The Impact of a Moving Mirror on False Vacuum Stability

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ABSTRACT

This thesis undertakes a comprehensive investigation into the calculation of tunnelling probability associated with false vacuum decay subject to a time-dependent boundary condition (a moving mirror), utilizing a specific example of a scalar field as a prototype. The study encompasses the derivation of the field equation, canonical quantization, in-vacuum state characteristics, and various Green's functions. The framework of a moving mirror is rigorously employed, and a conformal coordinate map is established between static and dynamic metrics. The formalism of the semiclassical bounce solutions is reviewed, and a toy potential, consisting of an inverted Liouville term and a massive free term, is studied. Hereafter, the modified field equation is solved through the Green's function approach equipped with the methodology of asymptotic matching to assess the tunnelling solution and infer the rate of this captivating phenomenon.

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Chapter 1

Introduction and Summary

1.1 A Qualitative Preview of False Vacuum Decay

False vacuum decay represents a quantum mechanical tunnelling process, in particular, the study of a metastable state decaying into the ground state.

Theoretical developments of the false vacuum decay phenomena stipulates plenty of possible phenomenological relevance. In particular, the electroweak vacuum determined by the Standard Model Higgs potential may not be absolutely stable[1, 2, 3, 4] and may decay in exponentially rare occurrences. Such an event would have catastrophic consequences, essentially destroying the universe the way we know it. Several other examples include: possible phase transitions in condensed matter physics, some phase transitions in the early universe, origin of the cosmic inflation, metastability of the Higgs vacuum, production of baryon asymmetry and dark matter of the universe.

False vacuum decay is also a subject of considerable theoretical significance. In the lines of its development, we expect to gauge insights into non-perturbative quantum field theory in curved spacetime [5], connections to the entropy of a Black hole [6] and an enterprising semiclassical notion of quantum gravity [7, 8].

In literature, plenty of efforts had been put to compute the false vacuum decay rate in a semiclassical setting. The tunnelling probability is determined by the *so-called* Euclidean bounce construction [9, 10, 11]. However, this elegant Euclidean method thoroughly restricts us to work in an equilibrium environment. In this thesis, we develop an alternate strategy to incorporate the non-equilibrium setting. The relevant challenges and enhancements associated with this methodology are analyzed discreetly in the subsequent chapters.

1.2 Road-Map of the Thesis

A synopsis of all the chapters in this thesis, is presented here as follows.

The chapter 2 introduces the setup and the methods we implement to our model. In section 2.1, we derive the field equations from the scalar field action. The section 2.2 describes the quantization scheme of the corresponding vacuum fields. In section 2.3, a thorough overview of all the required Green's functions is devised.

In chapter 3, we first pose the underlying problem of time-dependent boundary condition in the form of a so-called *moving mirror*. The case when the mirror is represented by a static curve at $x = 0$ is discussed in section 3.2; followed by the case of an arbitrarily moving mirror trajectory in section 3.3. In section 3.4 we introduce the technique of conformal mapping, that can reduce the complexities of our time-dependent boundary and leads us to a relatively more accessible, modified field equation in a transformed (still conformally-flat) spacetime. A comprehensive formulation is discussed in section 3.5.

In chapter 4, the existing semiclassical approach is reviewed in brevity. The Euclidean bounce solution in the context of single variable quantum mechanics is debriefed in section 4.2. The objective of this chapter, as prescribed in the subsequent sections (section 4.3 and section 4.4), is to infer the bounce solution in the field-theoretic problems that can explain the tunnelling from Minkowski vacuum, in imaginary (Euclidean) time.

In chapter 5, an application of the general formalism developed in chapter 2 and chapter 3 is illustrated on a toy model. First section 5.1 manifests the structure of the inverted Liouville potential with a mass term. The overall strategy to find the transition probability of decay of a false vacuum, in this toy model potential, subject to the moving mirror boundary condition is then treated in section 5.2. Some results identifying the structure of the bounce solution in the presence of a static mirror, is described in subsection 5.2.2. In the next and final section 5.3, we finally consider the challenges followed in the non-equilibrium domain; and attempt to implement our devised remedy to our toy model.

Chapter 2

The Setup & Methods

2.1 Field Equations

Consider a model of a real scalar field $\varphi(t, x)$ in a 2-dimensional spacetime[1] with the Lagrangian density[2]

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \tag{2.1}$$

Requiring the Lagrangian to be a Lorentz scalar and the resulting field equations to be linear leads us to construct the following action[3]

$$S[\varphi] = \int \mathcal{L}(x)\ d^2x \tag{2.2}$$

We demand that for variations of the action with respect to φ,

$$\delta S = 0 \tag{2.3}$$

To derive the field equations from the variational principle with the action (2.2), let $\delta\varphi(x)$ be the field variation; then

$$\delta S = \int d^2x \left[-g^{\mu\nu} \partial_\mu \varphi \delta(\partial_\nu \varphi) - \frac{\partial V}{\partial \varphi} \delta\varphi \right] \tag{2.4}$$

[1]The Lorentzian metric $g_{\mu\nu}$ is a *smooth* tensor field with signature $(-, +)$.
[2]System of units: $\hbar = c = 1$.
[3]The spacetime point $(t, x) = (x^0, x)$ is abbreviated as x.

Implementing $\delta(\partial_\nu\varphi) = \partial_\nu(\delta\varphi)$ and integration by parts in the first term in (2.4), we get

$$\delta S = \int d^2x \left[g^{\mu\nu}\partial_\mu\partial_\nu\varphi\delta\varphi - \frac{\partial V}{\partial\varphi}\delta\varphi \right] + \int d\Sigma^\nu \left(\partial_\nu\varphi\delta\varphi\right) \tag{2.5}$$

where the second integral is taken over the boundary of the overall volume. Without loss of generality, we suppose that $\delta\varphi = 0$ on the boundary of the two-dimensional volume element, then the second integral vanishes naturally. By imposing in (2.5), the condition (2.3); we finally obtain

$$\Box\varphi - \frac{\partial V}{\partial\varphi} = 0 \tag{2.6}$$

Where, $\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian, a linear second order differential operator.

2.1.1 Solutions Satisfying the Massive KG Equation

When the governing potential is[4] $V = m^2\varphi^2/2$, the corresponding scalar field equation is called the massive Klein-Gordon equation.

$$\left(\Box - m^2\right)\varphi = 0 \tag{2.7}$$

Mode decomposition

In search of the general solutions of (2.7), transforming to the Fourier space k-representation, the following is a set of modes

$$\varphi(t,x) = \int_{\omega\geq 0} d\omega \int dk \left[\tilde{\varphi}_{\omega,k}(t,x) + \tilde{\varphi}^*_{\omega,k}(t,x)\right] \tag{2.8}$$

$\tilde{\varphi}_{\omega,k}(t,x) \propto e^{-i\omega t + ikx}$ (upto normalization) and $\omega \equiv k^0$ is defined to be the energy component of the relativistic momentum, portraying the dispersion law for free waves. Plugging (2.8) into (2.7) residues

$$\left[\omega^2 - k^2 - m^2\right]\tilde{\varphi}_{\omega,k}(t,x) = 0 \tag{2.9}$$

[4] m^2 being the arbitrary parameter here; However, it shall be interpreted as the mass of the field quanta, post quantization. Relevance of this potential is discussed more explicitly in chapter 3.

So that (2.8) holds for any non-zero arbitrary $\tilde{\varphi}_{\omega,k}(t,x)$, provided

$$\omega = \sqrt{k^2 + m^2} \tag{2.10}$$

In other words, $\tilde{\varphi}_{\omega,k}(t,x)$ is proportional to the Dirac's δ-function,

$$\tilde{\varphi}_{\omega,k}(t,x) = \varphi_k(t,x)\;\delta\left(\omega - \sqrt{k^2 + m^2}\right) \tag{2.11}$$

.

