Square (Real numbers)

Symmetric Matrix $\rightarrow$ $a_{ij} = a_{ji}$ $\Rightarrow$ $A = A^t = A$

Skew-Symmetric Matrix $\rightarrow$ $a_{ij} = -a_{ji}$ $\Rightarrow$ $A = -A^t = A$

All diagonal are zero or imaginary

Recall:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 8 \\ -8 & 9 & 10 \end{bmatrix} \qquad A^t = \begin{bmatrix} 1 & 4 & -8 \\ 2 & 7 & 9 \\ 3 & 8 & 10 \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} 2 & 6 & -5 \\ 6 & 14 & 17 \\ -5 & 17 & 20 \end{bmatrix} \qquad A - A^t = \begin{bmatrix} 0 & -2 & 11 \\ 2 & 0 & -1 \\ -11 & 1 & 0 \end{bmatrix}$$

$$a_{ij} = a_{ji} \qquad a_{ij} = -a_{ji}$$

$A + A^t$ is always symmetric & $A - A^t$ is always skew-symmetric

Square   Complex numbers $(a+ib)$ i = imaginary number
$a, b \in \mathbb{R}$

$\overline{a+ib} = a - ib$

Hermitian Matrix :- $a_{ij} = \overline{a_{ji}} \Rightarrow A = \overline{A^t} = \overline{A}^t$

$$A = \begin{bmatrix} 1+7i & 2-3i & 4 \\ 8+9i & -8+i & 3-i \\ 7i & 6 & 78 \end{bmatrix} \qquad A^t = \begin{bmatrix} 1+7i & 8+9i & 7i \\ 2-3i & -8+i & 6 \\ 4 & 3-i & 78 \end{bmatrix}$$

$$\overline{A^t} = \begin{bmatrix} 1-7i & 8-9i & -7i \\ 2+3i & -8-i & 6 \\ 4 & 3+i & 78 \end{bmatrix}$$

$$A + \overline{A^t} = \begin{bmatrix} 2 & 10-12i & 4-7i \\ 10+12i & -16 & 9-i \\ 4+7i & 9+i & 156 \end{bmatrix} \Longrightarrow \text{Hermitian Matrix}$$

$a_{12} = \overline{a_{21}}$ $a_{ij} = \overline{a_{ji}}$

All diagonal elements will be real numbers.

Square (Complex Numbers)

Skew - Hermitian Matrix $\Rightarrow$ $a_{ij} = -\overline{a_{ji}} \implies A = -\overline{A^t} = -\overline{A}^t$

$$A = \begin{bmatrix} 1+7i & 2-3i & 4 \\ 8+9i & -8+i & 3-i \\ 7i & 6 & 78 \end{bmatrix} \qquad A^t = \begin{bmatrix} 1+7i & 8+9i & 7i \\ 2-3i & -8+i & 6 \\ 4 & 3-i & 78 \end{bmatrix}$$

$$\overline{A^t} = \begin{bmatrix} 1-7i & 8-9i & -7i \\ 2+3i & -8-i & 6 \\ 4 & 3+i & 78 \end{bmatrix}$$

$$A - \overline{A^t} = \begin{bmatrix} 14i & -6+6i & 4+7i \\ 6+6i & 2i & -3-i \\ -4+7i & 3-i & 0 \end{bmatrix}$$

$\overline{A^t} = A^\theta = \overline{A}^t$

$a_{ij} = -(\overline{a_{ji}})$

Skew - Hermitian

All diagonal elements will be purely imaginary or zero

1) $(A+B)^t = A^t + B^t$

2) $(AB)^t = B^t A^t$ — *

3) $(KA)^t = K A^t$

4) $(A^t)^t = A$

5) $\overline{(\overline{A})} = A$

6) $\overline{A+B} = \overline{A} + \overline{B}$

7) $\overline{(KA)} = \overline{K}\ \overline{A}$

8) $\overline{AB} = \overline{A}\ \overline{B}$

$\overline{2} = 2 \qquad \overline{2+3i} = 2-3i \qquad \overline{4-7i} = 4+7i$

$$A = \begin{bmatrix} -2 & 3i \\ 6-i & 5+i \end{bmatrix} \qquad iA = \begin{bmatrix} -2i & 3i^2 \\ 6i-i^2 & 5i+i^2 \end{bmatrix} = \begin{bmatrix} -2i & -3 \\ 1+6i & -1+5i \end{bmatrix}$$

Hermitian Matrix $\rightarrow$ A

iA $\rightarrow$ Skew - Hermitian

Skew Hermitian $\rightarrow$ B

iB $\rightarrow$ Hermitian

Theorem :- If A is any square matrix then A can be uniquely expressed as a sum of symmetric & skew-symmetric matrices

Proof :- Let A be a $n \times n$ square matrix

Consider $2A = (A + A^t) + (A - A^t)$ $(***)$

$$\therefore A = \frac{(A + A^t)}{2} + \frac{(A - A^t)}{2}$$

$\therefore A = P + Q$, where $P = \frac{1}{2}(A + A^t)$

$Q = \frac{1}{2}(A - A^t)$

T.P.T :- P is symmetric matrix & Q is skew-symmetric matrix

T.P.T: $P^t = P$ & $Q^t = -Q$

Consider $P^t = \left[\frac{1}{2}(A + A^t)\right]^t = \left(\frac{1}{2}\right)^t (A + A^t)^t$ $[(KA)^t = KA^t]$

$$= \frac{1}{2}\left(A^t + (A^t)^t\right) \qquad [(A+B)^t = A^t + B^t]$$

$$= \frac{1}{2}(A^t + A)$$

$$= \frac{1}{2}(A + A^t) = P$$

$\therefore P^t = P$

$\Longrightarrow P$ is a symmetric matric

Consider $Q^t = \left[\frac{1}{2}(A - A^t)\right]^t \qquad [(KA)^t = KA^t]$

$$= \left(\frac{1}{2}\right)^t (A - A^t)^t \qquad [(A+B)^t = A^t + B^t]$$

$$= \frac{1}{2}\left(A^t - (A^t)^t\right)$$

$$= \frac{1}{2}(A^t - A)$$

$$= -\frac{1}{2}(A - A^t) = -Q$$

Uniqueness:- Let $A = R + S$, where R is symmetric & S is skew symmetric

$R \neq P$, $S \neq Q$

$A^t = (R+S)^t = R^t + S^t$

Consider $A + A^t = R + S + R^t + S^t$

$= R + \cancel{S} + R - \cancel{S}$

$= 2R$

$\therefore R = \frac{1}{2}(A + A^t) = P$

But this is contradiction $(R \neq P)$

$A - A^t = (R+S) - (R^t + S^t) = \cancel{R} + S - \cancel{R} + S = 2S$

$\therefore S = \frac{1}{2}(A - A^t) = Q$

But this is contradiction

$\therefore A = P + Q$ is the only way possible

unique

Show that every square matrix can be uniquely expressed as the sum of a hermitian matrix and a skew-hermitian matrix.

Proof :- Let A be a $n \times n$ square matrix

Consider $2A = (A + \overline{A}^t) + (A - \overline{A}^t)$ (***)

$\therefore A = \dfrac{(A + \overline{A}^t)}{2} + \dfrac{(A - \overline{A}^t)}{2}$

$\therefore A = P + Q$, where $P = \frac{1}{2}(A + \overline{A}^t)$, $Q = \frac{1}{2}(A - \overline{A}^t)$

T.P.T :- P is Hermitian matrix & Q is skew-hermitian matrix

T.P.T: $\overline{P}^t = P$ & $\overline{Q}^t = -Q$

Consider $\overline{P}^t = \overline{\left[\frac{1}{2}(A + \overline{A}^t)\right]}^t = \overline{\left(\frac{1}{2}\right)^t (A + \overline{A}^t)^t}$

$= \overline{\frac{1}{2}\left(A^t + (\overline{A}^t)^t\right)}$

$[(kA)^t = kA^t]$

$[(A+B)^t = A^t + B^t]$

$$= \overline{\frac{1}{2}} \overline{\left(A^t + \overline{A}\right)}$$

$$= \frac{1}{2}\left(\overline{A^t} + \overline{\overline{A}}\right)$$

$$= \frac{1}{2}\left(\overline{A^t} + A\right)$$

$$= \frac{1}{2}\left(A + \overline{A^t}\right)$$

$$= P$$

$\therefore$ $P^t = P \Rightarrow P$ is a hermitian matrix

Consider $\overline{Q^t} = \overline{\left(\frac{1}{2}\left(A - \overline{A^t}\right)\right)^t}$

$$= \overline{\left(\frac{1}{2}\right)^t \left(A - \overline{A^t}\right)^t}$$

$$= \overline{\frac{1}{2}\left(A^t - \left(\overline{A^t}\right)^t\right)}$$

$$= \frac{1}{2}\left(\overline{A^t} - \overline{\overline{A}}\right) = \frac{1}{2}\left(\overline{A^t} - \overline{\overline{A}}\right) = \frac{1}{2}\left(\overline{A^t} - A\right)$$

$$= -\frac{1}{2}\left(A - \overline{A^t}\right)$$

$$= -Q$$

$\left((A^t)^t = A\right)$

$\left(\overline{kA} = \overline{k}\,\overline{A}\right)$

$\left(\overline{A+B} = \overline{A} + \overline{B}\right)$

$\left[(A+B)^t = A^t + B^t\right]$

$\left[(kA)^t = kA^t\right]$

$\left[\overline{(A+B)} = \overline{A} + \overline{B}\right]$

$\left[\overline{\overline{A}} = A\right]$

Uniqueness :- Let $A = R + S$, where R is hermitian & S is skew hermitian

$R \neq P$ and $S \neq Q$

$$A^t = (R+S)^t = R^t + S^t$$

Consider $A + \overline{A^t} = R + S + \overline{R^t + S^t} = R + S + \overline{R^t} + \overline{S^t}$

$$= R + \cancel{S} + R - \cancel{S}$$

$$= 2R$$

$$\therefore R = \frac{1}{2}\left(A + \overline{A^t}\right) = P$$

But this is contradiction $(R \neq P)$

$$A - \overline{A^t} = (R+S) - \left(\overline{R^t + S^t}\right) = \cancel{R} + S - \cancel{R} + S = 2S$$

$$\therefore S = \frac{1}{2}\left(A - \overline{A^t}\right) = Q$$

But this is contradiction

$\therefore$ $A = P + Q$ is the only way possible

unique

Show that every square matrix can be uniquely expressed as P+iQ where P and Q are hermitian matrices.

Proof :- Let A be any $n\times n$ square matrix.

Consider $2A = (A + \overline{A}^t) + (A - \overline{A}^t)$

$$A = \frac{A + \overline{A}^t}{2} + i\,\frac{(A - \overline{A}^t)}{2i}$$

$$A = \frac{1}{2}(A + \overline{A}^t) + i\left(\frac{1}{2i}(A - \overline{A}^t)\right)$$

$$A = P + iQ$$

where $P = \frac{1}{2}(A + \overline{A}^t)$ & $Q = \frac{1}{2i}(A - \overline{A}^t)$

T.P.T :- P and Q are hermitian matrices

T.P.T :- $\overline{P}^t = P$ and $\overline{Q}^t = Q$

Consider $\overline{P^t} = \overline{\left[\frac{1}{2}(A+\overline{A^t})\right]^t} = \overline{\left(\frac{1}{2}\right)^t (A+\overline{A^t})^t}$

$[(kA)^t = kA^t]$

$= \overline{\frac{1}{2}\left(A^t + (\overline{A^t})^t\right)}$

$[(A+B)^t = A^t + B^t]$

$= \overline{\frac{1}{2}}\ \overline{\left(A^t + \overline{A}\right)}$

$\left\{ \begin{array}{l} (A^t)^t = A \\ \overline{kA} = \overline{k}\,\overline{A} \end{array} \right.$

$(\overline{A+B} = \overline{A} + \overline{B})$

$= \frac{1}{2}\left(\overline{A^t} + \overline{\overline{A}}\right)$

$= \frac{1}{2}\left(\overline{A^t} + A\right)$

$= \frac{1}{2}\left(A + \overline{A^t}\right)$

$= P$

$\therefore \overline{P^t} = P \Rightarrow P$ is a hermitian matrix

Consider $\overline{Q^t} = \overline{\left(\frac{1}{2i}\left(A - \overline{A^t}\right)\right)^t}$

$$= \overline{\left(\frac{1}{2i}\right)^t\left(\left(A - \overline{A^t}\right)\right)^t}$$

$$= \overline{\frac{1}{2i}\left(A^t - \left(\overline{A^t}\right)^t\right)}$$

$$= \frac{1}{2i}\,\overline{\left(A^t - \overline{A}\right)}$$

$$= \frac{1}{2(-i)}\left(\overline{A^t} - \overline{\overline{A}}\right)$$

$$= -\frac{1}{2i}\left(\overline{A^t} - A\right)$$

$$= \frac{1}{2i}\left(A - \overline{A^t}\right) = Q \implies \overline{Q^t} = Q$$

$(KA)^t = KA^t$

$(A+B)^t = A^t + B^t$

$\left(\overline{KA}\right) = \overline{K}\,\overline{A}$

$\left(\overline{\overline{A}}\right) = A$

Show that every hermitian matrix can be expressed as P+iQ where P is a real symmetric matrix and Q is a real skew- symmetric matrix

Proof :- Let A be any $n\times n$ Hermitian matrix

$$\overline{A^t} = A \implies \overline{\overline{A^t}} = \overline{A} \implies A^t = \overline{A}$$

Consider $~~2A = (A+A^t) + (A-A^t)~~$

$$2A = (A+\overline{A}) + (A-\overline{A})$$

$$A = \frac{1}{2}(A+\overline{A}) + i\,\frac{1}{2i}(A-\overline{A})$$

$\therefore A = P + iQ$, where $P = \frac{1}{2}(A+\overline{A})$

$Q = \frac{1}{2i}(A-\overline{A})$

T.P.T :- P is real symmetric matrix

Q is real skew-symmetric matrix

T.P.T :- $\overline{P} = P$ and $\overline{Q} = Q$ $\quad P^t = P$ & $Q^t = -Q$

Consider $\overline{P} = \overline{\frac{1}{2}(A+\overline{A})} = \frac{\overline{1}}{2}\overline{(A+\overline{A})} = \frac{1}{2}(\overline{A}+\overline{\overline{A}})$

$= \frac{1}{2}(\overline{A}+A)$

$= \frac{1}{2}(A+\overline{A})$

$= P$

$\therefore \overline{P} = P \Rightarrow P$ is real matrix

$\overline{Q} = \overline{\frac{1}{2i}(A-\overline{A})} = \overline{\frac{1}{2i}}\,\overline{(A-\overline{A})} = \frac{1}{2(-i)}(\overline{A}-\overline{\overline{A}})$

$= \frac{-1}{2i}(\overline{A}-A)$

$= \frac{1}{2i}(A-\overline{A})$

$= Q$

$\therefore \overline{Q} = Q \Rightarrow Q$ is real matrix

Consider $P^t = \left(\frac{1}{2}(A+\bar{A})\right)^t = \frac{1}{2}^t (A+\bar{A})^t$

$$= \frac{1}{2}\left(A^t + \bar{A^t}\right)$$

$$= \frac{1}{2}\left(\bar{A} + A\right)$$

$$= \frac{1}{2}(A+\bar{A}) = P$$

$\therefore P^t = P$

$\Rightarrow P$ is symmetric matrix

$$Q^t = \left(\frac{1}{2i}(A-\bar{A})\right)^t = \left(\frac{1}{2i}\right)^t (A-\bar{A})^t = \frac{1}{2i}\left(A^t - \bar{A}^t\right)$$

$$= \frac{1}{2i}\left(\bar{A} - A\right)$$

$$= -\frac{1}{2i}\left(A - \bar{A}\right)$$

$$= -Q$$

$\therefore Q^t = -Q$

$\Rightarrow Q$ is a skew-matrix matrix

Show that every skew-hermitian matrix can be expressed as P+iQ where P is a real skew-symmetric matrix and Q is a real symmetric matrix.

