An Introduction to the Extent of Mean Field Games in Economics

Candidate Number: JYPF6*

Department of Economics

University College London

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Abstract

This dissertation explores the application of mean-field game (MFG) theory in economics, aiming to serve as an introductory guide, bridging theory with practice and providing insights into strategic interactions within large populations. Through computational simulations and empirical analysis, it demonstrates the practical utility of MFGs in analyzing economic dynamics, particularly within the contexts of a long-run pure competitive framework of Cournot competition, and oil price coordination. The development of model classifications, their appropriate applications, and the integration of real-world data enhance the representativeness of MFG models. Opportunities for refinement and future research are identified, emphasizing the potential impact of MFGs on economic analysis and policy-making.

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Contents

1 Introduction

1.1 Aims

Game theory serves as a crucial tool in economic theory, but traditional models face limitations in capturing complex dynamics as the number of agents increases, leading to computational challenges. Mean-field game theory emerges as a promising extension, offering a novel approach to strategic interactions within large populations, leveraging the smoothing effect of large numbers and providing a versatile framework for modelling various economic scenarios.

The primary aim of this dissertation is to introduce mean-field game (MFG) theory in an accessible manner, particularly targeting students of economics. Beyond theory, the dissertation seeks to demonstrate the practical utility of MFGs through computational simulations, providing tangible results for economic analysis. Additionally, it aims to develop versatile model classifications for diverse economic contexts, synthesizing prior research into a cohesive guide for understanding MFGs in economics.

Ultimately, the dissertation aims to serve as a comprehensive guide, bridging theoretical insights with practical applications, to provide a clearer understanding of mean-field games in economics.

1.2 Methodology and Approach

In order to prime our research and discussion of MFGs in economics, section 2 will serve as a literature review, focusing on the practical applications of MFGs as a mathematical tool rather than abstract theory. Here, we will underscore the significance of MFGs in real-world economic scenarios. Following this, in Section 3, we will delve into classifications of MFG models, elucidating their applicability to diverse economic contexts and providing the requisite mathematical foundations. Subsequently, Section 4 will see the implementation of these model classifications to elucidate specific economic phenomena, supported by code references in Appendices A, B and C, and subsequent model analysis. Finally, Sections 5 and 6 will be dedicated to comprehensive discussions and conclusions drawn from our research findings, respectively.

2 Literature Review

2.1 Game Theory Landscape

2.1.1 Background of Game Theory

Game theory is a branch of mathematics pertaining to the study of strategic interactions between rational decision-makers [1]. It provides a framework, governed by a defined set of mathematical axioms, for analyzing the behaviour of individual entities that make decisions constrained by their perception of others' decisions. It can be divided into two distinct branches: cooperative games, which describe interactions assuming an alliance between any two parties, and non-cooperative games, which examine interactions in the absence of such phenomena [2]. For purposes of simplicity and future reference, we shall, for the remainder of this literature review, refer to games existing under the umbrella of the latter context.

These games are characterized by the existence of a Nash equilibrium, where a given player's strategic decision optimally responds to the combined strategies of others, a condition true for all players in the game [3]. A Nash equilibrium can also be thought of as a rationalizable equilibrium [4] because, for such a collection of strategies to exist, players must act with rationality; a defining characteristic of a rational agent is that they always make decisions that maximize their expected outcome constrained by their information set. This assumption is crucial when constructing mathematical modelling of economic phenomena since predictable behaviours of agents are required [5]. This condition is satisfied within the context of rational agents. Hence, game theory provides a suitable mathematical framework for the modelling and analysis of economic contexts.

Therefore, the utility of game theory in an economic context lies in its ability to construct a game-theoretic model for a given economic scenario. This allows us to identify the existence and uniqueness of a Nash equilibrium, serving as a rationalization equilibrium. In essence, we can determine if a representative game-theoretic model accurately captures the economic dynamics and if a rational equilibrium can be established. Unfortunately, the latter can be difficult to verify because a traditional deterministic game theoretic model scales exponentially in its interactions as the number of agents increases [6]. In light of this, traditional game theory encounters limitations in realistically describing various economic phenomena and consequently, these challenges reveal an oversimplification in using traditional game theory to capture the intricacies of such economic contexts.

2.1.2 Mean-Field Game Theory

Mean-field game theory serves as an extension of traditional game theory, addressing strategic interactions within large populations of rational decision-makers. In this framework, each player's decision is influenced by the aggregated decisions of the entire player population, moving away from direct consideration of individual interactions [7]. In essence, mean-field game theory capitalizes on the inherent smoothing effect of large numbers. Unlike traditional game theory, which faces challenges in analyzing systems with more than two players, mean-field games redefine the paradigm. Instead, the focus shifts to framing game theory as an interaction between a representative player and the collective mass of others, facilitating a more tractable analysis of strategic interactions within larger populations [8].

To grasp the intuition behind mean-field games, one can view their notion as a concept from physics that can be used in game theory and economics [7, 8]: derived from particle physics, this notion of a "mean-field" simplifies complex interactions in a large population by representing the collective impact of numerous agents, in economics. Similar to how physicists use a statistical idea with a media - the mean field - to study interactions between particles, mean-field game theory employs this approach to model strategic interactions between players. The mean-field, created by the agents, provides a representative framework for comprehending the dynamics of the entire system, akin to principles in quantum mechanics [7].

This logic seamlessly extends to various economic scenarios, as mean-field games generalize sequential and dynamic economic interactions with an increasing number of agents approaching infinity [9]. This versatility positions mean-field games as a unique game-theoretic framework adept at modelling economic phenomena. Notably, it circumvents the computational complexity limitations evident in traditional game theory [8].

Mean-field game theory relies on statistical approximation, leveraging the asymptotic properties of large numbers [10]. This approximation strategy enables us to constrain player strategies, limiting complex, player-specific strategies [8]. Subsequent research has focused on its error term [11], specifically how it contrasts with deterministic traditional N-player games. Consequently, it allows for the reduction in the finite dimension of the game to a computationally large N (instead of N approaching infinity), introducing a granularity effect that results in a common noise for the player group [8].

2.2 The MFG Framework

2.2.1 Mathematical representation

In 2006, mean-field game theory saw independent development by two groups: Minyi Huang, Roland Malhamé, and Peter Caines in Montreal, and Jean-Michel Lasry and Fields medalist Pierre-Louis Lions in Paris. Despite unique approaches, the methodologies of the two groups are interconnected, and we now outline their preliminary strategies for establishing the mean-field game framework.

Perspective from Lasry and Lions

The approach from Lasry and Lions [12] demonstrates the relationship between an individual player and the population mass using a coupled system of differential equations: the Fokker-Planck (FP) equation representing system dynamics forward in time, and the Hamilton-Jacobi-Bellman (HJB) equation representing dynamics backward in time.

The HJB equation plays a crucial role in dynamic optimization, specifically within optimal control theory. In this context, the equation provides a solution that minimizes the cost for a defined dynamical system, given a cost function [13]. In mean-field game theory, the HJB equation describes a player's optimal decision path by means of a continuous backwards induction from where they wish to be in the future and is mathematically formulated as the following partial differential equation (PDE):

$$\frac{\partial J}{\partial t} - \Delta J + H(x, \nabla J) = V(x, m), \quad J|_{t=0} = V_0(x, m(x, 0)) \tag{2.1}$$

where J is a scalar function associated with the cost [6], m gives us the probability distribution of players' states, x is the state of the agent and *H(x, p)* is a given convex function, in particular with respect to the last variable [14]. Mathematically rigorous definitions are well-documented in [12].

On the other hand, the FP equation describes the evolution of the player population mass over time, influenced by interactions with individual players. It represents the final positions of players based on their initial positions and is mathematically formulated as the following partial differential equation:

$$\frac{\partial m}{\partial t} + \Delta m + \nabla\left(\frac{\partial H}{\partial p}(x, \nabla u) \cdot m\right) = 0, \quad m|_{t=T} = m_0 \tag{2.2}$$

The MFG system's PDE coupling is marked by the HJB equation, portraying a representative player's optimal decision path and control at any time. This control informs the evolution of the collective player mass, represented as a distribution, which reciprocally impacts the representative firm's optimal decisions. This iterative process continues until convergence or, in other words, Nash equilibrium is reached.

Perspective from Huang et al.

On the other hand, Huang et al. [15] demonstrate the aforementioned dynamics and relationship by framing the mean-field game as a stochastic optimal control problem for an average player. They introduce the Nash-Certainty Equivalence (NCE) Methodology, requiring a McKlean-Vlasov (MV) type controlled stochastic process and an associated objective function.

An MV-type stochastic process, as defined by [15], has a drift rate coefficient determined at any time t by the corresponding probability distribution of players. In mean-field game theory, this process elucidates how an average player controls their movement to impact the population's player mass, aligning with the equation in (2.2). Mathematically, for N-players, it is formulated as:

$$dx_t = f[x_t, u_t, \mu_t^1, ..., \mu_t^K]dt + \sigma dW_t \tag{2.3}$$

where x_t denotes the representative player's state at time t, $f[\cdot]$ denotes the drift rate, u_t is the input control at time t and $\mu_{i,t}$ is the probability distribution of player i at time t for $1 \leq i \leq K \leq N$ [15]. Approximating N players by implementing K players is effective when K is adequately large and N approaches infinity [10].