Normalization

For the normalization scheme, firstly, we are required to define the scalar product of the field in the following sense

$$\langle \varphi_\alpha(t,x) \cdot \varphi_\beta(t,x) \rangle \equiv -i \int dx\;\; \left[\varphi_\alpha(t,x)\left[\partial_t \varphi^*_\beta(t,x)\right] - \left[\partial_t \varphi_\alpha(t,x)\right] \varphi^*_\beta(t,x)\right] \tag{2.12}$$

We would want to choose the normalization constant of the field $\varphi(t,x)$ for a given mode k, should they be normalized in the inner product (2.12) such that

$$\langle \varphi_k(t,x) \cdot \varphi_{k'}(t,x) \rangle = \delta(k - k') \tag{2.13}$$

Therefore we may choose the normalized mode functions

$$\varphi_k(t,x) = \frac{1}{\sqrt{(2\pi)(2\omega)}}\; e^{-i\omega t + ikx} \tag{2.14}$$

2.2 Vacuum Field Quantization

In the context of this study, the false vacuum is a quantum state defined by quantizing the classically decomposed set of mode functions ((2.8) along with the condition (2.10)) satisfying the massive Klein-Gordon equation in the canonical quantization scheme by treating the field φ as an operator. Hence we impose the following equal-time commutation relations

$$\begin{aligned}
[\hat{\varphi}(t,x),\hat{\varphi}(t,x')] &= 0 \\
[\hat{\pi}(t,x),\hat{\pi}(t,x')] &= 0 \\
[\hat{\varphi}(t,x),\hat{\pi}(t,x')] &= i\delta(x-x')
\end{aligned} \tag{2.15}$$

And $\hat{\pi}$ being the canonically conjugate variable to $\hat{\varphi}$ in (2.15); defined by

$$\hat{\pi}(t,x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \hat{\varphi})} = \partial_t \hat{\varphi}(t,x) \tag{2.16}$$

The modes (2.14) along with its complex conjugates form a complete orthonormal basis equipped with the inner product (2.12) and can be expressed as

$$\hat{\varphi}(t,x) = \int dk \left(\hat{a}_k \, \varphi_k(t,x) + \hat{a}_k^\dagger \, \varphi_k^*(t,x) \right) \tag{2.17}$$

Here $\hat{a}_k$, $\hat{a}_k^\dagger$ are the annihilation and creation operators for quanta in the mode with momentum k, satisfying the equal-time commutation relations equivalent to (2.15)

$$\begin{aligned}
[\hat{a}_k,\hat{a}_{k'}] &= 0 \\
[\hat{a}_k^\dagger,\hat{a}_{k'}^\dagger] &= 0 \\
[\hat{a}_k,\hat{a}_{k'}^\dagger] &= \delta(k-k')
\end{aligned} \tag{2.18}$$

For considerations that are relevant to us, the state annihilated by all the $\hat{a}_k$ operators is defined as the in-vacuum[5]

$$\hat{a}_k \, |0,\mathrm{in}\rangle = 0, \quad \forall\, k \tag{2.19}$$

2.3 Green's Functions

Green's functions of the field equation are defined as the vacuum expectation values of various products of free field operators. For a real scalar field operator $\hat{\varphi}(t,x)$, one of the Green's function is the expectation value of the commutator of the fields,

$$i\mathcal{G}(t,x;t',x') = \langle 0| \left[\hat{\varphi}(t,x),\hat{\varphi}(t',x')\right] |0\rangle \tag{2.20}$$

[5] In Fock representation, we can refer to $|0,\mathrm{in}\rangle$ as the no-particle state.

that can be decomposed in the positive and negative frequency components:

$$\begin{aligned} \mathcal{G}^{+}(t,x;t',x') &= \langle 0|\ \hat{\varphi}(t,x)\ \hat{\varphi}(t',x')\ |0\rangle \\ \mathcal{G}^{-}(t,x;t',x') &= \langle 0|\ \hat{\varphi}(t',x')\ \hat{\varphi}(t,x)\ |0\rangle \end{aligned} \tag{2.21}$$

These Green's functions in (2.21) are known as the Wightman functions.

The Feynman propagator $\mathcal{G}_F$ is identified as the time-ordered product of field operators

$$\begin{aligned} i\mathcal{G}_F &= \langle 0|\mathcal{T}\left(\hat{\varphi}(t,x),\hat{\varphi}(t',x')\right)|0\rangle \\ i\mathcal{G}_F &= \Theta(t-t')\ \mathcal{G}^{+}(t,x;t',x') + \Theta(t'-t)\ \mathcal{G}^{-}(t,x;t',x') \end{aligned} \tag{2.22}$$

Where, $\mathcal{T}(\cdot)$ is the time-ordering operator, and $\Theta(t)$ is the Heaviside step function. We should also note another two rather important definitions of Green's functions

$$\mathcal{G}_R(t,x;t',x') = -\Theta(t-t')\mathcal{G}(t,x;t',x') \tag{2.23}$$

$$\mathcal{G}_A(t,x;t',x') = \Theta(t'-t)\mathcal{G}(t,x;t',x') \tag{2.24}$$

Namely, $\mathcal{G}_R$ and $\mathcal{G}_A$ are the retarded and advanced Green's functions respectively. Clearly, $\mathcal{G}_R$ vanishes if $t < t'$ and $\mathcal{G}_A$ vanishes if $t > t'$.

Integral Representation

These Green's functions satisfy the homogeneous Klein-Gordon equation with δ function as a source term (modulo an overall factor of $\pm i$)

$$(\Box - m^2)\mathcal{G}_{F,R,A}(t,x;t',x') = \delta(t-t')\delta(x-x') \tag{2.25}$$

Substituting the mode decomposition of $\hat{\varphi}(t,x)$ in the definition of Green's functions yields its integral representation

$$\mathcal{G}_{F,R,A}(t,x;t',x') = \frac{1}{(2\pi)^2}\int d\omega\ dk\ \frac{\varphi_k(t,x)\ \varphi_k^*(t',x')}{\omega^2 - k^2 - m^2} \tag{2.26}$$

This is a general expression for Feynman, retarded and advanced Green's function of the field equation; describing the propagation of field disturbances subject to certain boundary conditions. To make it specific, notice that the integral (2.26) encounters singularities at $\omega = \pm\sqrt{k^2+m^2}$.

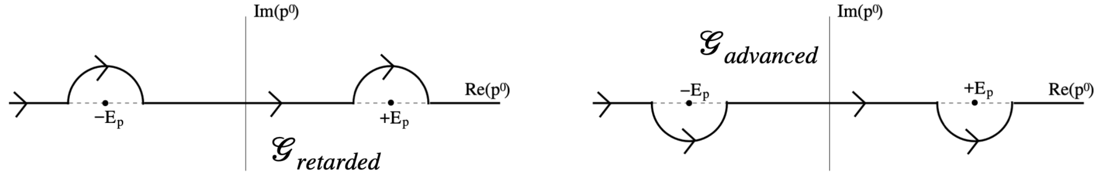

Figure 2.1: *(Left)* The contour associated with the retarded Green's function $\mathcal{G}_R$, in the complex $\omega = p^0$ plane. Both poles are enclosed in the lower half plane; *(Right)* The contour associated with the advanced Green's function $\mathcal{G}_A$; Both poles are enclosed in the upper half plane. Here, $\pm E_p = \pm\sqrt{k^2+m^2}$ as in (2.26)

Considering it as a contour integration performed on the ω-axis, we can deform the contour to circumnavigate the poles and the open-ends of the contour should be assumed as being closed by the infinite semicircles in the upper or lower half of the complex ω-plane. However, the exact forms of these contours depend on the boundary conditions imposed on the corresponding field and the various Green's functions are each associated with their descriptions of the contour deformations. The case of *Retarded* and *advanced* Green's functions is shown in Figure 2.1.[12]

The Retarded Green's function

Since the specific form of retarded Green's function, under a constraining boundary (i.e. a moving mirror), will be of later use in our study; it is customary to introduce the "$i\epsilon$ prescription" with $\epsilon > 0$, and infinitesimal real number. We want the denominator of (2.26) as

$$\left(\omega - \sqrt{k^2+m^2}\right)\left(\omega + \sqrt{k^2+m^2}\right) \mapsto \left(\omega - \sqrt{k^2+m^2} - i\epsilon\right)\left(\omega + \sqrt{k^2+m^2} - i\epsilon\right) \tag{2.27}$$

So the retarded Green's function gets the form

$$\mathcal{G}_R(t,x;t',x') \equiv \int \frac{d\omega}{2\pi}\int \frac{dk}{2\pi}\,\frac{e^{-i\omega(t-t')}\varphi_k(x)\varphi_k^*(x')}{\omega^2-k^2-m^2-i\epsilon\omega}, \tag{2.28}$$

where $\varphi_k(x)$ now only describes the spatial mode functions, since we extracted the temporal positive frequency modes explicitly. By construction, at $t > t'$, we can enclose the contour ω in the lower half-plane as the plane frequency modes suppress exponentially as $\omega \to -i\infty$.