Let A be any $n \times n$ skew-hermitian

$\overline{A^t} = -A \Rightarrow \overline{\overline{A^t}} = -\overline{A} \Rightarrow A^t = -\overline{A}$

$2A = (A + A^t) + (A - A^t)$

$A = \frac{(A+\overline{A})}{2} + i\frac{(A-\overline{A})}{2i}$

$= P + iQ$

$\therefore A = P + iQ$, where $P = \frac{1}{2}(A+\overline{A})$

$Q = \frac{1}{2i}(A-\overline{A})$

T.P.T :- P is real skew-symmetric matrix

Q is real symmetric matrix

T.P.T :- $\overline{P} = P$ and $\overline{Q} = Q$ $\quad P^t = -P$ & $Q^t = Q$

Consider $\overline{P} = \overline{\frac{1}{2}(A+\overline{A})} = \frac{\overline{1}}{2}\overline{(A+\overline{A})} = \frac{1}{2}(\overline{A}+\overline{\overline{A}})$

$= \frac{1}{2}(\overline{A}+A)$

$= \frac{1}{2}(A+\overline{A})$

$= P$

$\therefore \overline{P} = P \Rightarrow P$ is real matrix

$\overline{Q} = \overline{\frac{1}{2i}(A-\overline{A})} = \frac{\overline{1}}{\overline{2i}}\overline{(A-\overline{A})} = \frac{1}{2(-i)}(\overline{A}-\overline{\overline{A}})$

$= \frac{-1}{2i}(\overline{A}-A)$

$= \frac{1}{2i}(A-\overline{A})$

$= Q$

$\therefore \overline{Q} = Q \Rightarrow Q$ is real matrix

Consider $P^t = \left(\frac{1}{2}(A+\overline{A})\right)^t = \frac{1}{2}^t (A+\overline{A})^t$

$= \frac{1}{2}\left(A^t + \overline{A^t}\right)$

$= \frac{1}{2}\left(-\overline{A} + A\right)$

$= -\frac{1}{2}(A+\overline{A}) = -P$

$\therefore P^t = -P$

$\Rightarrow P$ is skew symmetric matrix

$Q^t = \left(\frac{1}{2i}(A-\overline{A})\right)^t = \left(\frac{1}{2i}\right)^t (A-\overline{A})^t = \frac{1}{2i}\left(A^t - \overline{A}^t\right)$

$= \frac{1}{2i}\left(-\overline{A} + A\right)$

$= \frac{1}{2i}\left(A-\overline{A}\right)$

$= Q$

$\therefore Q^t = -Q$

$\Rightarrow Q$ is a symmetric matrix

Theorem - 01 : *Show that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew symmetric matrix.*

Theorem - 02 : *Show that every square matrix can be uniquely expressed as the sum of a hermitian matrix and a skew-hermitian matrix.*

Theorem - 03 :- *Show that every square matrix can be uniquely expressed as P+iQ where P and Q are hermitian matrices.*

Theorem - 04 : *Show that every hermitian matrix can be expressed as P+iQ where P is a real symmetric matrix and Q is a real skew- symmetric matrix.*

$(\bar{a} = a)$
i.e a is a real number

Theorem - 05 : *Show that every skew-hermitian matrix can be expressed as P+iQ where P is a real skew-symmetric matrix and Q is a real symmetric matrix.*

Express the matrix $A=\begin{bmatrix}3 & 2 & 3\\ 4 & 5 & 3\\ 2 & 4 & 5\end{bmatrix}$ as the sum of a symmetric (real) and a skew symmetric matrix. (real)

Sol^n :-

$A + A^t$ symmetric

$A - A^t$ skew-symmetric

Given $A = \begin{bmatrix}3 & 2 & 3\\ 4 & 5 & 3\\ 2 & 4 & 5\end{bmatrix}$ $\quad A^t = \begin{bmatrix}3 & 4 & 2\\ 2 & 5 & 4\\ 3 & 3 & 5\end{bmatrix}$

$\frac{1}{2}(A + A^t) = \frac{1}{2}\begin{bmatrix}6 & 6 & 5\\ 6 & 10 & 7\\ 5 & 7 & 10\end{bmatrix} \Longrightarrow a_{ij} = a_{ji} \Rightarrow \frac{1}{2}(A+A^t)$ is symmetric matrix

$\frac{1}{2}(A - A^t) = \frac{1}{2}\begin{bmatrix}0 & -2 & 1\\ 2 & 0 & -1\\ -1 & 1 & 0\end{bmatrix} \Longrightarrow a_{ij} = -a_{ji} \Rightarrow \frac{1}{2}(A-A^t)$ is skew symmetric matrix

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & 6 & 5 \\ 6 & 10 & 7 \\ 5 & 7 & 10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Therefore A is expressed as a sum of symmetric and skew symmetric matrices

Express the matrix $A=\begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix}$ as the sum of a hermitian and a skew hermitian matrix.

Sol^n :- $A+\overline{A}^t$ hermitian , $A-\overline{A}^t$ skew-hermitian

$$A = \begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix} \Longrightarrow A^t = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\overline{A^t} = \begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix}$$

$$\frac{1}{2}(A+\overline{A}^t) = \frac{1}{2}\begin{bmatrix} 2 & 2+2i & 4+6i \\ 2-2i & 4 & -2i \\ 4-6i & 2i & 0 \end{bmatrix} \Longrightarrow a_{ij} = \overline{a_{ji}} \Longrightarrow \frac{1}{2}(A+\overline{A}^t) \text{ is a hermitian matrix}$$

$$\frac{1}{2}(A-\overline{A}^t) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow a_{ij} = -\overline{a_{ji}} \Longrightarrow \frac{1}{2}(A-\overline{A}^t) \text{ is a skew hermitian matrix}$$

$$\therefore \quad A = \frac{1}{2}\left(A + \overline{A^t}\right) + \frac{1}{2}\left(A - \overline{A^t}\right)$$

$$\begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 2+2i & 4+6i \\ 2-2i & 4 & -2i \\ 4-6i & 2i & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore A is expressed as a sum of hermitian and skew hermitian matrices

Express the matrix A= $\begin{bmatrix} 2 & 3-i & 1+2i \\ i & 0 & 1 \\ 1+2i & 1 & 3i \end{bmatrix}$ as P + iQ where P and Q are hermitian matrix.

$A - \overline{A}^t \Rightarrow$ Skew-Her

$\frac{A - \overline{A}^t}{i} \Rightarrow$ Hermitian

Soln :-

$$A = \underbrace{\frac{1}{2}(A + \overline{A}^t)}_{\text{Hermitian}} + i\left(\underbrace{\frac{1}{2i}(A - \overline{A}^t)}_{\text{Hermitian}}\right)$$

$$\frac{1}{i} \times \frac{(-i)}{(-i)}$$

$$\frac{-i}{-i^2} = \frac{i}{-1}$$

$$\frac{1}{i} = -i$$

$$A^t = \begin{bmatrix} 2 & i & 1+2i \\ 3-i & 0 & 1 \\ 1+2i & 1 & 3i \end{bmatrix} \qquad \overline{A}^t = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 1-2i & 1 & -3i \end{bmatrix}$$

Consider

$$P = \frac{1}{2}(A + \overline{A}^t) = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 2 \\ 3+2i & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \Rightarrow a_{ij} = \overline{a_{ji}} \Rightarrow \frac{1}{2}(A + \overline{A}^t) \text{ is a Hermitian matrix}$$

$$Q = \frac{1}{2i}(A - \overline{A}^t) = \frac{1}{2i}\begin{bmatrix} 0 & 3 & 4i \\ -3 & 0 & 0 \\ 4i & 0 & 6i \end{bmatrix}$$

$$\frac{1}{i} = -i \Rightarrow \frac{1}{2}\begin{bmatrix} 0 & -3i & -4i^2 \\ 3i & 0 & 0 \\ -4i^2 & 0 & -6i^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -3i & 4 \\ 3i & 0 & 0 \\ 4 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow \quad a_{ij} = \overline{a_{ji}}$$

$$\Rightarrow \frac{1}{2i}(A - \overline{A}^t) \text{ is a hermitian matrix}$$

$$\therefore A = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 2 \\ 3+2i & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} + i\,\frac{1}{2}\begin{bmatrix} 0 & -3i & 4 \\ 3i & 0 & 0 \\ 4 & 0 & 6 \end{bmatrix}$$

$$\therefore A = \underbrace{\frac{1}{2}(A + \overline{A}^t)}_{\text{Hermitian}} + i\left(\underbrace{\frac{1}{2i}(A - \overline{A}^t)}_{\text{Hermitian}}\right)$$

$$= P + iQ$$

Therefore A is expressed as P+iQ where P and Q are hermitian matrices

Express the matrix $A = \begin{bmatrix} 2 & 2+i & -2i \\ 2-i & 3 & i \\ 2i & -i & 1 \end{bmatrix}$ as P + iQ where P is a real symmetric matrix and Q is a real skew-symmetric matrix

$\bar{P} = P \Rightarrow P$ is real

Soln :-

$$A = \left(\frac{1}{2}(A+\bar{A})\right) + i\left(\frac{1}{2i}(A-\bar{A})\right)$$

$$A = \begin{bmatrix} 2 & 2+i & -2i \\ 2-i & 3 & i \\ 2i & -i & 1 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 2 & 2-i & 2i \\ 2+i & 3 & -i \\ -2i & i & 1 \end{bmatrix}$$

Consider $\frac{1}{2}(A+\bar{A}) = \frac{1}{2}\begin{bmatrix} 4 & 4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow a_{ij} = a_{ji} \Rightarrow \frac{1}{2}(A+\bar{A})$ is a real symmetric matrix

$$\frac{1}{2i}(A-\bar{A}) = \frac{1}{2i}\begin{bmatrix} 0 & 2i & -4i \\ -2i & 0 & 2i \\ 4i & -2i & 0 \end{bmatrix} \Rightarrow \frac{1}{2}\begin{bmatrix} 0 & -2i^2 & 4i^2 \\ 2i^2 & 0 & -2i^2 \\ -4i^2 & 2i^2 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & 2 \\ 4 & -2 & 0 \end{bmatrix}$$

$a_{ij} = -a_{ji}$

$\therefore \frac{1}{2i}(A-\bar{A})$ is a real skew-symmetric matrix

$$A = \frac{1}{2}\begin{bmatrix} 4 & 4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} + i\,\frac{1}{2}\begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & 2 \\ -4 & -2 & 0 \end{bmatrix}$$

Therefore A is expressed as P+iQ where P is a real symmetric and Q is a real skew symmetric matrices

Express the matrix A $= \begin{bmatrix} 0 & 2-3i & 1+i \\ -2-3i & 2i & 2-i \\ -1+i & -2-i & 3i \end{bmatrix}$ as P + iQ where P is a real skew-symmetric matrix and Q is a real symmetric matrix

linear Differentiation

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⇒ First order linear differentiation.

① find out independent and dependent Varible.

$\frac{dr}{dt}$ → dependent
→ Independent

② No dependent Variable in R.H.S

③ Along with the differential turm no Independent Variable is allowed

$x^2 \frac{dy}{dx}$

④ In LHS the power of dependent Variable should be one (without differential term)

Example

function of independent.

formula $\frac{dy}{dx} + Py = Q$

the Solution is given by

Dependent Variable $e^{\int P dx} = \int (Q e^{\int P.dx} dx + C$

→ Integrating factor.

$y e^{\int P dx} = \int Q e^{\int P.dx} \cdot dx + C$

$$\int (f(x))^n \cdot (f'(x) \cdot dx = \frac{(f(x))^{n+1}}{n+1}$$

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1) $x \log x \frac{dy}{dx} + y = 2\log x$

Divide both side by $x \cdot \log x$.

$$\frac{dy}{dx} + \cancel{x\log x} \frac{y}{x\log x} = \frac{2\log x}{x\log x}$$

$$\frac{dy}{dx} + \left(\frac{1 y}{x\log x}\right) = \left(\frac{2}{x}\right)$$

P, Q

Integrating factor $= e^{\int P \cdot dx} = e^{\int \frac{1}{x\log x} \cdot dx}$

$$= e^{\int \frac{1/x}{\log x} \cdot dx}$$

$$= e^{\log(\log x)}$$

$$= \log x$$

↑ formula

$$y \log x = 2\int \frac{1}{x} \cdot \log x \cdot dx$$

$$= \cancel{2} \frac{(\log x)^2}{\cancel{2}}$$

$$y(\log x) = (\log x)^2 + c$$

2) $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^3}(\log z)^3$

Divide by $z(\log z)^3$

$$\frac{1}{z(\log z)^3}\frac{dz}{dx} + \frac{1}{x}\cdot\frac{\cancel{z}(\log z)}{\cancel{z}(\log z)^{\cancel{3}\,2}} = \frac{1}{x^3}$$

$$\frac{1}{z(\log z)^3}\frac{dz}{dx} + \frac{1}{x(\log z)^2} = \frac{1}{x^3} \quad \text{--- (1)}$$

let $(\log z)^{-2} = y$

Differentiation w.r.t. x

$$-2(\log z)^{-3}\frac{d}{dx}(\log z) = \frac{dy}{dx}$$

$$-2(\log z)^{-3}\cdot\frac{1}{z}\frac{dz}{dx} = \frac{dy}{dx}$$

$$\frac{-2}{z(\log z)^3}\frac{dz}{dx} = \frac{dy}{dx}$$

$$\frac{1}{z(\log z)^3}\frac{dz}{dx} = \frac{-1}{2}\frac{dy}{dx}$$

from equation (1).

$$-\frac{1}{2}\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^3}$$

multiply by -2

$$\frac{dy}{dx} + \left(\frac{-2}{x}\right)y = \frac{-2}{x^3}$$

$$\frac{dy}{dx} + Py = Q$$

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Integrating factor $= e^{\int P\cdot dx} = e^{\int -\frac{2}{x}\cdot dx}$

$= e^{-2\log x}$

$= e^{\log x^{-2}}$

$= x^{-2}$

$y\, x^{-2} = \int \frac{-2}{x^3} \cdot x^{-2} \cdot dx$

$y x^{-2} = -2\int \frac{1}{x^5} \cdot dx$

$= -2\int (x^{-5})\cdot dx$

$= -2\left[\frac{x^{-5+1}}{-5+1}\right] + c$

$= \cancel{-2}\left[\frac{x^{-4}}{\cancel{-4}\,2}\right] + c$

$\frac{y}{x^2} = \frac{1}{2x^4} + c$

$y = \frac{x^2}{2x^{\cancel{4}\,2}} + c$

$y = \frac{1}{2x^2} + c$

$(\log z)^{-2} = \frac{x^{-2}}{2} + cx^2$

$\frac{1}{(\log z)^2} = \frac{1}{2x^2} + cx^2$

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3) $y\frac{dx}{dy} = x - yx^2\cos y$

$x \Rightarrow$ Dependent

$y \Rightarrow$ independent

Divide b.t.s by 'y' and 'x^2'

$$\frac{dx}{dy} = \frac{\cancel{x}}{y\cancel{x^2}} - \frac{\cancel{x^2}\cos y}{\cancel{x^2}}$$

$$\frac{1}{x^2}\frac{dx}{dy} = \frac{1}{xy} - \cos y$$

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{xy} = -\cos y \quad \text{—— (1)}$$

let $-\frac{1}{x} = t$.

