Instantons in the functional integral formalism

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Abstract

These notes offer a review of instantons in semiclassical quantum tunneling, introducing the concept in the context of single-variable quantum mechanics with an eye toward generalizations to multi-variable systems and scalar field theory. Emphasis is placed on a clear exposition within the (Euclidean) functional integral framework, following the foundational approaches of [1, 2].

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1 Introductory remarks

The term "instanton" was coined by Gerard 't Hooft to describe certain localized solutions in classical field theories. These solutions resemble solitons, or particle-like structures (i.e., interpreted as a localized excitation or, according to standard field-theoretical practice, a *particle*) - hence the suffix "-on." However, unlike solitons, which are spatially localized, instantons are localized in time (considering *Euclidean* time as akin to a spatial dimension), which justifies the prefix "instant-". A.M. Polyakov referred to these objects as "pseudoparticles", a term that sporadically appears in the literature.

Instantons are crucial not only for understanding solitons but also for addressing tunneling phenomena in quantum field theory and quantum mechanics. These localized solutions arise from the inherent nonlinearity of the field equations, and in theories with a

small coupling constant, they can be studied via semiclassical methods. In this framework, instanton solutions in Euclidean spacetime provide the leading semiclassical exponential contribution to tunneling probabilities.

Instanton effects play a key role in understanding the ground state, or vacuum, structure of gauge theories and in studying early universe phenomena. Although these techniques may initially seem cumbersome in simpler cases, such as one-dimensional quantum mechanics, their true strength lies in their seamless extension to quantum field theory and quantum chromodynamics.

In the next two sections, we are particularly interested in looking at the instanton methods in the context of particle mechanics.

2 Euclidean functional integral

2.1 Preliminary basis

Consider a spinless particle of unit mass moving in potential $V(x)$ in one dimension:

$$H = \frac{p^2}{2} + V(x). \tag{2.1}$$

Our guiding tool of interest is the Euclidean (Wick-rotated to imaginary time by analytic continuation) version of Feynman's path integral:

$$\langle x_f | e^{\frac{-HT}{\hbar}} | x_i \rangle = N \int D[x(t)] e^{-S_E[x(t)]/\hbar}. \tag{2.2}$$

Let's breakdown equation (2.2) systematically:

On left hand side, $|x_i\rangle$ and $|x_f\rangle$ are position eigenstates, H is the Hamiltonian as in (2.1) and T is a positive number, sometimes referred to as normalization time. Notice that upon expanding LHS in a complete set of energy eigenstates,

$$H |n\rangle = E_n |n\rangle , \tag{2.3}$$

We get

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = \sum_n e^{-E_n T/\hbar} \langle x_f | n \rangle \langle n | x_i \rangle . \tag{2.4}$$

So, for large T, leading term in (2.4) is the energy and wavefunction of the lowest energy eigenstate.

On right hand side, N is a normalization factor, S_E is the Euclidean action defined by,

$$S_E[x(t)] = \int_{-T/2}^{T/2} dt \left[\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + V \right], \tag{2.5}$$

and $D[x(t)]$ denotes functional integration measure over all functions $x(t)$, obeying boundary conditions:

$$x\left(-\frac{T}{2}\right) = x_i, \quad x\left(\frac{T}{2}\right) = x_f. \tag{2.6}$$

To be more precise, if $\bar{x}(t)$ is any function obeying B.C.s (2.6), then, in general, $x(t)$ can be written as:

$$x(t) = \bar{x}(t) + \sum_n c_n x_n(t), \tag{2.7}$$

where x_n's are a complete set of real orthonormal functions vanishing at the boundaries, i.e.,

$$\int_{-T/2}^{T/2} dt\, x_n(t)x_m(t) = \delta_{nm}, \quad x_n\left(\pm\frac{T}{2}\right) = 0; \quad \forall\, n, m. \tag{2.8}$$

Then by virtue of equations (2.7) and (2.8), the measure $D[x(t)]$ can be identified as,

$$D[x(t)] = \prod_n \left(\frac{1}{\sqrt{2\pi\hbar}}\right) dc_n. \tag{2.9}$$

Here, the pre-factor $1/\sqrt{2\pi\hbar}$ is simply chosen for later convenience. It is worth pointing out now, that the RHS of (2.2) is particularly of interest, as it can be readily evaluated in small-$\hbar$ (semiclassical) limit, where the functional integral is significantly dominated by the stationary points of S_E.

To this end, let's assume now, for simplicity, that there exists only one such stationary point: $\bar{x}(t)$, so that

$$\frac{\delta S_E}{\delta \bar{x}(t)} = 0 \implies -\frac{d^2\bar{x}}{dt^2} + V'(\bar{x}) = 0, \tag{2.10}$$

where prime denotes ∂_x. Further, let's choose x_n's to be the eigenfunctions of second variational derivative of $S_E[x(t)]$ at the stationary point $\bar{x}(t)$, such that

$$\left[-\frac{d^2}{dt^2} + V''(\bar{x})\right] x_n = \lambda_n x_n. \tag{2.11}$$