This $-i\epsilon\omega$ term shall shift the poles marginally off the real axis, so the integral along the real ω axis is equivalent to the integral along the *deformed* contour shown in Figure 2.1 (*Left*), by naturally avoiding the poles on the integration axis.

At $t < t'$, we can close the contour in the upper half-plane since the plane frequency modes suppress the integral exponentially as $\omega \to i\infty$ and no poles are enclosed by the contour. Thus, we get $\mathcal{G}_R = 0$ at $t < t'$, as desired.

The Feynman Green's function

Using the "$i\epsilon$ prescription" that we introduced earlier, the Feynman Green's function (contour of integration is shown in Figure 2.2) in integral representation gets the form

$$\mathcal{G}_F(t, x; t', x') \equiv \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \frac{e^{-i\omega(t-t')}\varphi_k(x)\varphi_k^*(x')}{\omega^2 - k^2 - m^2 - i\epsilon}, \tag{2.29}$$

where, similar to the retarded case, $\varphi_k(x)$ only describes the spatial mode functions, and temporal modes are extracted out explicitly.

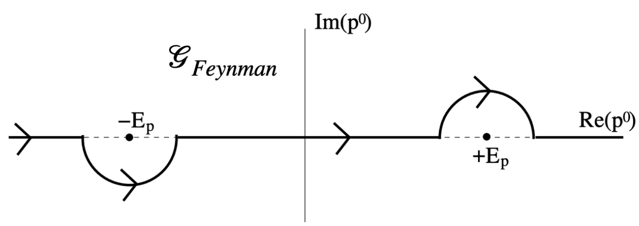

Figure 2.2: The contour associated with the Feynman Green's function $\mathcal{G}_F$, in the complex $\omega = p^0$ plane. Negative frequency pole shall be enclosed in the upper half plane, and positive frequency pole shall be enclosed in the lower one. $\pm E_p = \pm\sqrt{k^2 + m^2}$ as in (2.26)

Chapter 3

A Moving Mirror in Two Dimensional Spacetime

3.1 Overview

Our goal is to develop and extend the methods to the decay of a false vacuum under non-equilibrium situations, corresponding to time-dependent boundary conditions. In particular, the motion of a single reflecting boundary (a moving mirror, see figure 3.1) induces disturbance in the quantum state and leads to production of new field quanta. This is expected to affect the probability of the false vacuum decay [13].

In this section we will explore the possibilities offered by the arrangement of a time-dependent moving mirror where the conformal triviality of the flat spacetime shall be exploited.

Null Coordinates' chart

In what follows, an introduction to the null coordinates (3.1) (sometimes referred to as light-like coordinates) will be useful:

$$u \equiv t - x, \quad v \equiv t + x \tag{3.1}$$

In this (u, v) coordinate chart, in the 2D Minkowski spacetime, the *static* metric[1] gets the form

$$ds^2 = -dt^2 + dx^2 = -du\, dv \tag{3.2}$$

The Jacobian factor is

$$|\det \mathcal{J}| = \left| \frac{\partial(t, x)}{\partial(u, v)} \right| = \frac{1}{2} \tag{3.3}$$

And the static d'Alembertian operator ($\Box_0 \equiv \eta^{\mu\nu}\partial_\mu \partial_\nu$) in this chart is[2]

$$\Box_0 = -2\frac{\partial^2}{\partial u \partial v} - 2\frac{\partial^2}{\partial v \partial u} = 4\left(-\frac{\partial^2}{\partial u \partial v}\right) \tag{3.4}$$

3.2 The Static Mirror Case

For a massive real scalar field $\varphi(t, x)$ present in a two dimensional Minkowski spacetime, let's consider stationary (Dirichlet) boundary condition at $x = 0$ such that [14]

$$\varphi(t, 0) = 0 \tag{3.5}$$

We know that the scalar field φ satisfies the field equation (2.6). For our purposes we will study the dynamics of linear perturbations around the false vacuum. Thus, we replace the potential term in the field equation by the free-field part,

$$V(\varphi) \mapsto \frac{m^2 \varphi^2}{2} \tag{3.6}$$

where m is the mass of the field in the false vacuum state. Upon this substitution, we obtain the linearized field equation (2.7).

The system of equations (2.7) and (3.5) have a well-defined solution in terms of the following set of mode functions[3]

$$\varphi_k(t, x) = \frac{1}{\sqrt{\pi\omega}}\, e^{-i\omega t} \sin(kx), \quad k > 0 \tag{3.7}$$

along with the defining condition that relates ω and k: $\omega = \sqrt{k^2 + m^2}$.

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[1] $\eta_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$ in the (u, v) basis.

[2] Since $\frac{\partial^2}{\partial u \partial v} = \frac{\partial^2}{\partial v \partial u}$, $\Box = -4\frac{\partial^2}{\partial u \partial v} = -4\frac{\partial^2}{\partial v \partial u}$.

[3] Here, we labelled φ with the modes k; though we may equivalently label the modes using ω without any loss of generality.

Therefore, the *normalized* field operator subject to the Dirichlet boundary condition can be defined as

$$\hat{\varphi}_0(t,x) \equiv \int_0^\infty \frac{dk}{\sqrt{\pi\omega}} \left(\hat{a}_k\, e^{-i\omega t} \sin(kx) + \hat{a}_k^\dagger\, e^{i\omega t} \sin(kx)\right) \tag{3.8}$$

It is easy to check that $\hat{\varphi}_0(t,x)$ satisfies (3.5).

3.3 A Mirror Moving on an Arbitrary Trajectory

Consider the situation (shown in the figure 3.1) of a quantized vacuum state subject to a single reflecting (time-dependent) mirror in two dimensional spacetime (t,x). The mirror trajectory is given by the curve $x = \zeta(t)$. Given that space has only one

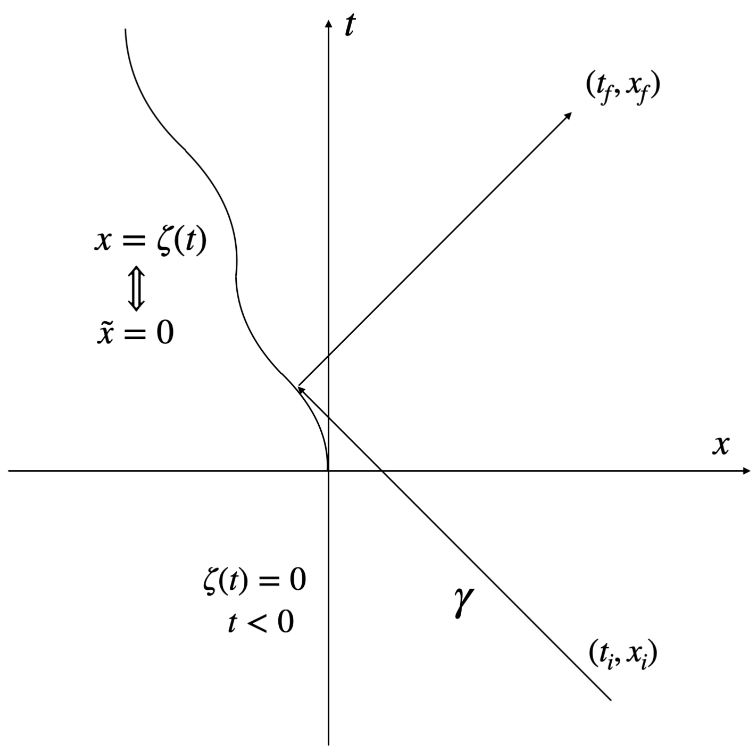

Figure 3.1: Incoming null rays γ from the in-vacuum shown in (t,x) spacetime. The mirror trajectory $x = \zeta(t)$ corresponds to $\tilde{x} = 0$ upon conformal mapping. We require the field to vanish at the mirror.

dimension, the mirror is actually just a reflecting point.

Constraints on $x = \zeta(t)$

At this point we assume that the mirror trajectory is a known function, but we do not specify its form. Still, there are some naturally arising constraints on the function $\zeta(t)$ we must point out: Firstly, we want $x = \zeta(t)$ to be a timelike curve.

$$\left|\dot{\zeta}(t)\right| < 1 \tag{3.9}$$

A slightly stronger condition would be that this trajectory has to be very slowly varying in time. This allows us to implement the adiabatic approximation to this problem. We also require the curve to be smooth and *at least* twice differentiable at the origin. Hence

$$\dot{\zeta}(0) = 0 \tag{3.10}$$

In the following section, we implement the technique of *conformal coordinate transformation* onto this spacetime plane containing the mirror at $x = \zeta(t)$ and formulate an inhomogeneous field equation that can exploit the properties and constraints of the known function.