Similarly, in direct correspondence with the equation in (2.1), the associated objective function represents what the average player aims to optimize through the control of their movement, influenced by u_t. This is mathematically formulated as:

$$J(u, \mu_t^1, ..., \mu_t^K) = \mathbb{E}\left[\int_0^T L[x_t, u_t, \mu_t^1, ..., \mu_t^K]\, dt\right] \tag{2.4}$$

where $L[\cdot]$ is a non-linear function that maps to $\mathbb{R}^+ = [0, \infty)$ [15]. The stochastic nature of this problem is an important feature, emphasizing that the average player cannot be certain about their future states x_t.

2.2.2 Economic applications

Mean-field games are efficient tools for modelling large-scale interactions. This is particularly effective when individual players become increasingly absorbed into a large population mass, simplifying the problem to the stochastic optimal control problem for a single representative player, as outlined in [15]. Consequently, economic contexts meeting these criteria are well-suited for the application of the mean-field game framework.

One primary consideration is to utilize the mean-field game framework for modelling competition in markets or industries. This leverages the inherent assumption of rational players and the aforementioned efficiency in scenarios where players become increasingly absorbed into a large population mass. The focus extends to rational players interacting in various forms of competition, notably in perfect competition, where no producer can individually influence prices. This aligns well with the mean-field game framework [16]. Subsequently, research has been conducted in various perfectly competitive markets, including environmental economics, specifically in the hypothetical context of perfect competition in oil production using a finite, exhaustible resource [16]. Pure mathematical research, exploring the mathematical properties of mean-field game models, has extended to various microeconomic competition structures such as Cournot [18], Bertrand [19], and particularly Stackelberg competition [20], due to the prevalence of leader-follower dynamics in economic competition. Notably, in this dissertation, we will extend the method outlined in [16] to exemplify the practical effectiveness of mean-field games in microeconomic competition scenarios. Specifically, we will apply it to a generalized microeconomic competition structure, focusing on Cournot Competition.

2.2.3 Limitations

Mean-field games, instrumental in modelling strategic interactions among numerous agents, exhibit limitations. A significant trade-off arises between modelling generality and problem solvability. Lasry and Lions' first method ([12]) offers broader applicability but poses challenges in analytical and numerical solutions, whereas the second approach ([15]) provides easier solutions but confines problem scope [6].

Hence, a fundamental hurdle in mean-field games is validating the existence, uniqueness, and regularity (EUR) of Nash equilibrium solutions, frequently considered untractable in numerous cases [7]. This challenge places constraints on the broader applicability of mean-field games, leading to a substantial body of literature focused on proving EUR or formulating master equations [11].

To maintain simplicity and coherence with our overarching theme of introducing mean-field game applications in economics, our discussion will primarily focus on the methodology outlined in [15], emphasizing the linear quadratic Gaussian (LQG) feedback controller. This approach allows us to define our stochastic optimal control problem with a linear quadratic objective function, leveraging the LQG feedback controller's established EUR. While our focus remains on deriving analytical solutions for simpler cases, we will also integrate the methodology outlined in [12] into our research on mean-field game applications. By maintaining an economic approach throughout, we ensure a comprehensive understanding of the extent of mean-field games and uphold the integrity of this dissertation's theme.

3 MFG Model Classifications

In Section 3, we delineate MFG model classifications, underlining their adaptability to diverse economic contexts and establishing the necessary mathematical groundwork. By employing the HJB-FP method from [12] and the NCE method from [15], we furnish a concise mathematical toolkit. This toolkit comprises crucial lemmas, streamlining the development of MFG model classifications and facilitating their practical implementation in economic scenarios.

3.1 Prerequisites

3.1.1 MFG approach using HJB-FP System Methodology

Let us consider the following problem where we wish to maximize:

$$\mathbb{E}\left[\int_0^T F(X_t, U_t, t)\, dt + G(X_T, T)\right] \tag{3.1}$$

where X_t is a state variable and U_t is a closed-loop[1] control variable. The function F may be interpreted as a running reward and G as a terminal reward. Furthermore, we must consider the dynamics of X_t, influenced by both a deterministic "drift" element f and stochastic "noise" element g:

$$dX_t = f(X_t, U_t, t)dt + g(X_t, U_t, t)dW_t, \quad X_0 = x_0 \tag{3.2}$$

where W_t is a standard Wiener process and the necessary initial state condition has been established.

Lemma 1 *Consider the stochastic optimal control problem characterized by assigning, to the value function $V(t, x)$, the objective function in (3.1) defined over the interval $[t, T]$ and additionally, by the associated Itô stochastic differential equation (SDE) in*

[1] The term "closed-loop" refers to a control policy where the control actions are continuously adjusted based on real-time feedback from the system's current state and observed uncertainties.

(3.2). Furthermore, $V(t,x)$ is defined for when an optimal policy is followed over the interval $[t,T]$, given $X_t = x$. Then, as per the principle of optimality,

$$V(x,t) = \max_u \mathbb{E}\left[F(x,u,t)\,dt + V(x+dX_t, t+dt)\right] \tag{3.3}$$

we obtain the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \max_u \left[F + V_t + V_x f + \frac{1}{2}V_{xx}g^2\right] \tag{3.4}$$

for the value function $V(t,x)$ with the boundary condition

$$V(x,T) = G(x,T). \tag{3.5}$$

Now, let us consider a time-dependent mass distribution $m(t,x)$ of the states for X_t, coupled with the equation presented in (3.2).

Lemma 2 *Consider for a single spatial dimension x, the Itô process governed by the SDE in (3.2), with drift $f(X_t,U_t,t)$ and diffusion coefficient $D(X_t,U_t,t) = g^2(X_t,U_t,t)/2$. Then, the Fokker–Planck (FP) equation is given by*

$$\frac{\partial}{\partial t}m(t,x) = -\frac{\partial}{\partial x}[f(x,u,t)m(t,x)] + \frac{\partial^2}{\partial x^2}[D(x,u,t)m(t,x)] \tag{3.6}$$

for the mass distribution $m(t,x)$ with the initial boundary condition

$$m(0,x) = m_0(x). \tag{3.7}$$

A more comprehensive treatment of stochastic optimal control, complete with rigorous proofs and definitions, is documented in [21]. Thus, we have demonstrated how a general stochastic optimization problem is translated into the HJB-FP system methodology.

3.1.2 MFG approach using NCE Methodology

Similarly to the approach outlined above, let's examine a specific instance explored in [22], where the stochastic optimal control problem utilizes the LQG feedback controller. This instance can be characterized by minimizing the cost function

$$J_i(u_i, \phi) = \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\left[q(X_i(t) - \phi)^2 + rU_i(t)^2\right]\, dt\right], \quad i = 1, ...N, \tag{3.8}$$

subject to the dynamic constraint

$$dX_i = (aX_i + bU_i)dt + \sigma_i dW_i \tag{3.9}$$

where ϕ is a function of all players' states for $i = 1, ...N$, the terms a, b and $\sigma_i \in R$ are constant variables defined for each player i, and the coefficients q and r are assigned to the state and control variables, respectively. The LQG feedback controller is utilized due to the linear-quadratic nature of the cost function in (3.8) and the utilization of a Wiener process in the corresponding SDE in (3.9), which follows a Gaussian distribution.

To simplify this problem further, considering its infinite horizon, we can opt for a long-run average cost instead of the discounted cost depicted in (3.8).

Lemma 3 *Examining the cost function depicted in (3.8), substituting the exponential discounting component with a representation of long-run average cost yields*

$$J_i(u_i, \phi) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left[q(X_i(t) - \phi)^2 + rU_i(t)^2\right]\, dt \tag{3.10}$$

For a rigorous mathematical exposition of this transformation, consult [23]. As a result, we now consider the stochastic optimal control problem characterized by (3.9)-(3.10) and the corresponding theorem presented and proved in [23]:

Lemma 4 *For the LQG optimal control problem characterized by (3.9)-(3.10), assume (i) $[a, b]$ is stabilizable*[2]*, (ii) $[a, \sqrt{q}]$ is detectable*[3]*, and (iii) $\phi(\cdot) \in C^b$. Then, we have:*

a) The algebraic Riccati equation $2aP - \frac{b^2}{r}P^2 + q = 0$ has a unique solution P

b) $\gamma := a - \frac{b}{2r}P$ is asymptotically stable

c) The differential equation $\frac{ds}{dt} = -\gamma s + q\phi$ has a unique solution in C^b:

$$s(t) = -\int_t^\infty e^{\gamma(\tau - t)} q\phi(\tau)\, d\tau \tag{3.11}$$

d) The optimal control law: $u^*(\cdot) := arg\inf_{u(\cdot)\in\mathcal{U}} J(u(\cdot)) = -\frac{b}{r}(Px(\cdot) + s(\cdot))$

This result, based on the parameters of the optimal control system, facilitates the computation of the "optimal control" for each player. The utilization of the algebraic Riccati equation establishes a connection between the method outlined in [12] and the method outlined in [24], allowing the HJB equation to be recapitulated by employing the Riccati equation.

As depicted, the optimal control $u^*(\cdot)$ depends on the specified variables and constants, along with a new function $s(t)$, interpreted as the Laplace transform of $\phi(t)$ with weight q [23]. The function $s(t)$ records the exponentially discounted value of $\phi(t)$ at each time point, requiring knowledge of the context-dependent function $\phi(t)$ to solve for $s(t)$ in the LQG optimal control problem.