Diff w.r.t. y.

$$+\frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dy}$$

from eq (1).

$$\frac{dt}{dy} + \underbrace{t\left(\frac{1}{y}\right)}_{P} = \underbrace{\boxed{-\cos y}}_{Q}$$

Integrating factor $= I.f. = e^{\int \frac{1}{y}\cdot dy} = e^{(\log y)} = y$

The solution is given by:-

~~$t \cdot y = \int -\cos y \cdot y \cdot dy + C$~~

~~$y^2 = -\sin y$~~. $(t)y = \int y(-\cos y)\cdot dy + C$

$$-\frac{1y}{x} = -\int y\cdot\cos y\cdot dy + c$$

$$-\frac{1(y)}{xy} = -\left[y\int\cos y\cdot dy - \int\left(\frac{d}{dx}(y)\int\cos y\cdot dx\right)dy\right]+c$$

$$= -\left[y\cdot \cancel{\sin y} - \int \sin y\,dy\right]+c$$

$$= -\left[y\cdot\sin y + \cos y\right]+c$$

$$-\frac{y}{xy} = -y\cdot\sin y - \cos y + c$$

—x—x—x— (Beta- Gamma functions)

Gamma Gamma Beta Beta

Exp log Alg Trigo Inverse

Gamma Gamma Beta Beta Beta

Beta

Gamma $\longrightarrow \Gamma$

Beta $\longrightarrow \beta$, B

* Beta Gamma only used for Simplification

* $$\int_0^\infty e^{-x}\cdot x^n\cdot dx = \Gamma n+1$$

$$\int_0^\infty e^{-x^7}\cdot dx$$

$$x^7 = t$$

$$x = t^{1/7}$$

$$dx = \frac{1}{7}t^{-6/7}\,dt$$

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$$= \frac{1}{7} \int_0^\infty e^{-t} \cdot t^{-6/7} \cdot dt$$

$$= \frac{1}{7} \Gamma\left(\frac{-6}{7}+1\right)$$

$$= \frac{1}{7} \Gamma\left(\frac{1}{7}\right)$$

1) $\Gamma n = (n-1)!$ if n = integer +ve

$\Gamma 2 = 1!$

2) $\Gamma n = (n-1)\Gamma n-1$ if $n = \frac{p}{q}$ $p > q$

$$\Gamma \frac{19}{8} = \frac{11}{8} \cdot \frac{3}{8} \Gamma \frac{3}{8}$$

$\Gamma 1/2 = \sqrt{\pi}$

$$(\Gamma p)(\Gamma 1-p) = \frac{\pi}{\sin p\pi}$$

$$\left(\Gamma \frac{\pi}{4}\right)\left(\Gamma 1 - \frac{\pi}{4}\right) = \frac{\pi}{\sin \left(\frac{\pi}{4}\right) \times \pi}$$

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1) $\int_0^1 (x \cdot \log x)^{11} \cdot dx$

$\int_0^1 (x)^{11} (\log x)^{11} \cdot dx$

$\int_0^\infty e^{-x} \cdot x^n \cdot dx$

let $\log x = -t$

$x = e^{-t}$

$dx = -e^{-t} dt$

x	0	1
t	∞	0

$\int_\infty^0 (e^{-t})^{11} (-t)^{11} (-e^{-t} \cdot dt)$

$= -\int_0^\infty e^{-12t} \cdot t^{11} \cdot dx$

let $12t = u$

$t = \frac{u}{12}$

$dt = \frac{du}{12}$

$-\int_0^\infty e^{-u} \left(\frac{u}{12}\right)^{11} \cdot \frac{du}{12}$

$= \frac{-1}{12} \int_0^\infty e^{-u} \cdot \frac{u^{11}}{(12)^{11}} \cdot du$

$= \frac{-1}{(12)^{12}} \int_0^\infty e^{-u} \cdot u^{11} \cdot du$

$= \frac{-1}{(12)^{12}} \cdot \overline{|11+1}$

$= \frac{-1}{(12)^{12}} \overline{|12}$

$= \frac{-1}{(12)^{12}} (12-1)!$

$= \frac{-1}{(12)^{12}} (11)!$

2) $\int_0^1 \sqrt[4]{x \cdot \log\left(\frac{1}{x}\right)} \cdot dx$

$\int_0^1 \left(x \log\left(\frac{1}{x}\right)\right)^{1/4} \cdot dx$

$\int_0^1 x\left(\log(1) - \log(x)\right)^{1/4} \cdot dx$

$\int_0^1 \left(x(-\log(x))\right)^{1/4} \cdot dx$

let ~~$-\log x = t$~~ $\quad -\log x = t$
~~$-x = e^{t}$~~ $\quad \log x = -t$
$x = e^{-t}$
$dx = -e^{-t} \cdot dt$

x	0	1
t	∞	0

$\int_\infty^0 (e^{-t})^{1/4} (t)^{1/4} \left(-e^{-t} \cdot dt\right)^{1/4}$ $\quad -\frac{1}{4} - \frac{1 \times 4}{1 \times 4}$

$= \int_0^\infty e^{-t/4} \cdot t^{1/4} \cdot e^{-t} \cdot dt$ $\quad \frac{-1-4}{4}$

$-5/4$

$= \int_0^\infty e^{-5t/4} \cdot t^{1/4} \cdot dt$

let $\frac{5t}{4} = u$

t	0	∞
u	0	∞

$t = \frac{4u}{5}$

$\boxed{dt = \frac{4\,du}{5}}$

$1+\frac{1}{4}$

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$$= \int_0^\infty e^{-u} \cdot \left(\frac{4u}{5}\right)^{1/4} \left(\frac{4\,du}{5}\right)$$

$$= \left(\frac{4}{5}\right) \left\{ \int_0^\infty e^{-u} \cdot \left(\frac{4}{5}\right)^{1/4} \cdot (u)^{1/4} \cdot du \right.$$

$$= \left(\frac{4}{5}\right)^1 \left(\frac{4}{5}\right)^{1/4} \int_0^\infty e^{-u} \cdot u^{1/4} \cdot du$$

$$= \left(\frac{4}{5}\right)^{5/4} \int_0^\infty e^{-u} \cdot u^{1/4} \cdot du$$

$$= \left(\frac{4}{5}\right)^{5/4} \Gamma\left(\frac{1}{4}+1\right)$$

$$= \left(\frac{4}{5}\right)^{5/4} \Gamma\left(\frac{5}{4}\right)$$

$$= \left(\frac{4}{5}\right)^{5/4} \Gamma\left(\frac{5}{4}-1\right) \left(\frac{5}{4}-1\right)$$

$$= \left(\frac{4}{5}\right)^{5/4} \left(\frac{1}{4}\right)\left(\Gamma\frac{1}{4}\right)$$

$\frac{5}{4}$ $\frac{1}{4}$

$\frac{19}{8}$ $\frac{11}{8}$ $\frac{3}{8}$ $\frac{3}{8}$

(3) $\int_0^{\infty} \frac{x^5}{5^x} \cdot dx$

$\int_0^{\infty} x^5 \cdot 5^{-x} \cdot dx$

$= \int_0^{\infty} e^{-x} \cdot x^n \cdot dx$

let $5^{-x} = e^{-t}$

Take log on b.t.s.

$\log(5)^{-x} = \log e^{-t}$

$-x \log 5 = -t \log e$

$\cancel{-} x \log 5 = \cancel{-} t$

$x = \frac{t}{\log 5}$

$dx = \frac{dt}{\log 5}$

x	0	∞
t	0	∞

$\int_0^{\infty} \left(\frac{t}{\log 5}\right)^5 \cdot e^{-t} \cdot \frac{dt}{(\log 5)}$

$\frac{1}{(\log 5)^6} \int_0^{\infty} t \cdot e^{-t} \cdot dt$

$\frac{1}{(\log 5)^6} \sqrt{5+1} = \frac{\sqrt{6}}{(\log 5)^6}$

$= \frac{5!}{(\log 5)^6}$

(completely algebric, trigo)
No Combination

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Beta function

$$\int_0^1 x^m (1-x)^n \cdot dx = \beta(m+1, n+1)$$

Beta $\longleftrightarrow$ Gamma

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$

1) $$\int_0^2 x^4 \sqrt[5]{8-x^3} \cdot dx$$

$$\int_0^2 x^4 (8-x^3)^{1/5} \cdot dx$$

$$\int_0^2 x^4 \left[8\left(1-\frac{x^3}{8}\right)\right]^{1/5} \cdot dx$$

let $\frac{x^3}{8} = t$

$x^3 = 8t$

$\cancel{3x^2 dx = 8 dt}$

$dx = \frac{8}{3x^2} dt$

$x = (8t)^{1/3}$

$dx = 8^{1/3} \cdot t^{1/3}$

$dx = 8^{1/3} \cdot \frac{1}{3} t^{-2/3} \cdot dt$

$dx = (2^3)^{1/3} \cdot \frac{1}{3} t^{-2/3} \cdot dt$

$dx = \frac{2}{3} t^{-2/3} \cdot dt$

x	0	2
t	0	1

$\frac{4}{3} - \frac{2}{3}$ $\qquad\qquad$ $\frac{5\times 4}{5\times 3} + \frac{1\times 3}{5\times 3}$

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$$\int_0^1 \left((8t)^{1/3}\right)^4 \left[8(1-t)\right]^{1/5} \cdot \frac{2}{3} t^{-2/3} \cdot dt$$

$$\frac{2}{3}\int_0^1 8^{4/3} \cdot t^{4/3} \left[8^{1/5}(1-t)^{1/5}\right] \cdot t^{-2/3} \cdot dt$$

$$\frac{2}{3}\int_0^1 8^{23/15} \cdot t^{2/3} (1-t)^{1/5} \cdot dt$$

$$\frac{2}{3} \times 8^{23/15} \int_0^1 t^{2/3} \cdot (1-t)^{1/5} \cdot dt$$

$$\int_0^1 x^m (1-x)^n \cdot dx$$

$m = \frac{2}{3}$ $\quad n = 1/5$

$$\frac{2}{3} \times (2)^{23/5} \beta\left(\frac{2}{3}+1, \frac{1}{5}+1\right)$$

$$\frac{2}{3} (2)^{23/5} \beta\left(\frac{5}{3}, \frac{6}{5}\right)$$

$\frac{23}{3}$ $\frac{}{69}$

$$\frac{2}{3}(2)^{23/5} \frac{\Gamma 5/3 \, \Gamma 6/5}{\Gamma\left(\frac{5}{3}+\frac{6}{5}\right)} = \frac{2}{3}(2)^{23/5} \frac{\Gamma 5/3 \, \Gamma 6/5}{\Gamma 43/15}$$

$$\frac{2}{3}(2)^{23/5} \frac{\left(\frac{2}{3}\Gamma\frac{2}{3}\right)\left(\frac{1}{5}\Gamma\frac{1}{5}\right)}{\frac{28}{15} \times \left(\frac{13}{15}\Gamma\frac{13}{15}\right)}$$

$$\frac{(2)^{28/5} \left(\frac{2}{3}\Gamma\frac{2}{3}\right)\left(\frac{1}{5}\Gamma\frac{1}{5}\right)}{3 \cdot \frac{28}{15}\left(\frac{13}{15}\Gamma\frac{13}{15}\right)}$$

2) $\int_0^a \sqrt{a^2-x^2}\cdot dx$

$$\int_0^a \left(a^2\left(1-\frac{x^3}{a^2}\right)\right)^{1/2}\cdot dx$$

let $\frac{x^2}{a^2}=t$

$x^2=a^2t$

$x=(a^2t)^{1/2}$

~~dx~~ $x=(a^2)^{1/2}\cdot t^{1/2}$

$x=a\cdot t^{1/2}$

$dx = a\cdot\frac{1}{2}\cdot t^{-1/2}\cdot dt$

x	0	a
t	0	1

$$\int_0^1 a(1-t)^{1/2}\cdot \frac{a}{2}t^{-1/2}\cdot dt$$

$$\int_0^1 \frac{a^2}{2}t^{-1/2}(1-t)^{1/2}\cdot dt$$

$$\frac{a^2}{2}\beta\left(\frac{3}{2},\frac{1}{2}\right)$$

$$\frac{a^2}{2}\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)} = \frac{a^2}{2}\frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 2} = \frac{a^2}{2}\pi$$

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3) $$\int_0^{\infty} \frac{x^3}{(4+x^6)^{5/2}}$$

Consider, $4+x^6$

$4+(x^3)^2$

let $x^3 = 2\tan\theta$

$x = (2)^{1/3} (\tan\theta)^{1/3}$

$dx = (2)^{1/3} \frac{1}{3}(\tan\theta)^{-2/3} \cdot \sec^2\theta \cdot d\theta$

$4+(x^3)^2 = 4+(2(\tan\theta))^2$

$= 4+4\tan^2\theta$

$= 4(1+\tan^2\theta)$

$= 4\sec^2\theta$

x	0	∞
θ	0	$\pi/2$

$$\int_0^{\pi/2} \frac{(2\tan\theta)(2)^{1/3} \frac{1}{3}(\tan\theta)^{-2/3} \cdot \sec^2\theta \cdot d\theta}{(4\sec^2\theta)^{5/2}}$$

$\frac{a-5}{3}$

$\frac{4-15}{3}$

$$\int_0^{\pi/2} \frac{(2\tan\theta)\cdot(2)^{1/3} \cdot 1/3\,(\tan\theta)^{-2/3} \cdot \sec^2\theta \cdot d\theta}{((2\sec\theta)^2)^{5/2}}$$

$$\frac{(2)^{1/3}\times 2}{3}\int_0^{\pi/2} \frac{(\tan\theta)^{1/3} \cdot \sec^2\theta \cdot d\theta}{(2^5)(\sec^5\theta)}$$

$$\frac{2^{4/3}}{3 \;\; 2^4}\int_0^{\pi/2} \frac{(\tan\theta)^{1/3} \cdot \sec^2\theta \cdot d\theta}{(\sec\theta)^3}$$

$$\frac{2^{1/3}}{3\times 2^4}\int_0^{\pi/2} \frac{(\sin\theta)^{1/3}}{(\cos^{1/3}\theta)} \times (\cos 3\theta)$$

$$\frac{2^{1/3}}{3\times 2^4}\int_0^{\pi/2} (\sin\theta)^{1/3} (\cos\theta)^{8/3} \cdot d\theta$$

$$\int_0^{\pi/2} (\sin\theta)^m (\cos\theta)^n \cdot d\theta$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

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$$= \frac{2^{1/3}}{3 \times 2^4} \times \frac{1}{2} \beta\left(\frac{4/3}{2}, \frac{11/3}{2}\right)$$

$$= \frac{2^{1/3}}{3 \times 2^5} \beta\left(\frac{4}{6}, \frac{11}{6}\right)$$

$$= \frac{2^{1/3}}{3 \times 2^5} \frac{\Gamma\frac{4}{6}\,\Gamma\frac{11}{6}}{\Gamma\left(\frac{4}{6}+\frac{11}{6}\right)}$$

4) $\displaystyle\int_0^\infty \frac{x^3}{1+x^8}\,dx$

Consider, $1+x^8$

$1+(x^4)^2$

let $x^4 = \tan\theta$

$1+x^8 = 1+\tan^2\theta$

$\sec^2\theta$

$x = \tan^{1/4}\theta$

$dx = \frac{1}{4}\tan^{-3/4}\theta \cdot \sec^2\theta \cdot d\theta$

x	0	θ
∞	0	$\pi/2$

$$\int_0^{\pi/2} \frac{((\tan\theta)^{1/4})^3}{\cancel{\sec^2\theta}} \cdot \frac{1}{4} \times \tan^{-3/4}\theta \cdot \cancel{\sec^2\theta} \cdot d\theta$$

$$\int_0^{\pi/2} \cancel{(\tan\theta)^{3/4}} \cdot \frac{1}{4} \times \cancel{\tan^{-3/4}\theta}$$

$$= \frac{1}{4}[1]_0^{\pi/2} = \frac{\pi}{8} //$$

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5) $\int_0^1 \underset{v}{x^5} \underset{u}{\cos^{-1}x} \cdot dx$

$u\int v - \int(u'\int v)$

$$\left[\left(\cos^{-1}x \int x^5 \cdot dx - \int -\frac{1}{\sqrt{1-x^2}} \int x^5 \cdot dx\right) \cdot dx\right]_0^1$$

$$= \left[\cos^{-1}x \cdot \frac{x^6}{6} + \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^6}{6} \cdot dx\right]_0^1$$

$$= \left[\cos^{-1}x \cdot \frac{x^6}{6}\right]_0^1 + \frac{1}{6}\left[\int \frac{1}{\sqrt{1-x^2}} \cdot x^6 \cdot dx\right]$$

$$= \left(\cos^{-1}(1)\left(\frac{1}{6}\right) - \cos^{-1}(0) \times \frac{0}{6}\right) + \frac{1}{6}\left[\int x^6(1-x^2)^{-1/2} \cdot dx\right]$$

$$= 0 + \frac{1}{6}\int_0^1 x^6(1-x^2)^{-1/2} \cdot dx$$

let $x^2 = t$

~~let~~

$x = t^{1/2}$

$dx = \frac{1}{2} t^{-1/2} \cdot dt$

x	0	1
t	0	1

$-\frac{1}{2}$

$$= \frac{1}{6}\int_0^1 t^3 (1-t)^{-1/2} \cdot \frac{1}{2} t^{-1/2} \cdot dt$$

$-\frac{1}{2}$

$\frac{5}{2}$

$$= \frac{1}{12}\int_0^1 t^{5/2}(1-t)^{-1/2} \cdot dt$$

$$= \frac{1}{12} \beta\left(\frac{7}{2}, \frac{1}{2}\right) = \frac{1}{6} \frac{\Gamma\frac{7}{2}\,\Gamma\frac{1}{2}}{\Gamma 4}$$

$$\int_0^{\pi/2} (\sin\theta)^m (\cos\theta)^n \cdot d\theta = \frac{1}{2}\beta\left[\left(\frac{m+1}{2}\right), \left(\frac{n+1}{2}\right)\right]$$

$$= \frac{1}{12} \frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}}}{3 \times 2}$$

$$= \frac{1}{12} \times \frac{1}{6\,2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \pi$$

$$= \frac{5\pi}{192}$$

6) $\int_0^{\pi} \frac{(\sin\theta)^4}{(1+\cos\theta)^2} \cdot d\theta$ (complefy trigo)

$$\int_0^{\pi} \frac{(2\sin\theta/2 \cos\theta/2)^4}{(2\cos^2\theta/2)^2} \cdot d\theta$$

$$\int_0^{\pi} \frac{2^{\cancel{4}\,2} \sin^4\theta/2 \cancel{\cos^4\theta/2}}{\cancel{2^2} \cdot \cancel{\cos^4\theta/2}}$$

$$\int_0^{\pi} 2^2 \cdot \sin^4\theta/2 \cdot d\theta$$

let $\frac{\theta}{2} = t$.