3.4 Conformal Mapping of an Arbitrarily Moving Mirror into a Static one

Now we would like to perform a conformal transformation of the coordinate chart (null coordinates, in particular) in the following manner [15]

$$\begin{aligned} (t, x) &\mapsto (\tilde{t}, \tilde{x}) \\ (u, v) &\mapsto (\tilde{u}, \tilde{v}) \end{aligned} \tag{3.11}$$

such that

$$\tilde{u} = f(u), \quad \tilde{v} = g(v) \tag{3.12}$$

Note that since (in (3.13)[4]) the metric in $\tilde{u}, \tilde{v}$ coordinates is conformally flat,

$$-d\tilde{u}\, d\tilde{v} = f'(u)g'(v)\,[-du\, dv] \tag{3.13}$$

[4]As in (3.13), a prime will always denote the differentiation of a function with respect to its own argument

In other words, the metric is just a conformal rescaling of the flat metric

$$\tilde{g}_{\mu\nu} \mapsto \mathcal{C}(t,x)\eta_{\mu\nu} \tag{3.14}$$

We want to choose the functions f and g in such a way that the curve $\tilde{x} = 0$ corresponds exactly to the mirror trajectory $x = \zeta(t)$. Acknowledge the whole point of this treatment is to cast the complications of a dynamic mirror ($x = \zeta(t)$) in a static metric environment (i.e. the (u, v) chart), to the situation of a static mirror ($\tilde{x} = 0$) present in the transformed, dynamic $(\tilde{u}, \tilde{v})$ chart.

Under this choices, we postulate that the function $g(v)$ is an identity map. Hence we suppose

$$\tilde{v} = v = g(v) \tag{3.15}$$

By this choice we do not perturb the incoming vacuum modes.

Now, let an incoming null ray (u) collide with the mirror at time $t = t_u$. So

$$u = t_u - \zeta(t_u) \tag{3.16}$$

Taking partial derivative with respect to u

$$\begin{aligned} \therefore 1 &= \frac{\partial t_u}{\partial u} - \dot{\zeta}(t_u)\frac{\partial t_u}{\partial u} \\ \therefore \frac{\partial t_u}{\partial u} &= \frac{1}{1-\dot{\zeta}(t_u)} \end{aligned} \tag{3.17}$$

Also, from (3.1) and (3.12), we know that

$$\tilde{x} = \frac{1}{2}(\tilde{v} - \tilde{u}) = \frac{1}{2}(g(v) - f(u)) \tag{3.18}$$

When we evaluate (3.18) on the mirror trajectory (so $\tilde{x} = 0$) along with our assumption (3.15), we obtain an expression for the mirror trajectory in terms of the outgoing null-line (See figure 3.2)

$$\begin{aligned} 0 &= g(v) - f(u) \\ 0 &= v - f(u) \\ \therefore v &= f(u) \end{aligned} \tag{3.19}$$

Equating (3.19) with its definition (evaluated at $t = t_u$): $v = t_u + \zeta(t_u)$; we arrive

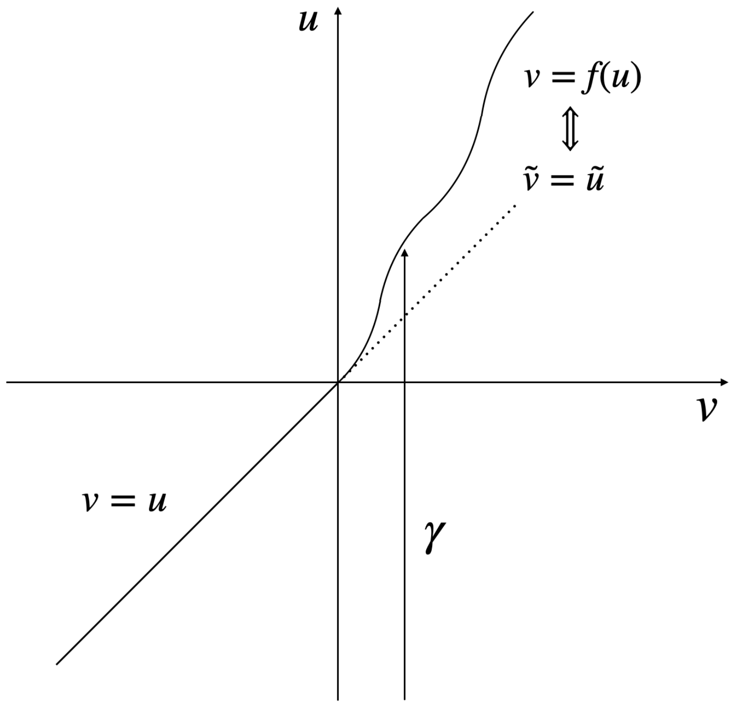

Figure 3.2: Conformal mapping depicted in the (u, v) plane. Here, the mirror trajectory becomes $v = f(u)$, that is further identified as $\tilde{v} = \tilde{u}$.

at

$$f(u) = t_u + \zeta(t_u) \tag{3.20}$$

Taking partial derivative of (3.20) with respect to u; and substituting (3.17) yields[5]

$$f'(u) = \frac{1 + \dot{\zeta}(t_u)}{1 - \dot{\zeta}(t_u)} \tag{3.21}$$

Let's point out that, the relation (3.20) implies, $t_u = \frac{u+f(u)}{2}$. This makes (3.21) an ODE defined in terms of the known function $\dot{\zeta}(t_u)$

$$f'(u) = \frac{1 + \dot{\zeta}(\frac{u+f(u)}{2})}{1 - \dot{\zeta}(\frac{u+f(u)}{2})} \tag{3.22}$$

[5]Dot denotes the time-derivative; and prime denotes the derivative with respect to its own argument.

This form of $f'(u)$ instead of (3.21), ensures that it is well-defined, non-trivial and indeed only dependent on the u null-coordinate. The equation is supplemented with the initial condition $f(u) = u$ at $u < 0$.

One must put particular emphasis on the function f being an invertible, holomorphic map defined as $f : u \to \tilde{u}$; And henceforth, we will be using the null-coordinate u and $\tilde{u}$ interchangeably, keeping in mind their synonymous relationship.

3.5 Modified Field Equation in the Presence of a Moving Mirror

In the conformally transformed background geometry equipped with the metric (3.13); we substitute our choice (3.15), to get

$$ds^2 = -dudv = \frac{1}{f'(u)}\left(-d\tilde{u}d\tilde{v}\right) \tag{3.23}$$

Notice that the metric is *not* static in the $(\tilde{u}, \tilde{v})$ chart; with a conformal scaling coefficient equal to $1/f'(u)$. Hence, the d'Alembertian (defined as $\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu$) makes use of

$$g^{\mu\nu} = \begin{pmatrix} 0 & -2f'(u) \\ -2f'(u) & 0 \end{pmatrix} \tag{3.24}$$

Observe that (3.24) is the "raised" metric expressed in the $(\tilde{u}, \tilde{v})$ coordinate basis (as appears in the d'Alembert operator). This is consistent, because $f(u)$ is an invertible function and $u = f^{-1}(\tilde{u})$.