Finally, we establish the existence of a Nash equilibrium in the LQG optimal control problem [23], concluding the demonstration of its translation into the NCE system methodology. Notably, as aforementioned, the NCE methodology limits the scope of MFG-representable optimal control problems but offers more accessible solutions.

[2] The term "stabilizable" indicates that all uncontrollable state variables can be adjusted to exhibit stable dynamics [6, 13].

[3] The term "detectable" signifies that all unobserved state variables maintain stability, where "unobserved" implies internal states of the system cannot be deduced from system outputs. These conditions on the selected parameters $[a, b, \sqrt{q}]$ ensure that the agent states will remain bounded [6, 13].

3.2 Classification I: Effort-Value model

3.2.1 Model formulation

In this scenario, envision a multitude of players, each endowed with individual energy resources. We posit that the player count is substantial enough to meet the continuum hypothesis, justifying the use of an MFG model to describe their strategic interactions. Furthermore, for simplicity, we presume a constant size of this player continuum, implying no entry or exit from the MFG.

In this classification model, players possess an initial quantifiable resource E_0 (state variable), distributed among them according to an initial mass distribution $m(0,\cdot)$. Assuming players consume energy resources at a rate of effort ϵ (control variable), it is logical to incorporate uncertainty about future energy states. Thus, we can model the dynamics of the state variable for each player i as

$$dE_i(t) = -\epsilon_i(t)dt + \nu E_i(t)dW_{t,i} \tag{3.12}$$

where ν measures the magnitude of uncertainty. For simplicity, we assume ν is uniform across all players, yet it scales proportionally with their individual energy resources.

Now, let us delve into the objective behind a player's effort, which aims to maximize the value of their effort exertion, denoted as $V_i(E_i(t), \epsilon_i(t), t) = R_i - C_i$, where R_i represents the reward and C_i signifies the cost incurred by exerting effort. We can reasonably assume that a player values their effort more in the present moment compared to the future. Therefore, we will incorporate exponential discounting, at a rate r, to reflect this dynamic in the running value of their effort exertion. Therefore, a player wishes to maximize the objective function

$$\mathbb{E}\left[\int_0^\infty (V(E(t), \epsilon(t), t))e^{-rt}\, dt\right] \quad s.t.\ \epsilon(t) \geq 0, E(t) \geq 0 \tag{3.13}$$

which is assigned to the value function

$$u(t,E) = \max_{\substack{(\epsilon(s))_{s\geq t} \\ \epsilon\geq 0}} \mathbb{E}\left[\int_t^{\infty} (V(E(s),\epsilon(s),s))e^{-r(s-t)}\,ds\right] \quad s.t.\ E(t)=E, \forall s\geq t, E(s)\geq 0 \tag{3.14}$$

Consequently, we can formulate a stochastic optimal control problem that represents the average player i. This problem is characterized by equations (3.12) to (3.14). From Lemma 1, we deduce that the value function $u(t,E)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{\partial u(t,E)}{\partial t} + \frac{\nu^2}{2}E^2\frac{\partial^2 u(t,E)}{\partial E^2} - ru(t,E) + \max_{\epsilon\geq 0}\left[V(E(t),\epsilon(t),t) - \epsilon\frac{\partial u(t,E)}{\partial E}\right] = 0 \tag{3.15}$$

The Hamiltonian[4] of this optimal control problem is $\max_{\epsilon\geq 0}\left[V(E(t),\epsilon(t),t) - \epsilon\frac{\partial u(t,E)}{\partial E}\right]$. We can derive the optimal control $\epsilon^*(t,E)$ by solving the Hamiltonian. This control represents the instantaneous optimal effort at time t for a player with an energy resource E at that time.

Now, let $m(t,E)$ represent the distribution of players' energy resources at time t. Initially, this distribution is determined by a density function $m_0(\cdot)$, which subsequently evolves based on players' optimal effort decisions $\epsilon^*(t,E)$. From Lemma 2, the evolution of this mass distribution is described by the Fokker-Planck (FP) equation:

$$\frac{\partial}{\partial t}m(t,E) = -\frac{\partial}{\partial E}[-\epsilon^*(t,E)m(t,E)] + \frac{\nu^2}{2}\frac{\partial^2}{\partial E^2}[E^2 m(t,E)] \tag{3.16}$$

These equations constitute the classical partial differential equations of the MFG methodology outlined in [12]. The HJB equation operates backward in time, while the

[4] The term "Hamiltonian" refers to the expression which represents the maximum instantaneous reward minus the cost associated with control action ϵ and its impact on the derivative of the value function with respect to the resource E at time t.

FP equation progresses forward and the connection between them lies in the double coupling mechanism.

3.2.2 Economic applications

The Effort-Value (EV) model, as outlined, offers a versatile MFG classification suitable for diverse economic scenarios characterized by strategic interactions among numerous entities with either exhaustible or inexhaustible resources. It therefore positions itself as a robust tool for modelling quantity or production competition observed in markets where firms aim to maximize profit by controlling the quantity of goods produced, based on available capital or resource levels. Additionally, the model integrates uncertainty to encompass factors like fluctuating market demand, technological progress, and economic shocks or geopolitical events, which are pertinent in various economic competition settings. As foreshadowed, an application of this model classification was illustrated in the hypothetical scenario of perfect competition within the oil production sector, focusing on a finite, exhaustible resource [16]. For this dissertation, we will explore the application of the EV model to Cournot competition contexts with inexhaustible and exhaustible stock in Sections 4.1.2 and 4.1.3, respectively.

3.3 Classification II: Trajectory-Follower model

3.3.1 Model formulation

In this scenario, envision a multitude of players, each with their own distinct states. Similar to the EV model, we maintain the initial assumptions, where the player count remains constant, indicating no entry or exit from the MFG, and satisfies the continuum hypothesis.

In this classification model, players start with independently distributed initial states $\{x_i(0) \in \mathbb{R} : 1 \leq i \leq n\}$, evolving over time under the influence of their current state $\{x_i \in \mathbb{R} : 1 \leq i \leq n\}$ and control $\{u_i \in \mathbb{R} : 1 \leq i \leq n\}$ for altering it. Considering uncertainty about future player states, we model the dynamics of the state variable for each player i as:

$$dx_i(t) = (ax_i(t) + bu_i(t))dt + \sigma_i dW_{t,i} \tag{3.17}$$

where σ_i measures the magnitude of uncertainty.

Now, let's explore the objective behind a player's control, which aims to minimize the distance between their state x_i and a shared trajectory path $\phi(t, x_i, x_{-i})$. Here, x_{-i} and u_{-i} represent the states and controls of all players except player i, and are defined as $x_{-i} := (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ and $u_{-i} := (u_1, ..., u_{i-1}, u_{i+1}, ..., u_n)$. This distance function, controlled by u_i and influenced by x_i, can be defined as the equation of an ellipse centered at $(0, \phi)$ in the u_i-x_i plane. Additionally, this distance function can be interpreted as a running cost, accumulating over the game duration. Considering that players prioritize immediate costs over future ones, we incorporate exponential discounting at a rate ρ, leading to the minimization of the objective function

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left[q(x_i(t) - \phi(t, x_i, x_{-i}))^2 + ru_i(t)^2\right]\, dt\right], \quad i = 1, ...N, \tag{3.18}$$

which is assigned to the cost function $J_i(u_i; u_{-i})$, where $r > 0$ represents the weight assigned to the significance of a player's control impact on the running cost, while $q > 0$ denotes the weight assigned to the significance of the disparity between a player's state and the trajectory function's impact on the running cost. Given the infinite horizon of this cost function, simplifying the model involves considering a long-run average cost rather than a discounted cost. From Lemma 3, we derive the long-run average cost function

$$J_i(u_i; u_{-i}) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T \left[q(x_i(t) - \phi(t, x_i, x_{-i}))^2 + ru_i(t)^2\right]\, dt. \tag{3.19}$$

As anticipated, we will now proceed to establish a trajectory function $\phi(t, x_i, x_{-i})$ tailored to this MFG model classification. The trajectory path players strive to adhere to consists of both a defined and a variable component. At time t, we interpret the defined component as a "reference trajectory" function $h(t)$ that is universally known to all players, while the variable component represents the "centroid" of the distribution of average player states $\psi(t, x_i, x_{-i}) := \frac{1}{n}\sum_{i=1}^{n} \bar{x}_i(t)$. Combining these two components in a convex combination, the trajectory function is expressed as

$$\phi(t, x_i, x_{-i}) := \mu h(t) + (1-\mu)\psi(t, x_i, x_{-i}) \tag{3.20}$$

where $h(\cdot)$ is a continuous function players follow with scalar weight $\mu \in (0, 1)$.

Now equipped with the structure of the trajectory path function $\phi(t, x_i, x_{-i})$, we can derive the corresponding optimal control function $u^*(\cdot)$ for the stochastic optimization problem representing the average player i, characterized by equations (3.17) to (3.19). For further simplicity, assume the equality of constants a and b in (3.17). From Lemma 4, and as demonstrated in [24], the resultant MFG system of equations is thus expressed as:

$$\frac{ds_i(t)}{dt} = -\gamma s(t) + q\phi(t) \tag{3.21}$$

$$\frac{d\bar{x}_i(t)}{dt} = \gamma \bar{x}(t) - \frac{b^2}{r} s(t) \tag{3.22}$$

$$\psi(t) = \frac{1}{n}\sum_{i=1}^{n} \bar{x}_i(t) \tag{3.23}$$

$$\phi(t) = \mu h(t) + (1-\mu)\psi(t) \tag{3.24}$$

These equations constitute the MFG methodology delineated in [15], akin to the classical partial differential equations described in [12]. Equation (3.21) functions as the HJB-backward equation, enabling computation of the optimal control function $u^*(\cdot)$. Similarly, equation (3.22) operates as the FP-forward equation, detailing the evolution of the "centroid" of the average player state distribution. Once more, their linkage is facilitated by the double coupling mechanism.

