Linearizing the GPE to obtain a wave-like equation

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1 Introduction & the linearization scheme

Goal: In the following notes, we will attempt to review the derivation of the acoustic metric (a more general case of an effective metric is considered in [1]) (i.e., a wave-like equation) for a BEC system and show that their dynamics (with appropriate approximations) can closely replicate a massless scalar field in a curved spacetime.

1.1 Mean-field approximation of the Gross-Pitaevskii equation

In the dilute gas approximation, a Bose gas is identified by a quantum field $\hat{\Psi}(t,x)$[1] such that

$$i\hbar\frac{\partial\hat{\Psi}}{\partial t} = \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mathbb{V}_{ext} + g\hat{\Psi}^\dagger\hat{\Psi}\right)\hat{\Psi}. \tag{1}$$

Here, $g = \frac{4\pi\hbar^2 a}{m}$ parameterises the strength of the interactions amongst bosons in the gas; with a being the s-wave scattering amplitude and m being the atomic mass.

Now following [2] as usual, the natural ansatz for studying such a system is Bogoliubov's mean-field approach, in which case the field operator $\hat{\Psi}(t,x)$ can be separated into a (macroscopic) condensate term ψ_0 and a small perturbation term ψ_1,

$$\hat{\Psi} = \psi_0 + \epsilon\psi_1; \tag{2}$$

where, $\psi_0 \equiv \langle\hat{\Psi}\rangle$ is often called the wave function of the condensate and plays the role of an order parameter. Notice that ϵ is introduced here as a smallness parameter for bookkeeping purposes.

Inserting (2) into (1) yields two equations; one for each terms ψ_0 (of leading order ϵ^0) and ψ_1 (of order ϵ^1). We will deal with the equation containing ϵ^1 terms later, by

[1] For simplicity, we only consider 1+1 spacetime dimensions in what follows.

linearizing the fluctuations around the background field $\psi_0(t,x)$. For the leading order ϵ^0, we obtain:

$$i\hbar\,\partial_t\psi_0(t,x) = \left(\frac{-\hbar^2}{2m}\partial_x^2 + \mathbb{V}_{ext}(t,x) + g|\psi_0(t,x)|^2\right)\psi_0(t,x). \tag{3}$$

Equation (3) is the Gross-Pitaevskii equation [3] describing the dynamics of the order parameter. This equation is closed (decoupled) because all fluctuations are of order ϵ and thus do not appear in this leading-order analysis.

Consider the Madelung representation [4] for the wave function of the condensate

$$\psi_0(t,x) = \sqrt{n_0(t,x)}\,\exp\left\{\frac{i}{\hbar}\theta_0(t,x)\right\}, \tag{4}$$

and define an irrorational and inviscid velocity flow of the condensate by

$$v_0(t,x) = \frac{\partial_x\theta_0(t,x)}{m}. \tag{5}$$

By virtue of (4), we can rewrite (3) as follows:

$$\partial_t n_0 + \frac{1}{m}\partial_x\left(n_0\,\partial_x\theta_0\right) = 0, \tag{6}$$

$$\partial_t\theta_0 + \left(\frac{(\partial_x\theta_0)^2}{2m} + \mathbb{V}_{ext} + gn_0 - \frac{\hbar^2}{4m}\left(\frac{\partial_x^2 n_0}{n_0} - \frac{(\partial_x n_0)^2}{2n_0^2}\right)\right) = 0. \tag{7}$$

In fact, the equation of continuity (6) and the equation (7) for the phase[2] provide a closed set of coupled equations, exactly equivalent to the original Gross-Pitaevskii equation (3).

