Mathematics Booklet of Various Computer Science Based Topics

A Collection of Mathematics Topics Studied

Original University Module: **Mathematics**

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Abstract

This paper is a collection of mathematical topics studied at university. Each chapter is a collection of various questions on a particular mathematics topic, followed by a series of worked-out answers. All of these topics were thoroughly studied to prepare for the end of year mathematics module exam.

Contents

Introduction

1. Place each of the following numbers in the appropriate set $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

   Note: identify the *smallest*set that he number belongs to. For example, -13 is an integer and a rational number and a real number but not a natural number, so you would answer $\mathbb{Z}$ in this case.

   $$24,\ 4/5,\ 8.3,\ -47,\ -22/3,\ \pi/4,\ \sqrt{5},\ \sqrt{49},\ 1.252,\ 152,\ -\sqrt{15},\ 7\pi/6.$$

2. Round each of the following to three significant figures.

(i)	12.3456	(iv)	10 497 452
(ii)	123 456	(v)	1.50298
(iii)	0.0123456	(vi)	-7.852

3. Evaluate each of the following (without using a calculator!).

(i)	2^5	(v)	5^{-2}
(ii)	3^4	(vi)	$16^{1/2}$
(iii)	4^3	(vii)	$25^{3/2}$
(iv)	4^{-1}	(viii)	$9^{-1/2}$

4. Simplify each of the following expressions.

(i)	$(a^2)^5$	(iv)	$1/a^{-4}$
(ii)	$(a^2b)^3$	(v)	$(a^2b^6)^{-1/2}$
(iii)	a^2/a^5	(vi)	$(a^2/b)^7$

5. Evaluate each of the following.

(i)	$\log_2 64$	(iv)	$\log_3\left(\frac{1}{81}\right)$
(ii)	$\log_2 2^{15}$	(v)	$\log_5 25^2$
(iii)	$\log_3 81$	(vi)	$\log_3\left(\frac{1}{9}\right)^3$

6. Write each of the following as a single logarithm.

   For example, $\log_a x^4 - \log_a x^2 = \log_a\left(\frac{x^4}{x^2}\right) = \log_a x^2$.

(i)	$\log_a x^2 + \log_a x^5$	(iii)	$2\log_a x^3 + 5\log_a y$
(ii)	$\log_a x^3 + \log_a y - \log_a z^2$	(iv)	$2\log_a x - \log_a(1/x)$

Introduction : Solutions

1. $24,\ \sqrt{49} = 7,\ 152$ belong to set $\mathbb{N}$,
   -47 belongs to set $\mathbb{Z}$,
   $4/5,\ 8.3 = \frac{83}{10},\ -22/3,\ 1.252 = \frac{1252}{1000}$ belong to set $\mathbb{Q}$, and
   $\pi/4,\ \sqrt{5},\ -\sqrt{15},\ 7\pi/6$ belong to set $\mathbb{R}$.

2. To three significant figures:

(i)	12.3	(iv)	10 500 000
(ii)	123 000	(v)	1.50
(iii)	0.0123	(vi)	-7.85

3. Evaluate each of the following.

(i)	$2^5 = 32$	(v)	$5^{-2} = 1/5^2 = 1/25$
(ii)	$3^4 = 81$	(vi)	$16^{1/2} = \sqrt{16} = 4$
(iii)	$4^3 = 64$	(vii)	$25^{3/2} = (25^{1/2})^3 = (\sqrt{25})^3 = 5^3 = 125$
(iv)	$4^{-1} = 1/4$	(viii)	$9^{-1/2} = 1/\sqrt{9} = 1/3$

4. Simplify each of the following expressions.

(i)	$(a^2)^5 = a^{10}$	(iv)	$1/a^{-4} = a^4$
(ii)	$(a^2b)^3 = a^6b^3$	(v)	$(a^2b^6)^{-1/2} = \dfrac{1}{(a^2)^{1/2}}\dfrac{1}{(b^6)^{1/2}} = \dfrac{1}{ab^3}$
(iii)	$a^2/a^5 = 1/a^3 = a^{-3}$	(vi)	$(a^2/b)^7 = a^{14}/b^7$

5. Evaluate each of the following.

(i)	$\log_2 64 = \log_2 2^6 = 6$	(iv)	$\log_3\left(\dfrac{1}{81}\right) = \log_3 3^{-4} = -4$
(ii)	$\log_2 2^{15} = 15$	(v)	$\log_5 25^2 = \log_5 5^4 = 4$
(iii)	$\log_3 81 = \log_3 3^4 = 4$	(vi)	$\log_3\left(\dfrac{1}{9}\right)^3 = \log_3\left(\dfrac{1}{3^6}\right) = -6$

6. Write each of the following as a single logarithm.

(i)	$\log_a x^2 + \log_a x^5 = \log_a x^7$	(iii)	$2\log_a x^3 + 5\log_a y$ $= \log_a x^6 + \log_a y^5 = \log_a x^6y^5$
(ii)	$\log_a x^3 + \log_a y - \log_a z^2$ $= \log_a\left(\dfrac{x^3y}{z^2}\right)$	(iv)	$2\log_a x - \log_a(1/x)$ $= \log_a x^2 + \log_a x = \log_a x^3$

Algebra 1

1. (i) Express each of the following binary numerals in decimal form.

    (a) 101101 (c) 1010 1010
    (b) 111111 (d) 1101 0110

    (ii) Express each of the following 'bicimals' as decimals.

    (a) 110.101 (c) 1010.0101
    (b) 0.1011 (d) 11.11011

    (iii) Express each of the following decimal numerals as binary numerals.

    (a) 19 (c) 71
    (b) 29 (d) 157

    (iv) [Harder.] Here we express decimals as bicimals. To do so, remember the decimal equivalents of powers of 2:

    $$\tfrac{1}{2} = 0.5,\ \tfrac{1}{4} = 0.25,\ \tfrac{1}{8} = 0.125,\ \tfrac{1}{16} = 0.0625,\ \tfrac{1}{32} = 0.03125, \ldots.$$

    So, for example, $0.21875 = 0.125 + 0.0625 + 0.03125 = 0.00111$.

    Express each of the following decimals as bicimals.

    (a) 13.375 (c) 14.09375
    (b) 0.6875 (d) 8.046875

    (v) [Harder.] Explain why any binary numeral of the form $111\ldots11$ (that is, just a sequence of 1s) represents a number of the form $2^n - 1$ for some n.

    Hints: You may wish to evaluate a few examples first – $111, 1111, 11111, 111111, \ldots$ – to 'see what is going on'.
    Although not absolutely necessary, you *may* find it useful to use the formula

    $$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

2. Expand each of the following brackets, simplifying where possible.

    (i) $(a + 3b)(b + 3c)$ (iv) $(a - b)^3$
    (ii) $(a + b + 2)^2$ (v) $(a + b)^2(a - b)$
    (iii) $(a - 2b)(a + 3b)$ (vi) $(2a + 3b)^3$

3. Simplify each of the following algebraic fractions.

(i)	$\dfrac{a+b}{2} \times \dfrac{3a}{a+b}$	(iv)	$\dfrac{a+b}{a-b} \times \dfrac{a^2-b^2}{2a+2b}$
(ii)	$\dfrac{2+a}{b} + \dfrac{a+1}{2b}$	(v)	$\dfrac{a}{b} + \dfrac{b}{a}$
(iii)	$\dfrac{a+1}{b} + \dfrac{a+2}{c}$	(vi)	$\dfrac{2}{a} + \dfrac{3}{b} + \dfrac{4}{c}$

4. Compute the value of each of the following.

   (i) $\dfrac{a^2+b}{2-a/b}$ when $a = 2$ and $b = -3$

   (ii) $\dfrac{2-a/b}{a^2+b}$ when $a = 2$ and $b = -3$

   (iii) $\dfrac{1/a+2/b}{3/c+4/d}$ when $a = 3$, $b = 4$, $c = 5$ and $d = 6$.

   (iv) $\dfrac{a(b+2c)}{c(2+3a^2)}$ when $a = -1$, $b = 5$ and $c = 2$.