3.3.2 Economic applications

The Trajectory-Follower (TF) model offers a versatile MFG classification tailored to a wide array of economic scenarios characterized by strategic interactions among entities coordinating or following a trajectory path concerning a state variable. It serves as a powerful tool for modelling price coordination dynamics in response to price setting in various markets, where firms aim to minimize costs by controlling their deviation from the trajectory path. Moreover, the model incorporates uncertainty in the trajectory path to address unforeseeable factors such as fluctuating market demand and economic shocks/geopolitical events. Applications of this model have been demonstrated in analyzing leader-follower dynamics [24], where the leader sets the trajectory path for followers, in contexts like Stackelberg competition [20], and in examining price coordination dynamics, including asset prices coordination [6]. For this dissertation, we will delve into the application of the TF model to the context of oil price coordination, an economic phenomenon observed in real-world oil cartels such as the "Organization of the Petroleum Exporting Countries" (OPEC), which will be explored in Section 4.2.

4 MFG Approach to Modelling Economic Phenomena

In Section 4, we apply the MFG model classifications outlined in Section 3 to describe various economic contexts, both hypothetical and realistic, demonstrating the versatility and efficacy of MFGs as a quantitative economic modeling tool. Through a combination of analytic and numerical/computational analyses, following from applied instances of both the EV model [16] and TF model [6], we illustrate their applicability in diverse contexts. Our approach involves defining the context, developing analytic and/or numerical descriptions through the MFG lens, and providing economic interpretations of the results.

4.1 Context I: Cournot Competition

4.1.1 Set-up

We expand the framework of the Cournot competition model to incorporate the MFG approach, utilizing the EV model as the designated classification. In this setup, we examine an industry comprising a fixed continuum of N identical firms, all involved in strategic and rational quantity competition within a long-run pure competitive framework of Cournot competition.

Utilizing the EV model, we envisage a multitude of firms, each possessing individual stock represented by the state variable S_i. As stock is depleted at a rate determined by the production level, denoted by the control variable q_i, we can depict the dynamics of individual stock, adhering to the EV model framework, as follows:

$$dS_i(t) = -q_i(t)dt + \nu S_i(t)dW_{t,i} \tag{4.1}$$

The aim of a firm's production is to maximize its value, or in simpler terms, its profit criterion $V_i(S_i(t), q_i(t), t) = p(t)q_i(t) - C(q_i(t))$ over time, considering an objective measure of exponential discounting, uniform across all firms within this pure competitive framework, alongside uncertainty in future stock holdings. If we define lifetime profit as the value function $u(t, S)$, derived from the EV model, the corresponding HJB equation for this infinite-horizon stochastic optimal control problem can be expressed as:

$$\frac{\partial u(t,S)}{\partial t} + \frac{\nu^2}{2} S^2 \frac{\partial^2 u(t,S)}{\partial S^2} - ru(t,S) + \max_{q \geq 0} \left[p(t)q - C(q) - q\frac{\partial u(t,S)}{\partial S} \right] = 0 \tag{4.2}$$

The corresponding Hamiltonian is given by $\max_{q\geq 0} \left[p(t)q - C(q) - q\frac{\partial u(t,S)}{\partial S} \right]$, allowing us to derive the optimal control $q^*(t, S)$. As previously noted, this control signifies the instantaneous optimal production quantity at time t for a firm with a stock level of S at that time.

To depict the collective firm mass distribution dynamics within the MFG framework, we denote $m(t, S)$ as the distribution of firms' stock at time t, initialized by $m_0(\cdot)$ at time 0. We model the evolution of this distribution, influenced by firms' optimal production quantities $q^*(t, S)$, through the FP equation derived from the EV model:

$$\frac{\partial}{\partial t} m(t,S) = -\frac{\partial}{\partial S}[-q^*(t,S)m(t,S)] + \frac{\nu^2}{2}\frac{\partial^2}{\partial S^2}[S^2 m(t,S)] \tag{4.3}$$

Therefore, we have established the coupled system of partial differential equations that delineate the generalized MFG approach to Cournot competition, aligning its context with the EV model.

4.1.2 Cournot Competition with inexhaustible stock

In the context of inexhaustible stock competition, we define it as having finite initial stock without depletion over time. Thus, the distribution of firms' stock $m_0(S_0)$ at the outset follows dynamics without drift or noise elements (= 0). Given the static nature of stock distribution over time, independent of firms' production decisions, the

FP equation aspect of the MFG approach is not considered. Consequently, solving for the optimal control via the HJB equation, unaffected by other firms' states, yields an immediate convergent solution, or Nash equilibrium. Hence, in line with Lemma 1 and the context, the pertinent HJB equation for this scenario is:

$$\frac{\partial u(t, S_0)}{\partial t} - ru(t, S_0) + \max_{q \geq 0} [p(t)q - C(q)] = 0 \tag{4.4}$$

The corresponding Hamiltonian is $\max_{q \geq 0} [p(t)q - C(q)]$, facilitating the derivation of the optimal control $q^*(t, S_0)$. To determine this quantity, we need to define a suitable cost function for the pure competitive Cournot framework, consistent across all firms, such as the quadratic cost function $C(q) = \alpha q + \frac{1}{2}\beta q^2$. Thus, applying the first order condition of the Hamiltonian, the optimal production quantity is given by:

$$q^*(t, S_0) = \left[\frac{p(t) - \alpha}{\beta}\right]_+ \tag{4.5}$$

Notably, with a general cost function $C(q)$, the first-order condition of the Hamiltonian yields an expression equating the firm's price at the optimum to its marginal cost $C'(q)$. Hence, within the realm of this pure competitive framework, the optimal production quantity $q^*(t, S_0)$ can be interpreted as each firm's supply curve [25].

We can now reformulate the HJB equation with a revised expression for the Hamiltonian, incorporating the optimal control $q^*(t, S_0)$:

$$\frac{\partial u(t, S_0)}{\partial t} - ru(t, S_0) + \frac{1}{2\beta} \left[(p_E(t) - \alpha)_+\right]^2 = 0 \tag{4.6}$$

Here, the HJB equation captures the evolution of lifetime profit for a representative firm. To solve this differential equation for the lifetime profit function, the equilibrium price $p_E(t)$ must first be determined, reflecting the industry price level established through the equilibrium between industry demand and supply.

The industry supply expression is obtained by horizontally summing the individual firms' supply curves. Given that all firms have identical individual supply curves, industry supply is derived by integrating this over the distribution of firms:

$$\text{Industry Supply} := \int q^*(t, S_0) m_0(S_0)\, dS_0 \tag{4.7}$$

The industry demand expression is derived as the inverse function of price and is represented by the following equation:

$$\text{Industry Demand} := Ap(t)^{-1} \tag{4.8}$$

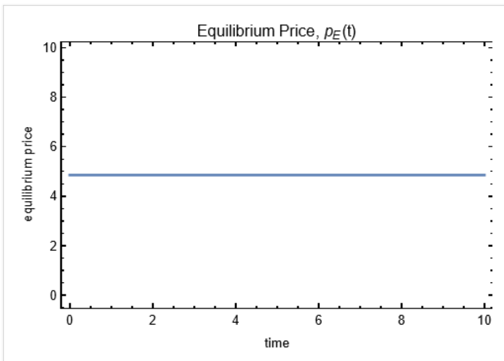
where A is a "demand" parameter that is proportional to industry demand, and price is inversely proportional to industry demand, implying that a 1% increase in quantity demanded results in a 1% decrease in price. Therefore, industry equilibrium is achieved by equating expressions (4.7) and (4.8), yielding the equilibrium price $p_E(t)$:

$$\begin{aligned}
\text{Industry Demand} &= \text{Industry Supply} \\
Ap_E(t)^{-1} &= \int q^*(t, S_0) m_0(S_0)\, dS_0 \\
Ap_E(t)^{-1} &= \int \left[\frac{p_E(t) - \alpha}{\beta}\right]_+ m_0(S_0)\, dS_0 \\
Ap_E(t)^{-1} &= \left[\frac{p_E(t) - \alpha}{\beta}\right]_+ \underbrace{\int m_0(S_0)\, dS_0}_{=1} \\
\Rightarrow p_E(t) &= \frac{1}{2}(\alpha + \sqrt{\alpha^2 + 4A\beta})
\end{aligned}$$

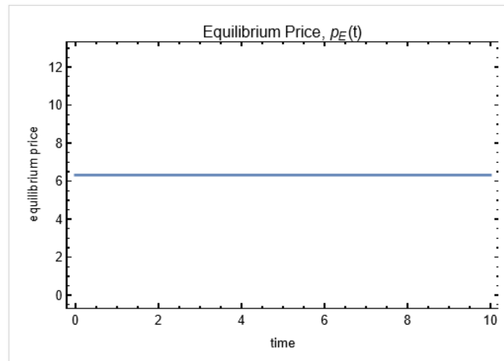
We can now reformulate the HJB equation with the expression for the equilibrium price level $p_E(t)$:

$$\frac{\partial u(t, S_0)}{\partial t} - ru(t, S_0) + \frac{1}{2\beta}\left[(\frac{1}{2}(\alpha + \sqrt{\alpha^2 + 4A\beta}) - \alpha)_+\right]^2 = 0 \tag{4.9}$$