$\theta = 2t$

$d\theta = 2dt$.

θ	0	π
t	0	$\pi/2$

$$4\int_0^{\pi/2} \sin^4 t \cdot 2\,dt$$

$$8\int_0^{\pi/2} (\sin t)^4\,dt$$

$m=4, \; n=0$

$$= \frac{1}{\cancel{2}} \times \cancel{8}^{4}\, \beta\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$= 4\, \beta\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$= 4 \times \frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{5}{2}+\frac{1}{2}\right)}$$

$$= 4 \times \frac{\frac{3}{2}\times\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= \frac{\cancel{4}^{2}}{8} \times \frac{3}{\cancel{2}} \times \frac{1}{2}\,\pi$$

$$= \frac{3\pi}{2}$$

$2!$

2×1

$\cos 2\theta =$

7) $\displaystyle\int_0^{\pi/4} (1+\cos 4\theta)^5 \cdot d\theta$

$\displaystyle\int_0^{\pi/4} (2\cos^2 2\theta)^5 \cdot d\theta$

let $2\theta = t$

$\theta = \frac{t}{2}$

$d\theta = \frac{1}{2}\, dt.$

θ	0	$\pi/4$
t	0	$\pi/2$

$\displaystyle (2)^5 \int_0^{\pi/\cancel{4}2} (\cos^2 t)^5 \cdot \frac{1}{2}\, dt$

$\displaystyle 2^4 \int_0^{\pi/2} \cos^{10} t \cdot dt$

$\displaystyle 2^4 \int_0^{\pi/2} (\cos t)^{10} \cdot dt.$

$m = 0, \quad n = 10$

$2^4 \frac{1}{2} \beta\left(\frac{11}{2}, \frac{1}{2}\right)$

$\displaystyle 2^4 \cdot \frac{\Gamma\frac{11}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{11}{2}+\frac{1}{2}\right)} = \cancel{2^4}\, \frac{\frac{9}{\cancel{2}} \times \frac{7}{2} \times \frac{5}{\cancel{2}} \times \frac{3}{\cancel{2}} \times \frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 6}$

$\displaystyle = \frac{9 \times 7 \times \cancel{5} \times \cancel{3} \cdot \pi}{2 \times \cancel{5} \times 4 \times \cancel{3} \times 2} = \frac{63\pi}{16}$

8) $\int_5^9 \sqrt[5]{(9-x)(x-5)}\,.\, dx$

$x-5 = 4t \quad \Rightarrow \quad x = 5+4t$

$dx = 4dt$

x	5	9
t	0	1

Integrant $\sqrt[5]{(9-x)(x-5)}$

$\sqrt[5]{(9-5-4t)(\cancel{5}+4t-\cancel{5})}$

$\sqrt[5]{(4-4t)(4t)}$

$\sqrt[5]{4(1-t)(4t)}$

$(16)^{1/5}\,(1-t)^{1/5}\,(t)^{1/5}$

$\int_0^1 (16)^{1/5}(1-t)^{1/5}\,(t)^{1/5}\,.\,4dt$

$(4)(16)^{1/5}\int_0^1 t^{1/5}\,(1-t)^{1/5}\,.\,dt$

$m = 1/5 \quad n = 1/5$

$4\times(16)^{1/5}\,\beta\left(\frac{6}{5},\frac{6}{5}\right)$

$(4)^{7/5}\,\frac{\Gamma\frac{6}{5}\,\Gamma\frac{6}{5}}{\Gamma\frac{12}{5}}$

$(4)^{7/5}\,\frac{\frac{1}{5}\Gamma\frac{1}{5}\quad\frac{1}{5}\Gamma\frac{1}{5}}{\frac{7}{5}\times\frac{2}{5}\Gamma\frac{2}{5}}$

$(4)^{7/5}\cdot\frac{\frac{1}{5}\cdot\frac{1}{5}\Gamma\frac{1}{5}\,\Gamma\frac{1}{5}}{\frac{7}{5}\times\frac{2}{5}\Gamma\frac{2}{5}}$

$\Gamma\frac{1}{2}$
2

$\frac{63\pi}{32}$

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9) $\int_4^6 (6-x)^7 (x-4)^{3/7} \cdot dx$

let $x-4 = 2t \Rightarrow x = 4+2t$.
$dx = 2dt$

x	4	6
t	0	1

Integrand $\int_4^6 (6-x)^7 (x-4)^{3/7}$.

$= (6-4-2t)^7 (\cancel{4}+2t-\cancel{4})^{3/7}$

$= (2-2t)^7 (2t)^{3/7}$

$= 2^7(1-t)^7 (2t)^{3/7}$

$= 2^7 (1-t)^7 (2)^{3/7} \cdot t^{3/7}$

$= 2^{52/7} (1-t)^7 \cdot (t)^{3/7} \cdot 2\,dt$

$\frac{7\times 7+3}{7}$ $= \int_0^1 2^{59/7} (1-t)^7 (t)^{3/7} \cdot dt$

$\frac{49+3}{7}$ $= 2^{59/7} \int_0^1 t^{3/7} (1-t)^7 \cdot dt$

$\frac{52}{7}+1$ $m = \frac{3}{7}$ $n = 7$.

$\frac{52+7}{7}$ $= 2^{59/7} \beta\left(\frac{10}{7}, 8\right)$

$59/7$ $= 2^{59/7} \frac{\Gamma 10/7 \, \Gamma 8}{\Gamma \frac{66}{7}} = 2^{59/7} \frac{\frac{3}{7}\Gamma\frac{3}{7} \; 7!}{\Gamma\frac{66}{7}}$

$\int_0^a f(x)\cdot dx = \int_0^a f(a-x)\cdot dx$, $\int_0^{2a} f(x)\cdot dx = \int_0^a f(x)\cdot dx + \int_0^a f(2a-x)\cdot dx$

Alg x trigo ↑ , (This requires)

10)

$I = \int_0^{\pi} x\cdot \cos^6 x\cdot \sin^4 x\cdot dx$

$\sin(\pi-\theta) = \sin\theta$

$\cos(\pi-\theta) = -\cos\theta$

$= \int_0^{\pi} (\pi-x)\cos^6(\pi-x)\cdot \sin^4(\pi-x)\cdot dx$

$= \int_0^{\pi} (\pi-x)\left[\cos(\pi-x)\right]^6 \left[\sin(\pi-x)\right]^4\cdot dx$

$= \int_0^{\pi} (\pi-x)\left[(-\cos x)^6\right]\left[\sin x\right]^4\cdot dx$

$= \int_0^{\pi} (\pi-x)\cos^6 x\cdot \sin^4 x\cdot dx$

$= \int_0^{\pi} \pi\cdot\cos^6 x\cdot \sin^4 x - x\cos^6 x\cdot\sin^4 x\cdot dx$

$I = \int_0^{\pi} \pi\cdot\cos^6 x\cdot\sin^4 x\cdot dx - \int_0^{\pi} x\cos^6 x\cdot\sin^4 x\cdot dx$

$I = \int_0^{\pi} \pi\cdot\cos^6 x\cdot\sin^4 x\cdot dx - I$

$2I = \pi\int_0^{\pi} \cos^6 x\cdot\sin^4 x\cdot dx.$

Second formula.

$2I = \pi\left[\int_0^{\pi/2} \cos^6 x\cdot\sin^4 x\cdot dx + \int_0^{\pi/2} (\cos(\pi-x))^6(\sin(\pi-x))^4\cdot dx\right.$

$2I = \pi\left[\int_0^{\pi/2} \cos^6 x\cdot \sin^4 x\cdot dx\right\} + \int_0^{\pi/2} (-\cos x)^6(\sin x)^4\cdot dx$

$$= \pi \left[\int_0^{\pi/2} \cos^6 x \cdot \sin^4 x \cdot dx + \int_0^{\pi/2} (\cos^6 x)(\sin^4 x) \cdot dx \right]$$

$$\cancel{2} I = \cancel{2}\pi \int_0^{\pi/2} \cos^6 x \cdot \sin^4 x \cdot dx$$

$$I = \pi \left[\int_0^{\pi/2} \cos^6 x \cdot \sin^4 x \cdot dx \right]$$

$$m = 6 \quad , \quad n = 4$$

$$\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$I = \pi \frac{1}{2}\beta\left(\frac{6+1}{2}, \frac{4+1}{2}\right)$$

$$I = \pi \frac{\Gamma\frac{7}{2}\,\Gamma\frac{5}{2}}{\Gamma\frac{\cancel{12}\,6}{\cancel{2}}}$$

$$I = \pi \times \frac{1}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\,\frac{\Gamma\frac{1}{2} \cdot \frac{3}{2} \times \frac{1}{2}\,\Gamma\frac{1}{2}}{5 \times 4 \times 3 \times 2 \times 1}$$

$\frac{3}{\cancel{128}}$ $\frac{2}{256}$

$$\cancel{I = \pi \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \frac{3}{2} \times \frac{1}{2} \times 5 \times 4 \times 3 \times 2 \times \pi}$$

$$I = \frac{1}{2}\pi \times \frac{\cancel{5}}{2} \times \frac{\cancel{3}}{\cancel{2}} \times \frac{1}{2} \times \pi \times \frac{3}{2} \times \frac{1}{2} \times \frac{1}{\cancel{5} \times 4 \times \cancel{3} \times 2}$$

$$I = \pi^2 \frac{3}{256} \times \frac{1}{2} = \frac{3\pi^2}{512}$$

11) $$\int_{-\pi/3}^{\pi/6} (\sin x + \sqrt{3}\cos x)^{1/6} \cdot dx$$

Integrant : $\sin x + \sqrt{3}\cos x$

$a=1,\ b=\sqrt{3}$ (coefficient)

$\sqrt{a^2+b^2} = \sqrt{4} = 2$

Multiply and Divide by 2

$2\left(\sin x \cdot \frac{1}{2} + \frac{\sqrt{3}}{2}\cos x\right)$

$2\left(\sin x \cdot \cos\frac{\pi}{3} + \cos x \cdot \sin\frac{\pi}{3}\right)$

$= 2\left(\sin\left(x+\frac{\pi}{3}\right)\right)$

second way

option (2)

$2\left(\sin x \sin\frac{\pi}{6} + \cos x \cos\frac{\pi}{6}\right)$

$= 2\left(\cos\left(x - \frac{\pi}{6}\right)\right)$

$$= \int_{-\pi/3}^{\pi/6} \left(2\sin\left(x+\frac{\pi}{3}\right)\right)^{1/6} \cdot dx$$

let $x + \frac{\pi}{3} = \theta \Rightarrow x = \theta - \frac{\pi}{3}$

$\cancel{x = \theta}$

$dx = d\theta$

x	$-\pi/3$	$\pi/6$
θ	0	$\pi/2$

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$$\int_0^{\pi/2} \left(2\sin(\theta)\right)^{1/6} \cdot d\theta$$

$$2^{1/6} \int_0^{\pi/2} (\sin\theta)^{1/6} \cdot d\theta$$

$m = \frac{1}{6}$ $\quad n = 0$

$$I = 2^{1/6} \times \frac{1}{2}\, \beta\left(\frac{7}{12}, \frac{1}{2}\right)$$

$$= 2^{1/6} \times 2^{-1} \, \frac{\Gamma\frac{7}{12}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{7}{12} + \frac{1\times 6}{2\times 6}\right)}$$

$\frac{1}{6} - 1$

$\frac{1-6}{6}$

$-\frac{5}{6}$

$$= 2^{-5/6} \frac{\Gamma\frac{7}{12}\,\Gamma\frac{1}{2}}{\Gamma\frac{13}{12}} = 2^{-5/6} \frac{\Gamma\frac{7}{12}\,\Gamma\frac{1}{2}}{\frac{1}{12}\Gamma\frac{1}{12}}$$

12) $$\int_{-\pi/4}^{\pi/4} (\cos x + \sin x)^7 \cdot dx$$

$a = 1$, $b = 1$

$\sqrt{a^2 + b^2} = \sqrt{2}$

Multiply and divide by $\sqrt{2}$

$\sqrt{2}\ \sqrt{2}\ \sqrt{2}\ \sqrt{2}$
$\sqrt{2}\ \sqrt{2}\ \sqrt{2}$
$2 \quad 2 \quad 2\sqrt{2}$

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Integrat $= \sqrt{2}\cos x + \sqrt{2}\sin x$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$

$= \sqrt{2}\left(\sin\frac{\pi}{4}\cos x + \sin x \cdot \cos\frac{\pi}{4}\right)$

$= \sqrt{2}\left(\sin\left(x + \frac{\pi}{4}\right)\right)$

$$\int_{-\pi/4}^{\pi/4} \left(\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right)^7 \cdot dx$$

$$\int_{-\pi/4}^{\pi/4} (\sqrt{2})^7 \left(\sin\left(x+\frac{\pi}{4}\right)\right)^7 \cdot dx$$

$$(\sqrt{2})^7 \int_{-\pi/4}^{\pi/4} \sin(x+\pi/4)^7 \cdot dx$$

let $x + \frac{\pi}{4} = \theta$

$dx = d\theta$

x	$-\pi/4$	$\pi/4$
θ	0	$\pi/2$

$\frac{\pi}{2}$

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$$(\sqrt{2})^7 \int_0^{\pi/2} (\sin\theta)^7 \cdot du$$

$m=7 \quad n=0$

$$(\sqrt{2})^7 \; \beta\left(\frac{\cancel{8}^{4}}{\cancel{2}}, \frac{1}{2}\right)$$

$$(\sqrt{2})^7 \frac{1}{2} \frac{\Gamma 4 \; \Gamma\frac{1}{2}}{\Gamma\frac{\cancel{4}\times 2+1}{\cancel{2}\,2}\frac{1}{2}}$$

$$(\sqrt{2})^7 \frac{1}{2} \frac{3! \; \sqrt{\pi}}{\Gamma\frac{9}{2}}$$

$$(\sqrt{2})^7 \times \frac{1}{\cancel{2}} \times \frac{3\cancel{!}\times\cancel{2}\times 1\;\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\frac{1}{2}}$$

$$\frac{(\sqrt{2})^7 \times \cancel{3} \times \cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}} \times \frac{2}{7} \times \frac{2}{5} \times \frac{2}{\cancel{3}} \times \frac{2}{1}$$

$$(\sqrt{2})^7 \frac{\times 2\times 2\times 2\times 2}{35}$$

$$\frac{\sqrt{2}\times\sqrt{2}\times\sqrt{2}\times\sqrt{2}\times\sqrt{2}\times\sqrt{2}\times\sqrt{2}\times 2\times 2\times 2\times 2}{35}$$

$$\frac{\sqrt{2}\times 2\times 2\times 2\times 16}{35} = \frac{128\sqrt{2}}{35}$$

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⇒ Differentiating under a integral sign with Constant limits of integration

Steps:-

differentiate w.r.t. 'a'

1) $\frac{d}{d(a)} \int f(x)\, dx = \int \frac{\partial}{\partial a} f(x) \cdot dx$

Parameters.

