Illinois Institute of Technology
Department of Computer Science

Solutions to Homework 2

CS 430 Introduction To Algorithms
Spring Semester, 2013

1. (a) False. Let $f(n) = n$ and $g(n) = n^2$, then for some constant c we have

$$0 \le n \le cn^2 \log n$$
$$f(n) = \hat{O}(g(n))$$

Suppose that $g(n) = \hat{O}(f(n))$, then for some constant c' we have

$$0 \le g(n) \le c' f(n) \log n$$
$$0 \le n^2 \le c' n \log n$$

However there exists n_0 such that $n^2 > c' n \log n$ when $n > n_0$. Thus $f(n) = \hat{O}(g(n))$ does not imply $g(n) = \hat{O}(f(n))$.

(b) True. Since $f(n) = \hat{O}(g(n))$, we have $0 \le f(n) \le cg(n) \log n$ and

$$0 \le \log f(n) \le \log(cg(n) \log n) = \log c + \log g(n) + \log \log n$$

Because

$$\log c + \log g(n) + \log \log n \le c' \log g(n) \log n,$$

for some constant c' when $n > n_0$, we have

$$0 \le \log f(n) \le c' \log g(n) \log n,$$

which implies $\log f(n) = \hat{O}(\log g(n))$

(c) Let $h(n) = o(f(n))$, then $\exists c > 0$, such that $h(n) < cf(n)$ and

$$f(n) \le f(n) + h(n) < (1 + c)f(n).$$

which implies

$$f(n) + o(f(n)) = \Theta(f(n))$$

(d) False. Let $f(n) = n$ and $g(n) = n^2$, then $\min(f(n), g(n)) = n$. However $f(n) + g(n) = n^2 \ne n$.

2. Let M_w be the maximum number of leaf nodes of cost w. Then we have

$$M_w = M_{w-3} + M_{w-1}$$

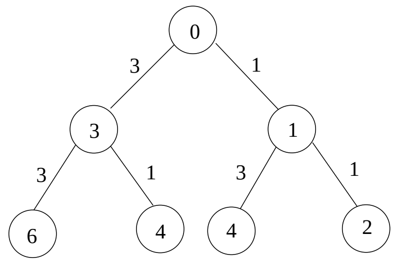

Figure 1: Lopsided Tree

3. With the first egg, we can go up say n floors at a step. If the first egg breaks, we can only go up one floor at a time from the previous step with the second egg. To minimize the number of tests, we need to balance the big steps and the small steps. We can formulate this process as a decision tree and make the tree balanced.

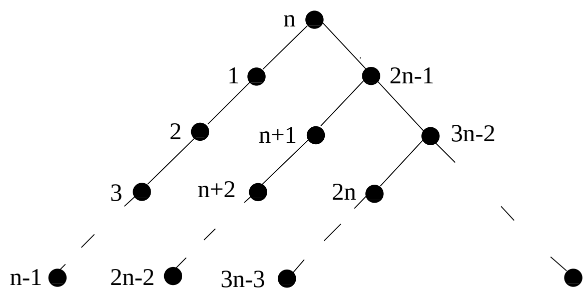

Figure 2: Decision Tree

As we can see from the above decision tree, if we go for big steps of sizes $n, 2n-1, 3n-2, \cdots$, then we balance the height of the tree to n. Let h be the height of the building, then we have

$$n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2} \geq h$$

By solving the above inequation we can get the optimal step sizes.