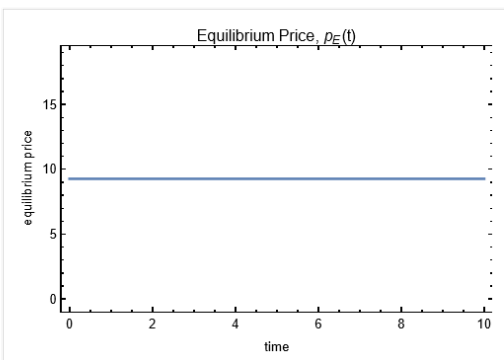
Figure 1: **Computational resolution***



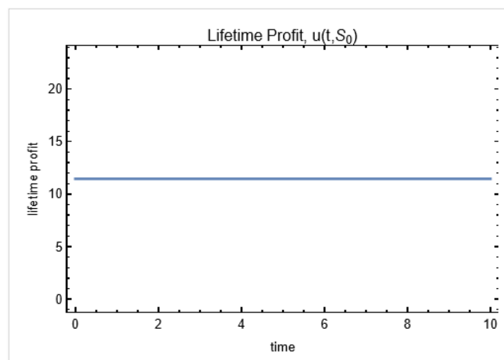
(a) $A = \alpha = \beta = 3$



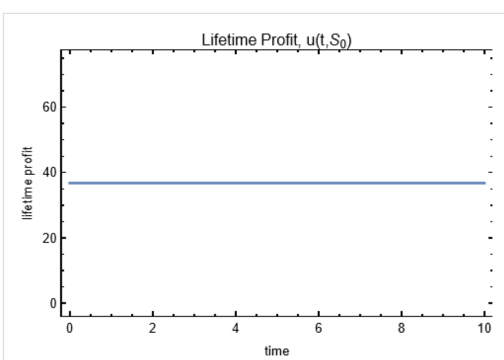
(b) $A = 7, \alpha = \beta = 3$



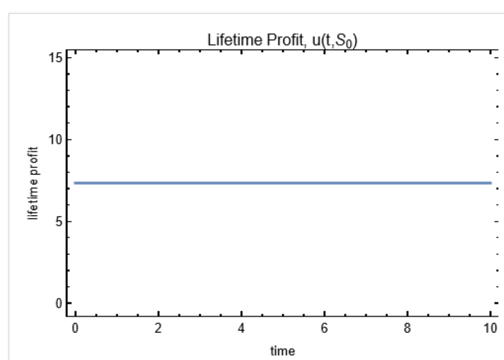
(c) $A = 3, \alpha = \beta = 7$



(d) $A = \alpha = \beta = 3, r = 0.05$



(e) $A = 7, \alpha = \beta = 3, r = 0.05$



(f) $A = 3, \alpha = \beta = 7, r = 0.05$

*Equations were solved and visualized using Mathematica, with the corresponding code provided in Appendix A for reference.

Economic Analysis

Our approach employs a Mean Field Game (MFG) framework to model the hypothetical economic phenomenon of long-run pure competitive Cournot competition within an inexhaustible resource context. Through this approach, we develop a descriptive model that elucidates the strategic interactions among agents inherent in this competitive setting, thus reinforcing established economic intuition and theory.

Examining the effects on equilibrium price, our model reveals that parameters such as the "demand" scaling parameter A and total cost $C(q^*)$ (represented by parameters α and β) positively influence equilibrium price. This finding is consistent with the economic intuition that demand increases lead to higher equilibrium prices, while cost increases shift the supply curve upwards, thereby raising equilibrium price levels.

Our model demonstrates immediate convergence to a Nash equilibrium solution, attributable to the stability conferred by the inexhaustible nature of the resource. This stability implies that equilibrium price and optimal quantity produced remain unchanged until there is a shift in parameters A, α, and β, underscoring the robustness of our model in capturing both comparative statics and long-run competitive dynamics.

Analyzing the impact on lifetime profit, we find that the "demand" scaling parameter A positively affects lifetime profit, while total cost $C(q^*)$ (determined by parameters α and β) has a negative impact. This economic rationale follows from demand-driven increases in revenue, outweighing simultaneous rises in costs, and conversely, cost-driven decreases in profit.

In our analysis, it is essential to acknowledge that our model operates under the assumption of unit elastic price elasticity of demand, making it particularly suitable for goods exhibiting such behavior. This assumption underscores the importance of considering the specific characteristics of the market when applying our model to describe real-world scenarios.

In our model, firms seek to maximize instantaneous profit subject to the inexhaustible resource proposition and a pure competitive industry environment. This is reflected in the boundary condition used in the particular solution of the Hamilton-Jacobi-Bellman (HJB) equation, indicating zero changes in lifetime profit in the long run. These insights offer a significant understanding of the dynamics inherent in long-term pure competitive Cournot competition scenarios, highlighting the effectiveness of employing the MFG framework and selecting suitable model classifications to elucidate intricate economic phenomena.

4.1.3 Cournot Competition with exhaustible stock

In exhaustible stock competition, we characterize it by finite initial stock undergoing depletion over time. The initial distribution of firms' stock, denoted as $m_0(S_0)$, follows dynamics outlined in (4.1), incorporating drift and noise elements. Following the setup based on the EV model classification, the evolution of this distribution over time is captured by the FP equation in (4.3). Similarly, in the HJB equation given by (4.2), we consider drift and noise elements to account for the dynamics of individual firms' decision-making processes.

The corresponding Hamiltonian is expressed as $\max_{q\geq 0}\left[p(t)q - C(q) - q\frac{\partial u(t,S)}{\partial S}\right]$. This formulation allows us to derive the optimal control $q^*(t,S)$, using the same cost function $C(q) = \alpha q + \frac{1}{2}\beta q^2$ as applied in the inexhaustible stock scenario:

$$q^*(t,S) = \left[\frac{p(t) - \alpha - \frac{\partial u(t,S)}{\partial S}}{\beta}\right]_+ \tag{4.10}$$

Significantly, for a general cost function $C(q)$, the first-order condition of the Hamiltonian results in an expression equating the firm's price at the optimum to the sum of its marginal cost $C'(q)$ and the derivative of the value function $u(t,S)$ with respect to stock S at time t. This derivative introduces a price "inflation" term, recognized in exhaustible resource economics as Hotelling rent [17].

We can now reformulate the HJB equation with a revised expression for the Hamiltonian, incorporating the optimal control $q^*(t,S)$:

$$\frac{\partial u(t,S)}{\partial t} + \frac{\nu^2}{2} S^2 \frac{\partial^2 u(t,S)}{\partial S^2} - ru(t,S) + \frac{1}{2\beta}\left[(p_E(t) - \alpha - \frac{\partial u(t,S)}{\partial S})_+\right]^2 = 0 \qquad (4.11)$$

Similar to our approach in the inexhaustible stock scenario, finalizing the forms of the HJB and FP equations necessitates determining the equilibrium price $p_E(t)$.

Considering the incorporation of changes in stock, the previous method of deriving equilibrium price from the industry supply expression shown in (4.7) is no longer applicable, as optimal production quantity now depends on the level of stock due to the Hotelling rent term in (4.10). Instead, we recognize that industry supply is equal to the rate at which total industry stock depletes:

$$\begin{aligned} \text{Industry Stock} &:= \int Sm(t,S)\,dS \\ \Rightarrow \text{Industry Supply} &:= -\frac{d}{dt}\int Sm(t,S)\,dS \end{aligned}$$

Equilibrium price is determined by setting industry demand equal to industry supply, considering the industry demand function $D(t, p(t))$:

$$\begin{aligned} &\text{Industry Demand} = \text{Industry Supply} \\ &\qquad \Rightarrow p_E(t) = D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int Sm(t,S)\,dS\right) \end{aligned}$$

We can now reformulate the HJB and FP equations with the expression for the equilibrium price level $p_E(t)$, respectively as:

$$\begin{aligned} &\frac{\partial u(t,S)}{\partial t} + \frac{\nu^2}{2} S^2 \frac{\partial^2 u(t,S)}{\partial S^2} - r u(t,S) \\ &+ \frac{1}{2\beta} \left[(D(t,\cdot)^{-1} \left(-\frac{d}{dt} \int S m(t,S)\, dS \right) - \alpha - \frac{\partial u(t,S)}{\partial S})_+ \right]^2 = 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} m(t,S) = -\frac{\partial}{\partial S} [- \left[\frac{D(t,\cdot)^{-1} \left(-\frac{d}{dt} \int S m(t,S)\, dS \right) - \alpha - \frac{\partial u(t,S)}{\partial S}}{\beta} \right]_+ m(t,S)] \\ + \frac{\nu^2}{2} \frac{\partial^2}{\partial S^2} [S^2 m(t,S)] \end{aligned} \tag{2}$$

Economic Analysis

Our economic analysis builds upon our previous study of long-run pure competitive Cournot competition within an inexhaustible resource context, and [16]. Now, we extend our investigation to an exhaustible stock scenario to compare results.

