Planetary Orbits Around Binary Star Systems

Introduction

Binary stars are systems containing two companion stars orbiting each other via gravitational attraction with a common centre of mass. It is generally believed that more than half of all star systems we observe contain multiple stars. The search for a habitable exoplanet is currently an active area of research in astronomy, and recent results confirm the existence of exoplanets that orbit binary star systems. Studying such planets is of great importance for exobiology. For life to exist, the planet on which it lives must keep to a stable orbit over long timescales. This is needed because the time for protoplanetary formation and for evolution would encompass billions of orbital periods.

There are two species of orbit possible for a planet in a binary system, namely P-type and S-type. P-type orbits will orbit the combined centre of mass of the two stars as if they are one single body whereas planets in an S-type orbit only travel around one of the stars with the other acting as a perturbation. In this project, we are only going to take P-type orbit into account.

To solve the differential equations involved in a 3-body system, the equations of motion must first be derived (as shown in the following section). This assumes Newtonian physics is appropriate and combines Newton's 2nd law with the gravitational attraction of the two stars. Satellites do not strictly follow this motion because of perturbations from other planets, comets, asteroids, and even effects such as accretion changing the gravitational field. However, for the scope of this paper, where we neglect such effects, they can be considered valid. Throughout, we assume that masses are in units of solar masses ($M_{\circledS}$) so that $M = 1M_{\circledS}$, distances are in terms of astronomical units (AU) so that $R = 1\ AU$ and time in units of years. It is further assumed that the system is confined to a 2D plane, as we observe in most solar systems.

Methodology:

Manipulation of differential equations:

In order to make the differential equations suitable for the computational leapfrog algorithm, first consider a pair of stars obeying Kepler's laws of motion and rotating with a period T around their common centre of mass on a circle of radius R. Their **Cartesian coordinates** will be given by,

$$X_1 = R\cos(\Omega t), \qquad Y_1 = R\sin(\Omega t),$$

$$X_2 = -R\cos(\Omega t), \qquad Y_2 = -R\sin(\Omega t),.$$

The planet's gravitational attraction to the two stars and Newton's laws must be combined.

Consider the gravitational force felt by a planet of negligible radius and mass, m, due to two stars (1 and 2) of equal mass M, a distance R from the origin, and at a distance r_1 and r_2 respectively from the planet.

Here,

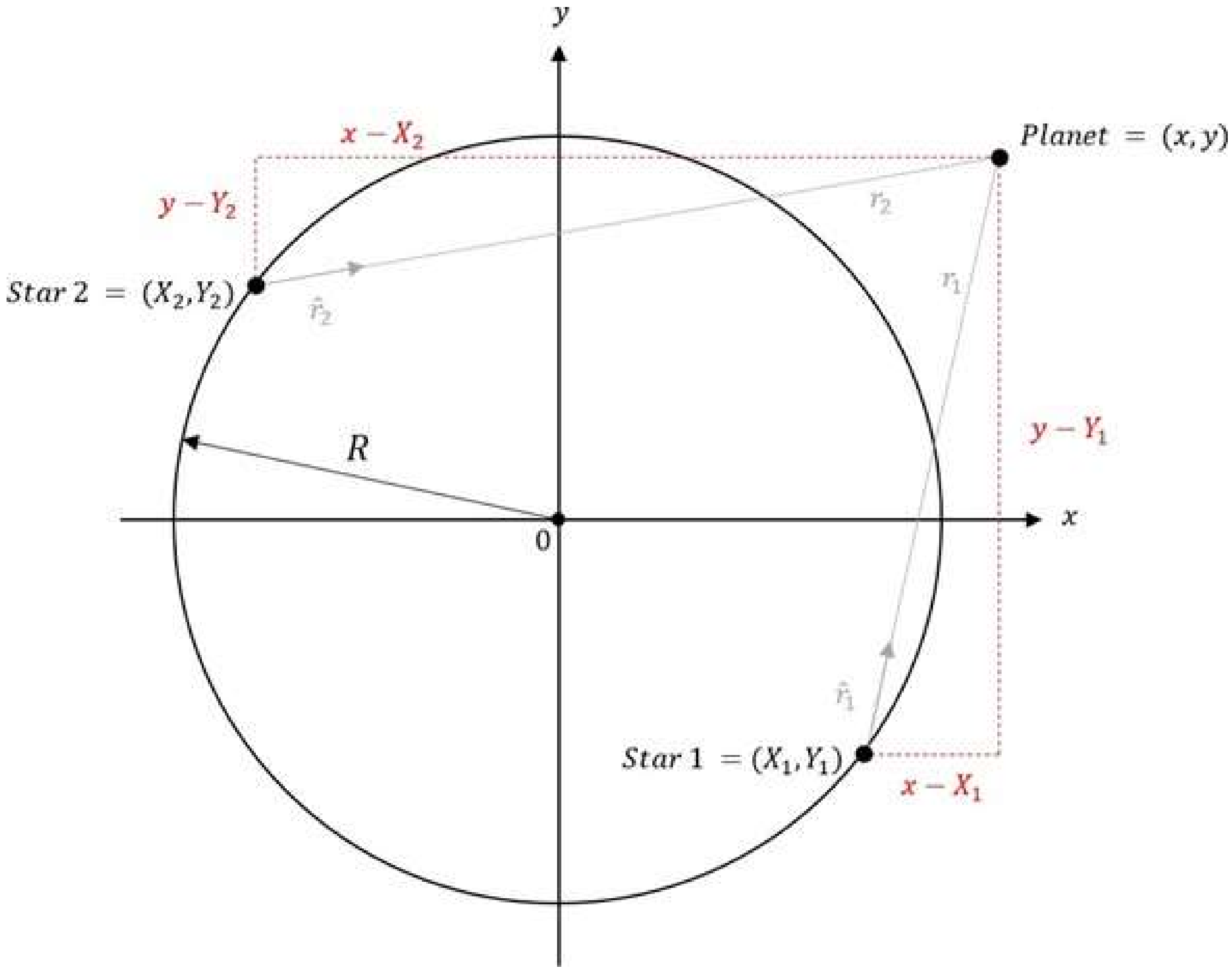
Introducing a planet of negligible mass into this scenario creates the 3-body system where the position $r_{1,2}$ of the planet from star 1 or star 2 is given by,

