (1-\alpha\eta)^{n_1}\right] \right\}^2$$

$$= \sum_{n_1=0}^{\infty} P(n_1) \left[1 - e^{-\lambda_2}(1-\alpha\eta)^{n_1}\right]^2$$

$$= 1 - 2\sum_{n_1=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^{n_1}}{n_1!} e^{-\lambda_2}(1-\alpha\eta)^{n_1} + \sum_{n_1=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^{n_1}}{n_1!} e^{-2\lambda_2}(1-\alpha\eta)^{2n_1}$$

$$= 1 - 2e^{-(\lambda_2 + \lambda_1\alpha\eta)} + e^{\lambda_1[(\alpha\eta)^2 - 2\alpha\eta] - 2\lambda_2}$$

hence, the total count rate $R_{tot} = \frac{N_{\Delta t}^{tot}}{\Delta t}$

$\lambda_1 = 0.05$, $\lambda_2 = 5\times10^{-7}$, $\alpha = 10^{-3.5}$, $\eta = 70\%$. $\Delta t = 5\,ns$.

so $R_{tot} \simeq 1860/h$.

Therefore, the false coincidence rate is $R_{tot} - R_1 \simeq 96/h$.

Problem 3

The degree of entanglement between two spin-1/2 particles can be assessed by performing a partial trace over one particle and evaluating the "purity" of the remaining state, where higher purity means lower entanglement.

(a) Show that, for a general pure state of two identical spin-1/2 particles, that the purity of the remaining state (and thus the measure of entanglement) does not depend on whether you trace over particle #1 or #2.

(b) What is the purity of the remaining state, and thus the degree of entanglement, for the two particle state $|\uparrow\downarrow\rangle$?

(c) What is the purity of the remaining state, and thus the degree of entanglement, for the two particle state $1/\sqrt{2}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]$?

(d) What is the purity of the remaining state, and thus the degree of entanglement, for the two particle state $1/2[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle]$?

(e) Explain the result you obtain in (d).

(a). for two identical spin-$\frac{1}{2}$ particles. there are four states:

$|00\rangle, |01\rangle, |1,0\rangle, |1,1\rangle$

a general pure state: $|\psi\rangle = \psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{10}|10\rangle + \psi_{11}|11\rangle$

the density matrix $\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \psi_{00}\psi_{00}^* & \psi_{00}\psi_{01}^* & \psi_{00}\psi_{10}^* & \psi_{00}\psi_{11}^* \\ \psi_{01}\psi_{00}^* & \psi_{01}\psi_{01}^* & \psi_{01}\psi_{10}^* & \psi_{01}\psi_{11}^* \\ \psi_{10}\psi_{00}^* & \psi_{10}\psi_{01}^* & \psi_{10}\psi_{10}^* & \psi_{10}\psi_{11}^* \\ \psi_{11}\psi_{00}^* & \psi_{11}\psi_{01}^* & \psi_{11}\psi_{10}^* & \psi_{11}\psi_{11}^* \end{pmatrix}$

if we trace out particle 1:

$$\rho_2 = \mathrm{Tr}_1 |\psi\rangle\langle\psi| = \begin{pmatrix} |\psi_{00}|^2 + |\psi_{10}|^2 & \psi_{00}\psi_{01}^* + \psi_{10}\psi_{11}^* \\ \psi_{01}\psi_{00}^* + \psi_{11}\psi_{10}^* & |\psi_{01}|^2 + |\psi_{11}|^2 \end{pmatrix}$$

trace out particle 2:

$$\rho_1 = \mathrm{Tr}_2 |\psi\rangle\langle\psi| = \begin{pmatrix} |\psi_{00}|^2 + |\psi_{01}|^2 & \psi_{00}\psi_{10}^* + \psi_{01}\psi_{11}^* \\ \psi_{10}\psi_{00}^* + \psi_{11}\psi_{01}^* & |\psi_{10}|^2 + |\psi_{11}|^2 \end{pmatrix}$$

hence.

$$\mathrm{Tr}\{\rho_1^2\} = (|\psi_{00}|^2 + |\psi_{01}|^2)^2 + 2(\psi_{00}\psi_{10}^* + \psi_{01}\psi_{11}^*)(\psi_{10}\psi_{00}^* + \psi_{11}\psi_{01}^*) + (|\psi_{10}|^2 + |\psi_{11}|^2)^2$$

$$\mathrm{Tr}\{\rho_2^2\} = (|\psi_{00}|^2 + |\psi_{10}|^2)^2 + 2(\psi_{00}\psi_{01}^* + \psi_{10}\psi_{11}^*)(\psi_{01}\psi_{00}^* + \psi_{11}\psi_{10}^*) + (|\psi_{01}|^2 + |\psi_{11}|^2)^2$$

and $\mathrm{Tr}\{\rho_1^2\} = \mathrm{Tr}\{\rho_2^2\}$

(b). $\psi_{00} = \psi_{10} = \psi_{11} = 0$. $\psi_{01} = 1$.

so we have $\mathrm{Tr}\{\rho_1^2\} = 1$. so there is no entanglement in state $|\uparrow\downarrow\rangle$

(c). $\psi_{00} = \psi_{11} = 0$, $\psi_{01} = \psi_{10} = \frac{1}{\sqrt{2}}$

so we have $\mathrm{Tr}\{\rho_1^2\} = \frac{1}{2}$. which is the maximum entangled state of two identical particles.

(d). $\psi_{00} = \psi_{11} = -\frac{1}{2}$, $\psi_{01} = \psi_{10} = \frac{1}{2}$.

so we have $Tr\{\rho_1^2\} = 1$. so there is no entanglement

(e). $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)/2$ can be expressed as a direct product between two particles:

$$\frac{i}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle) \otimes \frac{i}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

$$= \frac{-1}{2}|\uparrow\rangle\otimes|\uparrow\rangle - \frac{1}{2}|\downarrow\rangle\otimes|\downarrow\rangle + \frac{1}{2}|\uparrow\rangle\otimes|\downarrow\rangle + \frac{1}{2}|\downarrow\rangle\otimes|\uparrow\rangle$$

$$= (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)/2$$

so there is no entanglement