Algebra 1: Solutions

1. (i) (a) $101101 = 2^5 + 2^3 + 2^2 + 1 = 32 + 8 + 4 + 1 = 45$

   (b) $111111 = 2^5 + 2^4 + 2^3 + 2^2 + 2 + 1 = 32 + 16 + 8 + 4 + 2 + 1 = 63$

   (c) $1010\,1010 = 2^7 + 2^5 + 2^3 + 2 = 128 + 32 + 8 + 2 = 170$

   (d) $1101\,0110 = 2^7 + 2^6 + 2^4 + 2^2 + 2 = 128 + 64 + 16 + 4 + 2 = 214$

   (ii) (a) $110.101 = 2^2 + 2 + 2^{-1} + 2^{-3} = 4 + 2 + \frac{1}{2} + \frac{1}{8} = 6\frac{5}{8} = 6.625$

   (b) $0.1011 = 2^{-1} + 2^3 + 2^{-4} = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} = \frac{11}{16} = 0.6875$

   (c) $1010.0101 = 2^3 + 2 + 2^{-2} + 2^{-4} = 8 + 2 + \frac{1}{4} + \frac{1}{16} = 10\frac{5}{16} = 10.3125$

   (d) $11.11011 = 2+1+2^{-1}+2^{-2}+2^{-4}+2^{-5} = 3+\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{32} = 3\frac{27}{32} = 3.84375$

   (iii) (a) $19 = 16 + 2 + 1 = 2^4 + 0 \times 8 + 0 \times 4 + 2 + 1 = 10011$

   (b) $29 = 16 + 8 + 4 + 1 = 11101$

   (c) $71 = 64+4+2+1 = 1\times64+0\times32+0\times16+0\times8+1\times4+1\times2+1 = 1000111$

   (d) $157 = 128 + 16 + 8 + 4 + 1 = 10011101$

   (iv) (a) $13.375 = 8 + 4 + 1 + 0.25 + 0.0125 = 1101.011$

   (b) $0.6875 = 0.5 + 0.125 + 0.0625 = 0.1011$

   (c) $14.09375 = 8 + 4 + 2 + 0.0625 + 0.03125 = 1110.00011$

   (d) $8.046875 = 8 + 0.03125 + 0.015626 = 1000.000011$

   (v) Firstly, the examples:
   $111 = 7 = 2^3 - 1$, $1111 = 15 = 2^4 - 1$, $11111 = 31 = 2^5 - 1$, $111111 = 63 = 2^6 - 1, \dots$.

   There are a couple of ways to answer this.
   Suppose $111\dots11$ contains n 1s. Adding another 1 gives:

$$\underbrace{111\dots11}_{\longleftarrow\ n\ \longrightarrow} + 1 = 1\underbrace{000\dots00}_{\longleftarrow\ n\ \longrightarrow} = 2^n,$$

   so $\underbrace{111\dots11}_{\longleftarrow\ n\ \longrightarrow} = 2^n - 1$.

   Alternatively,

$$\underbrace{111\dots11}_{\longleftarrow\ n\ \longrightarrow} = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1 = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

   using the formula

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

2. (i) $(a+3b)(b+3c) = ab+3b^2+3ac+9bc.$

(ii) $(a+b+2)^2 = (a+b+2)(a+b+2) = a^2+2ab+b^2+4a+4b+4.$

(iii) $(a-2b)(a+3b) = a^2+ab-6b^2.$

(iv) $(a-b)^3 = (a-b)(a^2-2ab+b^2) = a^3-3a^2b+3ab^3-b^3.$

(v) $(a+b)^2(a-b) = (a+b)(a^2-b^2) = a^3+a^2b-ab^2-b^3.$

(vi) $(2a+3b)^3 = (2a)^3+3(2a)^2(3b)+3(2a)(3b)^2+(3b)^3 = 8a^3+36a^2b+54ab^2+27b^3,$ using the result $(a+b)^3 = a^3+3a^2b+3ab^2+b^3.$

3. (i) $\dfrac{a+b}{2} \times \dfrac{3a}{a+b} = \dfrac{3a}{2}.$

(ii) $\dfrac{2+a}{b} + \dfrac{a+1}{2b} = \dfrac{4+2a}{2b} + \dfrac{a+1}{2b} = \dfrac{3a+5}{2b}.$

(iii) $\dfrac{a+1}{b} + \dfrac{a+2}{c} = \dfrac{ac+c}{bc} + \dfrac{ab+2b}{bc} = \dfrac{ab+ac+2b+c}{bc}.$

(iv) $\dfrac{a+b}{a-b} \times \dfrac{a^2-b^2}{2a+2b} = \dfrac{(a+b)(a^2-b^2)}{2(a-b)(a+b)} = \dfrac{a+b}{2}.$

(v) $\dfrac{a}{b} + \dfrac{b}{a} = \dfrac{a^2}{ab} + \dfrac{b^2}{ab} = \dfrac{a^2+b^2}{ab}.$

(vi) $\dfrac{2}{a} + \dfrac{3}{b} + \dfrac{4}{c} = \dfrac{2bc}{abc} + \dfrac{3ac}{abc} + \dfrac{4ab}{abc} = \dfrac{2bc+3ac+4ab}{abc}$

4. (i) When $a=2$ and $b=-3$, we have $\dfrac{a^2+b}{2-a/b} = \dfrac{4-3}{2+2/3} = \dfrac{3}{58}$

(ii) This is the reciprocal of (i) so, when $a=2$ and $b=-3$, we have $\dfrac{2-a/b}{a^2+b} = \dfrac{8}{3}.$

(iii) When $a=3$, $b=4$, $c=5$ and $d=6$, we have

$$\frac{1/a+2/b}{3/c+4/d} = \frac{1/3+2/4}{3/5+4/6} = \frac{10+15}{18+20} = \frac{25}{38}$$

.

Note that the step, $\dfrac{1/3+2/4}{3/5+4/6} = \dfrac{10+15}{18+20}$, is obtained by multiplying top and bottom of the fraction by 30.

(iv) When $a=-1$, $b=5$ and $c=2$, we have

$$\frac{a(b+2c)}{c(2+3a^2)} = \frac{-(5+4)}{2(2+3)} = -\frac{9}{10}.$$

Algebra 2

1. Solve each of the following quadratic equations by factorising.

(i)	$x^2 - 11x + 24 = 0$	(iv)	$2x^2 + 9x + 4 = 0$
(ii)	$x^2 + 2x - 35 = 0$	(v)	$6x^2 - 11x + 3 = 0$
(iii)	$x^2 - 2x = 24$	(vi)	$5x^2 + 20 = 29x$

2. Solve each of the following quadratic equations using the quadratic formula. Where appropriate, give the answers to three decimal places.

(i)	$3x^2 - 5x - 8 = 0$	(iv)	$2x^2 - 9x + 8 = 0$
(ii)	$x^2 + 7x + 3 = 0$	(v)	$5x^2 + x - 7 = 0$
(iii)	$8x^2 - 38x + 45 = 0$	(vi)	$4x^2 + 11x + 1 = 0$

3. Express each of the following sums using sigma notation.

   (i) $7 + 8 + 9 + \cdots + 37$

   (ii) $3 + 6 + 9 + 12 + \cdots + 57$

   (iii) $2 + 6 + 12 + 20 + 30 + 42 + \cdots + 380$

   (iv) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{144}$

   (v) $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \cdots + \frac{29}{30}$

   (vi) $\frac{3}{2} + \frac{6}{5} + \frac{9}{10} + \frac{12}{17} + \cdots + \frac{39}{170}$

4. For each of the sums in question 3, write a for loop in Java to evaluate the expression.

5. Express each of the following sums using product and/or sigma notation.

   (i) $1 \times 3 \times 5 \times 7 \times \cdots \times 121$

   (ii) $\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{49}{50}$

   (iii) $1 \times (1+2) \times (1+2+3) \times (1+2+3+4) \times \cdots \times (1+2+3+\cdots+10)$

   (iv) $1^2 + (1^2 \times 2^2) + (1^2 \times 2^2 \times 3^2) + \cdots + (1^2 \times 2^2 \times 3^2 \times \cdots \times 10^2)$

6. For each of the sums in question 5, write a for loop in Java to evaluate the expression.

7. [Harder.] The following formulae are given:

$$\sum_{k-1}^{n} k = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1),$$
$$\sum_{k-1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1), \quad \text{and}$$
$$\sum_{k-1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Using these formulae, find expressions (formulae) for each of the following.

(i) $\displaystyle\sum_{k=1}^{2n} k = 1 + 2 + 3 + \cdots + 2n$

(ii) $\displaystyle\sum_{k=1}^{n^2} k = 1 + 2 + 3 + \cdots + n^2$

(iii) $\displaystyle\sum_{k=1}^{n} k(k+1) = 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1)$

(iv) $\displaystyle\sum_{k=1}^{n} k(k^2+1) = 1 \times 2 + 2 \times 5 + 3 \times 10 + \cdots + n(n^2+1)$

(v) $\displaystyle\sum_{k=n+1}^{2n} k = (n+1) + (n+2) + (n+3) + \cdots + 2n$

(vi) $\displaystyle\sum_{k=n+1}^{2n} k^3 = (n+1)^3 + (n+2)^3 + (n+3)^3 + \cdots + (2n)^3$

Algebra 2: Solutions

1. (i) $$\begin{aligned} x^2 - 11x + 24 = 0 &\Rightarrow (x-3)(x-8) = 0 \\ &\Rightarrow x-3 = 0 \text{ or } x-8 = 0 \\ &\Rightarrow x = 3 \text{ or } x = 8. \end{aligned}$$

(ii) $$\begin{aligned} x^2 + 2x - 35 = 0 &\Rightarrow (x-5)(x+7) = 0 \\ &\Rightarrow x-5 = 0 \text{ or } x+7 = 0 \\ &\Rightarrow x = 5 \text{ or } x = -7. \end{aligned}$$

(iii) $$\begin{aligned} x^2 - 2x = 24 &\Rightarrow x^2 - 2x - 24 = 0 \\ &\Rightarrow (x+4)(x-6) = 0 \\ &\Rightarrow x+4 = 0 \text{ or } x-6 = 0 \\ &\Rightarrow x = -4 \text{ or } x = 6. \end{aligned}$$

(iv) $$\begin{aligned} 2x^2 + 9x + 4 = 0 &\Rightarrow (2x+1)(x+4) = 0 \\ &\Rightarrow 2x+1 = 0 \text{ or } x+4 = 0 \\ &\Rightarrow x = -\tfrac{1}{2} \text{ or } x = -4. \end{aligned}$$