2) Integrate w.r.t. a

3) find Constant of integration

Example:-

1) Prove that $\int_0^\infty \frac{\log(1+ax^2)}{x^2} \cdot dx = \pi\sqrt{a}$, $a \geq 0$

let $I(a) = \int_0^\infty \frac{\log(1+ax^2)}{x^2} \cdot dx$

Differentiating w.r.t. 'a'

$$\frac{d}{d(a)} I(a) = \frac{d}{da} \int_0^\infty \frac{\log(1+ax^2)}{x^2} \cdot dx$$

$$= \int_0^\infty \frac{\partial}{\partial a} \left(\frac{\log(1+ax^2)}{x^2} \right) \cdot dx$$

$$= \int_0^\infty \frac{1}{\cancel{x^2}} \times \left(\frac{1}{(1+ax^2)} \cdot \cancel{x^2} \right) dx$$

$$= \int_0^\infty \frac{1}{(1+ax^2)} \cdot dx$$

$$= \int_0^{\infty} \frac{1}{(1) + (\sqrt{a}x)^2} \cdot dx$$

$$= \left[\tan^{-1}(\sqrt{a}\,x) \times \frac{1}{\sqrt{a}} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{a}} \left[\frac{\tan^{-1}(\cancel{\sqrt{a}}\sqrt{a} \times \infty)}{\cancel{\sqrt{a}}} - \frac{\tan^{-1}(0 \times \cancel{\sqrt{a}})}{\cancel{\sqrt{a}}} \right]$$

$$= \cancel{\frac{\pi}{2\sqrt{a}} - \frac{1 \times \sqrt{a} \times 2}{\sqrt{a} \times \sqrt{a} \times 2}}$$

$$= \cancel{\frac{\pi - \sqrt{a}\,2}{2\sqrt{a}}} \quad \frac{\pi}{2\sqrt{a}} - \frac{1}{\sqrt{a}}$$

~~Integrate w.r.t. a~~

$$= \frac{1}{\sqrt{a}} \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{\pi}{2\sqrt{a}}$$

Integrate w.r.t. 'a'

$$\int \frac{d}{da} I(a) \cdot da = \int \frac{\pi}{2\sqrt{a}} \cdot da$$

$$I(a) = \frac{\pi}{2} \int (a)^{-1/2} \cdot da$$

$$I(a) = \frac{\pi}{2} \frac{\left(a^{-1/2+1}\right)}{\left(\frac{-1}{2}+1\right)} + c$$

$$I(a) = \frac{\pi}{\cancel{2}} \frac{(a^{1/2})}{\frac{1}{\cancel{2}}} = \pi\sqrt{a} + c$$

let $a = 0$

$$I(0) = \pi\sqrt{0} + c$$

$$I(0) = 0 + c$$

$$c = I(0)$$

2) Prove that $\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x} \cdot dx = \pi \sin^{-1}(a)$

~~$a = 0$ to 1~~

$0 \leq a \leq 1$

let $I(a) = \int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}$

Differentiating w.r.t. 'a'

$$\frac{d}{d(a)} I(a) = \frac{d}{da} \int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x} \cdot dx$$

$$= \int_0^{\pi} \frac{\partial}{\partial a} \left(\frac{\log(1+a\cos x)}{\cos x} \right) \cdot dx$$

$$= \int_0^{\pi} \frac{1}{\cancel{\cos x}} \times \frac{1}{1+a\cos x} \times \cancel{\cos x} \cdot dx$$

$$= \int_0^{\pi} \frac{1}{1+a\cos x} \cdot dx$$

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$$= \int_0^{\pi} \frac{1}{1 + a\cos x} \cdot dx$$

let $x = 2\tan^{-1} t$

$$dx = \frac{2\,dt}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

x	0	π
t	0	∞

$$= \int_0^{\infty} \frac{1}{\frac{1+t^2}{\cancel{(1+t^2)}} + a\left(\frac{1-t^2}{\cancel{1+t^2}}\right)} \cdot \frac{2\,dt}{\cancel{(1+t^2)}}$$

$$= \int_0^{\infty} \frac{2}{t^2+1+a-at^2}\, dt$$

$$= \int_0^{\infty} \frac{2}{t^2(1-a)+(1+a)}\, dt$$

$$= 2\int_0^{\infty} \frac{dt}{(At)^2+(B)^2}$$

$$= A^2 = 1-a \qquad B^2 = (1+a)$$

$$= 2\left[\frac{1}{A B} \tan^{-1}\left(\frac{At}{B}\right)\right]_0^{\infty}$$

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$$= 2\left[\frac{1}{\sqrt{1-a}\times\sqrt{1+a}}\tan^{-1}\left(\frac{\sqrt{1-a}\times\sqrt{t}}{\sqrt{1+a}}\right)\right]_0^\infty$$

$$= \cancel{2}\left[\frac{1}{(\sqrt{1-a})(\sqrt{1+a})}\times\frac{\pi}{\cancel{2}}\right]$$

$$\frac{d}{da}I(a) = \frac{\pi}{\sqrt{1-a}(\sqrt{1+a})} = \frac{\pi}{\sqrt{1-a^2}}$$

Integrate w.r.t. 'a'

$$\int \frac{d}{da} I(a)\,.\,da = \int \frac{\pi}{\sqrt{1-a^2}}\,.\,da$$

$$= \cancel{\pi\int(1-a^2)^{-1/2}\,da}$$

$$= \cancel{\pi\frac{(1-a^2)^{-1/2+1}}{-\frac{1}{2}+1}\,da}$$

$$I(a) = \pi \sin^{-1}(a) + C$$

let $a = 0$.

$$I(0) = C$$

$$C = 0$$

If the D.E is non Exact then.

(1) If $M\,dx + N\,dy = 0$ is homogenous the I.F $= \dfrac{1}{Mx+Ny}$

(2) If $M\,dx + N\,dy = 0$ is of form $x f_1(xy)\,dx + y f_2(xy)\,dy = 0$

then I.F $= \dfrac{1}{Mx - Ny}$

(3) If $M\,dx + N\,dy = 0$ is more of the above two

$$\boxed{\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}} \quad \text{or} \quad \dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$$

= function of 'y' $\qquad$ = function of 'x'

$I.F = e^{\int f(y)\,dy}$ $\qquad$ $I.F = e^{\int f(x)\,dx}$

Example:-

1) $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

Comparing with $M\,dx + N\,dy = 0$

$M = 3x^2y^4 + 2xy \qquad N = 2x^3y^3 - x^2$

$\dfrac{\partial M}{\partial y} = 12x^2y^3 + 2x \qquad \dfrac{\partial N}{\partial x} = 6x^2y^3 - 2x$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ D.E is not Exact.

$$\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \dfrac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{3x^2y^4 + 2xy} = \dfrac{-6x^2y^3 - 4x}{3x^2y^4 + 2xy}$$

$$= \dfrac{-2\cancel{x}(3xy^3 + 2)}{\cancel{x}y(3xy^3 + 2)} = -\dfrac{2}{y}$$

$$I.F = e^{\int -\frac{2}{y}\, dy}$$

$$= e^{-2\int \frac{1}{y}\, dy}$$

$$= e^{-2\log y}$$

$$= e^{\log y^{-2}}$$

$$I.F = y^{-2}$$

Multiply by y^{-2}

$$y^{-2}\left[(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy\right] = 0$$

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 6x^2y - \frac{2x}{y^2}, \quad \frac{\partial N}{\partial x} = 6x^2y - \frac{2x}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{D.E is exact.}$$

The solution is given as

$$\int M\,dx + \int N\,dy = c$$

$$\cancel{3}y^2\frac{x^3}{\cancel{3}} + \frac{\cancel{2}x^2}{\cancel{2}y} + \int 0 = c$$

$$x^3y^2 + \frac{x^2}{y} = c$$

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2) $(x^4e^x - 2xy^2)\,dx + (2x^2y)\,dy = 0$

Comparing w.r.t

$M\,dx + N\,dy = 0$

$M = x^4e^x - 2xy^2 \qquad N = 2x^2y$

$\frac{\partial M}{\partial y} = -4xy \qquad \frac{\partial N}{\partial x} = 4xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ D.E is Not exact.

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4xy - 4xy}{2x^2y} = \frac{-\cancel{8}^{4}\cancel{x}\cancel{y}}{\cancel{2}x^{\cancel{2}}\cancel{y}}$

$= \frac{-4}{x}$

$I.F = e^{\int -\frac{4}{x}\,dx}$

$= e^{-4\log x}$

$= e^{\log x^{-4}}$

$= x^{-4}$

Multipy by x^{-4}

$x^{-4}\left[(x^4e^x - 2xy^2)\,dx + (2x^2y)\,dy\right] = 0$

$= \left(\frac{\cancel{x^4}e^x}{\cancel{x^4}} - \frac{2\cancel{x}y^2}{x^{\cancel{4}3}}\right)dx + \left(\frac{2\cancel{x^2}y}{x^{\cancel{4}2}}\right)dy = 0$

$= \left(e^x - \frac{2y^2}{x^3}\right)dx + \left(\frac{2y}{x^2}\right)dy = 0$

x^{-3+1} $\frac{x^{-2}}{-2}$

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$\frac{\partial M}{\partial y} = 0 - \frac{4y}{x^3}$, $\frac{\partial N}{\partial x} = \cancel{\frac{2y}{x^3}} \; -\frac{4y}{x^3}$

D.E is Exact

The Solution is given as:-

$\frac{x^{-2}}{-2}$

$$\int M \cdot dx + \int N \cdot dy = c$$

$\frac{x^{-3+1}}{-3+1}$ $\int e^x - \frac{2y^2}{x^3} + 0 = c$

$2y\left(\frac{x^{-2}}{-2}\right)$

$$e^x - \frac{\cancel{2}y^2}{-\cancel{2}x^2} = c$$

$$c = e^x + \frac{y^2}{x^2}$$

$$\boxed{cx^2 = x^2e^x + y^2}$$

3) $(x^4 + y^4)\,dx - (xy^3)\,dy = 0$

$M\,dx - N\,dy = 0$

Comparing w.r.t. $M\,dx + N\,dy = 0$

$M = x^4 + y^4$ $\quad N = -xy^3$

$\frac{\partial M}{\partial y} = 4y^3$ $\quad \frac{\partial N}{\partial x} = -y^3$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y^3 + y^3}{-xy^3} = \frac{5\cancel{y^3}}{-x\cancel{y^3}} = \frac{-5}{x}$$

$$I.F = e^{\int \frac{-5}{x} \cdot dx} = x^{-5}$$

$$x^{-5}(x^4 + y^4) \cdot dx - x^{-5}(xy^3)\,dy = 0$$

$$\left(\frac{\cancel{x^4}}{x^{\cancel{5}}} + \frac{y^4}{x^5}\right)dx - \left(\frac{\cancel{x}y^3}{x^{\cancel{5}4}}\right)dy = 0$$

$$\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx - \left(\frac{y^3}{x^4}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 0 + \frac{4y^3}{x^5} \qquad \frac{\partial N}{\partial x} = \frac{4y^3}{x^5}$$

$$\int \frac{1}{x} + \frac{y^4}{x^5} \cdot dx + \int 0 = c$$

$$\log x - \frac{y^4}{4x^4} = c$$

$$4x^4 \log x - y^4 = 4x^4 c$$

$\frac{x^{-4+1}}{-4+1}$

$\frac{x^{-5+1}}{-5+1}$

$\frac{x^{-4}}{-4}$

Iner integral $\Rightarrow$ Gives region.
Outer integral $\Rightarrow$ Gives boundary.

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5) Change to polar Co-ordinates and Evaluate

$$\int_0^2\int_y^{2+\sqrt{2^2-y^2}} \left(\frac{1}{2^2+x^2+y^2}\right)^2 \cdot dxdy$$

let, $x = r\cos\theta$
$y = r\sin\theta$

$dxdy = |J|\, drd\theta$

$dxdy = r \cdot drd\theta$

Integrant $\Rightarrow$

$$\left(\frac{1}{(2)^2+r^2\cos^2\theta+r^2\sin^2\theta}\right)^2 = \left(\frac{1}{4^2+r^2}\right)^2$$

Intermost integral depends on 'y' (region)

limits. x.

$x = y$ to $x = 2+\sqrt{2^2-y^2}$

$\cancel{r}\cos\theta = \cancel{r}\sin\theta$
$1 = \tan\theta$
$\theta = \tan^{-1}(1)$
$$\boxed{\theta = \frac{\pi}{4}}$$

$x - 2 = \sqrt{2^2-y^2}$
Squaring both side.
$(x-2)^2 = (2)^2 - y^2$
$x^2 - 4x + \cancel{4} = \cancel{4} - y^2$

$x^2+y^2 = 4x$
$r^{\cancel{2}} = 4\cos\theta \times \cancel{r}$
$$\boxed{r = 4\cos\theta}$$

limits of y.

$y = 0$ to $y = 2$.

$C \equiv (2,0)$. radius $= 2$

$$= -\frac{1}{32}\left[\frac{dt}{t^2+(\sqrt{2})^2} - \frac{\pi}{4}\right]$$

$$= -\frac{1}{32}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan\theta}{\sqrt{2}}\right)\right]_0^{\pi/4} - \frac{\pi}{4}$$

$$= -\frac{1}{32}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{4}\right]$$

6) Change to polar Co-ordinate and evaluate

$$\int_0^1\int_x^{\sqrt{2x-x^2}} x^2+y^2 \, dx\,dy$$

let $x = r\cos\theta$ $\quad$ x^2+y^2

$y = r\sin\theta$ $\quad$ $= r^2\cos^2 + r^2\sin^2\theta$

$= r^2$

$dx\,dy = |J|\, dr\,d\theta$.

$dx\,dy = r . dr . d\theta$

$\cancel{\int_0 \int_x x^2+y^2 . r . dr\,d\theta}$

Innermost integral depends on x (Region)

limits of y

$y = x$ to $y = \sqrt{2x-x^2}$

line

$\cancel{r}\sin\theta = \cancel{r}\cos\theta$ $\quad$ $y^2 = 2x-x^2$ / $x^2+y^2 = 2x$

$\tan\theta = 1$ $\quad$ $r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$.