Now, in the small-$\hbar$ limit, we know that the contribution of a stationary point to the functional integral is determined by a Gaussian integral over the perturbations about it (cf. equation (2.7)). So for paths $x(t)$ near $\bar{x}(t)$ (containing perturbations $\eta(t)$):

$$x(t) = \bar{x}(t) + \eta(t), \tag{2.12}$$

we obtain a quadratic action for fluctuations:

$$S_E^{(2)}[x(t)] = \int_{-T/2}^{T/2} dt\, \left[\frac{1}{2}\left(\frac{d\eta}{dt}\right)^2 + \frac{1}{2}V''(\bar{x})\,\eta^2\right]. \tag{2.13}$$

Here we integrate by parts the first term to throw away the vanishing boundary term therein (since all $\eta(t)$'s near $\bar{x}$ vanish at $\pm\, T/2$, according to (2.8)), to recast in the form

$$S_E^{(2)}[x(t)] = \int_{-T/2}^{T/2} dt\, \frac{1}{2}\eta(t)[-\partial_t^2 + V''(\bar{x})]\eta(t). \tag{2.14}$$

Earlier in (2.11) we introduced the eigenvalue equation of the corresponding operator (i.e., the Hessian operator $\mathcal{H}$ as introduced in appendix A) appearing here in (2.14). Combining them (in other words, expanding $\eta(t)$ in terms of $x_n(t)$) yields the following:

$$\begin{aligned} S_E^{(2)} &= \int_{-T/2}^{T/2} dt\, \left[\frac{1}{2}\left(\sum_n c_n x_n(t)\right)\left(\sum_m \lambda_m c_m x_m(t)\right)\right] \\ &= \frac{1}{2}\sum_{n,m} \lambda_m c_n c_m \int_{-T/2}^{T/2} dt\, x_n(t)x_m(t) \\ &= \frac{1}{2}\sum_n \lambda_n c_n^2. \end{aligned}$$

Where in the last equality, we made use of the orthonormality of x_n's (cf. equation (2.8)). Therefore, the contribution of this stationary point to the functional integral becomes a product of Gaussians, and it is summarized as follows:

$$\begin{aligned}
\langle x_f|e^{-HT/\hbar}|x_i\rangle &= N\int D[x(t)]e^{-S_E[x(t)]/\hbar} \\
&= Ne^{-S_E[\bar{x}(t)]/\hbar}\int D[x(t)]e^{-S_E^{(2)}/\hbar} \\
&= Ne^{-S_E[\bar{x}(t)]/\hbar}\int \prod_n \left[\frac{dc_n}{\sqrt{2\pi\hbar}}\right]\exp\left\{-\frac{1}{2\hbar}\sum_n \lambda_n c_n^2\right\} \\
&= Ne^{-S_E[\bar{x}(t)]/\hbar}\prod_n\left(\frac{1}{\sqrt{\lambda_n}}\right).
\end{aligned}$$

The quantity $\prod_n \lambda_n$ can be interpreted as the determinant of the operator $(-\partial_t^2 + V''(\bar{x}))$, since it is the product of its eigenvalues. So we can finally write equation (2.2) for our path integral (with a stationary point at $\bar{x}$) in its functional determinant form,

$$\langle x_f|e^{-HT/\hbar}|x_i\rangle = Ne^{-S_E[\bar{x}(t)]/\hbar}\left[\det\left(-\partial_t^2 + V''(\bar{x})\right)\right]^{-\frac{1}{2}} \tag{2.15}$$

A few comments, at this point, are in order:

- In deriving equation (2.15), we have been implicitly assuming that all the eigenvalues (λ_n's) are strictly positive; However, the cases where the eigenvalues can be zero or negative are subject to a more careful inspection (See appendix A).
- This method of computing functional integrals in the semiclassical limit can be easily generalized to a case of more than one stationary points, where one simply has to sum over all such stationary points.
- Notice that the equation (2.10) is indeed the equation of motion for a particle of unit mass moving in a potential *negative* V. Thus, the integral of motion is,

$$E = \frac{1}{2}\left(\frac{d\bar{x}}{dt}\right)^2 - V(\bar{x}), \tag{2.16}$$

  called the Euclidean energy. This can be used to determine the quantitative features of the solutions of (2.10), by inspection.

2.2 Computing functional determinant

To proceed further, we must first systematically compute $N\left[\det(-\partial_t^2 + V''(\bar{x}))\right]^{-\frac{1}{2}}$, in order to resolve the RHS of (2.15) for any pragmatic purposes. Thus, trailing the approach outlined in [1] (see appendix 1 therein), let's examine the following differential equation:

$$\left(-\partial_t^2 + \mathcal{W}(t)\right)\psi(t) = \lambda\,\psi(t), \tag{2.17}$$

where $\mathcal{W}(t)$ is some bounded function of t. The boundedness of $\mathcal{W}$ ensures that the operator $(-\partial_t^2 + \mathcal{W})$ is self-adjoint with a discrete, well-defined spectrum, allowing for a finite and regular determinant.

We then define $\psi_\lambda(t)$ to be the solution of this differential equation with initial conditions: $\psi_\lambda(-T/2) = 0$, and $\partial_t\psi_\lambda(-T/2) = 1$. Notice that (2.17) becomes an eigenvalue equation, i.e., the operator $(-\partial_t^2 + \mathcal{W}(t))$ has an eigenvalue λ_n, if and only if the corresponding eigenfunction vanishes at $+T/2$:

$$\psi_{\lambda_n}(T/2) = 0. \tag{2.18}$$

Now, we make use of the identity (see appendix B for a proof):

$$\det\left(\frac{-\partial_t^2 + \mathcal{W}^{(1)} - \lambda}{-\partial_t^2 + \mathcal{W}^{(2)} - \lambda}\right) = \frac{\psi_\lambda^{(1)}(T/2)}{\psi_\lambda^{(2)}(T/2)}, \tag{2.19}$$

where, the quantities are defined in the appendix as they appear.