Define the conformal scaling factor that appears in the metric as: $\frac{1}{f'(u)} \equiv 1 + \Omega(\tilde{u})$. Therefore, inside the adiabatic approximation regime $\left|\dot{\zeta}(t)\right| \ll 1$ (stronger than (3.9)), we may treat the term $\Omega(\tilde{u})$ as a perturbation. So using (3.22)

$$\Omega(\tilde{u}) = \frac{-2\dot{\zeta}(\tilde{u})}{1 + \dot{\zeta}(\tilde{u})} \tag{3.25}$$

And under the assumption: $\Omega(\tilde{u})$ term approximates to

$$\Omega(\tilde{u}) = -2\dot{\zeta}(\tilde{u}) \tag{3.26}$$

So by virtue of (3.24), $\Box = 4f'(u)\left(-\frac{\partial^2}{\partial\tilde{u}\partial\tilde{v}}\right)$;[6] And our field equation becomes

$$\left[f'(u)\left(-4\frac{\partial^2}{\partial\tilde{u}\partial\tilde{v}}\right) - m^2\right]\varphi = 0 \tag{3.27}$$

Dividing by $f'(u)$ both sides and substituting the definition of $\Omega(\tilde{u})$ finally reveals

$$\left(\Box_0 - m^2\right)\varphi = \left(m^2\Omega(\tilde{u})\right)\varphi \tag{3.28}$$

Where, on the left hand side, $\Box_0$ is the d'Alembertian operator for the case of a static mirror (as defined in (3.4)), however, evaluated in the $(\tilde{u}, \tilde{v})$ chart.

$$\Box_0 = -4\frac{\partial^2}{\partial\tilde{u}\partial\tilde{v}} \tag{3.29}$$

The right hand side of the equation (3.28) stands for the inhomogeneous part of our Klein-Gordon field equation, which we will refer to as the "massive source term".[7]

In the rest of the thesis we will work in the $(\tilde{u}, \tilde{v})$ coordinates, So to simplify notations we will omit tildes.

[6] Keeping in mind $\frac{\partial^2}{\partial\tilde{u}\partial\tilde{v}} = \frac{\partial^2}{\partial\tilde{v}\partial\tilde{u}}$.

[7] "Massive" because this term arises only in the case when mass term is present.

Chapter 4

Semiclassical Approximation: Euclidean Bounce Solution

4.1 Preliminary Considerations

In this chapter, we turn our attention to shed some light on the known semiclassical solutions describing the false vacuum decay phenomena. In this arena, the tunnelling probability is determined by the so-called *Euclidean bounce* solutions.[11] For pedagogical purposes, we begin this chapter by explaining the phenomena within the framework of quantum mechanics of single degree of freedom in section 4.2. Then, in the following section 4.3 (section 4.4), the natural extension of this method to many variables (to field theory) is supplied. Our main goal in this chapter is to derive the bounce solution describing the false vacuum decay in a theory of a real scalar field, in the empty Minkowski spacetime.

4.2 Decay of a Metastable State in Single Variable Quantum Mechanics

In dynamical systems, a metastable state is an intermediate energy state corresponding to a local minimum of the governing potential; However, not the true ground state of the system. Here, we want to describe the decay of such a metastable state (of single variable q) into the true ground state. For an example, let's consider the potential $V(q)$ of the form given in the figure 4.1.

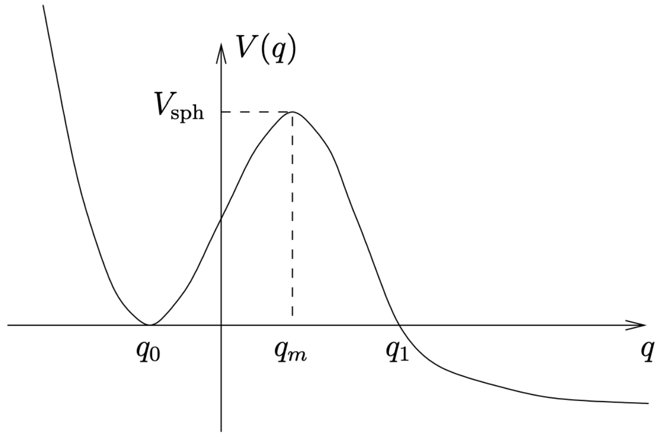

Figure 4.1: False vacuum decay in single variable quantum mechanics' framework. Particle is in the metastable state at $q = q_0$ and escapes to the *true* ground state at $q > q_1$, via tunnelling under the barrier.

In the semiclassical picture, the width of the metastable state Γ is related to the probability of tunnelling through the potential barrier and given by the expression

$$\Gamma = Ae^{-S_B}, \tag{4.1}$$

where the leading semiclassical exponent S_B can be estimated using the WKB approximation as

$$S_B = 2\int_{q_0}^{q_1} dq\sqrt{2MV(q)} \tag{4.2}$$

with q_1 being the turning point ($V(q_1) = 0$). M is the particle mass.

Needless to mention that the defining condition for the semiclassical WKB approximation to be valid, is when the wavelength of the particle is much smaller than the width of the potential barrier.

4.2.1 Connection with the Euclidean Action

Notice that S_B appearing in (4.2) coincides with the Euclidean action, calculated on the classical trajectory of a particle moving "in imaginary time" with energy $E = 0$ from the point $q = q_0$ to the point $q = q_1$ and backwards.

To make the connection concrete, scrutinize the classical action for a particle moving in the potential $V(q)$

$$S = \int dt \left(\frac{M}{2} \left(\frac{\mathrm{d}q}{\mathrm{d}t} \right)^2 - V(q) \right) \tag{4.3}$$

Now let's proceed via a formal substitution $t = -i\tau$ assuming τ is real; then (4.3) becomes iS_E

$$S_E = \int d\tau \left(\frac{M}{2} \left(\frac{\mathrm{d}q}{\mathrm{d}\tau} \right)^2 + V(q) \right) \tag{4.4}$$

We shall refer to S_E as the Euclidean action, and τ the Euclidean time. Upon this substitution, the Minkowski metric becomes Euclidean one, $ds_E^2 = d\tau^2 + dx^2$ modulo an overall sign. Hence the name "Euclidean time".

The equation of motion following from the action (4.4) is

$$M \frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} = -\frac{\partial(-V)}{\partial q} \tag{4.5}$$

Observe that this resembles the Newtonian equation of motion for a particle moving in the potential $(-V)$; and the integral of motion (i.e. Euclidean Energy, $\mathcal{E}$) is

$$\mathcal{E} = \frac{M}{2} \left(\frac{\mathrm{d}q}{\mathrm{d}\tau} \right)^2 - V(q) \tag{4.6}$$

We are interested in the solution with $\mathcal{E} = 0$; Without loss of generality, we may presume the solution to possess the following properties:

1. At $\tau \to -\infty$ the particle starts from the point $q = q_0$, goes to q_1, and returns back to q_0 as $\tau \to \infty$;
2. The choice of τ is made to accommodate that turning point $q = q_1$ at $\tau = 0$.

The solution satisfying this requirements will henceforth be called as "bounce", denoted as $q_B(\tau)$. On this solution, Euclidean action (4.4) becomes

$$S_E(q_B) = \int_{-\infty}^{\infty} d\tau \left(\frac{M}{2} \left(\frac{\mathrm{d}q_B}{\mathrm{d}\tau} \right)^2 + V(q_B) \right)$$

$$S_E(q_B) = 2 \int_{-\infty}^{0} d\tau \; 2V\left(q_B(\tau)\right) \tag{4.7}$$

Using $\mathcal{E} = 0$, for the bounce solution. Now changing the integration variables from τ to q_B using (4.6); we arrive at

$$S_B = S_E\left(q_B(\tau)\right) \tag{4.8}$$

Hence, the semiclassical leading exponent of the decay rate of a metastable state (as in (4.2)), is precisely equal to the Euclidean action evaluated on the bounce.

4.3 Generalization to Finitely Many Variables

A particle within a system of generalised coordinates $q^1, q^2, \ldots, q^N$ is defined by a Lagrangian[1]

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\mathrm{d}q^i}{\mathrm{d}t}\right)^2 - V(q^1, q^2, \ldots, q^N) \tag{4.9}$$

$$\mathcal{L} = \frac{1}{2}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)^2 - V(\mathbf{q}) \tag{4.10}$$

Here, $\mathbf{q}$ being the generalized vector coordinate.

Assuming the potential (similar to Figure 4.1, but in many variables) contains a local minimum at the point $\mathbf{q} = \mathbf{q_0}$; we fix the origin of energy such that $V(\mathbf{q_0}) = 0$. Further assuming that the *true* ground state of the system in this potential exists; However, separated by a classically impenetrable barrier.

The holistic connection between the leading semiclassical exponential for the probability of tunneling and classical Euclidean action carries forward to quantum mechanics of many variables. Isotropically, we start by looking for the bounce solution of classical equations of motion that arise from the Euclidean action

$$S_E = \int d\tau \left(\frac{M}{2}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\tau}\right)^2 + V(\mathbf{q})\right) \tag{4.11}$$

Now we impose our boundary conditions on the bounce $\mathbf{q_B}(\tau)$:

$$\mathbf{q_B}(\tau \to \pm\infty) = \mathbf{q_0} \tag{4.12}$$

[1]We use Lagrangian treatment rather than Hamiltonian approach to quantum mechanics because it is more naturally generalized to quantum field theory.

$$\left.\frac{\mathrm{d}q_B^i}{\mathrm{d}\tau}\right|_{\tau=0} = 0, \quad \forall\, i \tag{4.13}$$

Here, (4.13) ensures the turning point $\mathbf{q_B}(\tau) = \mathbf{q_1}$, precisely at $\tau = 0$.