$$r_{1,2}^2 = (x - X_{1,2})^2 + (y - Y_{1,2})^2.$$

The **gravitational force** has the functional form:

$$\mathbf{F} = -\frac{GMm}{r_1^2}\hat{\mathbf{r_1}} - \frac{GMm}{r_2^2}\hat{\mathbf{r_2}} = m\mathbf{a},$$

The following diagram illustrates all distances involved in the 3-body system of a planet and two binary stars:



Where, G is the gravitational constant, $\widehat{r_1}$ **and** $\widehat{r_2}$ are unit vectors in the direction of a straight line between the planet and star 1 and 2 respectively, and a is the acceleration of the planet. Newton II has been used to equate this gravitational force to the motion of the body **via** $\boldsymbol{F = ma}$. Writing the unit vectors in terms of x and y components we have,

$$\hat{\mathbf{r_1}} = \frac{x - R\cos(\Omega t)}{r_1}\hat{\mathbf{x}} + \frac{y - R\sin(\Omega t)}{r_1})\hat{\mathbf{y}},$$

$$\hat{\mathbf{r_2}} = \frac{x + R\cos(\Omega t)}{r_2}\hat{\mathbf{x}} + \frac{y + R\sin(\Omega t)}{r_2}\hat{\mathbf{y}}.$$

With these replacements,

$$\begin{aligned}\mathbf{a} = &-GM\left(\frac{x - R\cos(\Omega t)}{r_1^2} + \frac{x + R\cos(\Omega t)}{r_2^2}\right)\hat{\mathbf{x}}\\ &-GM\left(\frac{y - R\sin(\Omega t)}{r_1^2} + \frac{y + R\sin(\Omega t)}{r_2^2}\right)\hat{\mathbf{y}}.\end{aligned}$$

Decomposing this into x and y components, and defining the following variables,

$$\begin{aligned} X_1 &= R\cos(\Omega t), \qquad & Y_1 &= R\sin(\Omega t),\\ X_2 &= -R\cos(\Omega t), \qquad & Y_2 &= -R\sin(\Omega t),\end{aligned}$$

yields two equations for the acceleration in the x and y directions,

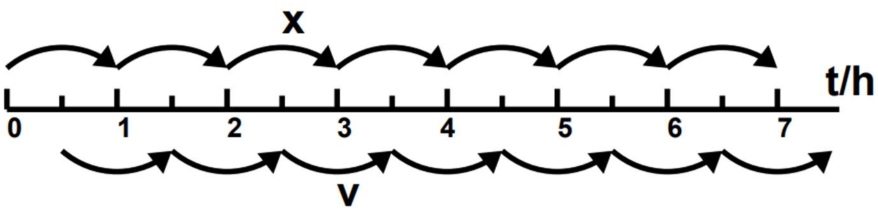
$$\ddot{x} = -GM\left(\frac{1}{r_1^2}\frac{x - X_1}{r_1} + \frac{1}{r_2^2}\frac{x - X_2}{r_2}\right),$$

$$\ddot{y} = -GM\left(\frac{1}{r_1^2}\frac{y - Y_1}{r_1} + \frac{1}{r_2^2}\frac{y - Y_2}{r_2}\right).$$

Hence, we have found the equations we require as inputs for our **numerical integration approach (leapfrog)** and how they have been utilized optimally are discussed in the following section.