In the context of the functional integral approach, determinants of differential operators, such as $\det(-\partial_t^2 + \mathcal{W})$, frequently arise. To handle these determinants in a well-defined way, it is essential to introduce normalization constants, like N, in a manner that simplifies the expressions while ensuring the results are finite and physically meaningful. To motivate a "natural" choice for N, let's consider the following definition:

$$N \equiv \frac{1}{\sqrt{\pi\hbar}}\left[\frac{\det(-\partial_t^2 + \mathcal{W})}{\psi_0(T/2)}\right]^{1/2}, \tag{2.20}$$

where $\psi_0(t)$ is a solution to the homogeneous equation

$$(-\partial_t^2 + \mathcal{W})\psi_0 = 0, \tag{2.21}$$

subject to the boundary conditions $\psi_0(-T/2) = 0$, and $\partial_t\psi_0(-T/2) = 1$. These boundary conditions ensure that $\psi_0(t)$ is uniquely defined, and its value at $t = T/2$, denoted by $\psi_0(T/2)$, becomes a quantity that characterizes the solution to this differential equation in a nontrivial[2.1] way that captures the influence of the potential $\mathcal{W}$. Hence, by virtue of (2.19), we ensure that the quantity N remains independent of the specific form of $\mathcal{W}$, as the dependence on $\mathcal{W}$ in $\det(-\partial_t^2 + \mathcal{W})$ is precisely canceled by the corresponding factor of $\psi_0(T/2)$.

This definition thus reduces our expression to a familiar form in evaluating the Gaussian functional integral,

$$N\left[\det\left(-\partial_t^2 + \mathcal{W}\right)\right]^{-\frac{1}{2}} = \left[\pi\hbar\,\psi_0(T/2)\right]^{-\frac{1}{2}}. \tag{2.22}$$

Next, we will be applying this method, firstly, in the simplest case of an ordinary harmonic oscillator; and then the nontrivial, yet prototypical case of the double well potential is considered. Hence it is in this latter case that we formally identify an instanton (and an anti-instanton) solution.

3 Application to various potential profiles

3.1 Good ol' simple harmonic oscillator

As a starting point, consider the potential:

$$V(x) = \frac{1}{2}\omega^2 x^2, \tag{3.1}$$

(a) $V(x)$ (b) $-V(x)$

Figure 1: *Potential* $V(x) = \frac{1}{2}\omega^2 x^2$.

as shown in the figure 1a; and the figure 1b shows the inverted potential, $-V(x)$. We simply choose $x_i = 0 = x_f$, so that the only (trivial) solution of (2.10) obeying the B.C.s is,

$$\bar{x}(t) = 0. \tag{3.2}$$

Also, we know that for this solution, the value of the Euclidean action: $S_E[\bar{x}(t)] = 0$. Thus, from (2.15),

$$\langle 0|e^{-HT/\hbar}|0\rangle = N\left[\det\left(-\partial_t^2 + \omega^2\right)\right]^{-\frac{1}{2}}. \tag{3.3}$$

In evaluating the right-hand side of (3.3), we need to compute the determinant of the operator $(-\partial_t^2 + \omega^2)$. This requires applying the precise methods developed in subsection 2.2, which provide a rigorous approach for handling the determinant of differential operators in the context of functional integrals. So in this case (cf. equations (2.21), (2.22))

$$\langle 0|e^{-HT/\hbar}|0\rangle = \left[\pi\hbar\,\psi_{\omega,0}(T/2)\right]^{-\frac{1}{2}}, \tag{3.4}$$

where, $\psi_{\omega,0}(T/2)$ is the solution of differential equation

$$(-\partial_t^2 + \omega^2)\psi_{\omega,0}(t) = 0, \quad \text{I.C.s: } \psi_{\omega,0}\left(-\frac{T}{2}\right) = 0, \quad \partial_t\psi_{\omega,0}\left(-\frac{T}{2}\right) = 1; \tag{3.5}$$

evaluated at $t = +T/2$. Of course, equation (3.5) admits a unique solution:

$$\psi_{\omega,0}(t) = \frac{\sinh\omega(T/2 + t)}{\omega}. \tag{3.6}$$

Thus, for large T,

$$\langle 0|e^{-HT/\hbar}|0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2}. \tag{3.7}$$

Therefore we can see, from equation (2.4), that equation (3.7) indeed produces the correct semiclassical results:

- The ground state energy of the particle in a harmonic oscillator:

$$E_0 = \frac{\hbar\omega}{2}. \tag{3.8}$$

- The probability of the particle being at origin, while in its ground state energy eigenstate:

$$|\langle x = 0|n = 0\rangle|^2 = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}. \tag{3.9}$$

[2.1]The latter initial condition $\partial_t\psi_0(-T/2) = 1$ fixes the initial rate of change of $\psi_0(t)$ at $t = -T/2$ to a specific, non-zero value. This choice avoids solutions that are trivially zero.