(v) $$\begin{aligned} 6x^2 - 11x + 3 = 0 &\Rightarrow (2x-3)(3x-1) = 0 \\ &\Rightarrow 2x-3 = 0 \text{ or } 3x-1 = 0 \\ &\Rightarrow x = \tfrac{3}{2} \text{ or } x = \tfrac{1}{3}. \end{aligned}$$

(vi) $$\begin{aligned} 5x^2 + 20 = 29x &\Rightarrow 5x^2 - 29x + 20 = 0 \\ &\Rightarrow (5x-4)(x-5) = 0 \\ &\Rightarrow 5x-4 = 0 \text{ or } x-5 = 0 \\ &\Rightarrow x = \tfrac{4}{5} \text{ or } x = 5. \end{aligned}$$

2. (i) $$\begin{aligned} 3x^2 - 5x - 8 = 0 &\Rightarrow x = \frac{5 \pm \sqrt{25 + 4 \times 3 \times 8}}{6} \\ &\Rightarrow x = \frac{5 \pm \sqrt{121}}{6} \\ &\Rightarrow x = \frac{5 \pm 11}{6} \\ &\Rightarrow x = \tfrac{5+11}{6} = \tfrac{8}{3} \text{ or } x = \tfrac{5-11}{6} = -1. \end{aligned}$$

(ii) $$x^2 + 7x + 3 = 0 \Rightarrow x = \frac{-7 \pm \sqrt{49 - 4 \times 1 \times 3}}{2}$$

$$\Rightarrow x = \frac{-7 \pm \sqrt{37}}{2}$$

$$\Rightarrow x = \frac{-7 + \sqrt{37}}{2} \text{ or } x = \frac{-7 - \sqrt{37}}{2}$$

$$\Rightarrow x = -0.459 \text{ or } x = -6.541.$$

(iii) $$8x^2 - 38x + 45 = 0 \Rightarrow x = \frac{38 \pm \sqrt{1444 - 4 \times 8 \times 45}}{16}$$

$$\Rightarrow x = \frac{38 \pm \sqrt{1444 - 1440}}{16}$$

$$\Rightarrow x = \frac{38 \pm 2}{16}$$

$$\Rightarrow x = \frac{38 + 2}{16} = \frac{5}{2} \text{ or } x = \frac{38 - 2}{16} = \frac{9}{4}.$$

(iv) $$2x^2 - 9x + 8 = 0 \Rightarrow x = \frac{9 \pm \sqrt{81 - 4 \times 2 \times 8}}{4}$$

$$\Rightarrow x = \frac{9 \pm \sqrt{17}}{4}$$

$$\Rightarrow x = \frac{9 + \sqrt{17}}{4} \text{ or } x = \frac{9 - \sqrt{17}}{4}$$

$$\Rightarrow x = 3.281 \text{ or } x = 1.219$$

(v) $$5x^2 + x - 7 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1 + 4 \times 5 \times 7}}{10}$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{141}}{10}$$

$$\Rightarrow x = \frac{-1 + \sqrt{141}}{10} \text{ or } x = \frac{-1 - \sqrt{141}}{10}$$

$$\Rightarrow x = 1.087 \text{ or } x = -1.287.$$

(vi) $$4x^2 + 11x + 1 = 0 \;\Rightarrow\; x = \frac{-11 \pm \sqrt{121 - 4 \times 4 \times 1}}{8}$$

$$\Rightarrow\; x = \frac{-11 \pm \sqrt{105}}{8}$$

$$\Rightarrow\; x = \frac{-11 + \sqrt{105}}{8} \text{ or } x = \frac{-11 - \sqrt{105}}{8}$$

$$\Rightarrow\; x = -0.094 \text{ or } x = -2.656.$$

3. (i) $7 + 8 + 9 + \cdots + 37 = \sum_{k=7}^{37} k$

(ii) $3 + 6 + 9 + 12 + \cdots + 57 = \sum_{k=1}^{19} 3k$

(iii) $2 + 6 + 12 + 20 + 30 + 42 + \cdots + 380 = \sum_{k=1}^{19} k(k+1) = \sum_{k=1}^{19} (k^2 + k)$

(iv) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{144} = \sum_{k=1}^{12} \frac{1}{k^2}$

(v) $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \cdots + \frac{29}{30} = \sum_{k=1}^{15} \frac{2k-1}{2k}$

(vi) $\frac{3}{2} + \frac{6}{5} + \frac{9}{10} + \frac{12}{17} + \cdots + \frac{39}{170} = \sum_{k=1}^{13} \frac{3k}{k^2+1}$

4. (i)
```
int sum=0;
for (int k=7; k<=37; k++){
     sum = k + sum;
}
```

(ii)
```
int sum=0;
for (int k=1; k<=19; k++){
     sum = (3*k) + sum;
}
```

(iii)
```
int sum=0;
for (int k=1; k<=19; k++){
     sum = k*(k+1) + sum;
}
```

(iv)
```
float sum=0.0;
```

```
for (int k=1; k<=12; k++){
     sum = 1/(k*k) + sum;
}
```

(v)
```
float sum=0.0;
for (int k=1; k<=15; k++){
     sum = (2*k-1)/(2*k) + sum;
}
```

(vi)
```
float sum=0.0;
for (int k=1; k<=13; k++){
     sum = (3*k)/(k*k+1) + sum;
}
```

5. (i) $1 \times 3 \times 5 \times 7 \times \cdots \times 121 = \prod_{k=1}^{61} (2k-1)$

(ii) $\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{49}{50} = \prod_{k=1}^{25} \frac{2k-1}{2k}$

(iii) $1 \times (1+2) \times (1+2+3) \times \cdots \times (1+2+3+\cdots+10) = \prod_{k=1}^{10} \left(\sum_{j=1}^{k} j \right)$

(iv) $1^2 + (1^2 \times 2^2) + (1^2 \times 2^2 \times 3^2) + \cdots + (1^2 \times 2^2 \times 3^2 \times \cdots \times 10^2) = \sum_{k=1}^{10} \left(\prod_{j=1}^{k} j^2 \right)$

6. (i)
```
int prod=1;
for (int k=1; k<=61; k++){
     prod = (2*k-1) * prod;
}
```

(ii)
```
float prod=1.0;
for (int k=1; k<=25; k++){
     prod = (2*k-1)/(2*k) * prod;
}
```

(iii)
```
int prod=1;
for (int k=1; k<=10; k++){
     int sum=0;
     for (int j=1; j<=k; j++){
          sum = j + sum;
     }
prod = sum * prod;
}
```

(iv)
```
int sum=0;
for (int k=1; k<=10; k++){
     int prod=1;
     for (j=1; j<=k; j++){
         prod = (j*j) * prod;
     }
sum = prod + sum;
}
```

7. (i) $\sum_{k=1}^{2n} k = 1+2+3+\cdots+2n$ is simply $\sum_{k=1}^{n} k$ but with '$2n$' substituted for 'n'.

Hence $\sum_{k=1}^{2n} k = \frac{1}{2}(2n)(2n+1) = n(2n+1)$.

(ii) Similarly $\sum_{k=1}^{n^2} k = 1+2+\cdots+n^2$ is $\sum_{k=1}^{n} k$ but with 'n^2' substituted for 'n'.

Hence $\sum_{k=1}^{n^2} k = \frac{1}{2}n^2(n^2+1)$.

(iii)
$$\begin{aligned}
\sum_{k=1}^{n} k(k+1) &= \sum_{k=1}^{n} (k^2+k) \\
&= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \\
&= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{1}{2}n(n+1) \\
&= \tfrac{1}{6}n(n+1)((2n+1)+3) \\
&= \tfrac{1}{6}n(n+1)(2n+4) \\
&= \tfrac{1}{3}n(n+1)(n+2).
\end{aligned}$$

(iv)
$$\begin{aligned}
\sum_{k=1}^{n} k(k^2+1) &= \sum_{k=1}^{n} (k^3+k) \\
&= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k \\
&= \tfrac{1}{4}n^2(n^2+1) + \tfrac{1}{2}n(n+1) \\
&= \tfrac{1}{4}n(n+1)\left(n(n+1)+2\right) \\
&= \tfrac{1}{4}n(n+1)\left(n^2+n+2\right)
\end{aligned}$$

(v)
$$\begin{aligned}
\sum_{k=n+1}^{2n} k &= (1+2+\cdots+n+(n+1)+\cdots+2n)-(1+2+\cdots+n)\\
&= \left(\sum_{k=1}^{2n} k\right)-\left(\sum_{k=1}^{n} k\right)\\
&= n(2n+1)-\tfrac{1}{2}n(n+1) \qquad \text{using part (i)}\\
&= \tfrac{1}{2}n\left(2(2n+1)-(n+1)\right)\\
&= \tfrac{1}{2}n(3n+1)
\end{aligned}$$