The Leapfrog (or Velocity Verlet Method) Algorithm:

Here, we will briefly discuss the structure of the leapfrog method.



Aforementioned illustration visualizes x (position) and v (velocity) **leapfrogging** over each other as shown.

$$\boxed{v_{n+1/2} = v_n + \tfrac{1}{2} h F(x_n),}$$

$$\boxed{x_{n+1} = x_n + h v_{n+1/2}} \qquad \text{(velocity Verlet),}$$

$$\boxed{v_{n+1} = v_{n+1/2} + \tfrac{1}{2} h F(x_{n+1})\,,}$$

```
for (i=0; i<(2*n); i+=2) {

    j = int(i/2);

    // kick

    X1[j] = pos_Mx (dt*j);  // Position of Stars at time = t
    X2[j] = (-1)*(pos_Mx (dt*j));
    Y1[j] = pos_My (dt*j);
    Y2[j] = (-1)*(pos_My (dt*j));

    r1[j] = dist_P (x[j], X1[j], y[j], Y1[j]);
    r2[j] = dist_P (x[j], X2[j], y[j], Y2[j]);

    ax[j] = acceleration (x[j], X1[j], X2[j], r1[j], r2[j]); // Acceleration at time = t
    ay[j] = acceleration (y[j], Y1[j], Y2[j], r1[j], r2[j]);

    vx[i+1] = vx[i] + ((dt)/2)*(ax[j]); // Velocity at time = t + dt/2
    vy[i+1] = vy[i] + ((dt)/2)*(ay[j]);

    // drift

    x[j+1] = x[j] + (dt)*(vx[i+1]); // Position of planet drifted forward at t + dt
    y[j+1] = y[j] + (dt)*(vy[i+1]);

    // kick

    X1[j+1] = pos_Mx (dt*(j+1));    // Position of Stars at time = t + dt
    X2[j+1] = (-1)*(pos_Mx (dt*(j+1)));
    Y1[j+1] = pos_My (dt*(j+1));
    Y2[j+1] = (-1)*(pos_My (dt*(j+1)));

    r1[j+1] = dist_P (x[j+1], X1[j+1], y[j+1], Y1[j+1]);
    r2[j+1] = dist_P (x[j+1], X2[j+1], y[j+1], Y2[j+1]);

    ax[j+1] = acceleration (x[j+1], X1[j+1], X2[j+1], r1[j+1], r2[j+1]); // Acceleration at time = t + dt
    ay[j+1] = acceleration (y[j+1], Y1[j+1], Y2[j+1], r1[j+1], r2[j+1]);

    vx[i+2] = vx[i+1] + ((dt)/2)*(ax[j+1]); // Velocity at time = t + dt
    vy[i+2] = vy[i+1] + ((dt)/2)*(ay[j+1]);

}
```

A screen capture of the Leapfrog (velocity Verlet method) code (**in cpp**, originally executed in the actual project on phys-ugrad) has been put up, to elicit the concept of **"Kick-Drift-Kick".**

In addition to combining great simplicity with second order accuracy, the leapfrog algorithm has several other desirable features:

1. *Time Reversal Invariant*
2. *Conserves Angular Momentum*
3. *Symplectic (area preserving)*

<u>Testing (demo):</u>

In order to do testing on the workings of the integrator, we can provide solutions for simpler cases to sketch the path of the orbit of a planet (like Earth) orbiting a single star (like Sun) in a uniform circular trajectory.