Most Probable Tunnelling Path

The bounce solution minimizes the Euclidean action (4.11), among all virtual paths describing tunnelling. Because of the boundary condition (4.12) it has zero Euclidean energy, $\mathcal{E} = 0$. Following the same procedure as in section 4.2, we can show that

$$S_B = 2\int_{\mathbf{q_0}}^{\mathbf{q_1}} d\lambda \sqrt{2MV(\mathbf{q})} \tag{4.14}$$

where,

$$d\lambda \equiv \sqrt{\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\tau}\right)^2} d\tau \tag{4.15}$$

is the differential length element of the trajectory of bounce solution. This exactly coincides with the exponential suppression of the tunnelling probability within the WKB approximation, and the bounce solution is nothing but the most probable tunnelling path.

Evolution in real time *after* tunnelling

Once in the classically allowed region, the most probable path for the particle to emerge from under the barrier at point $\mathbf{q_1}$ (according to the boundary condition (4.13)) is with zero momentum. the motion after evolution can now be estimated by solving the classical equations of Newtonian mechanics in conventional time with zero classical energy. And the boundary conditions for the tunnelling path becomes the initial conditions of the particle in real time.

We can check that this indeed corresponds to the classical trajectory, by analytically continuing the known bounce solution, upon substitution: $\tau = it$ (t real). So

$$\mathbf{q}_B(it) = \mathbf{q}(t) \tag{4.16}$$

satisfies the conventional Newtonian equations

$$M\frac{\mathrm{d}^2\mathbf{q}}{\mathrm{d}t^2} = -\frac{\partial V}{\partial \mathbf{q}} \tag{4.17}$$

as desired.

4.4 Extension to the Scalar Field Theory

In the context of field theory, quantum mechanical tunnelling of a metastable state decaying to a ground state goes by the name of "False Vacuum Decay".

The formalism in the previous sections was developed for a system with finitely many degrees of freedom. To avoid the complications of many fields of real world, we can think of it as put on a lattice, whereupon it becomes just quantum mechanics with large number of degrees of freedom. So it is rather instructive to generalise and identify the direct resemblance of the developments of the preceding sections to a system with continuously many degrees of freedom, labelled as a scalar field. For our study, these *semiclassical* considerations shall suffice [9].

4.4.1 Euclidean Bubble Nucleation

Consider the following scalar field action in D-dimensional spacetime

$$S = \int dt d^{D-1}\mathbf{x}\left[\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 - \frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\mathbf{x}}\right)^2 - V(\varphi)\right] \tag{4.18}$$

Now let's *Euclideanize* (4.18) by substituting $t = -i\tau$ to get iS_E; where,

$$S_E = \int d\tau d^{D-1}\mathbf{x}\left[\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\mathbf{x}}\right)^2 + V(\varphi)\right] \tag{4.19}$$

Abbreviating the coordinates as $x^\mu = (\tau, \mathbf{x})$ and summing over repeated indices $\mu = 0, 1, \ldots, (D-1)$ with the underlying Euclidean metric δ^μ_ν; we have,

$$S_E = \int dx^\mu \left[\frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi\right) + V(\varphi)\right] \tag{4.20}$$

The field equations of the action (4.20) are

$$-\partial_\mu \partial^\mu \varphi + \frac{\partial V}{\partial \varphi} = 0 \tag{4.21}$$

As before, we now seek the bounce solution of the equation (4.21), demanding the boundary conditions equivalent to (4.12) and (4.13):

$$\varphi\left(\tau, \mathbf{x}\right) = \varphi_- \text{ as } \tau \to \pm\infty \tag{4.22}$$

$$\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right|_{\tau=0} = 0, \quad \forall\, i \tag{4.23}$$

where φ_- is the false vacuum state. It is this latter condition (4.23) that ensures the analytic continuation of the bounce onto the real time axis is indeed real and describes the progression of the field after tunneling.

At this point, notice that both these boundary conditions ((4.22) and , (4.23)) can be simultaneously satisfied if we consider smooth, spherically-symmetric fields[2] $\varphi\left(\rho\right)$ (see figure 4.2 (*Left*)) such that the bounce depends only on $\rho = \sqrt{x_\mu x^\mu} = \sqrt{\tau^2 + \mathbf{x}^2}$ that reaches φ_- asymptotically as $\rho \to \infty$; this assertion has been proved under various assumptions in literature [16, 17]. The field equation (4.21) becomes

$$\partial_\rho^2 \left(\varphi(\rho)\right) + \frac{1}{\rho}\partial_\rho \left(\varphi(\rho)\right) - \frac{\partial V}{\partial \varphi} = 0 \tag{4.24}$$

The problem of finding the bounce solution with the smallest Euclidean action, giving the *most probable* trajectory for the vacuum transition is to simply find the solution to the equation (4.24); which indeed exists, from the analogy of (4.24) with Newtonian mechanics for a particle with its coordinate being φ, moving in ρ, (that is equivalent to the notion of time in Newtonian mechanics) under the potential $-V(\varphi)$ in the presence of a friction force, proportional to the particle speed, where the friction coefficient is inversely proportional to ρ.

When we analytically continue this bounce solution to real time, by substituting $\tau = it$, defined only outside the light cone in Minkowski spacetime; The field of the

[2] The idea of a "bubble" of the true vacuum in the false one, comes from this spherically symmetric form of the bounce solution, in scalar field theory.

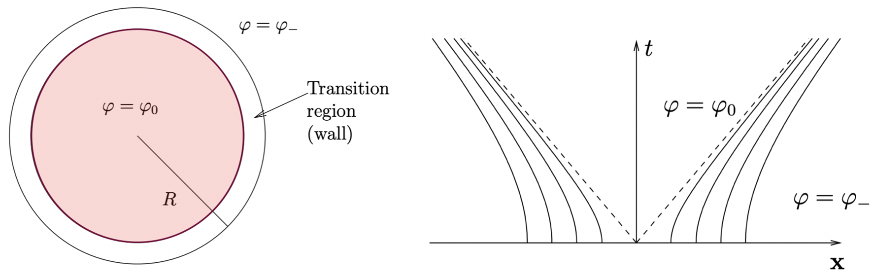

Figure 4.2: (*Left*) Spherically symmetric Euclidean bubble (bounce) solution, in Euclidean spacetime; (*Right*) Minkowski spacetime diagram of the classical growth of the "bubble" of true vacuum after its materialization. Hyperbolas are the path traced out by the bubble wall.

bubble in conventional time is of the form:

$$\varphi(\mathbf{x}, t) = \varphi_B \left(\sqrt{\mathbf{x}^2 - t^2} \right) \tag{4.25}$$

The surfaces of constant φ trace the hyperboloids $\mathbf{x}^2 - t^2 = \text{constant}$. This is depicted in Figure 4.2 (*Right*). It clearly indicates that the field present inside the light cone reaches its true vacuum state, as shown in Figure 4.2. From the reference frame of a stationary observer, the size of transition wall decreases with time, and the velocity of the wall approaches the speed of light.

Virtue of the Euclidean Spacetime

Note that tunnelling processes in scalar field theory are described by the solutions of field equations arising in Euclidean spacetime. Later when we derive Feynman Green's function[3] of the real scalar field, we shall see that the Wick-rotated "Euclidean" Green's function does not have poles on the Euclidean-time axis. Therefore it is often mathematically convenient to carry the computations in Euclidean spacetime, and Wick-rotate back to the pseudo-Euclidean spacetime at the end. The corresponding boundary conditions are naturally built-up for the Feynman propagator by the virtue of this procedure.

[3] Feynman Green's function is the unique propagator whose contour of integration can be analytically continued to perform Wick rotation without intersecting the poles.

Chapter 5

The Toy Model

5.1 Inverted Liouville Potential with a Mass Term

In this chapter, we introduce a toy model[18] with a scalar potential of the following form, as shown in figure 5.1:

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 - 2\kappa\left(e^{\varphi} - 1\right) \tag{5.1}$$

where m^2 and κ are positive constants.