(vi)
$$\begin{aligned}
\sum_{k=n+1}^{2n} k^3 &= (1^3+\cdots+n^3+(n+1)^3+\cdots+(2n)^3)-(1^3+\cdots+n^3)\\
&= \left(\sum_{k=1}^{2n} k^3\right)-\left(\sum_{k=1}^{n} k^3\right)\\
&= \tfrac{1}{4}(2n)^2(2n+1)^2-\tfrac{1}{4}n^2(n+1)^2\\
&= \tfrac{1}{4}n^2\left(4(2n+1)^2-(n+1)^2\right)\\
&= \tfrac{1}{4}n^2\left(4(4n^2+4n+1)-(n^2+2n+1)\right)\\
&= \tfrac{1}{4}n^2(15n^2+14n+3)
\end{aligned}$$

Functions 1

1. Which of the following define functions? Explain your answers and, if necessary, state any assumptions that you make.

   (i) $A = \{\text{university students studying in the UK}\}$, $B = \{\text{universities in the UK}\}$;
   $f : A \to B,\ f(X) = \text{university where } X \text{ studies.}$

   (ii) $A = \{\text{countries of the world}\}$, $B = \{\text{cities of the world}\}$;
   $f : A \to B,\ f(X) = \text{capital city of } X.$

   (iii) $A = \{\text{countries of the world}\}$, $B = \{\text{cities of the world}\}$;
   $f : A \to B,\ f(X) = \text{largest city of } X.$

   (iv) $A = \{\text{people}\}$, $B = \{\text{songs}\}$; $f : A \to B,\ f(X) = X\text{'s favourite songs.}$

   (v) $A = \{\text{premier league football teams}\}$;
   $f : A \to \mathbb{N},\ f(X) = X\text{'s position in the league today.}$

2. Three functions f, g and h are defined as follows.

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = |x - 1|$$

$$g : \mathbb{Z} \to \mathbb{R}, \quad g(x) = \frac{2x}{x^2 - 3}$$

$$h : \mathbb{R} \to \mathbb{R}, \quad h(x) = 2x + 1$$

   For each of the following, either evaluate the expression or explain why it is not defined.

(i)	$f(-3)$	(vi)	$(f \circ f)(-2)$
(ii)	$f\left(\frac{1}{2}\right)$	(vii)	$(g \circ g)(3)$
(iii)	$g\left(\frac{1}{2}\right)$	(viii)	$(h \circ h)(x)$
(iv)	$h(\pi)$	(ix)	$(f \circ h)(x)$
(v)	$(g \circ h)\left(\frac{1}{2}\right)$	(x)	$(f \circ g)(x)$

3. Two functions f and g are defined as follows.

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x - 1$$

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \frac{1}{x^2 + 1}$$

   Find expressions for each of the following.

(i)	$(f \circ g)(x)$	(iv)	$(g \circ g)(x)$
(ii)	$(g \circ f)(x)$	(v)	$(f \circ f \circ f)(x)$
(iii)	$(f \circ f)(x)$		

Functions 1: Solutions

1. (i) $A = \{\text{university students studying in the UK}\}$, $B = \{\text{universities in the UK}\}$;
$f : A \to B,\ f(X) = \text{university where } X \text{ studies.}$

This is a function because every student studies at a unique university. This assumes that no student studies at more than one university at a time.

(ii) $A = \{\text{countries of the world}\}$, $B = \{\text{cities of the world}\}$;
$f : A \to B,\ f(X) = \text{capital city of } X.$

This is a function. Every country has a one and only one capital city.

(iii) $A = \{\text{countries of the world}\}$, $B = \{\text{cities of the world}\}$;
$f : A \to B,\ f(X) = \text{largest city of } X.$

This is **not** a function, at least, as currently defined. What does 'largest' mean? It could mean largest in population terms, land area or something else. Without a more precise definition of 'largest', this is not a function.

(iv) $A = \{\text{people}\}$, $B = \{\text{songs}\}$; $f : A \to B,\ f(X) = X\text{'s favourite songs.}$

This is **not** a function. A person may have many different favourite songs or, indeed, a person may not have any favourite song.

(v) $A = \{\text{premier league football teams}\}$;
$f : A \to \mathbb{N},\ f(X) = X\text{'s position in the league today.}$

This is a function (although it will change over time).
On the day I typed these solutions (22 February), $f(\text{Manchester City}) = 1$, $f(\text{Burnley}) = 7$ (yay!), etc.

2. (i) $f(-3) = |-4| = 4$.

(ii) $f\left(\frac{1}{2}\right) = f\left|-\frac{1}{2}\right| = \frac{1}{2}$.

(iii) $g\left(\frac{1}{2}\right)$ is not defined; g has domain $\mathbb{Z}$, so $g(n)$ is only defined when n is an integer.

(iv) $h(\pi) = 2\pi + 1$.

(v) $(g \circ h)\left(\frac{1}{2}\right) = g\left(h\left(\frac{1}{2}\right)\right) = g\left(2 \times \frac{1}{2} + 1\right) = g(2) = 4$.

(vi) $(f \circ f)(-2) = f(f(-2)) = f(|-3|) = f(3) = |2| = 2$.

(vii) $(g \circ g)(3) = g\left(\dfrac{6}{9-3}\right) = g(1) = \dfrac{2}{1-3} = -1$.

(viii) $(h \circ h)(x) = h(2x+1) = 2(2x+1) + 1 = 4x + 3$.

(ix) $(f \circ h)(x) = f(h(x)) = f(2x+1) = |(2x+1) - 1| = |2x|$.

(x) $(f \circ g)(x) = f\left(\dfrac{2x}{x^2-3}\right) = \left|\dfrac{2x}{x^2-3} - 1\right| = \left|\dfrac{2x - (x^2-3)}{x^2-3}\right| = \left|\dfrac{2x - x^2 + 3}{x^2-3}\right|$.

3. (i) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2+1}\right) = \frac{2}{x^2+1} - 1 = \frac{2-(x^2+1)}{x^2+1} = \frac{1-x^2}{x^2+1}$.

(ii) $(g \circ f)(x) = g(f(x)) = g(2x-1) = \frac{1}{(2x-1)^2+1} = \frac{1}{4x^2-4x+2}$.

(iii) $(f \circ f)(x) = f(f(x)) = f(2x-1) = 2(2x-1) - 1 = 4x - 3$.

(iv)
$$(g \circ g)(x) = g(g(x)) = g\left(\frac{1}{x^2+1}\right) = \frac{1}{\left(\frac{1}{x^2+1}\right)^2 + 1}$$
$$= \frac{(x^2+1)^2}{1+(x^2+1)^2} = \frac{x^4+2x^2+1}{x^4+2x^2+2}.$$

(v) $(f \circ f \circ f)(x) = f(f(f(x))) = f(f(2x-1)) = f(4x-3) = 2(4x-3) - 1 = 8x - 7$.

Functions 2

1. Define simple methods in Java that implements each of the following (mathematical) functions. If you wish to use pseudo code instead, that's fine.

   (i) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = 3 - 7n.$

   (ii) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \dfrac{x}{x^2 + 1}.$

   (iii) $f : \mathbb{Z} \to \mathbb{R},\ f(n) = \sqrt{n^2 + n + 1}.$

   (iv) $f : \mathbb{Z} \to \mathbb{Q},\ f(n) = \dfrac{n + 1}{n^2 + 2}.$

   (v) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} 2x - 1 & \text{if } x \geq 0 \\ 1 - 3x & \text{if } x < 0. \end{cases}$

   (vi) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} \dfrac{1}{x - 1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1. \end{cases}$

   (vii) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} 2n - 1 & \text{if } n \text{ is even} \\ 3n + 2 & \text{if } n \text{ is odd.} \end{cases}$

   (viii) $f : \mathbb{R} \to \mathbb{R},\ f(n) = \begin{cases} x^2 + x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ 1/x & \text{if } x < 0. \end{cases}$

2. Find $\operatorname{im} f$, the image of f, for each of the following functions.

   (i) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = 2n + 10$

   (ii) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = n^2$

   (iii) $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, \ldots, 10\},\ f(n) = |1 - 2n|$

   (iv) $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, \ldots, 25\},\ f(n) = n^2 - n + 2$

   (v) $A = \{\text{countries of the world}\}$, $B = \{\text{cities of the world}\}$
   $f : A \to B$, $f(X) =$ the capital city of X.

   (vi) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 + 1$

   (vii) $f : \mathbb{R} \to \mathbb{R},\ f(x) = (x + 1)^2$

   (viii) $f : \mathbb{R} \to \mathbb{R},\ f(x) = 5x + 7$

3. For each of the following functions f which are defined 'in pieces', find $\operatorname{im} f$, the image of f.

   Note: you are advised to evaluate $f(n)$ for various values of n to 'see' how the function is behaving.

