Maxwell Fish-eye Lens

Introduction

Maxwell's lens entails a situation where the refractive index n varies as $n(r)$,

$$n(r) = \frac{n_0}{1 + (r/R)^2} \tag{1}$$

where r is the radial distance from origin and the refractive index at the point of origin is n_0. The constant R is the radius of the sphere that scales how quickly the refractive index decreases with distance (radially outward) from this point. This lens is a spherically symmetric inhomogeneous system. It examines the situation from the viewpoint of geometrical optics; Showing that all rays starting from the source point will travel in circular arcs in such a medium and end up passing through an image point.

In order to transform the refractive index of Maxwell Fish-eye lens to its corresponding potential profile, light rays may be assumed as trajectories of particles in the refractive index $n(r)$. Replace it with the potential $V(r)$ as in equation (2).

$$V(r) = E - \frac{\omega}{2R^2 \left[1 + \left(\frac{r}{R}\right)^2\right]^2} \tag{2}$$

where ω is the strength parameter of the potential ($\omega > 0$) and E is a fixed constant of motion. Thus, the zero-energy Hamiltonian describes

$$\boxed{V(r) = -\frac{\omega}{2R^2 \left[1 + \left(\frac{r}{R}\right)^2\right]^2}} \tag{3}$$

Classical trajectories

For the Maxwell fish-eye potential (in general, any central potential), the angular momentum is a conserved quantity and orbits are flat curves. We therefore adapt to polar coordinates in the classical picture since it is convenient when we work with zero-energy Hamiltonian. Classically, the energy

$$E = \frac{1}{2}(m\dot{r}^2 + mr^2\dot{\varphi}^2) + U(r) \tag{4}$$

and the angular momentum is $L = mr^2\dot{\varphi}$. In the dimensionless coordinate $\rho = r/R$, setting $E = 0$ in (4), we get (Makowski & Górska, 2009)

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\varphi}\right)^2 + \rho^2 = \frac{m\omega}{L^2}\frac{\rho^2}{\left(\rho^{-1} + \rho\right)^2} \tag{5}$$

From the equation (5), we obtain the following general (classical) solution:

$$\rho - \frac{1}{\rho} = 2\left(\sqrt{\frac{m\omega}{4L^2}} - 1\right)\sin\left(\varphi - \varphi_0\right) \tag{6}$$

In (6), notice that solution is invariant under $\varphi \mapsto \varphi + \pi$.

- Family of trajectories:

$$\boxed{\left(x + R\left(\sqrt{\frac{m\omega}{4L^2}} - 1\right)\sin\varphi_0\right)^2 + \left(y - R\left(\sqrt{\frac{m\omega}{4L^2}} - 1\right)\cos\varphi_0\right)^2 = \frac{m\omega}{4L^2}R^2} \tag{7}$$

Paraxial Approximation

Consider the wave equation of the following form:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{U}(\vec{r}, t) = 0 \tag{8}$$

We are interested in the Helmholtz equation that is often represented as the time-independent part of the wave equation (8) upon imposing the technique of separation of variables.

$$\mathcal{U}(\vec{r}, t) = \mathcal{A}(\vec{r})\mathcal{T}(t) \tag{9}$$

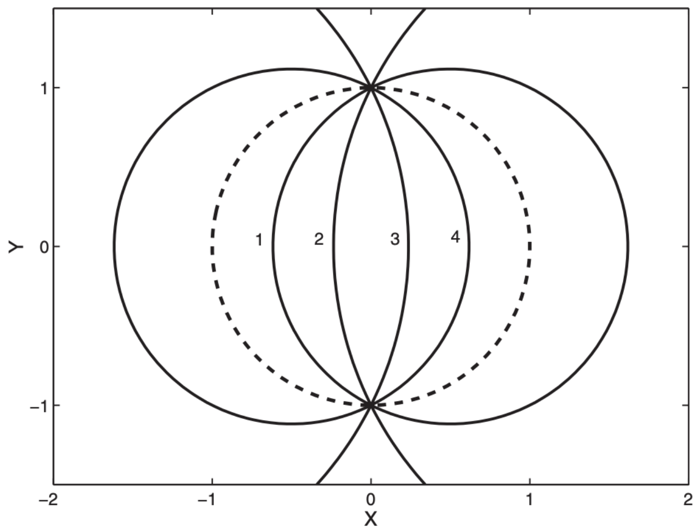

Figure 1: Classical (closed) trajectories in the MFEL potential profile for various φ_0 (multiples of $\pi/2$) & $a \equiv \frac{m\omega}{4L^2}$. The dashed circle is for $a = 1$.