For such orbits where the planet accelerates towards a single centre of mass at the origin following a circular path (**circular P-type orbits**), simplified acceleration equations are needed. This system is **analogous to Earth orbiting the Sun**. The equations of motion are therefore,

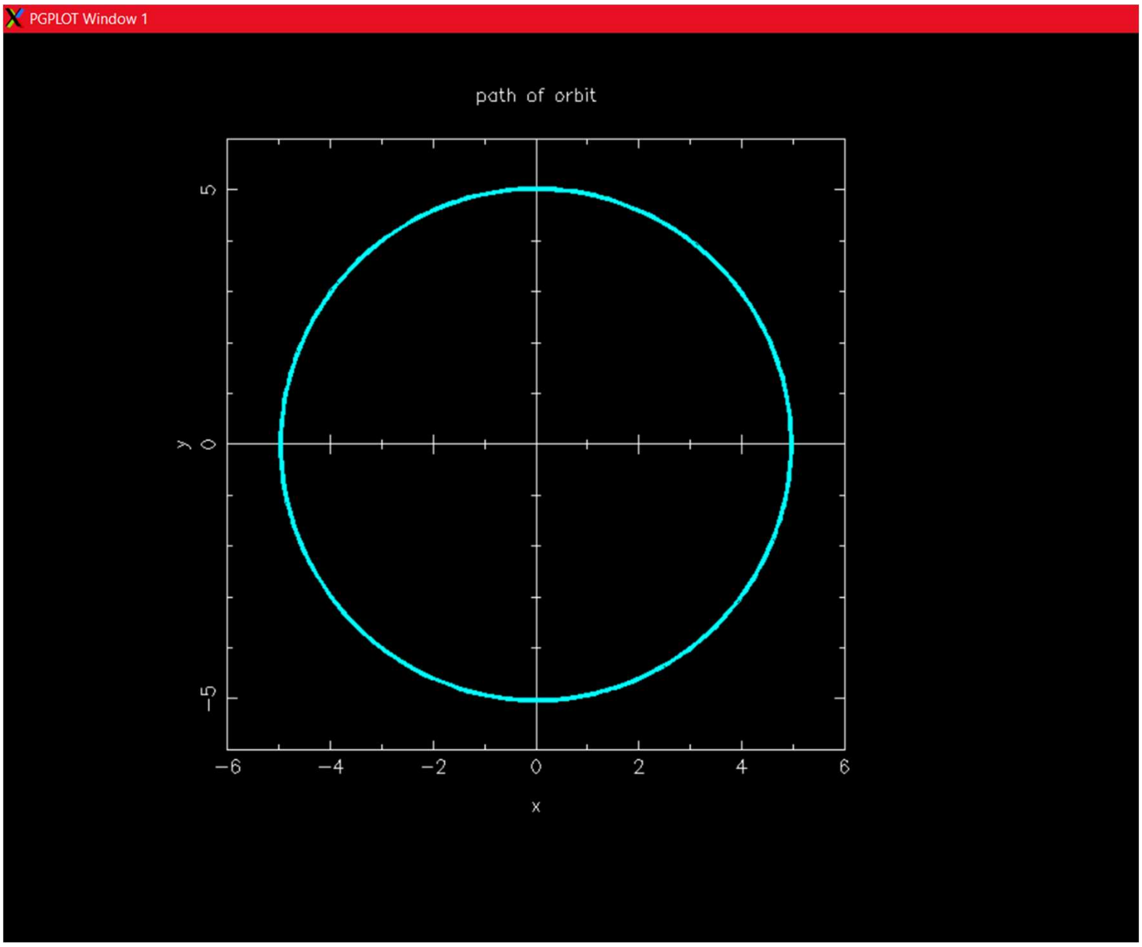
$$\ddot{x} = -\frac{2GM}{r^3}x,$$

$$\ddot{y} = -\frac{2GM}{r^3}y.$$

Using this and Kepler's third law to calculate the time period:

$$\frac{T^2}{4\pi^2} = \frac{r^3}{G(M_1 + M_2)}$$

Using the equations derived above and the appropriate initial conditions (such as $x_{position}, y_{position}, x_{velocity}, y_{velocity}$), we can model the graph of the orbiting planet and plot the co-ordinates using leapfrog integrator as following:



In the diagram shown, orbit of the planet has been plotted, while the only star has been situated at the origin (0,0). It is evident to notice that the planet follows the circular motion in a quite **stringent and conservative** manner.

Notation (dimensionally) of the quantities in the code are as follows:

```
// Notation: Mass => in Solar Masses, Distance => in AUs, Time => in Years

const float G = 39.62545507; // units defined by the notation
float period = (2*M_PI)*(sqrt(r*r*r))/(sqrt(2*G)); // Centre of Mass = 2 solar masses
float dt;
```

Initial values chosen in the code are as follows:

```
// initial values

x[0] = r;
y[0] = 0.0;
vx[0] = 0.0;
vy[0] = (sqrt(2*G/r));
```

Where, *r* is the distance of the planet from origin (star). And it's supposed to be the input from the user while running the program. *G* is the universal Gravitational constant (defined as given above).