3.2 Well well.. the double well

We now move our attention to a nontrivial problem; particle in a double well potential (depicted in the figure 2). Should the following assumptions be imposed without loss of generality, we need not be concerned with the specific form of the potential $V(x)$ here:

(a) $V(x)$ (b) $-V(x)$

Figure 2: Double well potential.

- The potential is even: $V(x) = V(-x)$.
- It contains two degenerate minima at $x = \pm a$, and an arbitrary constant may be added to $V(x)$, if necessary, such that $V(\pm a) = 0$.
- As in the SHO case (cf. (3.1)), we shall require: $V''(\pm a) = \omega^2$.

Now we wish to compute the following quantities:

$$\langle a|e^{-HT/\hbar}|a\rangle = \langle -a|e^{-HT/\hbar}|-a\rangle\,, \tag{3.10a}$$

and,

$$\langle -a|e^{-HT/\hbar}|a\rangle = \langle a|e^{-HT/\hbar}|-a\rangle\,; \tag{3.10b}$$

by our functional integral technique in the small-$\hbar$ limit (see equation (2.15)).

As a first step, we seek solutions to the classical Euclidean equation of motion, (2.10), subject to the boundary conditions (2.6). Two trivial solutions, analogous to the simple harmonic oscillator, correspond to the particle remaining stationary at the top of either of the two hills in figure 2b.

However, of greater interest is a non-trivial solution in which the particle starts at, say, the top of the left hill ($x = -a$) at $t = -T/2$ and moves to the top of the right hill ($x = a$) at $t = T/2$. In the limit $T \to \infty$, this solution describes the particle asymptotically reaching the hilltops as $t \to \pm\infty$. We denote this solution as $x_{\texttt{inst}}(t)$, representing an *instanton*. In computing the transition amplitudes, all aforementioned three types of path must be taken into account. In the instanton case, its locus is governed by the equation of motion under the condition of vanishing Euclidean energy ($E = 0$):

$$\frac{dx_{\texttt{inst}}(t)}{dt} = \sqrt{2V(x_{\texttt{inst}})}. \tag{3.11}$$

This can be rewritten as:

$$t = t_1 + \int_0^{x_{\texttt{inst}}(t)} \frac{d\tilde{x}}{\sqrt{2V(\tilde{x})}}, \tag{3.12}$$

where t_1 is an integration constant, representing the time at which the solution passes through $x = 0$, i.e., $x_{\texttt{inst}}(t_1) = 0$. For this reason, $x_{\texttt{inst}}(t)$ is referred to as 'an instanton centered at t_1'. This solution has been sketched in the figure 3.

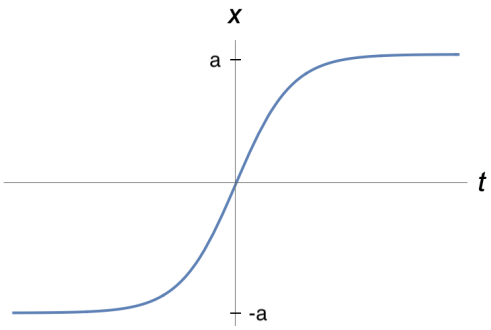

Figure 3: The instanton solution of a particle moving from $x = a$ to $x = -a$ in a double well potential.

We may, in principle, also construct a solution that moves from the right hill ($x = a$) to the left ($x = -a$) instead, merely by replacing $t \to -t$ in (3.12). Thus, they are called 'anti-instantons'.

Two key properties of instanton (or anti-instanton) solutions are important:

1. From (3.11), the action $S_E[x_{\texttt{inst}}(t)] =: S_{\texttt{inst}}$ of an instanton is given by,

$$\begin{aligned} S_{\texttt{inst}} &= \int dt \left[\frac{1}{2} \left(\frac{dx_{\texttt{inst}}}{dt} \right)^2 + V(x_{\texttt{inst}}) \right] = \int dt \left(\frac{dx_{\texttt{inst}}}{dt} \right)^2 \\ &= \int_{-a}^{a} dx_{\texttt{inst}} \sqrt{2V(x_{\texttt{inst}})}. \end{aligned} \tag{3.13}$$

   Where in the second equality, we use (3.11), and in the last equality we go from time integration to spatial integration using $dt = dx_{\texttt{inst}}/\sqrt{2V(x_{\texttt{inst}})}$.

   One may notice here that this expression coincides with the integral in the WKB formula. As we shall see, this is no coincidence.

2. For large t, as $x_{\texttt{inst}} \to a$, the equation of motion (3.11) can be approximated by

$$\frac{dx_{\texttt{inst}}}{dt} = \omega(a - x_{\texttt{inst}}). \tag{3.14}$$

   Solving this, we find:

$$(a - x_{\texttt{inst}}) \propto e^{-\omega t}, \tag{3.15}$$

   showing that for large t, the instanton approaches $x = a$ exponentially.

   Thus, instantons are localized objects (in Euclidean time, of course) with a characteristic size on the order of $1/\omega$.

