Homework #6

Problem 1

To simplify the calculation, we choose units such that $k = m = \hbar = 1$. Then we have

$$\hat{x} = \frac{1}{\sqrt{2}}(a^\dagger + a) \tag{1.1}$$
$$\hat{p} = \frac{i}{\sqrt{2}}(a^\dagger - a) \tag{1.2}$$
$$[\hat{x}, \hat{p}] = i \tag{1.3}$$
$$[a, a^\dagger] = 1 \tag{1.4}$$

For a coherent state $|\alpha\rangle$, we have

$$\begin{aligned}
\sigma_x^2 &= \langle\alpha|\hat{x}^2|\alpha\rangle - |\langle\alpha|\hat{x}|\alpha\rangle|^2 = \frac{1}{2}\langle\alpha|a^\dagger a^\dagger + aa + a^\dagger a + aa^\dagger|\alpha\rangle - \frac{1}{2}|\langle\alpha|a^\dagger + a|\alpha\rangle|^2 \\
&= \frac{1}{2}(\alpha^*\alpha^* + \alpha^2 + \alpha^*\alpha + \alpha\alpha^* + 1) - \frac{1}{2}(\alpha^* + \alpha)^2 \\
&= \frac{1}{2} \qquad (1.5) \\
\sigma_p^2 &= \langle\alpha|\hat{p}^2|\alpha\rangle - |\langle\alpha|\hat{p}|\alpha\rangle|^2 = -\frac{1}{2}\langle\alpha|a^\dagger a^\dagger + aa - a^\dagger a - aa^\dagger|\alpha\rangle - \frac{1}{2}|\langle\alpha|a^\dagger - a|\alpha\rangle|^2 \\
&= -\frac{1}{2}(\alpha^*\alpha^* + \alpha^2 - \alpha^*\alpha - \alpha\alpha^* - 1) - \frac{1}{2}(\alpha^* - \alpha)^2 \\
&= \frac{1}{2} \qquad (1.6) \\
\implies \quad \sigma_x\sigma_p &= \frac{1}{2} \qquad (1.7)
\end{aligned}$$

For the ground state, we have

$$\begin{aligned}
\sigma_x^2 &= \langle 0|\hat{x}^2|0\rangle - |\langle 0|\hat{x}|0\rangle|^2 = \frac{1}{2}\langle 0|a^\dagger a^\dagger + aa + a^\dagger a + aa^\dagger|0\rangle - \frac{1}{2}|\langle 0|a^\dagger + a|0\rangle|^2 = \frac{1}{2} \qquad (1.8) \\
\sigma_p^2 &= \langle 0|\hat{p}^2|0\rangle - |\langle 0|\hat{p}|0\rangle|^2 = -\frac{1}{2}\langle 0|a^\dagger a^\dagger + aa - a^\dagger a - aa^\dagger|0\rangle - \frac{1}{2}|\langle 0|a^\dagger - a|0\rangle|^2 = \frac{1}{2} \qquad (1.9) \\
\implies \quad \sigma_x\sigma_p &= \frac{1}{2} \qquad (1.10)
\end{aligned}$$

In fact, the ground state could be treated as a special coherent state with $\alpha = 0$.
For the first excited state, we have

$$\begin{aligned}
\sigma_x^2 &= \langle 1|\hat{x}^2|1\rangle - |\langle 1|\hat{x}|1\rangle|^2 = \frac{1}{2}\langle 1|a^\dagger a^\dagger + aa + a^\dagger a + aa^\dagger|1\rangle - \frac{1}{2}|\langle 1|a^\dagger + a|1\rangle|^2 = \frac{3}{2} \qquad (1.11) \\
\sigma_p^2 &= \langle 1|\hat{p}^2|1\rangle - |\langle 1|\hat{p}|1\rangle|^2 = -\frac{1}{2}\langle 1|a^\dagger a^\dagger + aa - a^\dagger a - aa^\dagger|1\rangle - \frac{1}{2}|\langle 1|a^\dagger - a|1\rangle|^2 = \frac{3}{2} \qquad (1.12) \\
\implies \quad \sigma_x\sigma_p &= \frac{3}{2} \qquad (1.13)
\end{aligned}$$

Returning to the regular units, we need to multiply a factor of $\hbar$ to the result.