Building up from the demo, we will now model the circumbinary planet ar various possibilities inclusive of extensions required analytically in the following section.

Results & Discussion:

A number of interesting planetary orbits are discovered as a result of differing initial conditions. The purpose of this section is to document and explain the physics behind them. The most simple case is that of a circular orbit around both stars – a so-called **P-type (planetary type) or circumbinary orbit**. P type orbits occur when the planet is sufficiently far away from the binary system as to see the system like a single stationary mass located at the centre of mass of the two stars which happens to be the origin in our case. The planet is simply too far away to feel the difference in acceleration from star 1 and star 2 and orbits in a circle around the two. In actual fact, due to collisions with protoplanetary material or asteroids/comets, a circular orbit would be unlikely. Instead, **protoplanets**, which are planets forming out of the nebula from a star which has recently undergone a substantial mass loss, orbit in ellipses. However for the conditions assumed throughout this document, a circular orbit is feasible. P-type circular orbits, starting from a **stable case at large radii**, and decreasing the radii until the boundary condition for the breakdown of these types of orbits is found for planets starting from the x and y axes.

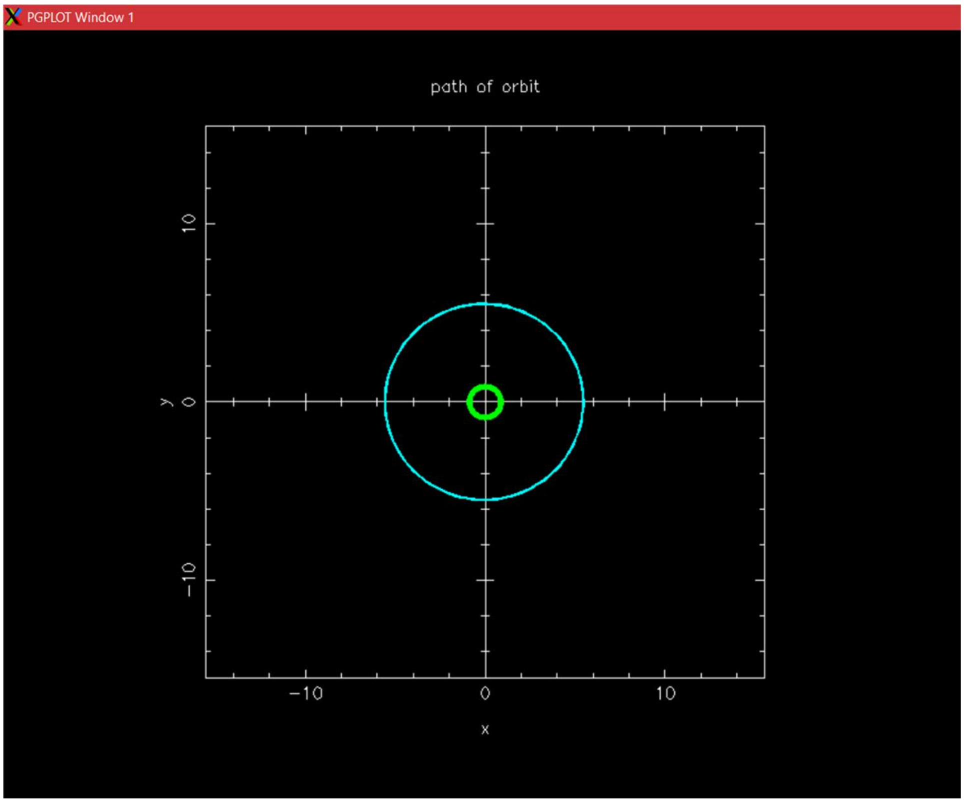
In our project, the source file is inclusive of plotting two different graphs of *"Path of the orbit of circumbinary planet"* and *"Energy of the planet varying with time"*.

Firstly, we are going to discuss the **boundary conditions** for the circular orbits at different initial positions, keeping the initial velocity uninterrupted.

Enter the no. of intervals to perform. ➔ *400*

Enter the distance of the planet from the centre.(in AUs) ➔ *5.5*

Enter the no. of orbits. ➔ *20*



The simulation evidently demonstrates the extremely stable circular orbit of the planet (of negligible mass) at a **distance of *5.5 AU***. For convenience, the orbits of the stars are also showed (in *Green*).