This property is nontrivial because, in the large T limit, the equation of motion admits not only individual instantons and anti-instantons as approximate solutions but also more complex solutions composed of an array of widely separated[3.1] instantons and anti-instantons. To account for all such possibilities, we evaluate the functional integral by

[3.1] This imposition of instantons being widely separated, is of cardinal importance; and hence bears a distinct name: 'dilute-gas approximation'. We shall later advocate this assumption upon rather mathematical constructs.

summing over configurations containing n such objects (instantons and anti-instantons), with their centers located at $t_1, t_2, \ldots, t_n$, subject to the ordering:

$$-\frac{T}{2} < t_n < \ldots < t_2 < t_1 < \frac{T}{2}.$$

This situation is depicted in figure 4. As advertised, T is assumed to be large, hence the apparent distinction in sharpness of instanton (and anti-instanton) solutions as compared to the solitary one appearing in figure 3.

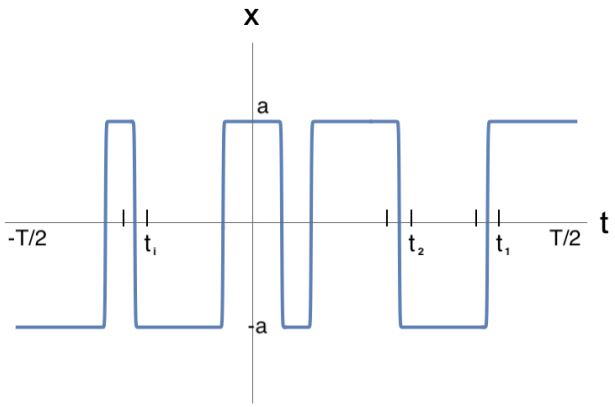

Figure 4: Dilute-gas approximation (i.e., summing over widely separated instantons). The vertical marks reflect the relative scale of localization encapsulating $i^{\texttt{th}}$ instanton ($i = 1, \ldots, n$) with center located slightly before $t = t_i$.

Now in order to actually calculate the propagators (transition amplitudes) (3.10) for each instanton configuration, as per equation (2.15), we need to multiply the following three ingredients as usual: (𝕀) Normalization constant: N, (𝕀𝕀) Exponential of the action: $e^{-S_E/\hbar}$, and, (𝕀𝕀𝕀) Functional determinant of Hessian: $[\det(-\partial_t^2 + V'')]^{-\frac{1}{2}}$. This task is, for n such (widely separated) objects, three-fold:

1. **Contribution of the exponential :** For n instantons (or anti-instantons), the total action is simply n times the action of a single instanton, $S_{\texttt{inst}}$. Hence, the exponential term becomes:

$$e^{-S_E/\hbar} \mapsto e^{-nS_{\texttt{inst}}/\hbar}. \tag{3.16}$$

2. **Integration over instanton centers :** To account for the positions of n instantons along the timeline $\left(-\frac{T}{2}, \frac{T}{2}\right)$, we integrate over their centers with the ordering constraint $t_1 > t_2 > \cdots > t_n$. This results in the time integral:

$$\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{t_1} dt_2 \ldots \int_{-T/2}^{t_{i-1}} dt_i \ldots \int_{-T/2}^{t_{n-1}} dt_n = \frac{T^n}{n!}. \tag{3.17}$$

   Also, there is an additional constraint in distributing n of these instantons and anti-instantons over the timeline $\left(-\frac{T}{2}, \frac{T}{2}\right)$: their arrangement must be alternating and respecting the boundary conditions. For instance, if the particle starts at $x = -a$, the sequence must begin with an instanton, followed by an anti-instanton, and so on. Physically, they are solutions with the particle repeatedly bouncing to-and-fro in the inverted potential. This restriction imposes that n must be even for terms like (3.10a), while n must be odd for terms like (3.10b).

3. Correction to the functional determinant : The functional determinant of the Hessian is affected by the presence of n instantons and anti-instantons. To understand this correction, consider the Euclidean time-evolution operator $e^{-HT/\hbar}$ as a product of contributions from the intervals between the instantons (vertical marks in figure 4). If these small intervals were absent, the second derivative of the potential, V'', would equal ω^2 over the entire time axis, resulting in the same expression as the single-well potential case (cf. equation (3.7)):

$$\left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2}.$$

The presence of instantons and anti-instantons introduces corrections to this determinant. The total correction for n widely separated instantons and anti-instantons modifies the formula as:

$$N\left[\det\left(-\partial_t^2 + V''\right)\right]^{-\frac{1}{2}} \mapsto \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2} K^n, \tag{3.18}$$

where K is a correction factor associated with a single instanton. Later, we explicitly calculate K to ensure that this formula aligns with the known result for a single instanton.

With this machinery, using equations (3.16), (3.17) and (3.18), we can readily compute the transition amplitudes (for now, although, in terms of the prefactor K)

$$\begin{aligned}
\langle -a|e^{-HT/\hbar}|-a\rangle &= Ne^{-S_E/\hbar}\left[\det\left(-\partial_t^2 + V''\right)\right]^{-\frac{1}{2}} \\
&= \sum_{\text{even } n}^{\infty} \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2} K^n\, e^{-nS_{\text{inst}}/\hbar} \overbrace{\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{t_1} dt_2 \ldots \int_{-T/2}^{t_{n-1}} dt_n}^{T^n/n!} \\
&= \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2} \sum_{\text{even } n}^{\infty} \frac{\left(TKe^{-S_{\text{inst}}/\hbar}\right)^n}{n!}.
\end{aligned} \tag{3.19}$$