Upon simplifying, we now obtain the spatial Helmholtz equation

$$(\nabla^2 + k^2)\mathcal{A}(\vec{r}) = 0 \tag{10}$$

Using $k^2 = -\frac{1}{c^2\mathcal{T}}\frac{\partial^2\mathcal{T}}{\partial t^2}$ from the temporal component of (9); where it is chosen, without loss of generality, the expression $-k^2$ to be the value of the constant.

In the paraxial approximation, we may demand the complex amplitude $\mathcal{A}$ as,

$$\mathcal{A}(\vec{r}) = \mathcal{F}(\vec{r})e^{\imath kz} \tag{11}$$

In this limit, $e^{\imath kz}$ is presumably the phase factor. Now assuming

$$\left|\frac{\partial^2\mathcal{F}}{\partial z^2}\right| << \left|k\frac{\partial\mathcal{F}}{\partial z}\right| \tag{12}$$

leads us to the following solution of the spatial Helmholtz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathcal{F}(\vec{r})e^{\imath kz} + \left(\frac{\partial^2}{\partial z^2}\mathcal{F}(\vec{r})\right)e^{\imath kz} + 2\imath k\left(\frac{\partial\mathcal{F}(\vec{r})}{\partial z}\right)e^{\imath kz} = 0$$

Under our assumption (12) and substituting back $\mathcal{A}(\vec{r})$ from equation (11), Eq. (10) becomes

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathcal{A}(\vec{r}) + 2\imath k\left(\frac{\partial \mathcal{A}(\vec{r})}{\partial z}\right) = 0} \tag{13}$$

Optical Conformal Mapping

As a starting point, consider the time independent Schrödinger equation in two-dimensional space:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x,y)\right]\Psi = E\Psi$$

Also, consider a dielectric medium that is uniform in one direction and light of wavenumber k that propagates orthogonal to that direction. The medium is characterized by the refractive-index profile $n(x, y)$.

To describe the spatial coordinates in the propagation plane we use complex numbers $z = x + iy$ with the partial derivatives $\partial_x = \partial_z + \partial_z^*$ and $\partial_y = i\partial_z - i\partial_z^*$ where the star symbolizes complex conjugation. In the case of a gradually varying refractive-index profile both amplitudes Ψ of the two polarizations of light obey the Helmholtz equation as follows:

$$(4\partial_z^*\partial_z + n^2k^2)\Psi = 0 \tag{14}$$

The equation (14) uses complex notation with the Laplace operator $\partial_x^2 + \partial_y^2 = 4\partial_z^*\partial_z$.

Now suppose we introduce new coordinates w described by an analytic function $w(z)$ (independent of z^*). Such functions define conformal maps that preserve the angles between the coordinate lines. We obtain an equivalent Helmholtz equation with the transformed refractive-index profile n' in the $w - space$ that is related to the original one as we implement the following substitutions:

$$z \mapsto w(z) \tag{15}$$

$$n \mapsto \frac{n'}{\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|} \tag{16}$$

$$\partial_z^* \partial_z \mapsto \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \partial_w^* \partial_w \tag{17}$$

Suppose that the medium is designed such that $n(z) = |g(z)|$ (modulus of an analytic function). The integral of g(z) defines a map $w(z)$ to new coordinates where, according to Eq. (17), the transformed index $n' = 1$. Consequently, in w coordinates the wave propagation is indistinguishable from empty space where light rays propagate along straight lines. The medium performs an optical conformal mapping to empty space. Using the optics of Riemann sheets, we only need to guide the light back to the exterior sheet from the interior sheet. Consequently, only two potentials have all trajectories closed. (Leonhardt, 2006)