In contrast to the scenario with inexhaustible stock, the exhaustible nature of the resource introduces a "scarcity" premium, leading to prices surpassing firms' marginal costs at equilibrium/optimum. This scarcity premium arises from diminishing stock over time, resulting in the optimal quantity produced and individual supply curves of firms gradually diminishing towards zero, prompting them to adjust prices accordingly.

Subsequently, unlike the constant equilibrium price and optimal quantity observed in the inexhaustible context, the exhaustible scenario sees non-zero effects of time and stock level on these variables. Depletion of the exhaustible resource necessitates dynamic adjustments in production and pricing strategies over time. Overall, our analysis highlights the significance of considering resource exhaustibility in economic modeling, providing insights into the dynamics of market competition and resource scarcity.

4.2 Context II: Oil Price Coordination

4.2.1 Set-up

We extend the application of the TF model classification to the realm of oil coordination, building upon MFG experiments and theory demonstrating its efficacy in describing asset price coordination [6, 26]. This extension further delves into real-world scenarios of oil price coordination, such as the OPEC cartel, to explore the applicability and insights offered by the TF model in understanding such economic phenomena.

In respect to the TF model classification, we define x_i as the state variable representing the oil price state of a country i, while u_i signifies their individual control or influence over this state. The dynamics of the oil price state for country i evolve over time, governed by the SDE detailed in (3.17) (with a and b assumed to be equal for simplicity). Furthermore, adhering to the TF model, these countries aim to minimize the cost function outlined in (3.19), working towards aligning with a predefined trajectory path over an infinite time horizon.

In our TF model framework, we define the trajectory path $\phi(t, x_i, x_{-i})$ as a convex combination of two components: a continuous "reference trajectory" $h(t)$ and a "variable" component $\psi(t, x_i, x_{-i})$, with a scalar weight $\mu \in (0, 1)$. This weight μ reflects the relative importance of adhering to the reference trajectory versus considering the states of all countries on average. As we adopt a MFG approach, $h(t)$ serves as a forecasting strategy universally employed by all countries as a reference for coordination, modulated by the weight μ.

The forecasting strategy outlined in [26] for asset price coordination, to which we specifically consider oil price, employs a linear auto-regressive process with two lags (AR(2)). The next period's forecast is determined as follows:

$$p^e_{h,t+1} = \alpha + \beta p_{t-1} + \delta(p_{t-1} - p_{t-2}) \tag{4.12}$$

where p_t represents the oil price at time t, $p^e_{h,t+1}$ denotes the forecasted oil price for the next period, and $\alpha, \beta, \delta \in \mathbb{R}$ are parameters of the time series process. The methodology delineated in [6, 27] enables us to express the aforementioned AR(2) process as an approximate continuous function. This is achieved by initially converting the AR(2) process into a second-order non-homogeneous linear difference equation with constant coefficients:

$$b_t = \beta_1 b_{t-1} + \beta_2 b_{t-2} + \alpha \tag{4.13}$$

Exploiting the stationary state condition $b^* = \frac{\alpha}{1-\beta_1-\beta_2}$ and setting $a_t = b_t - b^*$, the simplified homogeneous form of the equation in (4.13) becomes:

$$a_t = \beta_1 a_{t-1} + \beta_2 a_{t-2} \tag{4.14}$$

Then, consider the variable transformation $a_t = r^t$ and simplify; a continuous solution for $a(t)$ is obtained, given initial conditions a_1 and a_2:

$$a(t) = (-\beta_2)^{t/2}(E\cos(\theta t) + F\sin(\theta t)), \tag{4.15}$$

$$E = \frac{-\beta_1 a_1 + a_2}{\beta_2} \tag{4.16}$$

$$F = -i\frac{\beta_1^2 a_1 - \beta_1 a_2 + 2\beta_2 a_1}{\beta_2\sqrt{\beta_1^2 + 4\beta_2}} \tag{4.17}$$

$$\theta = \arccos(\frac{\beta_1}{2\sqrt{-\beta_2}}) \tag{4.18}$$

Following this, we employ the discrete-time parameter estimation technique described in [6] to estimate the parameters listed in (4.13). These estimates are then utilized to derive continuous-time counterparts, as presented in equations (4.16)-(4.18). Subsequently, these continuous-time parameters are substituted into the equation (4.15) to obtain our desired continuous "reference trajectory" function. The relevant code, introduced in [6], is provided in Appendix B for reference.

Returning to the MFG system of equations outlined by the TF model classification in (3.21)-(3.24), the approach introduced in [6, 24] simplifies the system and results in a second-order differential equation that characterizes the dynamics of $\psi(t)$:

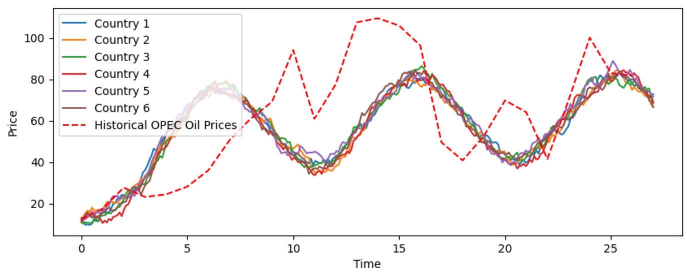
$$\frac{d^2\psi(t)}{dt^2} + 2\gamma\frac{d\psi(t)}{dt} - (a^2 + \frac{b^2}{r}q\mu)\psi(t) = -\frac{b^2}{r}q\mu h(t) \tag{4.19}$$

The general solution of this class of differential equations is the sum of the complementary function, given by $c_1e^{r_1t} + c_2e^{r_2t}$, and the particular integral $Y(t)$. To solve for the constants c_1 and c_2, we first consider the boundary condition at $t = 0$, $\psi(0) = \frac{1}{n}\sum_{i=1}^{n}\bar{x}_i(0)$. Additionally, to ensure the general solution is bounded, the roots of the characteristic equation corresponding to the homogeneous part of (4.19) must have strictly opposite signs [27]. As demonstrated in [6], the roots of the corresponding characteristic equation are:

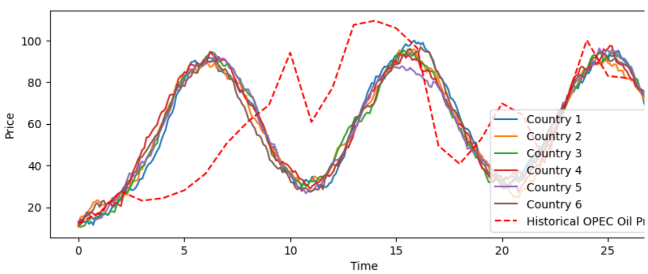
$$r_{1,2} = \sqrt{a^2 + \frac{b^2}{r}q} \pm \sqrt{2a^2 + \frac{b^2}{r}q(\mu+1)} \tag{4.20}$$

Since the two roots are strictly opposite, requiring $c_1 = 0$ to ensure the general solution's boundedness [27], we can determine a particular solution for $\psi(t)$. With solutions for $h(t)$ and $\psi(t)$, we can compute the trajectory path function $\phi(t)$. Subsequently, applying Lemma 4 allows us to derive the optimal control law $u^*(\cdot)$, representative for all countries, enabling the simulation of their oil-price states over time.