Problem 2

(a) Suppose we are given an unknown state $|\theta\rangle$ which could be $|\psi\rangle$ or $|\phi\rangle$. Using the device, we could determine whether it is $|\psi\rangle$ or $|\phi\rangle$ without destroying it. Since we have already known which state it is, what we need to do is just to construct a same state so that we copy the state without destroying it, which violates the no-cloning theorem.

(b) If we have a cloning device, we can just clone and measure a copy with POVM introduced in class corresponding to unambiguous state discrimination until we get a decisive result.

Problem 3

(a) Note that the zero net spin of singlet state implies

$$\vec{\sigma} \otimes I \, |S\rangle = -I \otimes \vec{\sigma} \, |S\rangle \tag{3.1}$$

Therefore,

$$\langle S|\sigma_v \otimes \sigma_u|S\rangle = -\langle S|I \otimes (\sigma_v \sigma_u)|S\rangle = -\frac{1}{2} \operatorname{tr}(\sigma_v \sigma_u) = -\vec{v} \cdot \vec{u} \tag{3.2}$$

(b)

$$\begin{aligned} \langle T_0|\sigma_v \otimes \sigma_u|T_0\rangle &= \langle \uparrow\downarrow|\sigma_v \otimes \sigma_u|\uparrow\downarrow\rangle + \langle \downarrow\uparrow|\sigma_v \otimes \sigma_u|\downarrow\uparrow\rangle - \langle T_0|\sigma_v \otimes \sigma_u|T_0\rangle \\ &= v_z(-u_z) + (-v_z)u_z + \vec{v} \cdot \vec{u} = v_x u_x + v_y u_y - v_z u_z \end{aligned} \tag{3.3}$$

Problem 4

Here we introduce a new linearly independent quantum state set $\{|\phi_1\rangle, \ldots, |\phi_m\rangle\}$ such that

$$\langle \psi_i|\phi_j\rangle = A_i \delta_{ij} \tag{4.1}$$

which means $|\phi_i\rangle$ is in the orthogonal space of $\{|\psi_1\rangle, \ldots, |\psi_{i-1}\rangle, |\psi_{i+1}\rangle, \ldots, |\psi_m\rangle\}$. We could always find such normalized $|\phi_i\rangle$ and $|A_i| > 0$ for any i since $\{|\psi_1\rangle, \ldots, |\psi_m\rangle\}$ are linearly independent. Also if $|\psi_i\rangle$ and $|\phi_i\rangle$ are normalized, we have

$$|A_i| \leq 1 \tag{4.2}$$

Then the POVM could be chosen as

$$E_i = c_i|\phi_i\rangle\langle\phi_i| \qquad 1 \leq i \leq m \tag{4.3}$$

$$E_i = I - \sum_{j=1}^{m} E_j \qquad i = m+1 \tag{4.4}$$

where $c_i > 0$ is some normalization factor. From the definition of $|\phi_i\rangle$, we have

$$\langle \psi|E_i|\psi\rangle > 0 \tag{4.5}$$

for any $1 \leq i \leq m$ and any state $|\psi\rangle$. And we could always choose the appropriate c_i, such as $c_i = 1/m$, so that for any state we have

$$\langle\psi|E_{m+1}|\psi\rangle = 1 - \sum_{j=1}^{m} \langle\psi|E_j|\psi\rangle \geq 0 \tag{4.6}$$

so that $\{E_i\}$ construct a POVM. Generally any c_i can be chosen as long as E_{m+1} is positive semidefinite. If outcome E_i occurs, the input state must be $|\psi_i\rangle$ since

$$\langle\psi_j|E_i|\psi_j\rangle = 0 \tag{4.7}$$

for any $i \neq j$ and $i, j \leq m$.