Introduction To Algorithms
Homework 4 Solutions

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1.Give an example where quicksort requires $O(n^2)$ steps.

Consider a list:

$$10, 9, 8, 7, 6, 5, 4, 3, 2, 1$$

We choose the last digit in the last as the pivot.Thus time complexity is given as:

$$T(n) \ = \ T(n-1) + T(0) + \Theta(n)$$

By using substitution method, we could get:

$$T(n) \ = \ \Theta(n^2)$$

If it is $\Theta(n^2)$, it is also a $O(n^2)$.

2.Problem 4-6 (Page 110) CLRS(3rd Edition).

a. Need to prove "if and only if", thus the proof will have to separate parts

Proof of 'Only if':

If A is a Monge array, by definition, we have:

$$A[i,j] + A[k,l] \leq A[i,l] + A[k,j] \ \ \forall i \ , \ j \ , \ k \ , \ l$$

$$where \ 1 \leq i < k \leq n \ , \ 1 \leq j < l \leq m$$

Let $k = i+1, \ l = j+1$, we will have:

$$A[i,j] + A[i+1,j+1] \leq A[i,j+1] + A[i+1,j] \ \ \forall i \ , \ j$$

$$where \ 1 \leq i < i+1 \leq n \ , \ 1 \leq j < j+1 \leq m$$

$$where \ 1 \leq i \leq n-1 \ , \ 1 \leq j \leq m-1$$

'Only if' has been proved.

Proof of 'if':
Induction method will be used separately on rows and columns.
For columns:
Let us consider a 2×2 submatrix of A. This is a base case and is True because $k = i + 1, l = j + 1$.
For a $2 \times n - 1$ matrix A', assuming it is true.
If $j, l \in A'$, then by induction method, $\forall i, j, k, l,\ A[i, j] + A[i + 1, j + 1] \leq A[i, j + 1] + A[i + 1, j]$ is True.
If $j \in A'$ and $l \in A$, let j=n-1,l=n,k=i+1,therefore we have:

$$A[i, n - 1] + A[i + 1, n] \leq A[i, n] + A[i + 1, n - 1] \quad (1)$$

This is True for $A[i, j] + A[i + 1, j + 1] \leq A[i, j + 1] + A[i + 1, j]$.
Let j=i,l=n-1,k=i+1,we have:

$$A[i, i] + A[i + 1, n - 1] \leq A[i, n - 1] + A[i + 1, i] \quad (2)$$

Sum (1) and (2),we have:

$$A[i, i] + A[i + 1, n] \leq A[i, n] + A[i + 1, i]$$

which is equal to:

$$A[i, j] + A[k, n] \leq A[i, n] + A[k, i]$$

Thus we proved $2 \times n$ for columns.

For rows:
Let us consider a base case $2 \times n$ which is True from above.
Assume a submatrix A', $m - 1 \times n$, in matrix A is True.
Assume k=n-1,l=i,r=m-1,we have:

$$A[m - 1, n - 1] + A[m, i] \leq A[m, n - 1] + A[m - 1, i]$$

$$A[i, n - 1] + A[m - 1, i] \leq A[i, i] + A[m - 1, n - 1]$$

sum these two up:

$$A[i, n - 1] + A[m, i] \leq A[i, i] + A[m, n - 1]$$

which is equal to:

$$A[i, k] + A[m, l] \leq A[i, l] + A[m, k]$$

which proves $m \times n$.Thus A is a Monge array.

b.