Problem 1

Calculate the value of the uncertainty (i.e. $\Delta x \Delta p$) for a coherent (Glauber) state with parameter α. Compare this with the uncertainty of the ground state of the harmonic oscillator, and with uncertainty of the first excited state of the harmonic oscillator.

the harmonic oscillator potential: $V(x) = \frac{1}{2} m w^2 x^2$

so, $H\psi = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m w^2 x^2\right)\psi = E\psi$

we have annihilation operator: $a = \frac{1}{\sqrt{2mw\hbar}}(mwx + ip)$

creation operator: $a^\dagger = \frac{1}{\sqrt{2mw\hbar}}(mwx - ip)$

so x and p can be expressed by a and a^+:

$$x = \sqrt{\frac{\hbar}{2mw}}(a + a^+). \quad p = -i\sqrt{\frac{mw\hbar}{2}}(a - a^+)$$

$$aa^\dagger = \frac{1}{\sqrt{2mw\hbar}}(mwx + ip)\frac{1}{\sqrt{2mw\hbar}}(mwx - ip) = \frac{1}{2mw\hbar}\left[(mwx)^2 - mwx\, ip + ip\, mwx + p^2\right]$$

$$= \frac{1}{2mw\hbar}\left[(mwx)^2 - mwx\, ip + ip\, mwx + p^2\right]$$

$$= \frac{1}{2mw\hbar}\left[(mwx)^2 + p^2 + mwi[p.x]\right]$$

$$a^\dagger a = \frac{1}{\sqrt{2mw\hbar}}(mwx - ip)\frac{1}{\sqrt{2mw\hbar}}(mwx + ip)$$

$$= \frac{1}{2mw\hbar}\left[(mwx)^2 + p^2 - mwi[p.x]\right]$$

$[p.x] = -i\hbar$. so $[a, a^+] = aa^+ - a^+a = 1$

define: $N = a^\dagger a$ occupation/particle number operator.

so $[N, a^+] = a^+$. $[N, a] = -a$.

$$x = \sqrt{\frac{\hbar}{2mw}}(a + a^+). \quad p = -i\sqrt{\frac{mw\hbar}{2}}(a - a^+)$$

$\Rightarrow \langle x\rangle = \langle \psi_n|x|\psi_n\rangle \propto \langle \psi_n|a+a^+|\psi_n\rangle \propto \langle \psi_n|\psi_{n-1}\rangle + \langle \psi_n|\psi_{n+1}\rangle = 0.$

$\langle x^2\rangle = \langle \psi_n|x^2|\psi_n\rangle = \frac{\hbar}{2m\omega}\langle \psi_n|a^2+aa^+ + a^+a + (a^+)^2|\psi_n\rangle$

because $[a,a^+] = aa^+ - a^+a = 1$. so $aa^+ = 1 + a^+a$

$\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle \psi_n|aa^+ + a^+a|\psi_n\rangle = \frac{\hbar}{2m\omega}\langle \psi_n|2a^+a+1|\psi_n\rangle = \frac{\hbar}{2m\omega}\langle \psi_n|2N+1|\psi_n\rangle$

$= \frac{\hbar}{m\omega}\langle \psi_n|N|\psi_n\rangle + \frac{\hbar}{2m\omega} = \frac{\hbar}{m\omega}(n+\frac{1}{2})$

so $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{m\omega}(n+\frac{1}{2})$

$\Rightarrow \langle p\rangle = \langle \psi_n|p|\psi_n\rangle \propto \langle \psi_n|a-a^+|\psi_n\rangle \propto \langle \psi_n|\psi_{n-1}\rangle - \langle \psi_n|\psi_{n+1}\rangle = 0$

$\langle p^2\rangle = \langle \psi_n|p^2|\psi_n\rangle \propto -\frac{m\omega\hbar}{2}\langle \psi_n|a^2 - aa^+ - a^+a + (a^+)^2|\psi_n\rangle$

$= -\frac{m\omega\hbar}{2}\langle \psi_n|-2N-1|\psi_n\rangle$

$= m\omega\hbar(n+\frac{1}{2})$

$\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 = m\omega\hbar(n+\frac{1}{2})$

so $\Delta x\Delta p = \hbar(n+\frac{1}{2}) \geq \frac{\hbar}{2}$

for ground state $(n=0)$. $|0\rangle$: $\Delta x\Delta p = \frac{\hbar}{2}$

for first excited state $|1\rangle$: $\Delta x\Delta p = \frac{3}{2}\hbar$

Problem 2

Explain how a device which upon input of one of two non-orthogonal quantum states $|\psi\rangle$ or $|\phi\rangle$ correctly identified the state, could be used to build a device which cloned the states $|\psi\rangle$ and $|\phi\rangle$, in violation of the no-cloning theorem. Conversely, explain how a device for cloning could be used to distinguish non-orthogonal quantum states.