Although the orbit has been overlapped multiple times (over a long period of time), it precisely orbits in its fundamental path.

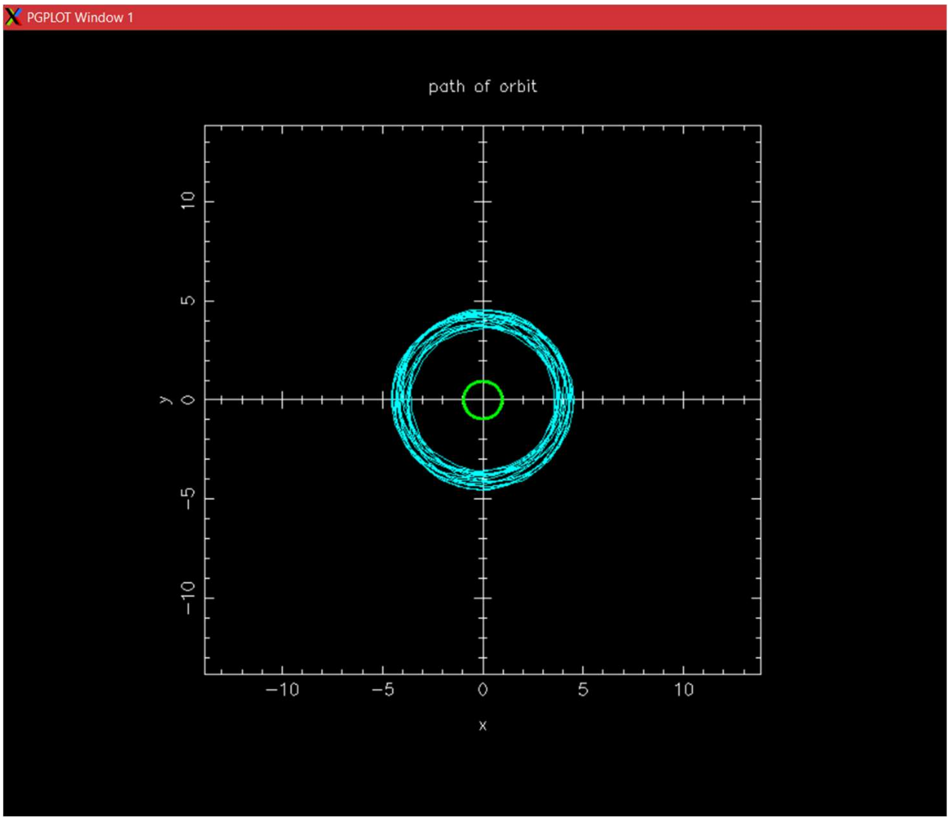
Boundary Condition:

The boundary is difficult to define for orbits. As one approaches the boundary, orbits may look stable but when looking at the next orbit on, it is clearly not as the planet has been ejected. The next orbit on might even be stable again and so on until it becomes fully stable. This shows that near boundaries chaos plays a role – with a very small change in variables producing a completely different result. Whilst some orbits are merely ejected, others are flung out to stable orbits at much greater radii from which they started.

Enter the no. of intervals to perform. ➔ *400*

Enter the distance of the planet from the centre.(in AUs) ➔ *3.82*

Enter the no. of orbits. ➔ *20*



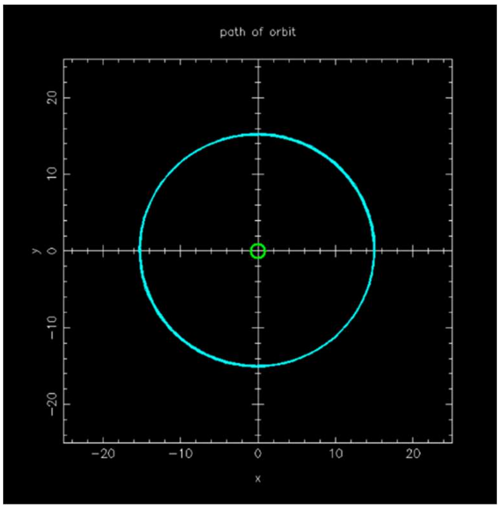
When we gradually decrement the radial distance **(accurate upto 3 sig. figures)** of the planet from the centre of mass of the binary system (origin), we can compare the differences to distinctly determine the **boundary condition** of the P-type orbits (for the chosen notation of units and initial values accordingly). In our case, it comes out to be ***3.82 AU*** from the origin.