$\theta = \tan^{-1}(1)$ $\quad$ $r^{\cancel{2}} = 2\cancel{r}\cos\theta$

$\boxed{\theta = \frac{\pi}{4}}$ $\quad$ $\boxed{r = 2\cos\theta}$

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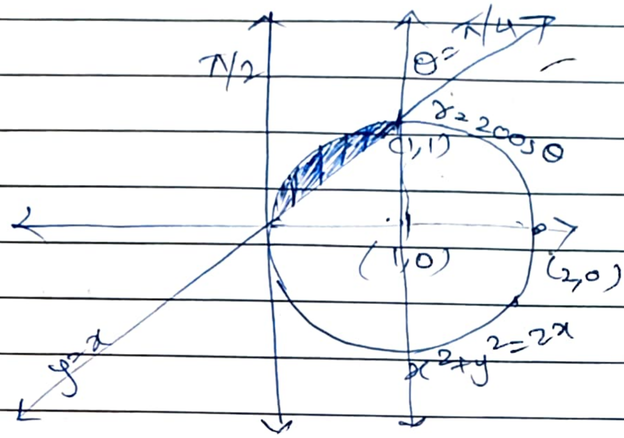
Outer integral (Boundary)

limits of x

$x=0$ to $x=1$



// To get point of intersection

$y=x$, $x^2+y^2=2x$

$x^2+x^2=2x$

~~$2x^2=2x$~~

$x=1\pm R$

$\cancel{2}x^2=\cancel{2}x$

$x^2=x$

$x^2-x=0$

$x(x-1)=0$

$\underline{\underline{x=0}}$ or $\underline{\underline{x=1}}$

limits of r and θ

$r=0$ to $r=2\cos\theta$

$\theta=\pi/4$ to $\theta=\dfrac{\pi}{2}$

$$\int_{\pi/4}^{\pi/2}\int_{0}^{2\cos\theta} r^2\cdot r\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\int_{0}^{2\cos\theta} r^3\cdot dr\right] d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\frac{r^4}{4}\right]_{0}^{2\cos\theta} d\theta$$

$$= \int_{\pi/4}^{\pi/2}\frac{(2\cos\theta)^4}{4}\cdot d\theta$$

$$= \frac{(2)^4}{4}\int_{\pi/4}^{\pi/2}(\cos\theta)^4\cdot d\theta$$

$$\cancel{= \frac{16}{4}\left[\frac{\cos\theta^5}{5}\right]_{\pi/4}^{\pi/2}}$$

$$\cancel{= \frac{4}{5}\left[\cos\frac{\pi}{2}\right]^5 - \left(\cos\frac{\pi}{4}\right)^5}$$

$$= 4\int_{\pi/4}^{\pi/2}(\cos^2\theta)^2\cdot d\theta$$

$$= 4\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2\cdot d\theta$$

$$= \frac{\cancel{4}^{1}}{\cancel{4}}\int_{\pi/4}^{\pi/2}(1+\cos 2\theta)^2\cdot d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left(1+2\cos 2\theta+(\cos 2\theta)^2\right)$$

$$= \left[\theta + \frac{\cancel{2}\sin 2\theta}{\cancel{2}}\right]_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2}(\cos 2\theta)^2$$

$$= \frac{\pi}{2} + \sin\pi - \frac{\pi}{4} - \sin\frac{\pi}{2}$$

$$\int_{\pi/4}^{\pi/2}\frac{1+\cos 4\theta}{2}\cdot d\theta$$

$$= \left[\frac{\pi}{4}-1\right] + \left[\frac{\theta}{2}+\frac{\sin 4\theta}{8}\right]_{\pi/4}^{\pi/2}$$

$$= \left[\frac{\pi}{4}-1\right] + \left[\frac{\pi}{4}+\frac{\sin\times\cancel{4}^{2}\times\frac{\pi}{\cancel{2}}}{8}\right] - \left[\frac{\pi}{8}+\frac{\sin 4\times\frac{\pi}{\cancel{4}}}{8}\right]$$

$$= \frac{\pi}{4} - 1 + \frac{\pi}{4} + \frac{\sin 2\pi}{8} - \frac{\pi}{8} - \frac{\sin\pi}{8} \quad (\sin\pi = 0)$$

$$= \frac{3\pi}{8} - 1$$

$$= \frac{3\pi-8}{8}$$

$\frac{\pi}{4}+\frac{\pi}{4}$

$\frac{2\pi}{4\times2}$

$-\frac{\pi}{8\times2}$

$\frac{8\pi-2\pi}{16}$

$\frac{3\cancel{6}\pi}{\cancel{16}8}$

(J.T) Double integration over a region ✓
hot polar co-ordinates ✗

3) a) Sperical and elipsoidal by cylindrical

$$\Rightarrow \int e^{ax}(af(x)) + f'(x)\,dx = e^{ax} f(x)$$

NOT IN Syllabus

$$\frac{1}{D-a} = e^{ax}\int e^{-ax}, \quad \frac{1}{D+a} = e^{-ax}\int e^{ax}$$

1) $(D^2-1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$ (~~method of Variation of parameters~~)

Solution: $f(D)=0$ (By Definition)

$$(D^2-1)=0$$
$$D^2=1$$
$$D=\pm 1$$

The C.F is

$$y_c = c_1e^{x} + c_2e^{-x}$$

$\cancel{y_c = c_1y_1 + c_2y_2}$

$\cancel{y_1 = e^{x}, \; y_2 = e^{-x}}$

By Definition

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -e^{x}\cdot e^{-x} - e^{x}\cdot e^{-x} = -e^{0} - e^{0} = -2$$

Particular integration

$$y_p = \frac{1}{f(D)} \times RHS$$

$$y_p = \frac{1}{(D^2-1)} \times (e^{-x}\sin(e^{-x}) + \cos(e^{-x}))$$

$$= \frac{1}{(D+1)(D-1)} \times \left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]$$ (Composite function)

$$\frac{1}{D-a} = e^{ax}\int e^{-ax} \quad \longleftarrow \begin{cases} \text{composite} \\ \text{log} \\ \text{inverse} \end{cases} \int \text{in R.H.S}$$

$$\frac{1}{D+a} = e^{-ax}\int e^{ax}$$

$$y_p = \frac{1}{(D-1)}\left[\frac{1}{(D+1)}\left(e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right)\right]$$

$$y_p = \frac{1}{(D-1)}\left[e^{-x}\int e^{x}\Big(\underbrace{e^{-x}\sin(e^{-x})}_{f'(x)} + \underbrace{\cos(e^{-x})}_{f(x)}\Big)\right] \cdot dx$$

from back formula

$$y_p = \frac{1}{(D-1)}\left[e^{x}\cos(e^{-x})\right]$$

$$y_p = \frac{1}{(D-1)}\left[\cos(e^{-x})\right]$$

$$y_p = e^{x}\int e^{-x}(\cos e^{-x}) \cdot dx$$

let $e^{-x} = t$

$$-e^{-x}\,dx = dt$$
$$e^{-x}\,dx = -dt$$

$$y_p = e^{x}\int -\cos(t) \cdot dt$$

$$y_p = -e^{x}\sin t + c$$

$$y_p = -e^{x}\sin(e^{-x}) + c$$

By Definition Method

2) $(D^2-D-2)y = \left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$

$f(D)=0$

$(D^2-D-2)=0$

$D = 2, -1$

$y_c = C_1 e^{2x} + C_2 e^{-x}$

Particular Integration $= \frac{1}{f(D)} \times R.H.S$

$= \frac{1}{(D-2)(D+1)} \times (2\log x + x^{-1} + x^{-2})$

$= \frac{1}{3}\left[\frac{1}{D-2} - \frac{1}{D+1}\right](2\log x + x^{-1} + x^{-2})$

$= \frac{1}{3}\left[\frac{1}{(D-2)} \times (2\log x + x^{-1} + x^{-2}) - \frac{1}{(D+1)}(2\log x + x^{-1} + x^{-2})\right]$

$= \frac{1}{3}\left[e^{2x}\int e^{-2x}(2\log x + x^{-1} + x^{-2}) - e^{-x}\int e^{x}(2\log x + x^{-1} + x^{-2})\right]dx$

$= \frac{1}{3}\left[e^{2x}\int e^{-2x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right) - e^{-x}\int e^{x}\left(2\log x + \frac{2}{x} - \frac{1}{x} + \frac{1}{x^2}\right)\right]$

$= \frac{1}{3}\left[e^{2x}\int e^{-2x}\left[2\log x + \frac{2}{x} - \frac{1}{x} + \frac{1}{x^2}\right] - e^{-x}\int e^{x}\left(2\log x - \frac{1}{x}\right) + \left(\frac{2}{x} + \frac{1}{x^2}\right)\right]$

$= \frac{1}{3}\left[\frac{e^{2x}}{(-1)}\int e^{-2x}\left(-2\left(\log x + \frac{1}{x}\right) + \left(\frac{1}{x} + \frac{1}{x^2}\right)\right) - e^{-x}\left(e^{x}\left(2\log x - \frac{1}{x}\right)\right)\right] + C$

$= \frac{1}{3}\left[e^{2x} \cdot e^{-2x}\left(-\left(\log x + \frac{1}{x}\right)\right) - e^{-x}\left(e^{x}\left(2\log x - \frac{1}{x}\right) + C\right)\right]$

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Method of Variation of Parameter

1) $(D^2+1)y = \dfrac{1}{1+\sin x}$ 2) $(D^2+6D+9)y = \dfrac{e^{3x}}{x^2}$

Consider,

$f(D)=0$

$D^2+1=0$

$D = +i, -i$

$(D+i)(D-i)$

$y_c = \cancel{c_1e^{x}+c_2e^{-x}}\ \cancel{(c_1+xc_2)}\ e^{0x}(c_1\cos x + c_2\sin x)$

$y_c = c_1\cos x + c_2\sin x$

$y_c = c_1y_1 + c_2y_2$

$y_1 = \cos x \quad , \quad y_2 = \sin x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$u = -\int \dfrac{y_2\, R.H.S}{W}$

$u = -\int \dfrac{\sin x}{1} \times \dfrac{1}{1+\sin x}$

$u = -\int \dfrac{\sin x}{1+\sin x} \times \dfrac{1-\sin x}{1-\sin x}$

$u = -\int \dfrac{\sin x - \sin^2 x}{\cos^2 x}$

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$u = -\int \tan x \cdot \sec x - \tan^2 x \cdot dx$

$u = -\int \tan x \cdot \sec x \cdot dx - \int (\sec^2 x - 1) \cdot dx$

$u = -\int \tan x \cdot \sec x \cdot dx - \int \sec^2 x \cdot dx + \int 1 \cdot dx$

$u = -\int \tan x \cdot \sec x \cdot dx - \tan x + x]$

$u = -[\sec x - \tan + x] + k_1$

$u = -\sec x + \tan x - x - k_1$

$V = \int \frac{y_1}{W} RHS$

$V = \int \cos x \times \frac{1}{(1+\sin x)} \cdot dx$

$V = \int \frac{\cos x}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)} \cdot dx$

$V = \int \frac{\cancel{\cos x}(1-\sin x)}{\cos^{\cancel{2}} x}$

$V = \int \sec x - \tan x \cdot dx.$

$V = \log|\sec x$

$V = \int \frac{y_1}{W} \times RHS$

$V = \int \frac{\cos x}{1+\sin x} \cdot dx$

let $1+\sin x = t$.

$V = \log|1+\sin x| + k_2$

$y_p = uy_1 + vy_2$

$y_p = [-\sec x + \tan x - x - k_1] \cos x + [\log|1+\sin x| + k_2] \sin x$

$y_p = -\cancel{\sec x} \cdot \cancel{\cos x} + \tan x \sin x \cos x - x\cos x - k_1 \cos x + \sin x \cdot \log|1+\sin x| + k_2 \sin x$

$y_p = -1 + \sin x - x\cos x - (k_1 - k_2)(\cos x - \sin x) + \sin x \log|1+\sin x|$

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• 4 products are available/possible.

1) exponential and trigo.
2) expo and algebric
3) algebric and trigo
4) exp. and Alg and trigo

2) $(D^2 - 6D + 9)\, y = \dfrac{e^{3x}}{x^2}$

$D^2 - 6D + 9 = 0$

$D = +3, +3$

complex and repeated

$y_c = c_1 e^{3x} + x c_2 e^{+3x}$

$y_1 = e^{+3x}, \; x e^{+3x}$

$$W = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & e^{3x} + 3x e^{3x} \end{vmatrix}$$

$W = e^{3x}(e^{3x} + 3x e^{3x}) - x e^{3x} \times (3e^{3x})$

$W = e^{6x} + \cancel{3x e^{6x}} - \cancel{3x e^{6x}}$

$\underline{W = e^{6x}}$

$u = -\int \dfrac{y_2 \times RHS}{W}$

$u = -\int \dfrac{x e^{3x}}{\cancel{e^{6x}}} \times \dfrac{e^{3x}}{x^2} \, dx$

$u = -\int \dfrac{1}{x} \, dx$

$u = -\log x + K_1$

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$$V = -\int \frac{y_1 \times RHS}{W}$$

$$V = -\int \frac{\cancel{e^{3x}}}{\cancel{e^{6x}}} \times \frac{\cancel{e^{3x}}}{x^2} \, dx$$

$$V = -\int \frac{1}{x^2} \, dx$$

$$V = -\frac{1}{x} + K_2$$

$$y_p = uy_1 + vy_2$$

$$y_p = \left(-\log|x| + K_1\right)e^{3x} + \left(-\frac{1}{x} + K_2\right)e^{3x}$$

$$y_p = -e^{3x}\log|x| + K_1 e^{3x} - \frac{e^{3x}}{x} + e^{3x}K_2$$

General Solution:

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1) If R.H.S = Expo × Trigo or Exp × Alg

$$y_p = \frac{1}{f(D)} \times (\text{Exp} \times \text{trigo})$$

e^{ax}

$$= \text{Exp} \times \frac{1}{f(D+a)} \times \text{trigo}$$

For example

$$y_p = \frac{1}{(D^2-6D+9)} \times (e^{-2x} \cdot \sin 7x)$$

$$y_p = e^{2x} \left[\times \frac{1}{((D-2)^2 - 6(D-2)+9)} \right] \times \sin 7x$$

1) $(D^2-4D+3)y = \cos 3x + e^x \cos 2x$

$D^2-4D+3=0$

$D = 1, 3$

C.F

$$y_c = C_1 e^x + C_2 e^{3x}$$

~~$y_c = C_1 y_1 C +$~~

$$y_p = \frac{1}{f(D)} \times RHS$$

$$y_p = \frac{1}{(D-1)(D-3)} \times (\cos 3x + e^x \cos 2x)$$

$D^2 = -b^2 = -9$

$$y_p = \frac{1}{\underset{D^2-4D+3}{\cancel{(D-1)(D+3)}}} \times \cos 3x + \left(\frac{1}{(D-1)(D-3)} \times e^x \cos 2x\right)$$

$$y_p = \frac{\cos 3x}{-9-4D+3} + \left(\frac{e^x \cos 2x}{(D-1)(D-3)}\right)$$

$$y_p = -\frac{1}{2}\frac{\cos 3x}{(2D+3)} + \left(\frac{e^x \cos 2x}{(D-1)(D-3)}\right)$$

$$y_p = -\frac{1}{2}\left[\frac{\cos 3x}{(2D+3)} \times \frac{(2D-3)}{(2D-3)}\right] + \left[\frac{e^x \cos 2x}{\underset{D^2-4D+3}{\cancel{(D-1)(D-3)}}}\right]$$

$$y_p = -\frac{1}{2}\left[\frac{2D\cos 3x - 3\cos 3x}{(2D)^2 - (3)^2}\right] + \left[e^x \cdot \frac{1}{(D+1)^2 - 4(D+1) + 3} \cdot \cos 2x\right]$$

$$y_p = -\frac{1}{2}\left[\frac{-2\sin 3x \cdot 3 - 3\cos 3x}{4D^2 - 9}\right] + e^x \cdot \frac{1}{D^2 + 2D + \cancel{1} - 4D \cancel{-4} \cancel{+3}} \cdot \cos 2x$$

$$y_p = -\frac{1}{2}\left[\frac{-6\sin 3x - 3\cos 3x}{4(-9) - 9}\right] + e^x \cdot \frac{1}{D^2 - 2D} \cdot \cos 2x$$

$$y_p = -\frac{1}{2}\left[\frac{-6\sin 3x - 3\cos 3x}{-45}\right] + \left[e^x \cdot \frac{1}{(D^2 - 2D)} \cdot \cos 2x\right]$$

$D = -b^2 = -2^2 = -4$

$$y_p = -\frac{1}{2}\left[\frac{-6\sin 3x - 3\cos 3x}{-45}\right] + \left[e^x \cdot \frac{1}{(-4-2D)} \cdot \cos 2x\right]$$

$$y_p = -\frac{1}{2}\left[\frac{-6\sin 3x - 3\cos 3x}{-45} + e^x \frac{1}{(D+2)} \cos 2x\right]$$

$$y_p = \frac{-1}{2}\left[\frac{\cancel{+}(6\sin 3x + 3\cos 3x)}{\cancel{+}45} + e^x\left(\frac{1}{D+2}\times\frac{D-2}{D-2}\right)\cos 2x\right]$$

$$y_p = -\frac{1}{2}\left[\frac{6\sin 3x + 3\cos 3x}{45} + e^x\left(\frac{D-2}{D^2-2^2}\right)\cos 2x\right]$$

$$y_p = \frac{-1}{2}\left[\frac{6\sin 3x + 3\cos 3x}{45} + e^x\left(\frac{D\cos 2x - 2\cos 2x}{-(2)^2 - 4}\right)\right]$$

$$y_p = \frac{-1}{2}\left[\frac{6\sin 3x + 3\cos 3x}{45} + e^x\left(\frac{\cancel{+}(2\sin 2x + 2\cos 2x)}{\cancel{+}\cancel{16}\,8}\right)\right]$$

$$y_p = \frac{-1}{2}\left[\frac{6\sin 3x + 3\cos 3x}{45} + e^x\left(\frac{2\sin 2x + 2\cos 2x}{\cancel{16}\,8}\right)\right]$$

$$y_p = \frac{-1}{2}\left[\frac{\cancel{3}(2\sin 3x + \cos 3x)}{\cancel{45}\ 15} + \frac{\cancel{2}e^x(\sin 2x + \cos 2x)}{\cancel{16}\ \cancel{8}\ 4}\right]$$

$$y_p = -\frac{(2\sin 3x + \cos 3x)}{30} - \frac{e^x(\sin 2x + \cos 2x)}{8}$$

General Solution,

$$y = y_c + y_p$$

$$y = C_1e^x + C_2e^{3x} - \frac{1}{30}(2\sin 3x + \cos 3x) - \frac{1}{8}\cdot e^x(\sin 2x + \cos 2x)$$