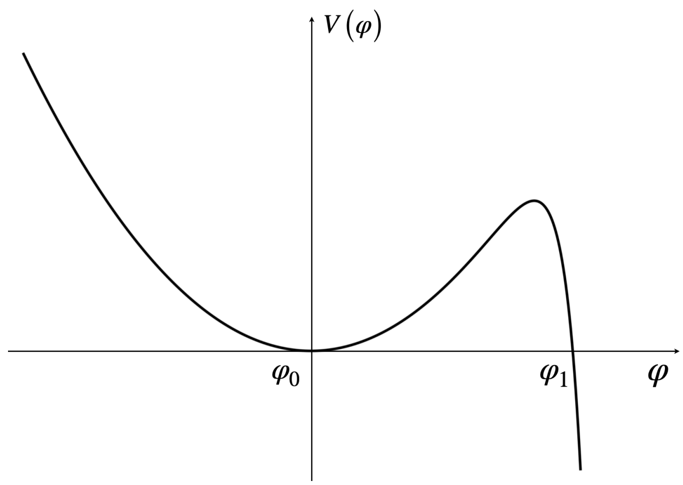

Figure 5.1: The scalar potential, implemented in the toy model consisting of a mass term and an inverted Liouville potential interaction term. φ_0 corresponds to the false vacuum and $\varphi \geq \varphi_1$ corresponds to the *unbounded* true vacuum.

The potential (5.1) contains two components: first one is the mass term, and second one is the interaction term. Recognize that the mass term accommodates false vacuum to be present at the origin and the interaction term introduces an inverted Liouville potential that is unbounded from below.

We now implement a technical assumption that the parameters in the potential m^2 and κ obey:

$$\ln \frac{m}{\sqrt{\kappa}} \gg 1 \tag{5.2}$$

This premise (5.2) is mandatory for the existence of a reasonable height of a potential barrier that separates false and true vacuum states of φ and allows relevant semiclassical solutions to be analytically devised in this domain.

Let us examine the dynamics of this toy model potential by considering the scalar field equation of motion in 2D Minkowski-flat spacetime as a starting point. So from (2.6) and (5.1), we obtain

$$\Box\varphi - m^2\varphi + 2\kappa e^{\varphi} = 0 \tag{5.3}$$

In search of its solutions, we find that the complexity is extensively reduced where the core of the bounce happens to be much smaller in size than the inverse mass m^{-1}. Then the solution outside this core (i.e. tail solution) is simply proportional to the Green's function (neglecting the third term). On the other hand, the core of the bounce can be found by neglecting the mass (second term). The full solution is obtained by matching the long-distance asymptotics of the core with the short-distance behavior of the Green's function. In the following section 5.2 we present this strategy in detail, to gauge solutions of the field equation (5.3).

5.2 Method of Asymptotic Expansion & Matching

The proposition is to split the solution into a core and a tail; and asymptotically match those two in the overlapping region. Now let's look at the equilibrium situation first and verify that this method correctly coincides with the known Euclidean solution.

To explore this specific example of a scalar field in the potential (5.1), we first look at the region where the mass term dominates, i.e. $\varphi \approx \varphi_0$ and $\frac{\partial V}{\partial \varphi} = m^2\varphi$. In this region, the third term in (5.3) can be neglected and its solution coincides with the solution in a free massive field theory subject to relevant boundary conditions.

On the other hand, when $\varphi \geq \varphi_1$, we may neglect the second (massive, free) term in (5.3) and the corresponding field equation simply reduces to the familiar Liouville equation of the form

$$\Box\varphi + 2\kappa e^{\varphi} = 0 \tag{5.4}$$

This equation (5.4) has a general solution[19, 20] in the (u, v) null coordinate chart

$$\varphi = \ln\left[\frac{4\mathcal{P}'(-u)\mathcal{Q}'(v)}{(1+\kappa\mathcal{P}(-u)\mathcal{Q}(v))^2}\right] \tag{5.5}$$

where the primes, as usual, imply the derivatives of these functions with respect to their own arguments. $\mathcal{P}(-u)$, $\mathcal{Q}(v)$ are arbitrary, meromorphic functions. To make it specific, these functions are further set by the relevant boundary conditions imposed on the field φ, and this specific form shall be designed to be matched by the tail.

5.2.1 Euclidean Vacuum Bounce

In two-dimensional empty Minkowski spacetime, the bounce solution (φ_B) obeys the field equation (4.24) (as derived in section 4.4)

$$\partial_\rho^2(\varphi_B) + \frac{1}{\rho}\partial_\rho(\varphi_B) - m^2\varphi_B + 2\kappa e^{\varphi_B} = 0 \tag{5.6}$$

Core Solution

Following our treatment, the core (neglecting the mass term at $\rho \ll m^{-1}$) is given by

$$[\varphi_B]_{\text{core}} = \ln\left(\frac{4C^2}{(1+\kappa C^2\rho^2)^2}\right) \tag{5.7}$$

Comparing with the general solution (5.5) yields our arbitrary functions $\mathcal{P}$ and $\mathcal{Q}$ to be linear:

$$\mathcal{P}(z) = Cz, \quad \mathcal{Q}(\bar{z}) = C\bar{z} \tag{5.8}$$

and the notations are consistent with the Euclidean signature,

$$u \mapsto -z = -(x + i\tau), \quad v \mapsto \bar{z} = (x - i\tau) \tag{5.9}$$

In the long-distance asymptotic limit ($\rho \gg (C\sqrt{\kappa})^{-1}$), (5.7) becomes

$$\varphi_B \approx -4\ln\sqrt{\kappa}\rho - 2\ln C + 2\ln 2 \tag{5.10}$$

Tail Solution

The tail, on the other hand, is found by solving the free massive equation and neglecting the Liouville term in (5.6). It is given by

$$[\varphi_B]_{\text{tail}} = A_{\text{tail}} K_0\,(m\rho) \tag{5.11}$$

where A_{tail} is another constant and K_0 is the modified Bessel function of the second kind.

At small ρ (as $\rho \to 0$),

$$\varphi_B \approx -A_{\text{tail}}\ln(m\rho) + A_{\text{tail}}(\ln 2 - \gamma_E) + \mathcal{O}(m\rho) \tag{5.12}$$

where, γ_E is the Euler constant. Matching (5.10) and (5.12) gives (upto first order):

$$A_{\text{tail}} = 4, \quad C = \frac{m^2}{2\kappa}\, e^{2\gamma_E} \tag{5.13}$$

(5.2) ensures the matching region exists.

Notice that the tail (5.11) is proportional to the analytic continuation to the Euclidean time ($\tau = -it$) of the Feynman Green's function (using "$i\epsilon$" prescription)

$$\mathcal{G}_F(t, x; 0, 0) = \frac{1}{2\pi} K_0\left(m\sqrt{x^2 - t^2 + i\epsilon}\right) \tag{5.14}$$

The tail solution, as in (5.11), is then obtained by

$$[\varphi_B]_{\text{tail}} = \mathcal{G}_F(t, x; 0, 0)|_{t \mapsto \tau} = \frac{1}{2\pi} K_0\left(m\sqrt{x^2 + \tau^2 + i\epsilon}\right) \tag{5.15}$$

As discussed in 4.4.1, the Feynman boundary conditions are correctly satisfied in (5.14) and its analytic continuation in to Euclidean time is illustrated in Figure 5.2.

Thus, the two forms of solutions (a core and a tail) shall be augmented together, in the asymptotic limit to unveil the solution for false vacuum decay of this toy model. The ancillary condition (5.2) ensures the existence of an overlap region.

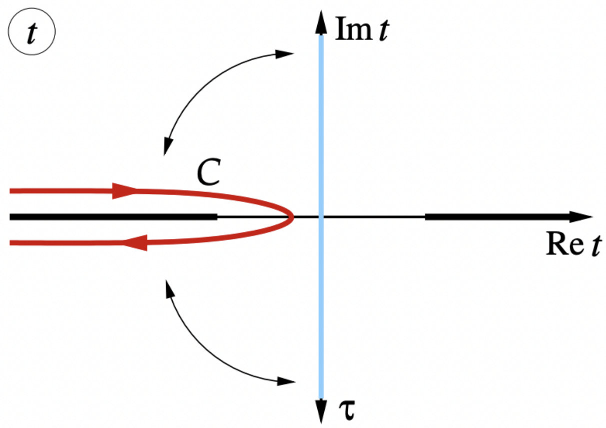

Figure 5.2: Analytic continuation of the contour C (*Red*) in complexified time plane. The standard Euclidean bounce trajectory is defined on the imaginary time (*Blue*) and contour satisfies Feynman Boundary conditions at $\mathrm{Re}(t) \to -\infty$.