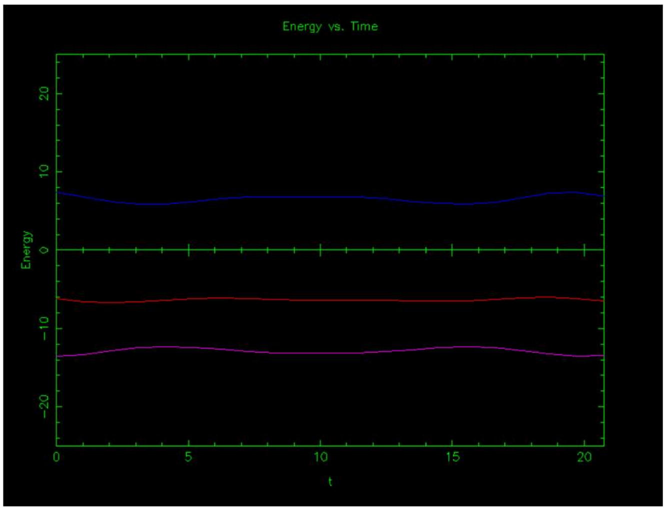
Some interesting captures of the orbits and the corresponding values of the energy of the planet is shown in the following section:

1. Enter the no. of intervals to perform. ➔ *500*

   Enter the distance of the planet from the centre.(in AUs) ➔ *15.0*

   Enter the no. of orbits. ➔ *20*





We can see the **Total Energy** of the planet being conserved in this context and it's exactly equal to $\boldsymbol{E} = -(\boldsymbol{K}.\boldsymbol{E}.)$ as expected theoretically.

Also, interestingly at these values of sufficiently large radii, KE and PE are seemingly **sinusoidal functions** (in harmony with the real world explorations).

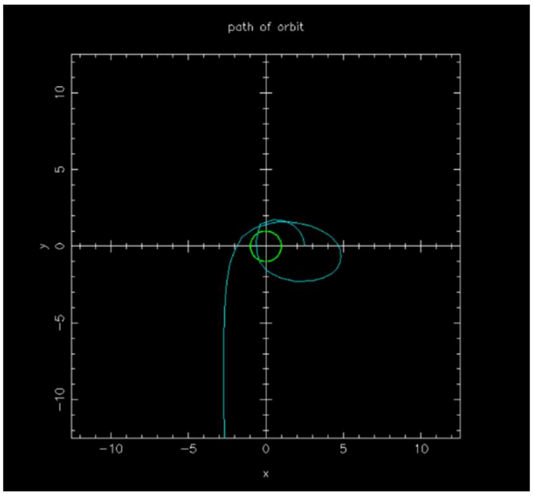
Chaotic Orbits:

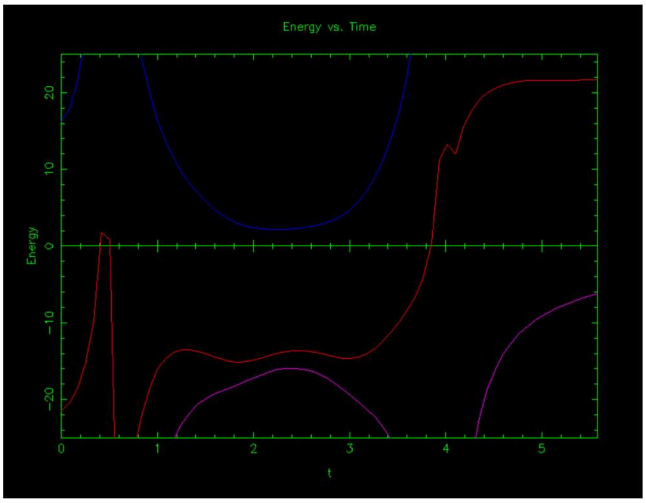
Chaotic orbits have the tendency to change their orbital period with no repetition and as such, often get ejected from the binary system. An orbit can be classified as chaotic when the gravitational effects from the binary star sum in unpredictable ways such that the orbital path is considered random. Resonance, the effect celestial bodies aligning to produce an enhanced gravitational pull, can cause some orbits to become chaotic. For example, in a true binary system model, many planets may orbit is such a way to create resonances and eventually one planet may be pulled from its stable orbit into a chaotic one which will in most cases lead to the ejection of the planet. This normally happens on long timescales, much longer than a human lifespan. Chaotic orbits are sensitive to very small changes in initial conditions which is why the boundary orbits mentioned in previous sections could be described as such. Figure below shows an example of chaotic orbits found via leapfrog method.