1.2 Linearizing fluctuations around the background equations

Now let's linearize the equations of motion (6) and (7) around some assumed background $(n_0(t,x),\,\theta_0(t,x))$ in the following sense:

$$n_0(t,x) \mapsto n_0(t,x) + \epsilon\,n_1(t,x) + \mathcal{O}\big(\epsilon^2\big), \tag{8a}$$

$$\theta_0(t,x) \mapsto \theta_0(t,x) + \epsilon\,\theta_1(t,x) + \mathcal{O}\big(\epsilon^2\big). \tag{8b}$$

[2] Equation (7) can also be expressed in terms of the parameter $v_0(t,x)$ by,

$$m\,\partial_t v_0 + \partial_x\left(\frac{1}{2}mv_0^2 + \mathbb{V}_{ext} + gn_0 - \frac{\hbar^2}{4m}\left(\frac{\partial_x^2 n_0}{n_0} - \frac{(\partial_x n_0)^2}{2n_0^2}\right)\right) = 0.$$

Substituting (8) into (4) and expanding in powers of ϵ gives,

$$\psi_0 \mapsto \psi_0 + \epsilon \, \exp\left\{\frac{i}{\hbar}\theta_0\right\} \left(\frac{n_1}{2\sqrt{n_0}} + \frac{i\sqrt{n_0}}{\hbar}\theta_1\right) + \mathcal{O}(\epsilon^2). \tag{9}$$

This linearization scheme naturally defines a representation of the perturbation term $\psi_1(t, x)$ introduced earlier in (2). Explicitly:

$$\psi_1 = \exp\left\{\frac{i}{\hbar}\theta_0\right\} \left(\frac{n_1}{2\sqrt{n_0}} + \frac{i\sqrt{n_0}}{\hbar}\theta_1\right). \tag{10}$$

To derive a closed set of coupled equations for these linearized perturbations, we substitute (8) into (6) and (7); Or alternatively, use the representation (9) in the original Gross-Pitaevskii equation. In both cases, we will retain terms only of the order ϵ^1.

At this linear order ϵ^1, we get:

$$\partial_t n_1 + \frac{1}{m}\partial_x \left(n_0 \, \partial_x\theta_1 + n_1 \, \partial_x\theta_0\right) = 0, \tag{11}$$

$$\partial_t\theta_1 + \frac{(\partial_x\theta_0)(\partial_x\theta_1)}{m} + g n_1 - \frac{\hbar^2}{4m}\hat{\mathfrak{D}}_2 n_1 = 0. \tag{12}$$

Here $\hat{\mathfrak{D}}_2$ represents a second-order differential operator defined as,

$$\hat{\mathfrak{D}}_2 n_1 \equiv \frac{1}{n_0}(\partial_x^2 n_1) - \frac{\partial_x n_0}{n_0^2}(\partial_x n_1) - \frac{\partial_x^2 n_0}{n_0^2} n_1 + \frac{(\partial_x n_0)^2}{n_0^3} n_1. \tag{13}$$

2 Derivation of a wave-like equation

From the equations (11) and (12), it is possible to derive a wave-like equation that describes the propagation of the linearized phase $\theta_1(t, x)$. To see that, we must first isolate for n_1 from (12) to get,

$$n_1 = -\left(g - \frac{\hbar^2}{4m}\hat{\mathfrak{D}}_2\right)^{-1} \left(\partial_t + v_0(t, x)\,\partial_x\right)\theta_1. \tag{14}$$

Now substituting (14) into (11) yields (modulo an overall sign):

$$\begin{aligned} &-\partial_t\left[\left(g - \frac{\hbar^2}{4m}\hat{\mathfrak{D}}_2\right)^{-1}\left(\partial_t + v_0(t, x)\,\partial_x\right)\theta_1\right] + \\ &\frac{1}{m}\partial_x\left[n_0\,\partial_x\theta_1 - (\partial_x\theta_0)\left(g - \frac{\hbar^2}{4m}\hat{\mathfrak{D}}_2\right)^{-1}\left(\partial_t + v_0(t, x)\,\partial_x\right)\theta_1\right] = 0. \end{aligned} \tag{15}$$