Now we would like to implement this strategy for computing the probability of tunnelling suppression in the presence of an arbitrarily moving, time-dependent mirror. This explicit time dependence enforces us out of the equilibrium regime, where the Euclidean construction is no longer valid. This shall be discussed in the next section 5.3.

However, before we treat this problem in all its generality, let's see what happens to the structure of the bounce, when we simply place scalar field of our toy model in the presence of a static mirror.

5.2.2 Bounce with Static Mirror

To begin with, we had already discussed the behaviour of the mode functions in the static mirror environment in 3.2. In the same lines of development, we would like to construct the Feynman Green's function that corresponds to the tail of our bounce. And we want this to vanish at $x = 0$ (the static mirror trajectory). Upon this boundary requirement, $\mathcal{G}_F$ is no longer given by (5.15). Its form may be found using the method of images:

$$\begin{aligned}\mathcal{G}_F(\tau, x; \tau', x') = &\frac{1}{2\pi} K_0\left(m\sqrt{(\tau-\tau')^2 + (x-x')^2 + i\epsilon}\right)\\ &+ \frac{1}{2\pi} K_0\left(m\sqrt{(\tau-\tau')^2 + (x+x')^2 + i\epsilon}\right)\end{aligned} \tag{5.16}$$

The short range behaviour demands $\rho \to 0$ (i.e. $\tau' \to \tau$ and $x' \to x$); So we have an extra contribution from (5.16) (in the first order) to the tail solution as follows:

$$[\varphi_B]_{\text{tail}} \approx -A'_{\text{tail}} \ln (m\rho) + A'_{\text{tail}}(\ln 2 - \gamma_E) - \frac{1}{2\pi} K_0 (2mx) + \mathcal{O}(m\rho) \tag{5.17}$$

where A'_{tail} is a constant; and it will further be fixed upon matching with the long-distance asymptotics of the core.

The additional third term in (5.17) is especially of our interest. The explicit appearance of x infers that the bounce now depends on its distance from the static mirror. In particular, it implies that the effective change in φ_B with respect to the empty vacuum case (5.12), should be exponentially suppressed if we move much further away from the mirror.

Thus, even in the equilibrium environment, we can see that the bounce solution is revised (*at least* in the close separation). Therefore, it shall be of much interest to look at some cases when there is a time-dependent mirror present in the background.

5.3 Challenge of Non-Equilibrium Dynamics

First, let's demand the solution in the region $\varphi \approx \varphi_0$, when only the mass term dominates.

5.3.1 Tail Solution: Massive Free Term

We begin with the modified field equation, that we developed in section 3.5. Rewriting the field equation (3.28),

$$\left(-4\frac{\partial^2}{\partial u \partial v} - m^2\right)\varphi = \left(m^2 \Omega\,(u)\right)\varphi \tag{5.18}$$

The equation (5.18) can be recast into an integral equation using the Green's function (in the (t, x) chart, *conformally-flat* spacetime),

$$\varphi(t, x) = \int \mathcal{G}(t, x; t', x') \left[m^2 \Omega\right] \varphi(t', x')\, dt' dx' \tag{5.19}$$

We can easily verify that (5.19) satisfies the modified field equation (5.18).

For our purposes, first, we will promote the fields in this integral to operators. Then, the static mirror boundary is imposed on our field operator in this conformally-flat spacetime, so the field operator (subject to the Dirichlet boundary conditions) is rewritten (derived in (3.8)) as

$$\hat{\varphi}_0(t,x) = \int_0^\infty \frac{dk}{\sqrt{\pi\omega}} \left(\hat{a}_k \, e^{-i\omega t} \sin(kx) + \hat{a}_k^\dagger \, e^{i\omega t} \sin(kx) \right) \tag{5.20}$$

where $\omega = \sqrt{k^2 + m^2}$.

The form of the Retarded Green's function, as reckoned in (2.28), with these spatial mode-functions (5.20) (vanishing at $x = 0$), becomes the following

$$\mathcal{G}_R(t,x;t',x') = \int_{-\infty}^\infty \frac{d\omega}{2\pi} \int_0^\infty \frac{dk}{2\pi} \, \frac{e^{-i\omega(t-t')} \sin(kx) \sin(kx')}{\omega^2 - k^2 - m^2 - i\epsilon\omega} \tag{5.21}$$

Putting all these together,

$$\begin{aligned} \hat{\varphi}(t,x) = \int dt' dx' \left[\int_{-\infty}^\infty \frac{d\omega}{2\pi} \int_0^\infty \frac{dk}{2\pi} \left(\frac{e^{-i\omega(t-t')} \sin(kx) \sin(kx')}{\omega^2 - k^2 - m^2 - i\epsilon\omega} \right) \right] \times \\ \left[m^2 \Omega(u) \right] \times \int_0^\infty \frac{d\tilde{k}}{\sqrt{\pi\tilde{\omega}}} \left(\hat{a}_{\tilde{k}} \, e^{-i\tilde{\omega}t'} \sin(\tilde{k}x') + \hat{a}_{\tilde{k}}^\dagger \, e^{i\tilde{\omega}t'} \sin(\tilde{k}x') \right) \end{aligned} \tag{5.22}$$

Hereafter, we observe that it may be possible to exploit the explicit dependence of the Ω term on u null-line, by first moving to the null coordinates' chart and performing the integration over the v axis, analytically.

Chapter 6

Conclusion & Outlook

In this thesis, we thoroughly investigate the question of computing tunnelling probability of the false vacuum decay phenomena, in the presence of a moving mirror, in a semiclassical setting. We explored a specific example of scalar field potential as our toy model (5.1) and attempt to probe the relevant semiclassical solutions by matching the solutions in distinctly corresponding domains, in their asymptotic limit.

In order to better understand the fascination of this problem, we considered each segments autonomously in each chapters of this course. The study commenced by an abstract, introductory derivation of the field equation in 2.1. without committing to any specificity of the governing potential, we posed its relevant solutions and corresponding mode functions with their defined normalization scheme. The promotion of the fields to operators via canonical quantization, properties of the in-vacuum state and the structure of various Green's functions (2.3) were also discussed.

We then implement the framework of a moving mirror, acting as a boundary value of the field operators. Both the cases: (i) static and, (ii) arbitrarily moving trajectory; Along with a section (3.4) dedicated to the construction of a conformal coordinate map between the two. We were able to transform the problem of characterizing a field present in a 2D Minkowski-flat spacetime (static metric) subject to an arbitrarily moving mirror boundary, to the problem of describing the field present in a conformally-flat spacetime (dynamic metric) subject to a static mirror boundary. This transformation naturally gives rise to the *massive source term* in the modified field equation (3.28).

We followed this by a brief review of the semiclassical Euclidean bounce construction, where the tunnelling solution lives in purely imaginary time. An explanation of this bounce solution was reasoned in three disciplines: (i) vacuum decay in quantum mechanics of one variable, (ii) generalization to many variables, and (iii) extension to the scalar field theory. In 4.2.1, we sensibly asserted the equivalence of the bounce suppression to the Euclidean action upon analytic continuation in complexified time plane.

We eventually model the scalar field with a toy potential, consisting of an inverted Liouville term and a massive free term, subject to a moving mirror boundary, in chapter 5. We attempt to analytically solve the modified massive free field equation (5.18) and look for the tunnelling bounce, to accommodate the *time-dependent* mirror boundary. We first calculate the Feynman Green's function from this modified field equation, ensuring the correct boundary conditions for the false vacuum field. Given the appropriate size of the parameters and the auxiliary condition (5.2), we are able to estimate the close-range behaviour of the Feynman Green's function, referring to this solution as a *tail*.

In subsection 5.2.2, we successfully derived the tail solution of the vacuum bounce associated with a static mirror, by explicitly solving for the Feynman Green's function and imposing the method of images. We observed interesting characteristics in the short-range asymptotics of this Green's function (5.17), by mere introduction of a static mirror boundary; that spiked our motivation to dig further into the non-equilibrium, time-dependent environment. The anticipation is to extend this strategy further in the arbitrarily moving mirror case and match that result with the general, non-linear, *core* solution of Liouville equation (5.5); to quantify the tunnelling solution in non-equilibrium domain and gauge predictions of this phenomena.

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