Illinois Institute of Technology
Department of Computer Science

Solutions to Homework 10

CS 430 Introduction To Algorithms
Spring Semester, 2013

1. (a) If an edge (u, v) is a back edge, then we have

$$StartDFS(v) < StartDFS(u) < FinishDFS(u) < FinishDFS(v)$$

If an edge (u, v) is a cross edge, then we have

$$StartDFS(v) < FinishDFS(v) < StartDFS(u) < FinishDFS(u)$$

There exist cycles in a graph if a back edge is present. Thus we can run DFS on the graph and if a back edge is found, then we can determine that there is a cycle in the graph, which takes time $O(|V| + |E|)$.

(b) If back edges are present in the graph; or if $|E| > |V| - 1$, then we can determine that the graph has cycles. Thus we can run DFS and mark each edge visited, if back edges are found or the number of marked edges is greater than $|V| - 1$, then a cycle is present in the graph.

(c) False. Assume u and v have the same parent w in the depth-first-tree. Then a visit sequence can be $w \to u \to v$ and hence $StartDFS(u) < StartDFS(v)$, but v is not a descendant of u.

2. (a) i. Suppose (u, v) is a back edge or a forward edge in a BFS of an undirected graph. Then one of u and v, say u, is a proper ancestor of the other v in the breadth-first tree. Since we explore all edges of u before exploring any edges of any of u's descendants, we must explore the edge (u, v) at the time we explore u. But then (u, v) must be a tree edge.

ii. In BFS, an edge (u, v) is a tree edge when we set $\pi[v] \leftarrow u$. But we only do so when we set $d[v] \leftarrow d[u] + 1$. Since neither $d[u]$ nor $d[v]$ ever changes thereafter, we have $d[v] = d[u] + 1$ when BFS completes.

iii. Consider a cross edge (u, v) where, without loss of generality, u is visited before v. At the time we visit u, vertex v must already be on the queue, for otherwise (u, v) would be a tree edge. Because v is on the queue, we have $d[v] \leq d[u] + 1$ by Lemma 22.3. By Corollary 22.4, we have $d[v] \geq d[u]$. Thus, either $d[v] = d[u]$ or $d[v] = d[u] + 1$.

(b) i. Suppose (u, v) is a forward edge. Then we would have explored it while visiting u, and it would have been a tree edge.

ii. Same as for undirected graphs.

iii. For any edge (u, v), whether or not it's a cross edge, we cannot have $d[v] > d[u] + 1$, since we visit v at the latest when we explore edge (u, v). Thus, $d[v] \leq d[u] + 1$.

iv. Clearly, $d[v] \geq 0$ for all vertices v. For a back edge (u, v), v is an ancestor of u in the breadth-first tree, which means that $d[v] \leq d[u]$. (Note that since self-loops are considered to be back edges, we could have $u = v$.)