Similarly, $\langle a|e^{-HT/\hbar}|-a\rangle$ is given by the same expression as (3.19); however, summed over odd n's. These sums are easy to evaluate:

$$\langle \pm a|e^{-HT/\hbar}|-a\rangle = \frac{1}{2}\left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2}\left[\exp\left\{TKe^{-S_{\text{inst}}/\hbar}\right\} \mp \exp\left\{-TKe^{-S_{\text{inst}}/\hbar}\right\}\right]. \tag{3.20}$$

Thus, upon comparison to equation (2.4) (i.e., expanding the LHS of (3.20) in a complete set of energy eigenstates), we register that the lowest two eigenenergies are,

$$E_{\pm} = \frac{1}{2}\hbar\omega \pm \hbar K e^{-S_{\text{inst}}/\hbar}, \tag{3.21}$$

And their respective eigenstates (namely, $|+\rangle$ and $|-\rangle$) imply:

$$\langle a|-\rangle\langle -|-a\rangle = |\langle \pm a|+\rangle|^2 = |\langle \pm a|-\rangle|^2 = \frac{1}{2}\left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}, \tag{3.22a}$$

$$\langle a|+\rangle\langle +|-a\rangle = -\frac{1}{2}\left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}. \tag{3.22b}$$

A Hessian of Euclidean action

In this appendix, we wish to study the spectral properties of the operator (also referred to as the Hessian) appearing in the second variational derivative of the Euclidean action evaluated at a stationary point ($\bar{x}(\tau)$).

The claim here is that, under standard conditions[A.1] [A.2], the eigenvalues of the Hessian operator are strictly positive, except for zero eigenvalues associated with the invariance group of the action (e.g., translational invariance for instantons).

To ensure the discussion is self-contained and accessible without requiring prior familiarity with the operator definition, we shall begin by introducing it from first principles. To that end, consider the Euclidean action $S_E[x(\tau)]$ for a path $x(\tau)$ (with fixed boundary conditions),

$$S_E[x(\tau)] = \int d\tau \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]. \tag{A.1}$$

Now assuming that there exists a stationary point $\bar{x}(\tau)$[A.3]; we know that it satisfies the (Euclidean analogue of) Euler-Lagrange equation of motion:

$$-\frac{d^2\bar{x}}{d\tau^2} + V'(\bar{x}) = 0, \tag{A.2}$$

and $\bar{x}(\tau)$ respects the same boundary conditions as our chosen paths $x(\tau)$.

Moreover, we are interested in looking at the variational derivatives of S_E at $\bar{x}(\tau)$. These are formally computed by evaluating the action on a general path $x(\tau)$ near the stationary point $\bar{x}(\tau)$ with perturbations $\eta(\tau)$:

$$x(\tau) = \bar{x}(\tau) + \epsilon\,\eta(\tau), \tag{A.3}$$

with ϵ being a book-keeping constant.

Thus by construction, the first variational derivative vanishes at $\bar{x}(\tau)$, and allows one to write:

$$S_E[\bar{x} + \epsilon\,\eta] = S_E[\bar{x}] + \epsilon^2 S_E^{(2)} + \mathcal{O}\big(\epsilon^3\big). \tag{A.4}$$

This residues, at order ϵ^2, the second variation ($S_E^{(2)}$) at a stationary point $\bar{x}(\tau)$ given explicitly by the quadratic action over perturbations $\eta(\tau)$:

$$S_E^{(2)}[x(\tau)] = \int d\tau\, \frac{1}{2}\,\eta\,\mathcal{H}\,\eta; \tag{A.5}$$

where, $\mathcal{H}$ is defined as the Hessian of the action,

$$\mathcal{H} \equiv -\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)). \tag{A.6}$$

The eigenvalue equation of $\mathcal{H}$, acting on an appropriate Hilbert space $L^2(\mathbb{R})$, is:

$$\mathcal{H}\ \psi_n(\tau) = \lambda_n \psi_n(\tau); \tag{A.7a}$$

[A.1] Assuming the potential is convex at its stationary points; in order to barricade those bound states with negative eigenvalues that otherwise correspond to non-trivial directions in configuration space where the action is locally depreciated.

[A.2] In field theory, negative eigenvalues may stem from instabilities in the vacuum or perturbative expansion (e.g., false vacuum decay).

[A.3] From this, the case of more than one stationary points is an easy generalization.

$$\left[-\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau))\right] \psi_n(\tau) = \lambda_n \psi_n(\tau). \tag{A.7b}$$

Here, λ_n are the eigenvalues, and $\psi_n(\tau)$ are the corresponding eigenfunctions. Their inner product is defined as

$$\langle \psi_m, \psi_n \rangle = \int_{-\infty}^{\infty} \omega(\tau)\, \psi_m(\tau)\psi_n(\tau)\, d\tau, \tag{A.8}$$

$\omega(\tau)$ is a weight function associated with the Sturm-Liouville form of the operator. In the case of $\mathcal{H}$, the weight function is typically $\omega(\tau) = 1$. This definition of the product makes the operator $\mathcal{H}$ self-adjoint in the standard $L^2(\mathbb{R})$ inner product space.