- Harmonic Oscillator Potential:

$$\left(n'_{HO}\right)^2 = 1 - \frac{|w - w_1|^2}{r^2}$$

- Kepler Potential:

$$\left(n'_{Kep}\right)^2 = \frac{r}{|w - w_1|} - 1$$

Regularized Potential Profiles

Now let's pivot our attention to the case where we can equate the dynamics of Luneburg Lens to the case of MFEL using a canonical mapping assisted by regularizing the potentials.

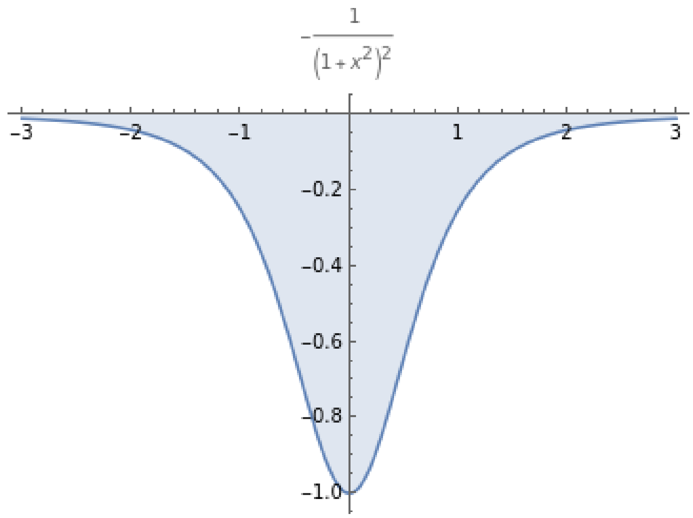

Figure 2: $V(r) = -\frac{\omega}{2R^2\left[1+\left(\frac{r}{R}\right)^2\right]^2}$.

Maxwell fish-eye lens potential

Pöschl-Teller potentials

In its symmetric form, $V_\lambda(x)$ is

$$V_\lambda(x) = -\frac{\lambda(\lambda+1)}{2}\operatorname{sech}^2 x$$

Potential: $V(x) = -2\tanh^2 x$

Solutions to the time independent Schrödinger equation with this potential can be found as follows:

$$\left[-\left(\frac{\partial^2}{\partial x^2}\right) + V(x)\right]\phi = E\phi$$

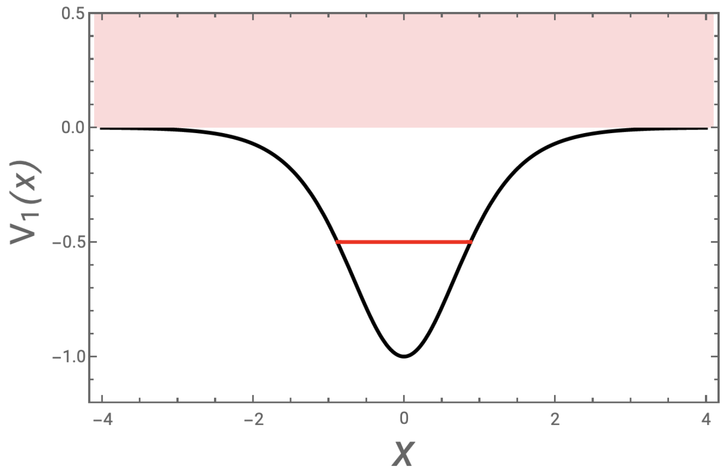

Figure 3: For $\lambda = 1$, $V_1(x) = -\operatorname{sech}^2 x$. The horizontal red line reflects the Energy eigenvalue ($E_1 = -0.5$) associated with the potential.