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2) $(D^2-1)y = x\cosh 7x$

$D^2-1=0$

$D=\pm 1$

$D=1,-1$

$y_c = c_1e^{x}+c_2e^{-x}$

$$y_p = \frac{1}{(D^2-1)} x\cdot\left(\frac{e^{7x}+e^{-7x}}{2}\right)$$

$$= \left(\frac{1}{2}e^{7x}\cdot x + \frac{1}{2}\cdot e^{-7x}\cdot x\right) \times \frac{1}{(D^2-1)}$$

$$= \frac{1}{2}\frac{x\cdot e^{7x}}{(D^2-1)} + \frac{1}{2}\frac{x\cdot e^{-7x}}{(D^2-1)}$$

$$= \frac{e^{7x}}{2}\cdot\frac{1}{(D^2-1)}x + \frac{e^{-7x}}{2}\cdot\frac{1}{(D^2-1)}\cdot x$$

$$= \left[-\frac{e^{7x}}{2}\cdot\frac{x}{(1-D^2)} - \frac{e^{-7x}}{2}\cdot\frac{x}{(1-D^2)}\right]$$

$$= -\frac{1}{2}\left[e^{7x}\cdot\frac{1}{(1-D^2)}\cdot x + e^{-7x}\cdot\frac{1}{(1-D^2)}\cdot x\right]$$

$$= -\frac{1}{2}\left[e^{7x}\left(\frac{1}{(1-(D+7)^2)}\cdot x\right) + e^{-7x}\left(\frac{1}{(1-(D-7)^2)}\cdot x\right)\right]$$

$$= -\frac{1}{2}\left[e^{7x}\left(\frac{1}{(1-(D^2+7D\times 2+7^2)}\cdot x\right) + e^{-7x}\left(\frac{1}{1-(D^2-7D\times 2+7^2)}\cdot x\right)\right.$$

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$$= \frac{-1}{2}\left[e^{7x} \cdot \frac{1}{1-D^2-14D-49} \cdot x\right) + e^{-7x} \cdot \frac{1}{1-D^2+14D-49} \cdot x$$

$$= \frac{-1}{2}\left[\left(e^{7x} \cdot \frac{1}{-D^2-14D-48} \cdot x\right) + \left(e^{-7x} \cdot \frac{1}{-D^2+14D-48} \cdot x\right)\right]$$

$$= \frac{1}{2}\left[\left(e^{7x} \cdot \frac{1}{D^2+14D+48} \cdot x\right) + \left(e^{-7x} \cdot \frac{1}{(D^2-14D+48)} \cdot x\right)\right]$$

$$= \frac{1}{2}\left(\frac{e^{7x}}{48}\left(\frac{1}{\left(\frac{D^2+14D}{48}\right)+1} \cdot x\right) + \frac{e^{-7x}}{48}\left(\frac{1}{\frac{D^2-14D}{48}+1} \cdot x\right)\right]$$

$$= \frac{1}{96}\left(e^{7x}\left(\frac{x}{\left(1+\left(\frac{D^2+14D}{48}\right)\right)}\right) + e^{-7x}\left(\frac{x}{1+\left(\frac{D^2-14D}{48}\right)}\right)\right]$$

$$= \frac{1}{96}\left(e^{7x}\left(\frac{x}{(1+r)}\right) + e^{-7x}\left(\frac{x}{(1+r)}\right)\right]$$

$$= \frac{1}{96}\left[e^{7x}(1-r)x + e^{-7x}(1-r)x\right]$$

$$= \frac{1}{96}\left(e^{7x}\left(1-\left(\frac{\cancel{D^2}^{\,0}+14D}{48}\right)\right)x + e^{-7x}\left(1-\left(\frac{\cancel{D^2}^{\,0}-14D}{48}\right)\right)x\right.$$

Power of x is 1, from there all zero.

$$= \frac{1}{96}\left(e^{7x}\left(1-\frac{14D}{48}\right)x + e^{-7x}\left(1+\frac{14D}{48}\right)x\right]$$

$$= \frac{1}{96}\left(e^{7x}\left(x-\frac{14Dx}{48}\right) + e^{-7x}\left(x+\frac{14Dx}{48}\right)\right]$$

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$$= \frac{1}{96}\left[e^{7x}\left(x - \frac{14}{48}\right) + e^{-7x}\left(x + \frac{14}{48}\right)\right]$$

$$= \frac{1}{96}\left[x\cdot e^{7x} - e^{7x}\cdot\frac{\cancel{14}^{\,7}}{\cancel{48}_{\,24}} + x\cdot e^{-7x} + e^{-7x}\cdot\frac{\cancel{14}^{\,7}}{\cancel{48}_{\,24}}\right]$$

$$= \frac{1}{96}\left[x\left(e^{7x} + e^{-7x}\right) - \frac{7}{24}\left(e^{7x} - e^{-7x}\right)\right]$$

$$= \frac{1}{96}\left[x\cdot 2\cosh 7x - \frac{7}{24} \times 2\sin 7x\right]$$

$$= \frac{\cancel{2}^{\,1}}{\cancel{96}_{\,48}}\left(x\cdot\cosh 7x - \frac{7}{24}\sin 7x\right]$$

$$= \frac{1}{48}\left(x\cdot\cosh 7x - \frac{7}{24}\sin 7x\right)$$

$\Rightarrow$ $e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta$ $\downarrow$ Real part

$i\sin\theta$ $\downarrow$ imagenry part.

If RHS is algebric into trigo

Alg x trigo

sin ax — IP (e^{iax})

cos ax — RP (e^{iax})

Alg $\times$ I.P (e^{iax}) / Alg $\times$ RP (e^{iax})

IP $(\text{Alg} \times e^{iax})$ / R.P $(\text{Alg} \times e^{iax})$

1) $(D^2-1)y = x^2\sin x + e^{-x}$

Complementry function is

$y_c = C_1e^{x} + C_2e^{-x}$

$y_p = \dfrac{1}{f(D)} \times \text{RHS}$

$= \dfrac{1}{(D^2-1)} \times (x^2\sin x + e^{-x})$

$= \dfrac{x^2\sin x}{(D^2-1)} + \dfrac{e^{-x}}{(D^2-1)}$

$= \dfrac{x^2\sin x}{(D^2-1)} + \dfrac{x \cdot e^{-x}}{2D}$

$$= \frac{x^2 \sin x}{(D^2-1)} + \frac{x}{2} \cdot \frac{1}{D} e^{-x}$$

$$= \frac{x^2 \sin x}{(D^2-1)} - \frac{x}{2} e^{-x}$$

$$= \frac{1}{(D^2-1)} \times x^2 \sin x - \frac{x}{2} e^{-x}$$

$$= \frac{1}{(D^2-1)} \left[x^2 (I.P e^{ix}) \right] - \frac{x}{2} e^{-x}$$

$$= I.P \left[e^{ix} \frac{1}{(D^2-1)} x^2 \right] - \frac{x}{2} e^{-x}$$

$$= I.P \left[e^{ix} \cdot \frac{1}{(D+i)^2 - 1} \cdot x^2 \right] - \frac{x}{2} e^{-x}$$

$$= I.P \left[e^{ix} \cdot \frac{1}{D^2 + 2Di + i^2 - 1} \cdot x^2 \right] - \frac{x}{2} e^{-x}$$

$$= I.P \left[e^{ix} \cdot \frac{1}{D^2 + 2Di - 2} \cdot x^2 \right] - \frac{x}{2} e^{-x}$$

$$= I.P \left[\frac{e^{ix}}{-2} \cdot \frac{1}{\left(\frac{D^2 + 2Di}{-2}\right) + 1} \cdot x^2 \right] - \frac{x}{2} e^{-x}$$

$$= I.P \left[\frac{e^{ix}}{-2} \cdot \frac{1}{(1+r)} \cdot x^2 \right] - \frac{x}{2} e^{-x}$$

$$= I.P \left[\frac{e^{ix}}{-2} (1 - r + r^2) x^2 \right] - \frac{x}{2} e^{-x}$$

$\frac{D^2 x^2}{2}$

x^2;
$2x$
2

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$$= I.P\left[\frac{e^{ix}}{-2}\left(1-\left(\frac{D^2+2iD}{-2}\right)+\left(\frac{D^2+2Di}{-2}\right)^2\right)x^2\right] - \frac{xe^{-x}}{2}$$

$$= I.P\left[\frac{e^{ix}}{-2}\left(1+\frac{D^2}{2}+\frac{\cancel{2}iD}{\cancel{2}}+\cancel{\frac{D^4}{4}}+\cancel{\frac{4iD^3}{4}}+\frac{\cancel{4}i^2D^2}{\cancel{4}}\right)x^2\right]-\frac{xe^{-x}}{2}$$

$$= I.P\left[\frac{e^{ix}}{-2}\left(1+\frac{D^2}{2}+iD-D^2\right)x^2\right]-\frac{xe^{-x}}{2}$$

$$= I.P\left[\frac{e^{ix}}{-2}\left(x^2+\frac{D^2x^2}{2}+iDx^2-D^2x^2\right)\right]-\frac{xe^{-x}}{2}$$

$$= I.P\left[\frac{e^{ix}}{-2}\left(x^2+\frac{\cancel{2}}{\cancel{2}}+i2x-2\right)\right]-\frac{xe^{-x}}{2}$$

$$= I.P\left(\frac{e^{ix}}{-2}\left(x^2+2xi-1\right)\right)-\frac{xe^{-x}}{2}$$

$$= I.P\left(\frac{\cos x+i\sin x}{-2}\left(x^2+2xi-1\right)\right]-\frac{xe^{-x}}{2}$$

$$= \frac{I.P}{-2}\left(\begin{array}{l}x^2\cos x+2xi\cos x-\cos x\\ +x^2 i\sin x+2xi^2\sin x-i\sin x\end{array}\right)-\frac{xe^{-x}}{2}$$

$$= \frac{1}{-2}\left(2x\cos x+x^2\sin x-\sin x\right)-\frac{xe^{-x}}{2}$$

Therefore, General Solution is

$$y = y_c + y_p$$

$$y = C_1 e^{x} + C_2 e^{-x} - \frac{1}{2}(2x\cos x + x^2 \sin x - \sin x) - \frac{1}{2}(x e^{-x})$$

2) $(D^2 - 2D + 1)y = x^2 \cos 2x$

y_c: $D^2 - 2D + 1 = 0$

$D = 1, 1.$ $\Rightarrow (D-1)^2 = 0$

$$y_c = (C_1 + x C_2)e^{x}$$

$$y_p = \frac{1}{f(D)} \times R.H.S.$$

$$y_p = \frac{1}{(D^2 - 2D + 1)} \cdot x^2 \cos 2x$$

$$= \frac{1}{(D-1)^2} \cdot (x^2 \cos 2x)$$

$$= \frac{1}{(D-1)^2} \left[\left(x^2 (R.P\, e^{i2x} \right) \right]$$

$$= \frac{1}{(D-1)^2}\, R.P \left[e^{i2x} \cdot \frac{1}{(D-1)^2} \cdot x^2 \right]$$

$$(D^2+49)y = e^{-3x} \cdot x \cdot \cos 8x$$

4) $\frac{1}{D^2+49} = e^{-3x} \cdot x \cos 8x$

$= e^{-3x}\left[\frac{1}{D^2+49} \cdot x \cdot \cos 8x\right]$

$= e^{-3x}\left[\frac{1}{(D-3)^2+49} x \cdot \cos 8x\right]$

$= e^{-3x}\left[\frac{1}{D^2-6D+58} \cdot x \cdot R.P \cdot e^{i8x}\right]$

$= e^{-3x} \cdot RP\left[\frac{1}{D^2-6D+58} \cdot x e^{i8x}\right]$

$= e^{-3x} \cdot R.P\left[e^{i8x} \cdot \frac{1}{(D+8i)^2-6(D+8i)+58} \cdot x\right]$

$= e^{-3x} \cdot R.P\left[e^{i8x} \cdot \frac{1}{D^2+16Di-64-6D-48i+58} \cdot x\right]$

$\begin{array}{r} 48 \\ 16 \\ \hline 32 \end{array}$

$= e^{-3x} \cdot R.P\left[e^{i8x} \frac{1}{\cancel{D^2+32iD-6D+6}} \cdot x\right]$

$D^2+16Di-6D-6-48i$

5

$5 \; 6 \; \frac{48}{5} \; 6$ $= e^{-3x} \cdot R.P\left[\frac{e^{i8x}}{-6-48i} \frac{1}{\left(\frac{D^2+16Di-6D}{-6-48i}\right)+1} \cdot x\right]$

$$= e^{-3x} \cdot R.P \left[\frac{e^{i8x}}{-6-48i} \cdot \frac{1}{(r+1)} \cdot x \right]$$

$$= e^{-3x} \cdot R.P \left[\frac{e^{i8x}}{-6-48i} (1-r)\, x \right]$$

$$= e^{-3x} \cdot R.P \left[\frac{e^{8x}}{-6-48i} \left(1 - \left(\frac{\cancel{D^2}+16Di-6D}{-6-48i}\right)\right) x \right]$$

$$= e^{-3x} \cdot R.P \left[\frac{e^{8x}}{-6-48i} \left(1 + \frac{16Di-6D}{6+48i}\right) x \right]$$

$$= e^{-3x} \cdot R.P \left[\frac{e^{8x}}{-6-48i} \left(x + \frac{16Dxi-6Dx}{6+48i}\right) \right]$$

$$= e^{-3x} \cdot R.P \left[\frac{e^{8x}}{-6-48i} \left(x + \frac{1}{(6+48i)} (16i-6)\right) \right]$$

$$= e^{-3x} \cdot R.P \left[\frac{e^{8x}}{-6-48i} \left(x + \left(\frac{16i-6}{6+48i} \times \frac{6-48i}{6-48i}\right)\right) \right]$$

$$= e^{-3x} \cdot R.P \left[\frac{e^{8x}}{-6-48i} \left(x + \frac{(16i-6)(6-48i)}{(6^2+48i^2)}\right) \right]$$

$$\cancel{\left(x + \frac{6(16i-6)-48i(16i-6)}{6^2+2\times 6\times 48i+(48i)^2}\right)}$$

$$\cancel{\left(x + \frac{96i-36+768+288i}{36+576i-2304}\right)}$$

$$= e^{-3x}\, R.P \left[\frac{e^{8x}}{-6-48i} \left(x + \frac{384i+732}{576-2268}\right) \right]$$

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$$= e^{-3x} \cdot R.P \left[\frac{e^{8x}}{-6-48i} \left(x + \frac{384i + 732}{2340} \right) \right]$$

$$= \frac{e^{-3x}}{-6} \cdot R.P \left(\frac{e^{8x} \cdot (1-8i)}{(1+8i)(1-8i)} \right) \left(x + \frac{384i + 732}{2340} \right)$$

$$= \frac{e^{-3x}}{-6} \cdot RP \left(\frac{(\cos 8x + i \sin 8x)(1-8i)}{1^2 - 8i^2} \right) \left(x + \frac{384i + 732}{2340} \right)$$

$$= \frac{e^{-3x}}{-6} \cdot R.P \left(\frac{\cos 8x + i \sin 8x - 8i \cos 8x + 8 \sin 8x}{65} \right) \left(x + \frac{384i + 732}{2340} \right)$$

$$= \frac{e^{-3x}}{2340 \times -6 \times 65} \cdot R.P \left((\cos 8x + 8 \sin 8x) - 8i \cos 8x + i \sin 8x \right) \left(2340x + 384i + 732 \right)$$

$$\cancel{= \frac{e^{-3x}}{912600} \left[(\cos 8x + 8 \sin 8x)(2340x + 732) \right]}$$

$$= \frac{-e^{-3x}}{912600} \left(2340x \cos 8x + 18720 \sin 8x + 3072 \cos 8x - 384 \sin 8x + 732 \cos 8x + 5856 \sin 8x \right)$$

Rectification (finding the length)

→ Formulas

1) $\displaystyle\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx \quad , \quad y = f(x)$

2) $\displaystyle\int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy \quad , \quad x = f(y)$

3) $\displaystyle\int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \cdot d\theta \quad , \quad r = f(\theta)$

4) $\displaystyle\int_{r_1}^{r_2} \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2} \cdot dr \quad , \quad \theta = f(r)$