2.1 Quasi-classical approximation

At this point, the most straightforward approximation (namely, "quasi-classical" approximation, as in [1]) is to simply neglect the terms containing $\hat{\mathfrak{D}}_2$. This is justified in the sense that terms containing $\hat{\mathfrak{D}}_2$ is always multiplied by $\hbar^2$. This exhibits its *truly* quantum nature and hence it shall be suitably small compared to other macroscopic parameters. Within this approximation, equation (15) simplifies to:

$$\partial_t^2\theta_1 + (-c_s^2 + v_0^2)\,\partial_x^2\theta_1 + 2v_0\,\partial_t\partial_x\theta_1 + \left(\partial_t v_0 \frac{\partial_x\theta_1}{m} + \partial_x v_0\,(\partial_t + 2v_0\,\partial_x)\,\theta_1\right) = 0, \tag{16}$$

where, $c_s \equiv \sqrt{gn_0/m}$ is the *so-called* speed of sound.

Here, the fourth and fifth terms contain either $\partial_t v_0(t,x)$ or $\partial_x v_0(t,x)$; so if we additionally assume that the gradients of the velocity flow of the condensate (i.e., $\partial_\mu v_0(t,x)$) are negligible compared to the gradients of θ_1, then they disappear; and we are left with a **wave equation** containing only second derivatives in both space and time! Explicitly:

$$\partial_t^2\theta_1 + (-c_s^2 + v_0^2)\,\partial_x^2\theta_1 + 2v_0\,\partial_t\partial_x\theta_1 = 0. \tag{17}$$

2.2 In case of slowly varying density and velocity flow of the condensate

Let's now consider an interesting case where the aforementioned approximation does not hold. In the quasi-classical (hydrodynamical) regime, we completely neglected the terms containing $\hat{\mathfrak{D}}_2$; however, if we keep those terms with an additional assumption that the density $n_0(t,x)$ and the velocity flow $v_0(t,x)$ of the condensate is slowly varying compared to the perturbation, the differential operator $\hat{\mathfrak{D}}_2$ (cf. equation (13)) can be identified as,

$$\hat{\mathfrak{D}}_2 n_1 \equiv \frac{1}{n_0}(\partial_x^2 n_1). \tag{18}$$

Notice the last three terms in (13) are dropped. To this end, incorporating (18) in (12), we rewrite equation (11) and (12) more conveniently as follows:

$$(\partial_t + v_0\partial_x)\,n_1 + \frac{n_0}{m}\partial_x^2\,\theta_1 = 0, \tag{19}$$

$$(\partial_t + v_0\partial_x)\,\theta_1 + g\left(1 - \frac{\xi^2}{4}\partial_x^2\right)n_1 = 0; \tag{20}$$

where, $\xi(t,x) = \hbar/mc_s = \hbar/\sqrt{gmn_0(t,x)}$ is the characteristic healing length. Now, to get a closed from differential equation for θ_1, apply $(\partial_t + v_0\partial_x)$ on (20) and use (19) to obtain,

$$(\partial_t + v_0\partial_x)^2\,\theta_1 = c_s^2\partial_x^2\left(1 - \frac{\xi^2}{4}\partial_x^2\right)\theta_1. \tag{21}$$

Defining $\bar{k}(t,x) = 2/\xi(t,x)$, equation (21) becomes,

$$\left(\frac{\partial}{\partial t} + v_0(t,x)\frac{\partial}{\partial x}\right)^2\theta_1(t,x) = c_s^2\frac{\partial^2}{\partial x^2}\left(1 - \frac{1}{\bar{k}^2(t,x)}\frac{\partial^2}{\partial x^2}\right)\theta_1(t,x). \tag{22}$$

References

[1] Carlos Barcelo, Stefano Liberati, and Matt Visser. "Analogue gravity from Bose-Einstein condensates". In: *Classical and Quantum Gravity* 18.6 (2001), p. 1137.

[2] N Bogoliubov. "On the theory of superfluidity". In: *J. Phys* 11.1 (1947), p. 23.

[3] Lev Pitaevskii and Sandro Stringari. *Bose-Einstein condensation and superfluidity.* Vol. 164. Oxford University Press, 2016.

[4] Erwin Madelung. "Quantum theory in hydrodynamical form". In: *z. Phys* 40 (1927), p. 322.