If there is a device that can correctly identify the state, then we could find the properties assigned to both states, and clone the two states as required. Conversely, if we have a cloning device that can be used to generate multiple replicas of the two states, then we may apply POVM theory with measurement operators $\{E_1 = |\psi^-\rangle\langle\psi^-|,\ E_2 |\phi^-\rangle\langle\phi^-|,\ E_3 = 1 - E_1 - E_2\}$.

if we measure E_1, so the state is bound to be $|\phi\rangle$, if we measure E_2, then the state is $|\psi\rangle$. Since we have many replicas, it's likely to get outcomes of either E_1 or E_2.

for instance, $|\uparrow_z\rangle$ and $|\uparrow_y\rangle$ are non-orthogonal.

$$\hat{E}_1 = \frac{|\downarrow_z\rangle\langle\downarrow_z|}{1+\frac{1}{\sqrt{2}}}.\quad \hat{E}_2 = \frac{|\downarrow_y\rangle\langle\downarrow_y|}{1+\frac{1}{\sqrt{2}}}.\quad \hat{E}_3 = \hat{1} - \hat{E}_1 - \hat{E}_2 = \hat{1} - \frac{|\downarrow_z\rangle\langle\downarrow_z|}{1+\frac{1}{\sqrt{2}}} - \frac{|\downarrow_y\rangle\langle\downarrow_y|}{1+\frac{1}{\sqrt{2}}}$$

$\hat{E}_1 = \frac{|\downarrow_z\rangle\langle\downarrow_z|}{1+\frac{1}{\sqrt{2}}}$. if $\hat{E}_1$ is the resule, so it can't be $|\uparrow_z\rangle$. another possibility is $|\uparrow_y\rangle$.

$\hat{E}_2 = \frac{|\downarrow_y\rangle\langle\downarrow_y|}{1+\frac{1}{\sqrt{2}}}$. if $\hat{E}_2$ is the resule, so it can't be $|\uparrow_y\rangle$. another possibility is $|\uparrow_z\rangle$.

$\hat{E}_1$. $\hat{E}_2$ are deterministic.

Problem 3

(a) Show that the average value of the observable $\sigma_v\sigma_u$ for a two-qubit system in the state $|S\rangle$ is $-\vec{v}\cdot\vec{u}$.

(b) Calculate the average value of the same observable for the state $|T_0\rangle$. (Hint: you can utilize the fact that $|S\rangle\langle S| + |T_0\rangle\langle T_0| = |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|$)

(a). $|S\rangle = |0,0\rangle = \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}$

the density matrix for the quantum state is $\rho = |S\rangle\langle S|$

$$|S\rangle = \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

$$\text{so } \rho = |S\rangle\langle S| = \frac{1}{2}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{pmatrix}$$

if $v \neq u$. for example. $v = y$. $u = z$.

$$\sigma_v\sigma_u = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&-i&0&0\end{pmatrix}$$

the average value of $\sigma_v\sigma_u$ is given by $\mathrm{Tr}\{\rho\sigma_v\sigma_u\} = 0 = -\vec{v}\cdot\vec{u}$

if $v = u$. for example. $v = u = z$

$$\sigma_v\sigma_u = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

$$\mathrm{Tr}\{\rho\sigma_v\sigma_u\} = \mathrm{Tr}\begin{pmatrix}0&0&0&0\\0&-1&1&0\\0&1&-1&0\\0&0&0&0\end{pmatrix} = -2 = -\vec{v}\cdot\vec{u}$$

it's the same with other scenarios.

(d) $|T_0\rangle = |10\rangle = \dfrac{(\uparrow\downarrow + \downarrow\uparrow)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle\otimes|\downarrow\rangle + |\downarrow\rangle\otimes|\uparrow\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

so $\rho = |T_0\rangle\langle T_0| = \frac{1}{2}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}$

if $v \neq u$. for example. $v = y$. $u = z$.

$$\sigma_v\sigma_u = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&-i&0&0\end{pmatrix}$$

$$\begin{matrix}0&0&0&0\\i&0&0&i\\i&0&0&i\\0&0&0&0\end{matrix}$$

the average value of $\sigma_v\sigma_u$ is given by $Tr\{\rho\,\sigma_v\sigma_u\} = 0$

if $v = u$. for example. $v = u = z$

$$\sigma_v\sigma_u = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

$$Tr\{\rho\,\sigma_v\sigma_u\} = Tr\begin{pmatrix}0&0&0&0\\0&-1&1&0\\0&1&-1&0\\0&0&0&0\end{pmatrix} = -2$$

if $\nu = u = x$

$$\sigma_\nu \sigma_u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Tr}\{\rho \sigma_\nu \sigma_u\} = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2$$

if $\nu = u = y$

$$\sigma_\nu \sigma_u = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Tr}\{\rho \sigma_\nu \sigma_u\} = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2$$

Problem 4

Suppose someone at a node on a quantum network receives quantum state from a set $|\psi_1\rangle, \ldots, |\psi_m\rangle$ of linearly independent states. Construct a POVM $\{E_1, E_2, \ldots, E_{m+1}\}$ such that if outcome E_i occurs, $1 \le i \le m$, then it is known with certainly that the state arriving at the node is state $|\psi_i\rangle$. The POVM must be such that $\langle\psi_i|E_i|\psi_i\rangle > 0$ for each i.

according to measurement theory, we need to find a series of POVM operators,

such that $\mathrm{Tr}\{|\psi_i\rangle\langle\psi_i| E_j\} = 0$

for $1 \le i \le m$. and $j \ne i$. $E_{m+1} = \mathbb{1} - \sum_{i=1}^{m} E_i$

for $1 \le i \le m$. E_i are constructed by a projective measurement

$E_j = \left(\sum_{i=1}^{m} \alpha_{ji} |\psi_i\rangle\right)\left(\sum_{i=1}^{m} \alpha_{ji} |\psi_i\rangle\right)^\dagger$, so we have $\sum_i \alpha_{ji} \psi_{ik}^* = \delta_{jk}$

where $\psi_{ik} = \langle\psi_i|\psi_k\rangle$. and $\alpha\psi_{ik} = \mathbb{1}$.

so POVM $\{E_1, E_2 \cdots E_{m+1}\}$ can be constructed by $\hat{\alpha}$.