1) find the length of the Cardiod $r = a(1-\cos\theta)$ lying inside the Circle $r = a\cos\theta$

Solution:- The given Curves are $r = a(1-\cos\theta)$ and $r = a\cos\theta$

Integrant $= \left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^2$

$= a^2(1-\cos\theta)^2 + \left(\frac{da(1-\cos\theta)}{d\theta}\right)^2$

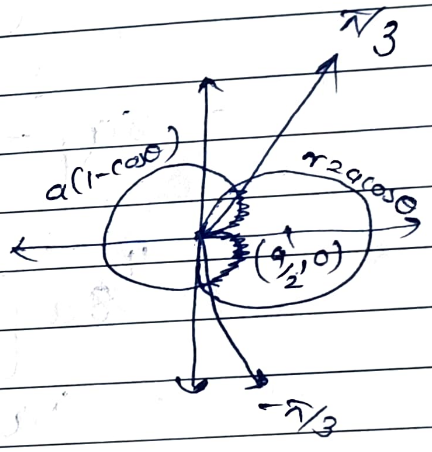
$= a^2(1-2\cos\theta+\cos^2\theta) + (a(\sin\theta))^2$

$= a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)$

$= a^2(2-2\cos\theta)$

$= 2a^2(1-\cos\theta)$

$= 2a^2(2\sin^2\theta/2)$



$= 4a^2\sin^2\theta/2$

$= (2a\sin\theta/2)^2$

Interfection, $r = r$

$a(1-\cos\theta) = a\cos\theta$

$1-\cos\theta = \cos\theta$

$1 = 2\cos\theta$

$\cos^{-1}\left(\frac{1}{2}\right) = \theta$

$\theta = \frac{\pi}{3}$ to $-\pi/3$

for calculating half-Part $\theta = 0$ to $\pi/3$

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length

$$= 2\int_0^{\pi/3} \sqrt{(2a\sin\theta/2)^2} \cdot d\theta$$

$$= 2\int_0^{\pi/3} 2a \cdot \sin\theta/2 \cdot d\theta$$

$$= 4a \int_0^{\pi/3} \sin\theta/2 \cdot d\theta$$

$$= 4a\left[\left(-\frac{\cos\theta/2}{1/2}\right)\right]_0^{\pi/3}$$

$$= 8a\left[-\cos\frac{\pi}{6} + \cos 0\right]$$

$$= 8a\left[-\frac{\sqrt{3}}{2} + 1\right]$$

$$= \cancel{8}^{4} a\left[\frac{-\sqrt{3}+2}{\cancel{2}}\right]$$

$$= 4a\left[-\sqrt{3}+2\right]$$

2) Find the ~~length~~ perimeter of Cardiode / length

$r = a(1 - \sin\theta)$

Solution:-

The given Curve is $r = a(1-\sin\theta)$

Integrant $= r^2 + \left(\frac{dr}{d\theta}\right)^2$

$$a^2(1-\sin\theta)^2 + \left(\frac{d}{d\theta}(a(1-\sin\theta))\right)^2$$

$$a^2(1 - 2\sin\theta + \sin^2\theta) + [a(0-\cos\theta)]^2$$

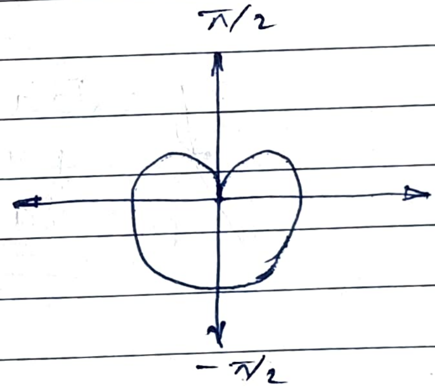
$$= a^2(1 - 2\sin\theta + \sin^2\theta + \cos^2\theta)$$

$$= a^2(2 - 2\sin\theta)$$

$$= 2a^2(1-\sin\theta)$$

$$= \cancel{2a^2(\cos^2\theta/2 + \sin^2\theta/2 - \sin\theta)}$$

$$= 2a^2(\cos\theta/2 - \sin\theta/2)^2$$



$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{2a^2\left(\cos\frac{\theta}{2} - \sin\theta/2\right)^2}$$

$$= 2\sqrt{2}a \int_{-\pi/2}^{\pi/2} \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right) \cdot dx$$

$$= 2\sqrt{2}a\left[\int_{-\pi/2}^{\pi/2} \cos\theta/2 \cdot d\theta - \int_{-\pi/2}^{\pi/2} \sin\theta/2 \cdot d\theta\right]$$

$$= 2\sqrt{2}a\left[\frac{\sin\theta/2}{1/2}\right]_{-\pi/2}^{\pi/2} - \left[\frac{-\cos\frac{\theta}{2}}{1/2}\right]_{-\pi/2}^{\pi/2}$$

$$= 2\sqrt{2}a \times 2\left[\sin\theta/2 + \cos\theta/2\right]_{-\pi/2}^{\pi/2}$$

$$= 4\sqrt{2}a\left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin\left(-\frac{\pi}{4}\right) - \cos\left(-\frac{\pi}{4}\right)\right]$$

$$= 4\sqrt{2}a\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \cancel{\frac{\sqrt{2}}{2}} - \cancel{\frac{\sqrt{2}}{2}}\right]$$

$$= \overset{2}{\cancel{4}}\sqrt{2}a\left[\frac{\sqrt{2}+\sqrt{2}}{\cancel{2}}\right]$$

$$= 2\sqrt{2}a\,(2\sqrt{2})$$

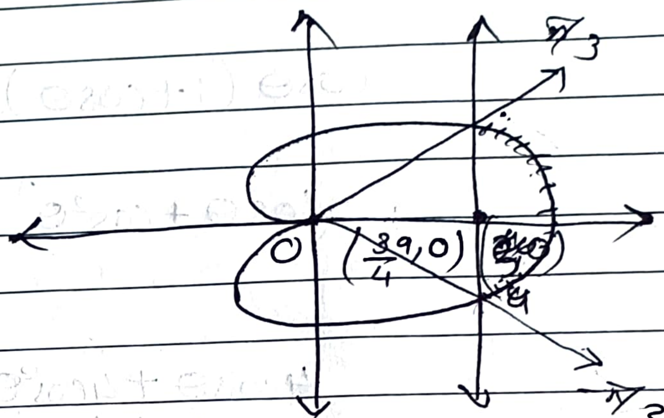
$$= 4 \times 2a$$

$$= 8a$$

3) Show that the length of the arc of a Cardiod $r = a(1+\cos\theta)$ which lies on a side of the line $4r = 3a\sec\theta$ away from the pole is $4a$.

Solution:-

The given curve are $r = a(1+\cos\theta)$, $4r = 3a\sec\theta$



$4r = 3a\sec\theta$

$4r = \frac{3a}{\cos\theta}$

$4r\cos\theta = 3a$

$4x = 3a$

$$\boxed{x = \frac{3a}{4}}$$

This represent a line parallel to y-axis and passing through $\left(\frac{3a}{4}, 0\right)$

Consider,

~~[illegible]~~ $r^2 + \left(\frac{dr}{d\theta}\right)^2$

$$= a^2(1+\cos\theta)^2 + \left(\frac{d\,a(1+\cos\theta)}{d\theta}\right)^2$$

$$= a^2(1+2\cos\theta+\cos^2\theta) + a^2(\sin^2\theta)$$

$$= a^2(2+2\cos\theta)$$

$$= 2a^2(1+\cos\theta)$$

$$= 2a^2(2\cos^2\theta/2)$$

$$= 4a^2(\cos^2\theta/2)$$

$$= (2a\cos\theta/2)^2$$

Intersection

$$r = r$$

$$\cancel{a}(1+\cos\theta) = \frac{3}{4}\cancel{a}\sec\theta$$

$$(1+\cos\theta) = \frac{3}{4\cos\theta}$$

$$\cos\theta(1+\cos\theta) = \frac{3}{4}$$

$$(\cos\theta + \cos^2\theta) = \frac{3}{4}$$

$$4\cos\theta + 4\cos^2\theta - 3 = 0$$

Let $\cos\theta = t$.

$$4t^2 + 4t - 3 = 0$$

$\times 6$ -2

$$4t^2 + 6t - 2t - 3 = 0$$

$$2t(2t+3) - 1(2t+3) = 0$$

$$2t + 3 = 0 \qquad 2t - 1 = 0$$

$$t = -\frac{3}{2} \qquad t = \frac{1}{2}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\theta = \pm\frac{\pi}{3}}$$

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$$= 2\int_0^{\pi/3} \sqrt{(2a\cos\theta/2)^{\cancel{2}}}\cdot d\theta$$

$$= 2\int_0^{\pi/3} 2a\cos\theta/2 \cdot d\theta$$

$$= 4a\int_0^{\pi/3} \cos\theta/2 \cdot d\theta$$

$$= 4a\left[\frac{\sin\theta/2}{1/2}\right]_0^{\pi/3}$$

$$= 8a\left[\sin\frac{\pi}{6} - \sin 0\right]$$

$$= \overset{4}{\cancel{8}}a\left[\frac{1}{\cancel{2}}\right]$$

$$= 4a$$

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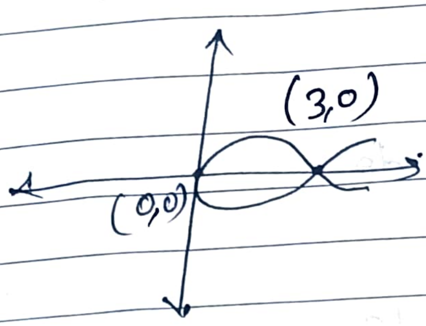
4) Find the length of a loop $9y^2 = x(x-3)^2$



put $y=0$.

$0 = x, (x-3)(x-3)=0$

$x=0$ $\qquad$ $x=3,3$

$(0,0)$ $\qquad$ $(3,0)(3,0)$

$$9y^2 = x(x-3)^2$$

Differentiating w.r.t. 'x'

$$9\left(2y\frac{dy}{dx}\right) = x\frac{d}{dx}(x-3)^2 + (x-3)^2\frac{d}{dx}(x)$$

$$= x \cdot 2(x-3) + (x-3)^2$$

$$18y\cdot\frac{dy}{dx} = (x-3)\left[2x + (x-3)\right]$$

$$= (x-3)(3x-3)$$

$$18y\cdot\frac{dy}{dx} = 3(x-3)(x-1)$$

$$\frac{dy}{dx} = \frac{\cancel{3}(x-3)(x-1)}{_6\cancel{18}y}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x-3)^2(x-1)^2}{36y^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x^2 - 2\times3\times x + 9)(x^2 - 2\times x + 1)}{36y^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{\cancel{(x-3)^2}(x-1)^2}{4\cancel{36}\times\frac{x}{\cancel{9}}\cancel{(x-3)^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\frac{(x-1)^2}{x}$$

Consider,

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x-1)^2}{4x}$$

$$= \frac{4x + (x-1)^2}{4x}$$

$$= \frac{4x + x^2 - 2x + 1}{4x}$$

$$= \frac{2x + x^2 + 1}{4x}$$

$$= \frac{x^2 + 2x + 1}{4x}$$

$$= \frac{(x+1)^2}{4x}$$

length of loop.

$$\int_0^3 2\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= 2\int_0^3 \sqrt{\frac{(x+1)^2}{(2\sqrt{x})^2}}$$

$$= 2\int_0^3 \frac{x+1}{2\sqrt{x}}$$

$$= \cancel{2}\times\frac{1}{\cancel{2}}\int_0^3 \frac{\sqrt{x}\times\sqrt{x}}{\cancel{\sqrt{x}}} + \frac{1}{\sqrt{x}}$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^3 \sqrt{x} + \frac{1}{\sqrt{x}} \cdot dx$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^3 (x)^{1/2} + (x)^{-1/2} \cdot dx$$

$\frac{1}{2}+1$

$$= \frac{\cancel{2}}{\cancel{2}}\left\{\left[\frac{x^{3/2}}{3/2}\right]_0^3 + \left[\frac{x^{1/2}}{1/2}\right]_0^3\right\}$$

$$= \left[\frac{(3)^{3/2}}{3/2} + \frac{(3)^{1/2}}{1/2}\right]$$

$$= \frac{2}{3}\times(3)^{3/2} + \frac{2\times}{3}(3)^{1/2}\times 3$$

$$= \frac{2\times 3^{3/2} + 6\times 3^{1/2}}{3}$$

$$= = \frac{2\times 3^{1/2}\times 3^{(3)} + 6\times 3^{1/2}}{3} = \frac{\sqrt{3}(}{3}$$

$$\int\sqrt{a^2+x^2} = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right).$$

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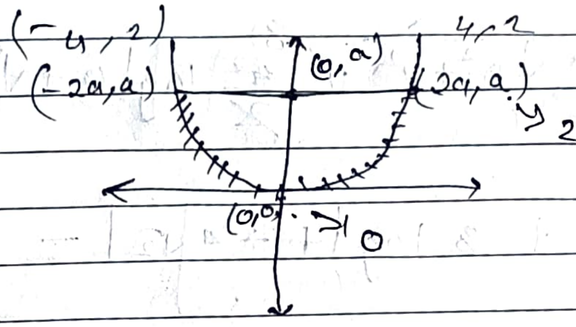
5) find the length of a parabola $x^2=8y$ cutoff by its latice rectum



$x^2 = 8y$

$x^2 = 4ay$

$\underline{\underline{a=2}}$

$x^2 = 8y$

Differentiating w.r.t. 'x'

$$2x\,\cancel{8}\cdot = 8\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{4}$$

length of arc of a parabola.

$$\int_{x_1, y_1}^{x_2, y_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= 2\int_0^4 \sqrt{1+\frac{x^2}{4^2}}$$

$$\frac{\cancel{2}\times 1}{\cancel{4}2}\int_0^4 \sqrt{4^2+x^2}$$

$$= \frac{1}{2}\int_0^4 \sqrt{4^2+x^2}$$

$$= \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+4^2} + \frac{4^2}{2}\log\left|x+\sqrt{x^2+4^2}\right|\right]_0^4$$

$-\int^{32}$

$= \frac{1}{2}\left[\frac{\cancel{4}^{2}}{\cancel{2}}\sqrt{4^2+4^2} + \frac{\cancel{4\times4}^{8}}{\cancel{2}}\log\left|4+\sqrt{4^2+4^2}\right|\right] -$

$\left[\frac{0}{2}\sqrt{0^2+4^2} + \frac{\cancel{4^2}^{8}}{\cancel{2}}\log\left|0+\sqrt{0^2+4^2}\right|\right]$

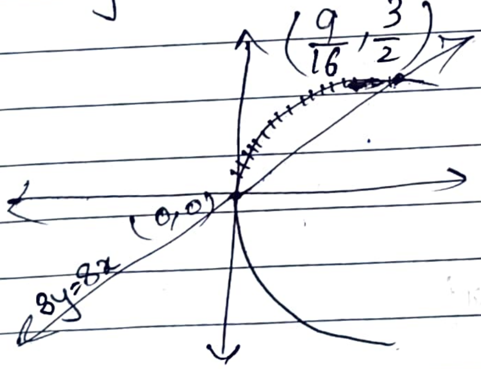
$= \frac{1}{2}\left[2\times4\sqrt{2} + 8\log|4+4\sqrt{2}| - 8\log 4\right]$

$= \frac{1}{2}\left[8\sqrt{2} + 8\left[\log\left(\frac{|4+4\sqrt{2}|}{4}\right)\right]\right]$

$= \frac{\cancel{8}^{4}}{\cancel{2}}\left[\sqrt{2} + \log(1+\sqrt{2})\right]$

$= 4\sqrt{2} + 4\log|1+\sqrt{2}|$

6) Find the length of parabola. $y^2 = 4x$ cutoff by the line $3y = 8x$



$3y = 8x$

$y = \frac{8}{3}x$

$y^2 = 4x$ and $3y = 8x$

$y = \frac{8}{3}x$

$\frac{64x^{\cancel{2}}}{9} = 4\cancel{x}$

$x = \frac{\cancel{4}^{1}\times 9}{\cancel{64}_{\cancel{32}_{16}}}$

$x = \frac{9}{16}$ $\quad y = \frac{\cancel{8}}{\cancel{3}}\times\frac{\cancel{9}^{3}}{\cancel{16}_{2}}$ $= \frac{3}{2}$