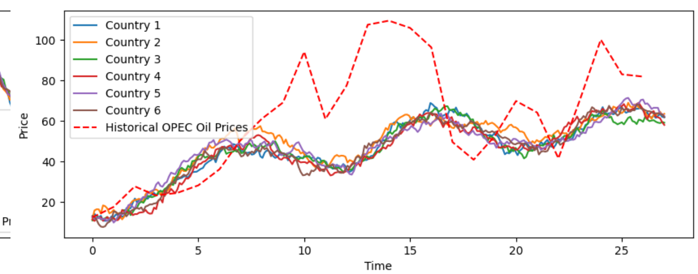
Figure 2: **Computational resolution***



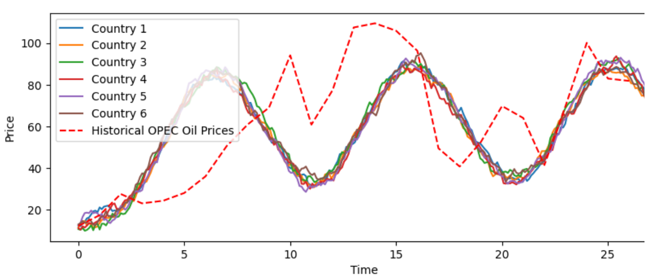
(a) $q = r = 1, \mu = 0.5$



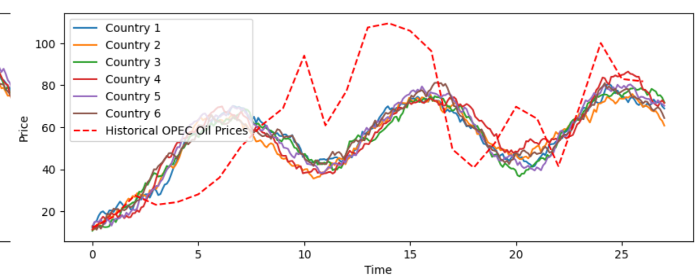
(b) $q = r = 1, \mu = 0.8$



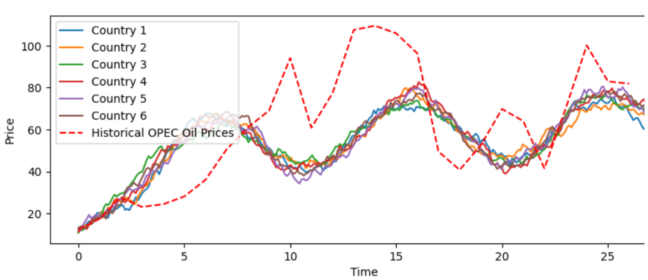
(c) $q = r = 1, \mu = 0.2$



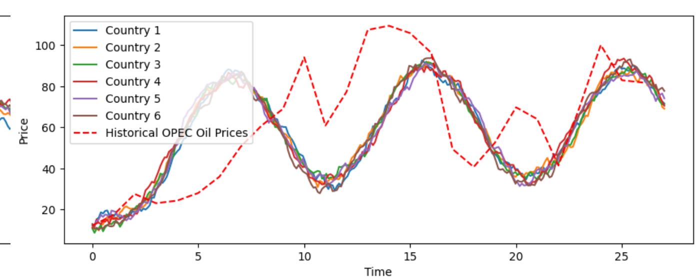
(d) $q = 2, r = 1, \mu = 0.5$



(e) $q = 0.5, r = 1, \mu = 0.5$



(f) $q = 1, r = 2, \mu = 0.5$



(g) $q = 1, r = 0.5, \mu = 0.5$

*Oil-price state simulations were solved for and visualized using Python algorithms from [6], with the corresponding code provided in Appendix C for reference. OPEC oil price data was obtained from the OPEC basket price (1998-2024).

4.2.2 Model analysis

The TF MFG model classification offers a systematic approach to investigate oil price coordination through a game theoretic lens. This framework provides a structured economic model capable of elucidating various behavioural patterns exhibited by countries, incorporating empirical data such as the OPEC basket price spanning from 1998 to 2024. Time 0 is anchored at the onset of 1998, initiating a simulation of oil-state prices for six representative OPEC nations—Saudi Arabia, Iran, Iraq, Kuwait, UAE, and Algeria—extended over a 27-year horizon until 2024. Notable global events like the 2008 financial crisis and the early 2010s Arab Spring uprisings are acknowledged as potential externalities that could impact the model's efficacy.

A pivotal aspect of the model setup involves establishing a "reference trajectory," $h(t)$, which mirrors the OPEC basket price dynamics. Parameter estimation methodologies, as outlined in [6, 26], are employed to model this trajectory using an AR(2) process. The estimated parameters, α and β_1, are determined to be 26.85 and 1.57 respectively, with β_2 set to -1 to facilitate sinusoidal[5] modeling. These parameters are crucial as they govern the behavior of the reference trajectory and, consequently, the simulated trajectories of the oil-price states of the countries under examination.

Parameter weights (r, q, μ) play a significant role in shaping the optimal control law $u^*(\cdot)$ and thereby influence the oil-price state simulations. The weight r scales a country's cost accrued from influencing large changes in their oil-state price with their optimal control, while q scales the cost accrued from the disparity between a country's state and the trajectory path. In parallel, μ dictates the degree of adherence to the reference trajectory. By adjusting these parameter weights, the model encapsulates various behavioral tendencies observed among countries, as illustrated in Figure 2, enabling a nuanced analysis of oil price coordination dynamics.

[5] The term "sinusoidal" refers to a periodic wave whose waveform is the trigonometric sine function

Despite identical simulated oil-price state dynamics owing to the MFG approach, variations among countries' trajectories stem from their individual stochastic dynamics (3.17). This stochasticity introduces heterogeneity, reflecting the diverse economic conditions and policy choices of the simulated nations. The interplay of parameter weights manifests in computational resolution graphs, offering insights into how changes in r, q, and μ affect countries' behaviors and subsequent oil price coordination dynamics. These insights contribute to a deeper understanding of the complexities inherent in global oil markets, informing policymakers and stakeholders alike.

5 Discussion

The model results presented in Section 4 underscore the efficacy of employing a Mean Field Game (MFG) approach in analyzing the dynamics of economic phenomena within a long-run pure competitive framework, particularly in the context of Cournot competition and oil price coordination. Notably, the MFG framework offers a comprehensive game theoretic perspective that aligns with conclusions drawn from traditional models, such as those based on $N \to \infty$ player game theory and representative agent-based game theoretic models. This consistency in outcomes across diverse modelling paradigms reinforces the robustness of MFGs in capturing the underlying dynamics of economic systems [9, 14].

In the case of Cournot competition with inexhaustible stock, the MFG model effectively converges to the theory of static pure competition, highlighting its ability to capture equilibrium outcomes in dynamic contexts [25]. This finding corroborates our economic analysis, demonstrating that the MFG framework, and EV model classification in particular, provides a descriptive economic model capable of elucidating the results of firm behaviour in competitive markets.

Similarly, our analysis of oil price coordination using the MFG approach reveals promising insights. By integrating real-world data, such as the OPEC basket price from 1998 to 2024, into the TF model, we were able to simulate the oil-price state dynamics of multiple countries over time. The model's parameter estimation techniques, based on discrete-time parameter estimation methods, allowed for the calibration of model parameters to real-world data, enhancing the model's representativeness and descriptive power. Moreover, an MFG approach offers a notable advantage by circumventing the need for extensive supercomputer computation, which has been a requirement in traditional game theoretic modelling approaches, in this context of oil cartel price coordination [28].

Looking ahead, there are significant opportunities for further exploration and refinement of our MFG-based models. In the context of Cournot competition with exhaustible stock, future research could leverage advanced numerical methods to solve coupled systems of partial differential equations. By doing so, we can elucidate the convergence to equilibrium and corresponding dynamic behaviour, for the context of an exhaustible stock. Such a method is outlined in [16].

Moreover, in the domain of oil price coordination, there exists scope for hyper-parameter tuning of the MFG model. By optimizing the vector of weight parameters $(r,\ q,\ \mu)$, which represent key aspects of country behaviour, we can better align the model with real-world dynamics. This refinement process holds promise for enhancing the model's predictive capabilities and its ability to capture the intricacies of oil market dynamics.

Nevertheless, it is crucial to recognize certain limitations concerning the application of MFG theory. While undeniably powerful and robust as a descriptive tool for economic modelling, its predictive capacity is inherently constrained by its game-theoretic nature. Therefore, while MFGs offer valuable quantitative insights into describing the behaviour of complex economic systems, their predictive utility is primarily qualitative rather than quantitative.

6 Conclusion

In conclusion, this dissertation has provided a comprehensive exploration of mean-field game (MFG) theory and its applications in economics. Our primary aim was to introduce MFG theory in an accessible manner while demonstrating its practical utility through computational simulations. Additionally, we aimed to develop versatile model classifications for diverse economic contexts, synthesizing prior research into a cohesive guide for understanding MFGs in economics.

Through a thorough literature review, we highlighted the practical significance of MFGs as a mathematical tool for understanding complex economic dynamics. The development of model classifications in Section 3 has furthered our objectives by elucidating the applicability of MFGs to diverse economic contexts and providing the requisite mathematical foundations. These model classifications not only contribute to our theoretical understanding but also serve as practical tools for economic analysis.

The model results presented in Section 4 underscore the effectiveness of employing an MFG approach to analyze economic phenomena. Specifically, our analysis of Cournot competition and oil price coordination showcases the ability of MFGs to capture equilibrium outcomes and simulate real-world dynamics. With regards to the latter application, by integrating real-world data into our model, we enhanced their representativeness and descriptive power, allowing for nuanced analysis of economic behaviour.

In essence, this dissertation serves as an all-encompassing compendium on mean-field game theory in economics, providing a blend of theoretical underpinnings and real-world implications. Through the development of model classifications and the integration of theory into practical scenarios, we have advanced our understanding of MFGs and their potential ramifications for economic analysis and policy-making.

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A Appendix

```
In[∘]:= Clear[u, p, f]

        eq1 = {A p^-1 == (p - α)/β}; (*industry equilibrium condition*)

        p[t_] = p /. Solve[eq1, p][[2]] (*solve for equilibrium price*)

Out[∘]= 1/2 (α + Sqrt[α^2 + 4 A β])

        Manipulate[Plot[1/2 (α + Sqrt[α^2 + 4 A β]), {t, 0, 10}, PlotLabel → "Equilibrium Price, pE(t)",
          Frame → True, FrameLabel → {"time", " equilibrium price"}], {A, 0, 10}, {α, 0, 10}, {β, 0, 10}]

        HJB = {D[u[t], t] - r u[t] + 1/(2 β) (p[t] - α)^2 == 0}; (*Corresponding HJB equation*)

        DSolve[{HJB, u'[0] == 0}, u[t], t, Assumptions → {p[t] - α > 0}][[1]][[1]] (*Solve HJB for value function,
        with an appropriate boundary condition*)

Out[∘]= u[t] → (α^2 + 2 A β - α Sqrt[α^2 + 4 A β])/(4 r β)

        Manipulate[Plot[(α^2 + 2 A β - α Sqrt[α^2 + 4 A β])/(4 r β), {t, 0, 10}, PlotLabel → "Lifetime Profit, u(t,S0)",
          Frame → True, FrameLabel → {"time", "lifetime profit"}], {A, 0.01, 10}, {α, 0.01, 10}, {β, 0.01, 10}, {r, 0.01, 1}]
```