2. Enter the no. of intervals to perform. ➔ *500*

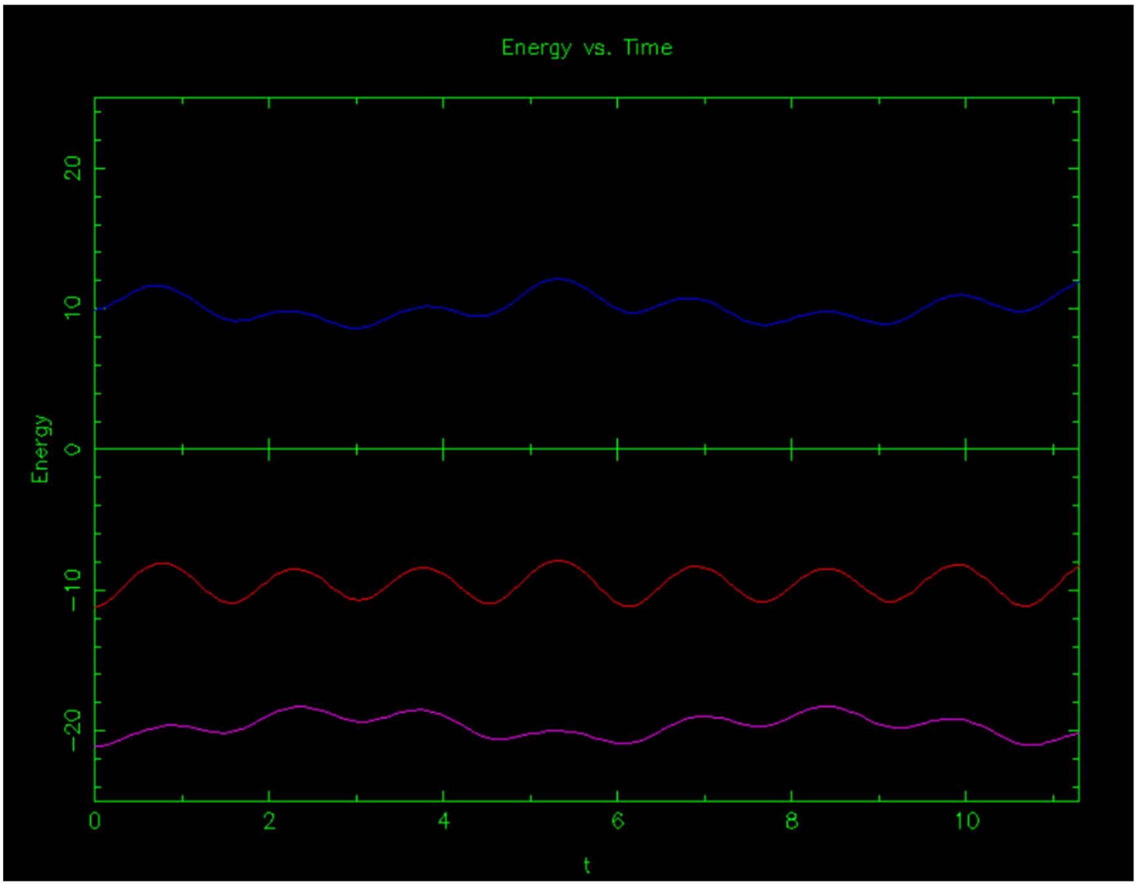
Enter the distance of the planet from the centre.(in AUs) ➔ *2.5*

Enter the no. of orbits. ➔ *15*





3. Another intriguing energy diagram showing the **total energy to be sinusoidal** (at obviously smaller radii = *4.0 AU*) :



Conclusion:

Using the Leapfrog algorithm, many **orbital scenarios** for a planet in a binary star system are explored. The vast majority of the orbits produced are so-called **P-type**, meaning that the star orbits both stars as if they are one. Many circular forms of this type of orbit are found, and the boundary conditions for these tested. It is found that for anticlockwise orbits starting on the x axis, the boundary is ***3.82 AU.***

The point where boundary conditions break is looked at, and it is concluded that orbits become more chaotic as this region is approached due to the planet's increased proximity to the stars. As the planet approaches the stars, they pull it from a stable orbit and it is often ejected. Some cases are found where a seemingly stable orbit suddenly ceases as the planet is ejected from the system. As an extension, chaotic orbits are discussed, and an example presented showing the slight change in initial conditions producing a completely different orbital path.

A huge range of planetary orbits are available when all position and velocity variables are allowed to change on small scales visually illustrated in this document.