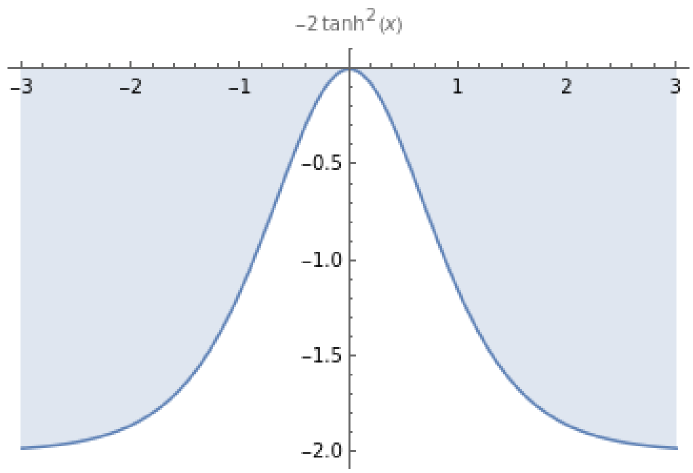

Figure 4: $V(x) = -2\tanh^2 x$.

Substituting $V(x) = -2\tanh^2 x$;

$$\left[-\left(\frac{\partial^2}{\partial x^2}\right) - 2\tanh^2 x\right]\phi = E\phi$$

By virtue of the substitution $u = \tanh x$,

$$-\left[(1-u^2)\frac{\partial}{\partial u}\left((1-u^2)\frac{\partial}{\partial u}\right) - 2u^2\right]\phi = E\phi$$

Simplifying,

$$-\left[(1-u^2)\left(-2u\frac{\partial}{\partial u} + (1-u^2)\frac{\partial^2}{\partial u^2}\right) - 2u^2\right]\phi = E\phi$$

$$\therefore \quad \left[-(1-u^2)^2\frac{\partial^2}{\partial u^2} + 2u(1-u^2)\frac{\partial}{\partial u} + 2u^2\right]\phi = E\phi$$

$$\therefore \quad \left[-\frac{\partial^2}{\partial u^2} + \left(\frac{2u}{1-u^2}\right)\frac{\partial}{\partial u} - \frac{-2u^2}{(1-u^2)^2}\right]\phi = E\phi \tag{18}$$

Now in order to further transform the aforementioned second order differential equation (18) into a more familiar form, we demand that $\phi = e^{\int \varrho(u)du}\psi$; where $\varrho(u) = \frac{2u}{1-u^2}$.

$$\left[-\frac{\partial^2}{\partial u^2} - \left(\frac{2(1+u^2)}{(1-u^2)^2} + \frac{-2u^2+E}{(1-u^2)^2}\right)\right]\psi = 0$$

$$\therefore \quad \left[-\frac{\partial^2}{\partial u^2} - \left(\frac{2}{(1-u^2)^2}\right)\right]\psi = E\psi \tag{19}$$

Observe that Eq. (19) is a time independent Schrödinger like equation for some fixed constant value of eigenenergy E. We again consider the zero-energy Hamiltonian (consistent with Eq. (2)) with the region of interest being $0 < u < 1$ (remember, $u = \tanh x$); So the Eq. (19) becomes:

$$\left[-\frac{\partial^2}{\partial u^2} - \left(\frac{2}{(1-u^2)^2}\right)\right]\psi = 0 \tag{20}$$

For our purposes, we may regard the independent variable (u) to be the dimensionless coordinate $u = r/R$, extending in the radial direction. Hence, the choice of the region $u \in (0, 1)$ makes sense. And the potential profile is:

$$V(r) = -\frac{2}{(1-(r/R)^2)^2} \tag{21}$$

Therefore, now we have a map that links the regularized version of the SHO potential (corresponding to the Lunebürg lens) canonically to the Maxwell's Fish-eye Lens upon analytically continuing $R \mapsto \iota R$ in (21).

References

Leonhardt, U. (2006). Optical conformal mapping. *science*, *312*(5781), 1777–1780.

Makowski, A., & Górska, K. (2009). Quantization of the maxwell fish-eye problem and the quantum-classical correspondence. *Physical Review A*, *79*(5), 052116.