   (i) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$

   (ii) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$

   (iii) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n^2 & \text{if } n \geq 0 \\ -n & \text{if } n < 0. \end{cases}$

   (iv) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n-1 & \text{if } n < 0. \end{cases}$

   (v) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} \dfrac{n}{3}+1 & \text{if } n = 3k \text{ for some integer } k \\ n^2+1 & \text{otherwise.} \end{cases}$

Functions 2: Solutions

1. (i)
```
public  int function1(int n){
return  3 - 7*n;
        }
```

(ii)
```
public  float function2(float x){
return  x/(x*x+1);
        }
```

(iii)
```
public  float function3(int n){
return  Math.sqrt(n*n+n+1);
        }
```

(iv)
```
public  float function4(int n){
return  (n+1)/(n*n+2);
        }
```

(v)
```
public  float function5(float x){
float result=0;
        if (x>=0)
             result = 2*x-1;
        if (x<0)
             result = 1-3*x;
return result;
          }
```

(vi)
```
public  float function6(float x){
float result=0;
        if (x != 1)
             result = 1/(x-1);
        if (x == 1)
             result = 0;
return result;
          }
```

(vii)
```
public  int function7(int n){
int result=0;
        if (n % 2 = 0)
             result = 2*n-1;
        if (n % 2 = 1)
             result = 3*n+2;
return result;
          }
```

(viii)
```
public  float function8(float x){
        float result=0;
              if (x>0)
                  result = x*x+x;
              if (x == 0)
                  result = 0;
              if (x<0)
                  result = 1/x;
        return result;
                }
```

2. (i) $\text{im}\, f = \{m \in \mathbb{Z} : m = 2n + 10 \text{ for some } n \in \mathbb{Z}\} = \{\text{even integers}\}$.

   (ii) $\text{im}\, f = \{m \in \mathbb{Z} : m = n^2 \text{ for some } n \in \mathbb{Z}\}$
   $= \{0, 1, 4, 9, 16, 25, \ldots\} = \{\text{squares of integers}\}$.

   (iii) We have: $1 \mapsto |1-2| = 1,\ 2 \mapsto |1-4| = 3,\ 3 \mapsto |1-6| = 5,\ 4 \mapsto |1-8| = 7$, and $5 \mapsto |1-10| = 9$.

   Hence $\text{im}\, f = \{1, 3, 5, 7, 9\}$.

   (iv) We have: $1 \mapsto 2,\ 2 \mapsto 4,\ 3 \mapsto 8,\ 4 \mapsto 14$ and $5 \mapsto 22$.

   Hence $\text{im}\, f = \{2, 4, 8, 14, 22\}$.

   (v) $\text{im}\, f = \{\text{capital cities of the world}\}$.

   (vi) $\text{im}\, f = \{y \in \mathbb{R} : y = x^2 + 1 \text{ for some } x \in \mathbb{R}\} = \{y \in \mathbb{R} : y \geq 1\}$.

   (vii) $\text{im}\, f = \{y \in \mathbb{R} : y = (x+1)^2 \text{ for some } x \in \mathbb{R}\} = \{y \in \mathbb{R} : y \geq 0\}$.

   (viii) $\text{im}\, f = \{y \in \mathbb{R} : y = 5x + 7 \text{ for some } x \in \mathbb{R}\} = \mathbb{R}$.

   This is because, for *every* $y \in \mathbb{R}$, there is an $x \in \mathbb{R}$ (namely $x = (y-7)/5$) such that $f(x) = y$.

3. (i) We have the following.

   $$\ldots, -3 \mapsto -2,\ -2 \mapsto -2,\ -1 \mapsto 0,\ 0 \mapsto 0,\ 1 \mapsto 2,\ 2 \mapsto 2,\ 3 \mapsto 4, \ldots.$$

   Only the even integers are images of something; no odd integer is mapped to.

   Therefore $\text{im}\, f = \{\text{even integers}\} = \{2n : n \in \mathbb{Z}\}$.

   (ii) We have the following.

   Even integers: $\ldots,\ -4 \mapsto -2,\ -2 \mapsto -1,\ 0 \mapsto 0,\ 2 \mapsto 1,\ 4 \mapsto 2,\ 6 \mapsto 3, \ldots$;
   odd integers: $\ldots,\ -5 \mapsto -2,\ -3 \mapsto -1,\ -1 \mapsto 0,\ 1 \mapsto 1,\ 3 \mapsto 2,\ 5 \mapsto 3, \ldots$.

   Hence, **every** integer is the image of something (twice, in fact).

   Therefore $\text{im}\, f = \mathbb{Z}$.

(iii) We have the following:

$$\ldots, -3 \mapsto 3,\ -2 \mapsto 2,\ -1 \mapsto 1,\ 0 \mapsto 0,\ 1 \mapsto 1,\ 2 \mapsto 4,\ 3 \mapsto 9,\ 4 \mapsto 16, \ldots.$$

Every $n \geq 0$ is the image of something; for example, if $n \geq 0$ then $-n$ maps to n. However, no negative integer is 'hit by an arrow'.

Therefore $\operatorname{im} f = \mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

(iv) We have the following:

$$\ldots, -3 \mapsto 5,\ -2 \mapsto 3,\ -1 \mapsto 1,\ 0 \mapsto 0,\ 1 \mapsto 2,\ 2 \mapsto 4,\ 3 \mapsto 6,\ 4 \mapsto 8, \ldots.$$

Hence every $n \geq 0$ is the image of something; the even non-negative integers are the images of non-negative integers and the odd positive integers are images of negative integers. However, no negative integer is an image.

Therefore $\operatorname{im} f = \mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

(v) We have the following.

Multiples of 3:

$$\ldots,\ -9 \mapsto -2,\ -6 \mapsto -1,\ -3 \mapsto 0,\ 0 \mapsto 1,\ 3 \mapsto 2,\ 6 \mapsto 3,\ 9 \mapsto 4, \ldots;$$

non-multiples of 3:

$$\ldots,\ -4 \mapsto 17,\ -2 \mapsto 5,\ -1 \mapsto 2,\ 1 \mapsto 2,\ 2 \mapsto 5,\ 4 \mapsto 17,\ 5 \mapsto 26, \ldots.$$

Since each integer is the image of some the multiple of 3, it does not really matter about the images of the non-multiples of 3.

Hence we have $\operatorname{im} f = \mathbb{Z}$.

Differentiation

1. Write down the equations of each of the following straight lines.

   (i) The straight line with gradient 3 and y-intercept -4.

   (ii) The straight line with gradient -4 and y-intercept 3.

   (iii) The straight line with y-intercept 2 that passes through the points $(1, 7)$ and $(3, 17)$.

   (iv) The straight line that passes through the points $(0, -4)$ and $(2, 2)$.

   (v) The straight line with gradient -2 that passes through the point $(1, 4)$.

2. Find the gradient function $\dfrac{dy}{dx}$ in each of the following cases.

   (i) $y = x^8$

   (ii) $y = x^{12}$

   (iii) $y = \dfrac{1}{x^2}$

   (iv) $y = \dfrac{1}{\sqrt{x}}$

   (v) $y = 3x^4 - 5x^2 - 3x$

   (vi) $y = x^2(3x^4 - 7)$

   (vii) $y = 5x^3\left(1 - \dfrac{2}{x^7}\right)$

   (viii) $y = \dfrac{8x^5 + 7x^2}{x^3}$

   (ix) $y = \sqrt{x}(x^2 + 2x - 1)$

   (x) $y = (x^2 + 2)^2$

3. (i) If $y = 5x^4 - 12x^2 + 3$, find $\dfrac{dy}{dx}$ when $x = 1$.

   (ii) If $y = \dfrac{3x^2 + 7x + 1}{x^2}$, find $\dfrac{dy}{dx}$ when $x = -1$.

   (iii) If $y = 5\sqrt{x} + \dfrac{5}{\sqrt{x}}$, find $\dfrac{dy}{dx}$ when $x = 9$.

   (iv) If $y = 2x^3 + 3x^2 - 12x + 1$, find the points where $\dfrac{dy}{dx} = 0$.

Differentiation: Solutions

1. Recall that $y = mx + c$ had gradient m and y-intercept c.

   (i) $y = 3x - 4$

   (ii) $y = -4x + 3$

   (iii) The gradient is $m = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{17-7}{3-1} = \dfrac{10}{2} = 5$.
   Hence (since the y-intercept is 2) the equation is $y = 5x + 2$.

   (iv) The gradient is $m = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{2-(-4)}{2-0} = \dfrac{6}{2} = 3$.
   The y-intercept is -4.
   Hence the equation is $y = 3x - 4$.

   (v) The equation is $y = -2x + c$ where c is the y-intercept.
   Substituting $x = 1$ and $y = 4$ (since the line passes through $(1, 4)$) gives $4 = -2 + c$ so $c = 6$.
   Hence the equation is $y = -2x + 6$.

2. (i) $\dfrac{dy}{dx} = 8x^7$

   (ii) $\dfrac{dy}{dx} = 12x^{11}$

   (iii) $y = \dfrac{1}{x^2} = x^{-2}$ so $\dfrac{dy}{dx} = -2x^{-3} = -\dfrac{2}{x^3}$

   (iv) $y = \dfrac{1}{\sqrt{x}} = x^{-1/2}$ so $\dfrac{dy}{dx} = -\tfrac{1}{2}x^{-\frac{3}{2}} = -\dfrac{1}{2x^{3/2}}$

   (v) $\dfrac{dy}{dx} = 12x^3 - 10x - 3$

   (vi) $y = x^2(3x^4 - 7) = 3x^6 - 7x^2$ so $\dfrac{dy}{dx} = 18x^5 - 14x$.