This inner product allows the eigenfunctions $\{\psi_n(\tau)\}$ to form an orthonormal basis for the space of square-integrable functions $L^2(\mathbb{R})$. This orthonormality of the eigenfunctions is expressed as

$$\int_{-\infty}^{\infty} \psi_m(\tau)\psi_n(\tau)\, d\tau = \delta_{mn}, \tag{A.9}$$

where δ_{mn} is the Kronecker delta. It also implies the expansion of any square-integrable test function $\eta(\tau)$ as a sum in the eigenbasis:

$$\eta(\tau) = \sum_{n=0}^{\infty} c_n \psi_n(\tau), \tag{A.10}$$

where the coefficients are given by:

$$c_n = \langle \psi_n, \eta \rangle = \int_{-\infty}^{\infty} \psi_n(\tau)\eta(\tau)\, d\tau. \tag{A.11}$$

Finally, the expression of the operator $\mathcal{H}$ in diagonal form in this eigenbasis is:

$$\mathcal{H}\,\eta(\tau) = \sum_{n=0}^{\infty} \lambda_n c_n \psi_n(\tau), \tag{A.12}$$

Now, we need to show:

1. The eigenvalues of $\mathcal{H}$ are non-negative.
2. Any zero eigenvalues are associated with invariance of the action under transformations (e.g., time-translation for instantons).

A.1 Strict positivity of eigenvalues away from zero

To analyze the spectrum of $\mathcal{H}$, we first observe that equation (A.7) is analogous to the time-independent Schrödinger equation. So the operator $-\frac{d^2}{d\tau^2}$ is kinetic energy-like term and $V''(\bar{x}(\tau))$ is an effective potential-like term.

- The former kinetic term (one-dimensional negative Laplacian) contribution is semi-positive definite in the following sense: for any smooth, square-integrable test function $\psi(\tau)$, the quadratic form is:

$$\begin{aligned} \langle \psi, -\frac{d^2}{d\tau^2}\psi \rangle &= \int d\tau\, \psi(\tau) \left(-\frac{d^2}{d\tau^2}\right) \psi(\tau) \\ &= \int d\tau \left(\frac{d\psi}{d\tau}\right)^2 \geq 0. \end{aligned} \tag{A.13}$$

The second equality is obtained using integration by parts; And the boundary term vanishes due to the assumption that $\psi(\tau)$ vanishes sufficiently rapidly as $|\tau| \to \infty$.

Thus, $-\frac{d^2}{d\tau^2}$ is semi-positive definite because the integral evaluates to zero if and only if $\frac{d\psi}{d\tau} = 0$, implying $\psi(\tau) = \text{const}$. Therefore, under reasonable[A.4] boundary conditions (Dirichlet boundary conditions in our case), its eigenvalue spectrum is non-negative.

- Assuming $V(x)$ to be convex, the latter effective potential term is: $V''(\bar{x}(\tau)) \geq 0$, on all stationary points of S_E (since $\bar{x}(\tau)$ minimizes the action, $V(x)$ must have a local minimum at $x = \bar{x}(\tau)$).

Thus, for any test function $\eta(\tau) \in L^2(\mathbb{R})$:

$$\langle \eta, \mathcal{H}\eta \rangle = \int d\tau\, \eta(\tau) \left(-\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)) \right) \eta(\tau) \geq 0. \tag{A.14}$$

This ensures that all eigenvalues λ_n are non-negative.

A more rigorous treatment requires treating the equation (A.7b) as a Sturm-Liouville problem and invoking the variational principle to minimize the associated Rayleigh quotient. In that context, the oscillation theorem also implies that all the eigenvalues are real, discrete and ordered as follows:

$$\lambda_0 < \lambda_1 < \lambda_2 \cdots \tag{A.15}$$

where, λ_0 is the smallest eigenvalue and the corresponding ground-state eigenfunction $\psi_0(x)$ has no zeros, reflecting the stability of the ground state.

A.2 Symmetry-induced zero modes

Let's examine the zero eigenvalues' case $\lambda_0 = 0$. The previous subsection A.1 ensures, by virtue of (A.15), that they are indeed the lowest eigenvalues ($\mathcal{H}$ is bounded from below). And the corresponding eigenfunctions (also called zero modes) $\psi_0(\tau)$ satisfy

$$\mathcal{H}\,\psi_0(\tau) = \left[-\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)) \right] \psi_0(\tau) = 0. \tag{A.16}$$

Now the claim is that those zero eigenvalues arise due to the invariance group of the action. For example for instantons, the action is invariant under time translation:

$$\tau \to \tau + \epsilon.$$

This symmetry implies a zero mode of the operator $\mathcal{H}$, given by

$$\psi_0(\tau) \propto \frac{dx_{\texttt{inst}}}{d\tau}.$$

To verify, substituting this into the left hand side of equation (A.16) to get

$$\mathcal{H}\,\frac{dx_{\texttt{inst}}}{d\tau} = -\frac{d^3 x_{\texttt{inst}}}{d\tau^3} + V''(x_{\texttt{inst}})\frac{dx_{\texttt{inst}}}{d\tau};$$

[A.4] Boundary conditions ensure that $-\frac{d^2}{d\tau^2}$ is a well-defined, self-adjoint operator.

And using the instanton equation of motion (cf. equation (A.2) differentiated with respect to τ once), we obtain the right hand side (zero):

$$\mathcal{H}\,\frac{dx_{\texttt{inst}}}{d\tau} = \frac{d}{d\tau}\left(-\frac{d^2x_{\texttt{inst}}}{d\tau^2} + V'(x_{\texttt{inst}})\right) = 0. \tag{A.17}$$

Hence, $\frac{dx_{\texttt{inst}}}{d\tau}$ is indeed an eigenfunction with eigenvalue $\lambda_0 = 0$. However, at face value, this form of zero mode is not yet normalized.