Figure 3

B Appendix

```python
import numpy as np

# Number of parameters to estimate for
numParam = 2

## Data extraction and set up
# Assuming 'OPEC_historical_data.txt' contains the historical OPEC basket price data

with open('OPEC_historical_data.txt', 'r') as pFile:
    pInter = pFile.readlines()

p = [float(value.strip().strip('"')) for value in pInter]

# Get number of terms in time series
M = len(pInter)

# Convert price data to floating point numbers
p = np.array([np.float64(val) for val in pInter])

## If estimating 3 parameters, alpha, beta_1, beta_2
if numParam == 3:
    # f_M column calculation
    f_M = np.zeros(numParam)
    for t in range(2, M):
        f_M += np.array([p[t], p[t-1]*p[t], p[t-2]*p[t]])
    f_M = 1 / (M - 2) * f_M

    # R_M matrix calculation
    r_M = np.zeros((numParam, numParam))
    for t in range(2, M):
        r_M += np.outer(np.array([1, p[t-1], p[t-2]]), np.array([1, p[t-1], p[t-2]]).transpose())
    r_M = 1 / (M - 2) * r_M

    # Calculate optimal parameters
    theta = np.linalg.inv(r_M).dot(f_M)

## If estimating 2 parameters, alpha and beta_1
## Gamma set to -1 (forces sinusoid)
if numParam == 2:
    # f_M column calculation
    f_M = np.zeros(numParam)
    for t in range(2, M):
        f_M += np.array([p[t] + p[t-2], p[t-1]*(p[t] + p[t-2])])
    f_M = 1 / (M - 2) * f_M
```

Figure 4

```python
# R_M matrix calculation
r_M = np.zeros((numParam, numParam))
for t in range(2, M):
    r_M += np.outer(np.array([1, p[t-1]]), np.array([1, p[t-1]]).transpose())
r_M = 1 / (M - 2) * r_M

# Calculate optimal parameters
theta = np.linalg.inv(r_M).dot(f_M)
```

✓ 0.0s

```python
theta
```

✓ 0.0s

```
array([26.85499962,  1.56551665])
```

Figure 5

C Appendix

```python
import numpy as np
import scipy as sc
import matplotlib.pyplot as plt
from scipy.stats import norm

# Historical OPEC average oil price data
historical_prices = [
    12.28, 17.44, 27.6, 23.12, 24.36, 28.1, 36.05, 50.59, 61, 69.04, 94.1, 60.86,
    77.38, 107.46, 109.45, 105.87, 96.29, 49.49, 40.76, 52.51, 69.78, 64.04, 41.47,
    69.89, 100.08, 82.95, 81.83
]

#######################################################
# Solve recurrence relation
#######################################################

# Given parameters
a1 = 13.05
a2 = 16.45
beta_1 = 1.56551665
beta_2 = -1
alpha = 26.85499962

# Calculate continuous parameters
pStar = alpha / (1 - beta_1 - beta_2)
a1 -= pStar
a2 -= pStar
E = (-beta_1 * a1 + a2) / beta_2
F = 1j * (np.power(beta_1, 2) * a1 - beta_1 * a2 + 2 * beta_2 * a1) / (beta_2 * np.sqrt(np.power(beta_1, 2) + 4 * beta_2 + 0j))
theta = np.arccos(beta_1 / (2 * np.sqrt(-beta_2 + 0j)))

# Handle imaginary components
if F.imag == 0:
    F = float(F.real)
if theta.imag == 0:
    theta = float(theta.real)
```

Figure 6

```
##########################################################
# Solve homogenous differential equation for psi
##########################################################

# Homogenous system parameters
a = 0.01
b = 1
c = [4, 4, 4, 4, 4, 4]
q = 1
r = 1
lam = 0.5
zNaught = [12.20, 11.45, 10.77, 11.26, 12.67, 13.02]
n = len(zNaught)

# Simulation time parameters
totalT = 27
deltaT = 0.1
T = int(totalT / deltaT)
t = np.linspace(0, 27, num=T)

# Calculate values in ODE
P = (a + np.sqrt(a**2 + b**2 / r * q)) / (b**2 / r)
gam = -np.sqrt(np.power(a, 2) + np.power(b, 2) / r * q)
aODE = 1
bODE = 2 * gam
cODE = -(np.power(a, 2) + np.power(b, 2) / r * lam * q)
refCoeff = -np.power(b, 2) / r * q * lam
r1 = (-bODE + np.sqrt(np.power(bODE, 2) - 4 * aODE * cODE)) / (2 * aODE)
r2 = (-bODE - np.sqrt(np.power(bODE, 2) - 4 * aODE * cODE)) / (2 * aODE)

# Note: Particular solutions were solved analytically outside of this code.

##########################################################
# Solve heterogenous DE for sinusoidal solutions
##########################################################

sinMatrix = np.array([[cODE - np.power(theta, 2), -bODE * theta],
                      [bODE * theta, cODE - np.power(theta, 2)]])
solutionSin = np.array([refCoeff * F, refCoeff * E])
x = np.linalg.solve(sinMatrix, solutionSin)
psiPart = x[0] * np.sin(theta * t) + x[1] * np.cos(theta * t) + pStar
h = F * np.sin(theta * t) + E * np.cos(theta * t) + pStar
homoC = np.average(zNaught) - psiPart[0]
psi = x[0] * np.sin(theta * t) + x[1] * np.cos(theta * t) + pStar + homoC * np.exp(r2 * t)
```

Figure 7

```python
#####################################################
# Solve heterogenous DE for exponential solutions
#####################################################

refA1 = ((E - F * 1j) / 2).real
refA2 = ((E + F * 1j) / 2).real
refGam1 = ((1j * theta + np.log(-beta_2) / 2))
refGam2 = ((-1j * theta + np.log(-beta_2) / 2))

# Define functions for exponential solutions
def psiExpSol(t):
    return (refCoeff * refA1 / (np.power(refGam1, 2) - bODE * refGam1 + cODE) * np.exp(refGam1 * t) +
            refCoeff * refA2 / (np.power(refGam2, 2) - bODE * refGam2 + cODE) * np.exp(refGam2 * t) +
            refCoeff * pStar / cODE).real

def hExp(t):
    return (np.power((-beta_2), t / 2) * ((E - F * 1j) / 2 * np.exp(1j * theta * t) +
                                           (E + F * 1j) / 2 * np.exp(-1j * theta * t)) + pStar).real

homoCExp = np.average(zNaught) - psiExpSol(0)

#############################################################
# Solve heterogenous DE for exponential sinusoidal solutions
#############################################################

rSin = np.log(-beta_2) / 2
sinExpMatrix = np.array([[np.power(rSin, 2) - np.power(theta, 2) + bODE * rSin + cODE, 2 * theta * rSin + bODE * theta],
                         [-2 * theta * rSin - bODE * theta, np.power(rSin, 2) - np.power(theta, 2) + bODE * rSin + cODE]])
solutionSinExp = np.array([refCoeff * F, refCoeff * E])
xSinExp = np.linalg.solve(sinExpMatrix, solutionSinExp)
psiSinExpPart = np.exp(rSin * t) * (x[0] * np.sin(theta * t) + x[1] * np.cos(theta * t)) + pStar
hSinExp = np.exp(rSin * t) * (F * np.sin(theta * t) + E * np.cos(theta * t)) + pStar
homoCSinExp = np.average(zNaught) - psiSinExpPart[0]
```

Figure 8

```python
################################################################
# Simulations
################################################################

# Define functions for s variable integrals
def sIntegralSin(tau, tConst):
    return np.exp(gam * (tau - tConst)) * q * ((1 - lam) * (x[0] * np.sin(theta * tau) + x[1] * np.cos(theta * tau) +
                                                        pStar + homoC * np.exp(r2 * tau)) +
                                               lam * (F * np.sin(theta * tau) + E * np.cos(theta * tau) + pStar))

# Compute s variable
s = []
for timeStep in range(0, T):
    tea = deltaT * timeStep
    integral, err = sc.integrate.quad(sIntegralSin, tea, np.inf, args=tea)
    s.append(-integral)

# Initialize variables for Brownian motion
dz = []
zAll = []
uAll = []
delta = 1

# Simulate agent behaviors
for i in range(0, n):
    dz = []
    z = [zNaught[i]]
    u = []
    w = [np.float64(0)]  # Initial condition for Brownian motion

    # Compute Brownian motion steps
    for timeStep in range(1, T):
        w.append(w[timeStep - 1] + np.float64(norm.rvs(scale=delta ** 2 * np.sqrt(deltaT))))

    # Compute predicted prices of agents
    for timeStep in range(0, T - 1):
        tea = deltaT * timeStep
        u.append(-b / r * (P * z[timeStep] + s[timeStep]))
        dz.append((a * z[timeStep] + b * u[timeStep]) * deltaT +
                  c[i] * (w[timeStep + 1] - w[timeStep]))
        z.append(z[timeStep] + dz[timeStep])

    uAll.append(u)
    zAll.append(z)
```

Figure 9

```python
# Plot results
plt.figure(figsize=(10, 8))


plt.subplot(2, 1, 2)
for i in range(n):
    plt.plot(t, zAll[i], label=f"Country {i+1}")
plt.plot(range(len(historical_prices)), historical_prices, label="Historical OPEC Oil Prices", linestyle='--', color='red')
plt.legend()
plt.ylabel("Price")
plt.xlabel("Time")
plt.show()
```

Figure 10