   (vii) $y = 5x^3\left(1 - \dfrac{2}{x^7}\right) = 5x^3 - 10x^{-4}$ so $\dfrac{dy}{dx} = 15x^2 + 40x^{-5} = 15x^2 + \dfrac{40}{x^5}$

   (viii) $y = \dfrac{8x^5 + 7x^2}{x^3} = 8x^2 + 7x^{-1}$ so $\dfrac{dy}{dx} = 16x - 7x^{-2} = 16x - \dfrac{7}{x^2}$

   (ix) $y = \sqrt{x}(x^2 + 2x - 1) = x^{5/2} + 2x^{3/2} - x^{1/2}$ so

$$\frac{dy}{dx} = \tfrac{5}{2}x^{3/2} + 3x^{1/2} - \tfrac{1}{2}x^{-1/2} = \frac{1}{\sqrt{x}}\left(\tfrac{5}{2}x^2 + 3x - \tfrac{1}{2}\right)$$

(x) $y = (x^2+2)^2 = x^4 + 4x^2 + 4$ so $\dfrac{dy}{dx} = 4x^3 + 8x = 4x(x^2+2)$

3. (i) If $y = 5x^4 - 12x^2 + 3$, then $\dfrac{dy}{dx} = 20x^3 - 24x$. When $x = 1$, $\dfrac{dy}{dx} = -4$.

(ii) If $y = \dfrac{3x^2+7x+1}{x^2} = 3 + 7x^{-1} + x^{-2}$, then $\dfrac{dy}{dx} = -7x^{-2} - 2x^{-3} = -\dfrac{7}{x^2} - \dfrac{2}{x^3}$.

When $x = -1$, $\dfrac{dy}{dx} = -7 + 2 = -5$.

(iii) If $y = 5\sqrt{x} + \dfrac{5}{\sqrt{x}} = 5x^{1/2} + 5x^{-1/2}$ then $\dfrac{dy}{dx} = \tfrac{5}{2}x^{-1/2} - \tfrac{5}{2}x^{-3/2} = \dfrac{5}{2x^{1/2}} - \dfrac{5}{2x^{3/2}}$.

When $x = 9$ then $\dfrac{dy}{dx} = \dfrac{5}{2\sqrt{9}} - \dfrac{5}{2\times 9\sqrt{9}} = \tfrac{5}{6} - \tfrac{5}{54} = \tfrac{20}{27}$.

(iv) If $y = 2x^3 + 3x^2 - 12x + 1$, then $\dfrac{dy}{dx} = 6x^2 + 6x - 12$.

Therefore
$$\begin{aligned} \frac{dy}{dx} = 0 \quad &\Rightarrow \quad 6x^2 + 6x - 12 = 0 \\ &\Rightarrow \quad x^2 + x - 2 = 0 \\ &\Rightarrow \quad (x-1)(x+2) = 0 \\ &\Rightarrow \quad x = 1 \text{ or } x = -2. \end{aligned}$$

When $x = 1$, $y = 2 + 3 - 12 + 1 = -6$.

When $x = -2$, $y = -16 + 12 + 24 + 1 = 21$.

The two points where $\dfrac{dy}{dx} = 0$ are $(1, -6)$ and $(-2, 21)$.

Integration

1. Evaluate each of the following indefinite integrals.

   (i) $\displaystyle\int x^6 + x^3 + 7 \; dx$

   (ii) $\displaystyle\int \frac{2}{x^2} + \frac{9}{x^4} \; dx$

   (iii) $\displaystyle\int 5x^4 + 6x^2 + \frac{1}{x^3} \; dx$

   (iv) $\displaystyle\int x^3 \left(5x^6 - x^2\right) \; dx$

   (v) $\displaystyle\int (x^2 + 1)(3x^3 + 5x) \; dx$

   (vi) $\displaystyle\int x^{5/2} + \tfrac{1}{2}x^{3/2} \; dx$

   (vii) $\displaystyle\int \sqrt{x}\left(x^3 + 1\right) \; dx$

   (viii) $\displaystyle\int \frac{x^3 + 2}{3\sqrt{x}} \; dx$

2. Find the area under each of the following curves between the points specified.

   (i) The curve $y = x^2 + 1$ between $x = -1$ and $x = 2$.

   (ii) The curve $y = x^3 - x$ between $x = 0$ and $x = 1$.

   (iii) The curve $y = 3\sqrt{x} + x$ between $x = 1$ and $x = 4$.

   (iv) The curve $y = \dfrac{1}{x^2}$ between $x = 1$ and $x = 3$.

   (v) The curve $y = (x + 1)^2$ between $x = -2$ and $x = 2$.

   (vi) The curve $y = \dfrac{2}{\sqrt{x}}$ between $x = 4$ and $x = 9$.

Integration: Solutions

1. (i) $\displaystyle\int x^6 + x^3 + 7\ dx = \tfrac{1}{7}x^7 + \tfrac{1}{4}x^4 + 7x + c$

(ii) $\displaystyle\int \frac{2}{x^2} + \frac{9}{x^4}\ dx = -\frac{2}{x} - \frac{3}{x^3} + c$

(iii) $\displaystyle\int 5x^4 + 6x^2 + \frac{1}{x^3}\ dx = x^5 + 2x^3 - \frac{1}{2x^2} + c$

(iv) $\displaystyle\int x^3\left(5x^6 - x^2\right)\ dx = \int 5x^9 - x^5\ dx = \tfrac{1}{2}x^{10} - \tfrac{1}{6}x^6 + c$

(v) $\displaystyle\int (x^2+1)(3x^3+5x)\ dx = \int 3x^5 + 8x^3 + 5x\ dx = \tfrac{1}{2}x^6 + 2x^4 + \tfrac{5}{2}x^2 + c$

(vi) $\displaystyle\int x^{5/2} + \tfrac{1}{2}x^{3/2}\ dx = \tfrac{2}{7}x^{7/2} + \tfrac{1}{5}x^{5/2} + c$

(vii) $\displaystyle\int \sqrt{x}\left(x^3+1\right)\ dx = \int x^{7/2} + x^{1/2}\ dx = \tfrac{2}{9}x^{9/2} + \tfrac{2}{3}x^{3/2} + c$

(viii) $\displaystyle\int \frac{x^3+2}{3\sqrt{x}}\ dx = \int \tfrac{1}{3}x^{5/2} + \tfrac{2}{3}x^{-1/2}\ dx = \tfrac{2}{9}x^{3/2} + \tfrac{4}{3}x^{1/2} + c$

2. (i) $\displaystyle\int_{-1}^{2} x^2 + 1\ dx = \left[\tfrac{1}{3}x^3 + x\right]_{-1}^{2} = (\tfrac{8}{3}+2) - (-\tfrac{1}{3}-1) = 6.$

(ii) $\displaystyle\int_{0}^{1} x^3 - x\ dx = \left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_{0}^{1} = (\tfrac{1}{4}-\tfrac{1}{2}) - (0) = -\tfrac{1}{4}.$

(iii) $\displaystyle\int_{1}^{4} 3\sqrt{x} + x\ dx = \left[2x^{3/2} + \tfrac{1}{2}x^2\right]_{1}^{4} = (16+8) - \left(2+\tfrac{1}{2}\right) = \tfrac{43}{2}.$

(iv) $\displaystyle\int_{1}^{3} \frac{1}{x^2}\ dx = \left[-\frac{1}{x}\right]_{1}^{3} = -\tfrac{1}{3} + 1 = \tfrac{2}{3}.$

(v) $\displaystyle\int_{-2}^{2} (x+1)^2\ dx = \int_{-2}^{2} x^2 + 2x + 1\ dx = \left[\tfrac{1}{3}x^3 + x^2 + x\right]_{-2}^{2}$

$\displaystyle = (\tfrac{8}{3}+4+2) - (-\tfrac{8}{3}+4-2) = \tfrac{28}{3}.$

(vi) $\displaystyle\int_{4}^{9} \frac{2}{\sqrt{x}}\ dx = \left[4\sqrt{x}\right]_{4}^{9} = 12 - 8 = 4.$

Vectors 1

1. If $\mathbf{a} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find:

   (i) $\mathbf{a} + \mathbf{b}$

   (ii) $\mathbf{a} + \mathbf{b} + \mathbf{c}$

   (iii) $\mathbf{a} + 5\mathbf{c}$

   (iv) $\mathbf{b} - 2\mathbf{c}$

   (v) $\|\mathbf{b}\|$

   (vi) $\|\mathbf{a} - \mathbf{b}\|$

   (vii) $\|\mathbf{a} - \mathbf{b} + \mathbf{c}\|$

   (viii) the value of k such that $k\mathbf{a} + \mathbf{b} = \begin{pmatrix} 1 \\ 18 \end{pmatrix}$

   (ix) the value of k such that $\mathbf{a} + \mathbf{b} = k\mathbf{c} + \begin{pmatrix} 7 \\ 6 \end{pmatrix}$

   (x) the value of $\|k\mathbf{a} + \mathbf{b}\|$ in terms of k.

2. If $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}$, find:

   (i) $\mathbf{a} + \mathbf{b}$

   (ii) $\mathbf{a} + \mathbf{b} + \mathbf{c}$

   (iii) $\mathbf{a} + 5\mathbf{c}$

   (iv) $\mathbf{b} - 2\mathbf{c}$

   (v) $\mathbf{a} + k\mathbf{b} + \ell\mathbf{c}$ in terms of k and ℓ

   (vi) $\|\mathbf{b}\|$

   (vii) $\|\mathbf{a} - \mathbf{b}\|$

   (viii) $\|\mathbf{a} - \mathbf{b} + \mathbf{c}\|$

   (ix) $\|\mathbf{a} + k\mathbf{b}\|$ in terms of k

   (x) $\|k\mathbf{b} + \ell\mathbf{c}\|$ in terms of k and ℓ.