The zero mode $\psi_0(\tau)$ is normalized according to a standard condition for eigenfunctions (cf. equation (A.9)). This normalization involves computing the integral

$$\int_{-\infty}^{\infty} |\psi_0(\tau)|^2 d\tau = 1.$$

Substituting our ansatz $\psi_0(\tau) = \frac{dx_{\texttt{inst}}}{d\tau}$, the normalization condition becomes

$$\int_{-\infty}^{\infty} \left(\frac{dx_{\texttt{inst}}}{d\tau}\right)^2 d\tau = S_{\texttt{inst}}, \tag{A.18}$$

where $S_{\texttt{inst}}$ is the classical action for the instanton. This follows from:

$$S_{\texttt{inst}} = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{dx_{\texttt{inst}}}{d\tau}\right)^2 + V(x_{\texttt{inst}})\right] d\tau,$$

and by integrating the equation of motion,

$$\frac{1}{2}\left(\frac{dx_{\texttt{inst}}}{d\tau}\right)^2 = V(x_{\texttt{inst}}),$$

we find equation (A.18). Thus, $\psi_0(\tau)$ ought to be normalized by dividing by $\sqrt{S_{\texttt{inst}}}$, yielding

$$\psi_0(\tau) = (S_{\texttt{inst}})^{-\frac{1}{2}}\,\frac{dx_{\texttt{inst}}}{d\tau}. \tag{A.19}$$

Finally, note that such a zero mode contributes to a flat direction in the appropriate configuration space corresponding to the underlying symmetry. And the number of zero modes corresponds to the number of symmetries of the action. For a single instanton, there is one (time-) translational zero mode.

B Proof of an identity

The goal here is to prove the following identity:

$$\det\left(\frac{-\partial_t^2 + \mathcal{W}^{(1)} - \lambda}{-\partial_t^2 + \mathcal{W}^{(2)} - \lambda}\right) = \frac{\psi_\lambda^{(1)}(T/2)}{\psi_\lambda^{(2)}(T/2)}, \tag{B.1}$$

where $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ are two bounded functions of t, and $\psi_\lambda^{(1)}$ and $\psi_\lambda^{(2)}$ are the associated solutions as defined in 2.2. The proof is as follows:

Step 1 : For the left hand side, define the eigenvalues of the operators $(-\partial_t^2 + \mathcal{W}^{(1)})$ and $(-\partial_t^2 + \mathcal{W}^{(2)})$ as $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$, respectively. We can write the determinants as products over these eigenvalues:

$$\det\big(-\partial_t^2 + \mathcal{W}^{(\nu)} - \lambda\big) = \prod_n (\lambda_n^{(\nu)} - \lambda), \quad \nu \in \{1, 2\}.$$

Step 2 : For the right hand side, the eigenvalues $\lambda_n^{(1)}$ of $(-\partial_t^2 + \mathcal{W}^{(1)})$ correspond to the values of λ for which $\psi_\lambda^{(1)}(T/2) = 0$ (see equation (2.18)); And similarly, $\lambda_n^{(2)}$ are the values for which $\psi_\lambda^{(2)}(T/2) = 0$.

Step 3 : Using the results of steps 1 and 2, the idea is to identify that both sides of equation (B.1) are meromorphic functions of λ with the following properties:

1. Zeros at each $\lambda = \lambda_n^{(1)}$,
2. Poles at each $\lambda = \lambda_n^{(2)}$.
3. As $\lambda \to \infty$ in any direction in the complex plane except along the positive real axis, both the determinant ratio and $\frac{\psi_\lambda^{(1)}(T/2)}{\psi_\lambda^{(2)}(T/2)}$ approach one. This can be seen by examining the asymptotic behavior of the differential equation solutions as $\lambda \to \infty$, where $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ become negligible compared to $-\lambda$.

These shared characteristics confirm that both sides have the same *skeleton* structure as functions of λ.

Step 4 : By the uniqueness theorem for meromorphic functions, if two meromorphic functions have the same zeros, poles, and asymptotic behavior at infinity, they must be identical.

The reason that two meromorphic functions, say, $f(z)$ and $g(z)$, with the same skeleton structures must be equal, is because: (i) Their difference, $h(z) \equiv f(z) - g(z)$, is holomorphic everywhere (since poles and zeros cancel out). (ii) $h(z)$ is bounded (as it vanishes at infinity). (iii) By Liouville's Theorem, any bounded entire function (holomorphic on the whole complex plane) must be constant. (iv) Since $h(z) \to 0$ as $z \to \infty$, the constant must be zero, so $h(z) = 0$, $\forall z$; or $f(z) = g(z)$.

Therefore,

$$\det\left(\frac{-\partial_t^2 + \mathcal{W}^{(1)} - \lambda}{-\partial_t^2 + \mathcal{W}^{(2)} - \lambda}\right) = \frac{\psi_\lambda^{(1)}(T/2)}{\psi_\lambda^{(2)}(T/2)}.$$

This concludes the proof.

References

[1] Sidney Coleman. "Aspects of symmetry: selected Erice lectures". In: Cambridge University Press, 1988. Chap. 7.

[2] Valery Rubakov. "Classical theory of gauge fields". In: Princeton University Press, 2009. Chap. Appendix.