Vectors 1 : Solutions

1. (i) $\mathbf{a}+\mathbf{b}=\begin{pmatrix}3\\8\end{pmatrix}$

(ii) $\mathbf{a}+\mathbf{b}+\mathbf{c}=\begin{pmatrix}5\\7\end{pmatrix}$

(iii) $\mathbf{a}+5\mathbf{c}=\begin{pmatrix}9\\0\end{pmatrix}$

(iv) $\mathbf{b}-2\mathbf{c}=\begin{pmatrix}0\\5\end{pmatrix}$

(v) $\|\mathbf{b}\|=5$

(vi) $\|\mathbf{a}-\mathbf{b}\|=\left\|\begin{pmatrix}-5\\2\end{pmatrix}\right\|=\sqrt{29}$

(vii) $\|\mathbf{a}-\mathbf{b}+\mathbf{c}\|=\left\|\begin{pmatrix}-3\\1\end{pmatrix}\right\|=\sqrt{10}$

(viii) when $k=3$ we have $k\mathbf{a}+\mathbf{b}=\begin{pmatrix}1\\18\end{pmatrix}$

(ix) when $k=-2$ we have $\mathbf{a}+\mathbf{b}=k\mathbf{c}+\begin{pmatrix}7\\6\end{pmatrix}$.

(x)
$$\begin{aligned}\|k\mathbf{a}+\mathbf{b}\| &= \left\|\begin{pmatrix}4-k\\5k+3\end{pmatrix}\right\| \\ &= \sqrt{(4-k)^2+(5k+3)^2} \\ &= \sqrt{(16-8k+k^2)+(25k^2+30k+9)} \\ &= \sqrt{26k^2+22k+25}.\end{aligned}$$

2. (i) $\mathbf{a}+\mathbf{b}=\begin{pmatrix}3\\-1\\2\end{pmatrix}$

(ii) $\mathbf{a}+\mathbf{b}+\mathbf{c}=\begin{pmatrix}7\\0\\-3\end{pmatrix}$

(iii) $\mathbf{a}+5\mathbf{c}=\begin{pmatrix}22\\4\\-22\end{pmatrix}$

(iv) $\mathbf{b} - 2\mathbf{c} = \begin{pmatrix} -7 \\ -2 \\ 9 \end{pmatrix}$

(v) $\mathbf{a} + k\mathbf{b} + \ell\mathbf{c} = \begin{pmatrix} 2 + k + 4\ell \\ -1 + \ell \\ 3 - k - 5\ell \end{pmatrix}$

(vi) $\|\mathbf{b}\| = \sqrt{2}$

(vii) $\|\mathbf{a} - \mathbf{b}\| = \left\| \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \right\| = \sqrt{18}$

(viii) $\|\mathbf{a} - \mathbf{b} + \mathbf{c}\| = \left\| \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} \right\| = \sqrt{26}$

(ix)
$$\begin{aligned} \|\mathbf{a} + k\mathbf{b}\| &= \left\| \begin{pmatrix} 2 + k \\ -1 \\ 3 - k \end{pmatrix} \right\| \\ &= \sqrt{(2+k)^2 + 1 + (3-k)^2} \\ &= \sqrt{(k^2 + 4k + 4) + 1 + (k^2 - 6k + 9)} \\ &= \sqrt{2k^2 - 2k + 14} \end{aligned}$$

(x)
$$\begin{aligned} \|k\mathbf{b} + \ell\mathbf{c}\| &= \left\| \begin{pmatrix} k + 4\ell \\ \ell \\ -k - 5\ell \end{pmatrix} \right\| \\ &= \sqrt{(k+4\ell)^2 + \ell^2 + (k+5\ell)^2} \\ &= \sqrt{(k^2 + 8k\ell + 16\ell^2) + \ell^2 + (k^2 + 10k\ell + 25\ell^2)} \\ &= \sqrt{2k^2 + 18k\ell + 41\ell^2}. \end{aligned}$$

Vectors 2

1. If $\mathbf{a} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, find:

   (i) $\mathbf{a} \cdot \mathbf{b}$;

   (ii) $\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$;

   (iii) the value of θ in degrees.

2. Let $\mathbf{a} = \begin{pmatrix} k \\ 4 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

   (i) Find $\mathbf{a} \cdot \mathbf{b}$ in terms of k.

   (ii) Hence determine the value of k for which $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

3. If $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, find:

   (i) $\mathbf{a} \cdot \mathbf{b}$;

   (ii) $\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$;

   (iii) the value of θ in degrees.

4. Let $\mathbf{a} = \begin{pmatrix} -1 \\ k \\ 3 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

   (i) Find $\mathbf{a} \cdot \mathbf{b}$ in terms of k.

   (ii) Hence determine the value of k for which $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

5. Let $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ -1 \\ 3 \end{pmatrix}$ be two 4-dimensional vectors.

   (i) Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$.

   (ii) Find $\mathbf{a} \cdot \mathbf{b}$.

   (iii) Hence evaluate $\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

   (iv) Determine θ.

6. Three documents D_1, D_2 and D_3 in some document collection are represented by the 5-dimensional vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 4 \\ -5 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 5 \\ 6 \\ -1 \end{pmatrix}, \text{ and } \mathbf{v}_3 = \begin{pmatrix} -2 \\ 3 \\ 1 \\ -1 \\ 4 \end{pmatrix},$$

respectively.

(i) Calculate $\|\mathbf{v}_1\|$, $\|\mathbf{v}_2\|$ and $\|\mathbf{v}_3\|$.

(ii) Evaluate $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_2 \cdot \mathbf{v}_3$, and $\mathbf{v}_1 \cdot \mathbf{v}_3$.

(iii) Calculate the cosine similarity between *each pair* of documents.

(iv) Which pair of documents are most similar? Justify your answer.

Vectors 2 : Solutions

1. (i) $\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix}-1\\5\end{pmatrix}\cdot\begin{pmatrix}4\\3\end{pmatrix} = -4+15 = 11.$

   (ii) Firstly, $\|\mathbf{a}\| = \sqrt{26}$ and $\|\mathbf{b}\| = 5$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{11}{5\sqrt{26}}.$

   (iii) Hence $\theta = \cos^{-1}\left(\dfrac{11}{5\sqrt{26}}\right) = \cos^{-1}(0.4314\ldots) = 64.4°.$

2. (i) $\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix}k\\4\end{pmatrix}\cdot\begin{pmatrix}2\\-3\end{pmatrix} = 2k-12.$

   (ii) Note that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular when the angle between them is 90° and, in this case, $\cos 90° = 0$.
   Hence $\mathbf{a}$ and $\mathbf{b}$ are perpendicular when $\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow 2k-12 = 0 \Rightarrow k = 6.$

3. (i) $\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix}3\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\4\\2\end{pmatrix} = 3-4-4 = -5$

   (ii) Firstly, $\|\mathbf{a}\| = \sqrt{9+1+4} = \sqrt{14}$ and $\|\mathbf{b}\| = \sqrt{1+16+4} = \sqrt{21}.$

   Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{-5}{\sqrt{14}\sqrt{21}} = -\dfrac{5}{\sqrt{2}\sqrt{7}\sqrt{3}\sqrt{7}} = -\dfrac{5}{7\sqrt{6}}.$

   (iii) Hence $\theta = \cos^{-1}\left(-\dfrac{5}{7\sqrt{6}}\right) = \cos^{-1}(-0.2916\ldots) = 106.95°.$

4. (i) $\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix}-1\\k\\3\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix} = -2-3k+3 = -3k+1.$

   (ii) Note that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular when the angle between them is 90° and, in this case, $\cos 90° = 0$.
   Hence $\mathbf{a}$ and $\mathbf{b}$ are perpendicular when $\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow -3k+1 = 0 \Rightarrow k = \frac{1}{3}.$

5. (i) $\|\mathbf{a}\| = \sqrt{4+1+0+9} = \sqrt{14}$ and $\|\mathbf{b}\| = \sqrt{16+4+1+9} = \sqrt{30}.$

   (ii) $\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix}2\\-1\\0\\3\end{pmatrix}\cdot\begin{pmatrix}4\\2\\-1\\3\end{pmatrix} = 8-2+0+9 = 15.$

(iii) $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{15}{\sqrt{14}\sqrt{30}} = \dfrac{15}{\sqrt{2}\sqrt{7}\sqrt{2}\sqrt{3}\sqrt{5}} = -\dfrac{5}{2\sqrt{105}}.$

(iv) Hence $\theta = \cos^{-1}\left(\dfrac{15}{2\sqrt{105}}\right) = \cos^{-1}(0.7319\ldots) = 42.95°.$

6. (i) $\|\mathbf{v}_1\| = \sqrt{1+1+4+16+25} = \sqrt{47},$

$\|\mathbf{v}_2\| = \sqrt{4+9+25+36+1} = \sqrt{75} = 5\sqrt{3}$ and

$\|\mathbf{v}_3\| = \sqrt{4+9+1+1+16} = \sqrt{31}.$

(ii) $$\mathbf{v}_1\cdot\mathbf{v}_2 = \begin{pmatrix}1\\-1\\2\\4\\-5\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\5\\6\\-1\end{pmatrix} = 2+3+10+24+5 = 44,$$

$$\mathbf{v}_2\cdot\mathbf{v}_3 = \begin{pmatrix}2\\-3\\5\\6\\-1\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\1\\-1\\4\end{pmatrix} = -4-9+5-6-4 = -18, \text{ and}$$

$$\mathbf{v}_1\cdot\mathbf{v}_3 = \begin{pmatrix}1\\-1\\2\\4\\-5\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\1\\-1\\4\end{pmatrix} = -2-3+2-4-20 = -27.$$

(iii) Between $\mathbf{v}_1$ and $\mathbf{v}_2$, the cosine similarity is

$$\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} = \frac{44}{\sqrt{47}\times 3\sqrt{5}} = \frac{44}{3\sqrt{235}} = 0.957.$$

Between $\mathbf{v}_2$ and $\mathbf{v}_3$, the cosine similarity is

$$\frac{\mathbf{v}_2\cdot\mathbf{v}_3}{\|\mathbf{v}_2\|\|\mathbf{v}_3\|} = \frac{-18}{3\sqrt{5}\sqrt{31}} = -\frac{6}{\sqrt{155}} = -0.241.$$

Between $\mathbf{v}_1$ and $\mathbf{v}_3$, the cosine similarity is

$$\frac{\mathbf{v}_1\cdot\mathbf{v}_3}{\|\mathbf{v}_1\|\|\mathbf{v}_3\|} = \frac{-27}{\sqrt{47}\sqrt{31}} = -\frac{27}{\sqrt{1547}} = -0.707.$$

(iv) Documents D_1 and D_2 are most similar as they have the largest cosine similarity; ie cosine similarity closest to 1.

Matrices

1. If $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}$, find:

   (i) $\mathbf{A} + 2\mathbf{B}$;

   (ii) $2\mathbf{A} - \mathbf{B}$;

   (iii) $\mathbf{AB}$;

   (iv) $\mathbf{BA}$;

   (v) $\det \mathbf{A}$;

   (vi) $\det \mathbf{B}$;

   (vii) $\det \mathbf{AB}$.

2. If $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$, find:

   (i) $\mathbf{A} + k\mathbf{B}$ where k is a real number;

   (ii) $\mathbf{AB}$;

   (iii) $\mathbf{BA}$;

   (iv) $\det \mathbf{A}$;

   (v) $\det \mathbf{B}$;

   (vi) $\det \mathbf{AB}$.

3. For each of the following matrices, $\mathbf{A}$, describe geometrically the mapping $\mathbb{R}^2 \to \mathbb{R}^2$ that is given by $\mathbf{x} \mapsto \mathbf{Ax}$.

   (i) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

   (ii) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$;

   (iii) $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$;

   (iv) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$;

   (v) $\mathbf{A} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$.

4. Each of the following matrices, $\mathbf{A}$, determines a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$.

   In each case, determine the image of the given triangle PQR. Show both the triangle PQR and its image $P'Q'R'$ on a diagram.

   (i) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$; $P(1,0)$, $Q(2,0)$ and $R(1,1)$.

   (ii) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$; $P(1,0)$, $Q(1,1)$ and $R(0,1)$.

   (iii) $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$; $P(1,0)$, $Q(1,1)$ and $R(0,1)$.

   (iv) $\mathbf{A} = \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix}$; $P(1,0)$, $Q(1,1)$ and $R(0,1)$.

Matrices Solutions

1. (i) $\mathbf{A} + 2\mathbf{B} = \begin{pmatrix} 7 & 4 \\ 1 & 9 \end{pmatrix}$;

   (ii) $2\mathbf{A} - \mathbf{B} = \begin{pmatrix} -6 & 3 \\ 12 & 3 \end{pmatrix}$;

   (iii) $\mathbf{AB} = \begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -8 & 5 \\ 14 & 14 \end{pmatrix}$;

   (iv) $\mathbf{BA} = \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 11 \\ 17 & 5 \end{pmatrix}$;

   (v) $\det \mathbf{A} = -3 - 10 = -13$;

   (vi) $\det \mathbf{B} = 12 - (-2) = 14$;

   (vii) $\det \mathbf{AB} = \det \begin{pmatrix} -8 & 5 \\ 14 & 14 \end{pmatrix} = 14 \times (-13) = -182 = \det \mathbf{A} \det \mathbf{B}$.

2. (i) $\mathbf{A} + k\mathbf{B} = \begin{pmatrix} -1+2k & 2-k \\ 1+3k & -1+2k \end{pmatrix}$;

   (ii) $\mathbf{AB} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ -1 & -3 \end{pmatrix}$;

   (iii) $\mathbf{BA} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 5 \\ -1 & 4 \end{pmatrix}$;

   (iv) $\det \mathbf{A} = 1 - 2 = -1$;

   (v) $\det \mathbf{B} = 4 - (-3) = 7$;

   (vi) $\det \mathbf{AB} = \det \begin{pmatrix} 4 & 5 \\ -1 & -3 \end{pmatrix} = -12 - (-5) = -7 = \det \mathbf{A} \det \mathbf{B}$.

3. (i) $\mathbf{Ax} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

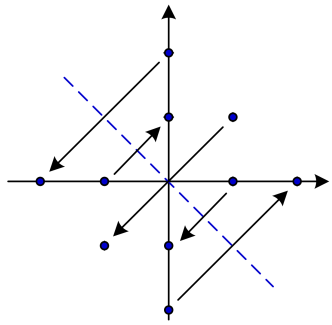
   The mapping $\mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix}$ is reflection in the x-axis.

   (ii) $\mathbf{Ax} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$.

   The mapping $\mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \end{pmatrix}$ is a rotation about the origin by $180°$.

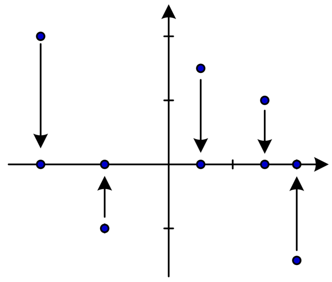
(iii) $\mathbf{Ax} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$.

The mapping $\mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ -x \end{pmatrix}$ is a reflection in the line $y = -x$.



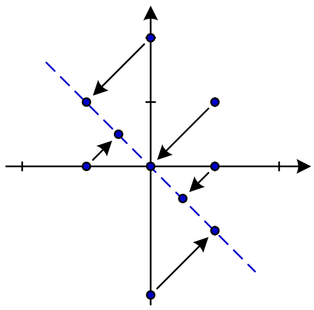
(iv) $\mathbf{Ax} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$.

The mapping $\mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$ collapses $\mathbb{R}^2$ onto the x-axis.



(v) $\mathbf{Ax} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x-y)/2 \\ (y-x)/2 \end{pmatrix}$.

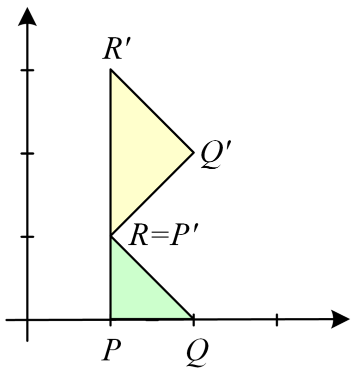
The mapping $\mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (x-y)/2 \\ (y-x)/2 \end{pmatrix}$ collapses $\mathbb{R}^2$ onto the line $y = -x$.



4. (i) P': $\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$, so P' is $(1,1)$.

Q': $\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}\begin{pmatrix}2\\ 0\end{pmatrix} = \begin{pmatrix}2\\ 2\end{pmatrix}$, so Q' is $(2,2)$.

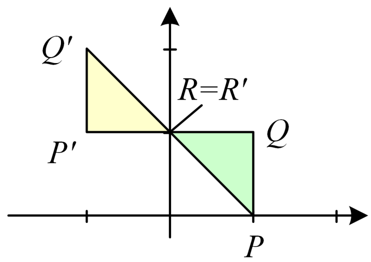
R': $\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 3\end{pmatrix}$, so R' is $(1,3)$.



(ii) P': $\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}-1\\ 1\end{pmatrix}$, so P' is $(-1,1)$.

Q': $\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}-1\\ 2\end{pmatrix}$, so Q' is $(-1,2)$.

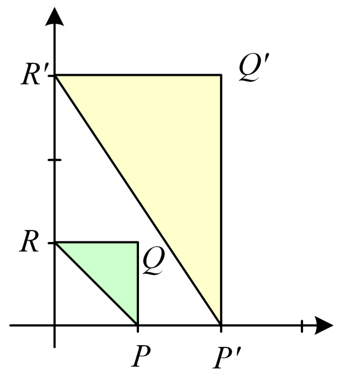
R': $\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$, so R' is $(0,1)$.



(iii) P': $\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}2\\ 0\end{pmatrix}$, so P' is $(2,0)$.

Q': $\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}2\\ 3\end{pmatrix}$, so Q' is $(2,3)$.

R': $\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 3\end{pmatrix}$, so R' is $(0,3)$.



(iv) P': $\begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$, so P' is $(-2, 0)$.

Q': $\begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, so Q' is $(-2, -2)$.

R': $\begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$, so R' is $(0, -3)$.