Machine Learning for Signal Processing
Sparse and Overcomplete Representations

Bhiksha Raj
(slides from Sourish Chaudhuri and Abelino Jimenez)

1

So far

Can we use linear composition to identify **basic units** that compose the signal?

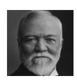
2

So far

Weights

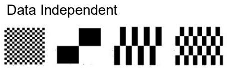
$$D \cdot \alpha = X$$
$$\approx$$

Data



Basis

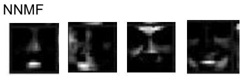
Data Independent



ICA

PCA

NNMF



basis <= dim X

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Just in case you missed it..

- Remember, #(Basis Vectors)= #unknowns

$$D \cdot \alpha = X$$

Basis Vectors

Weights

Input data

Standard representations: number of bases <= dimension of data

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A limitation we saw earlier

- Mathematical restrictions on the number of bases have no connection to reality
  - Universe does not respect your mathematical representations of the data
  - In reality: number of building blocks that compose any kind of data is unlimited
- One solution we saw earlier: picking *one* “closest” building block to represent any input
- Today: Learning linear compositional representations without restrictions on the number of basic units

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Poll 1

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Key Topics in this Lecture

- Basics – Component-based representations
  - *Overcomplete* and Sparse Representations,
  - *Dictionaries*
- Pursuit Algorithms
- How to learn a dictionary
- Why is an overcomplete representation powerful?

Sparse and Overcomplete Representations 7

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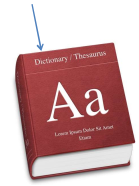
Representing Data

Dictionary (codebook)



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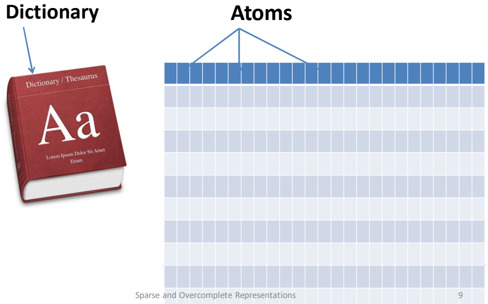
Representing Data



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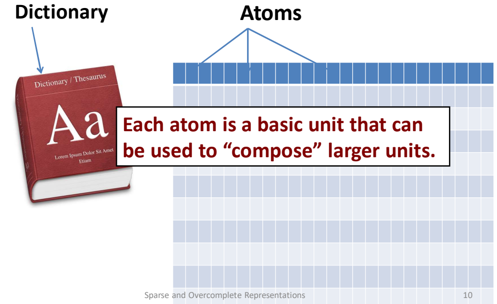
Representing Data



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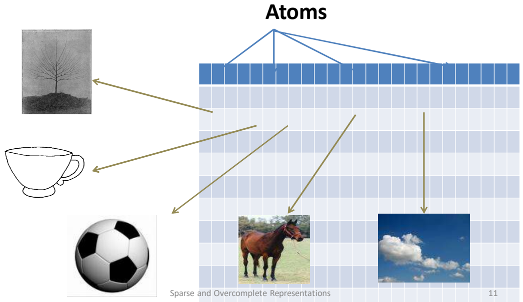
Representing Data



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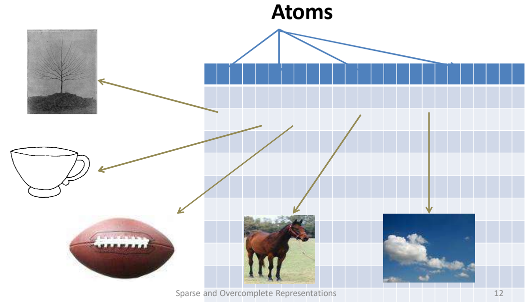
Representing Data



Sparse and Overcomplete Representations 12

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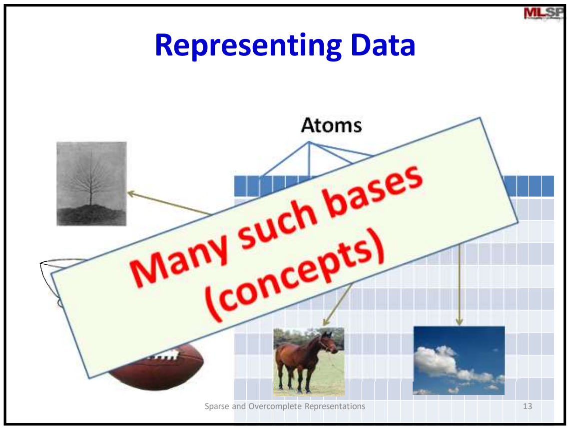

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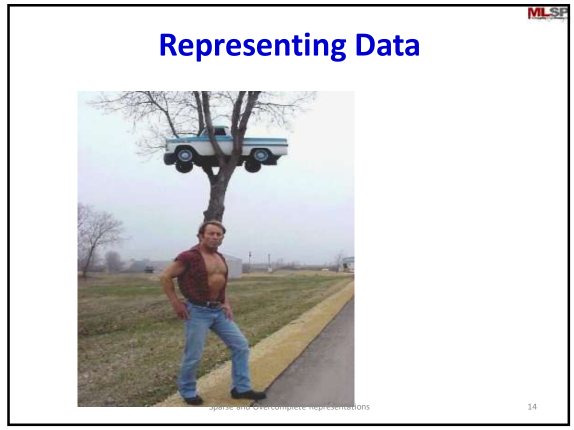

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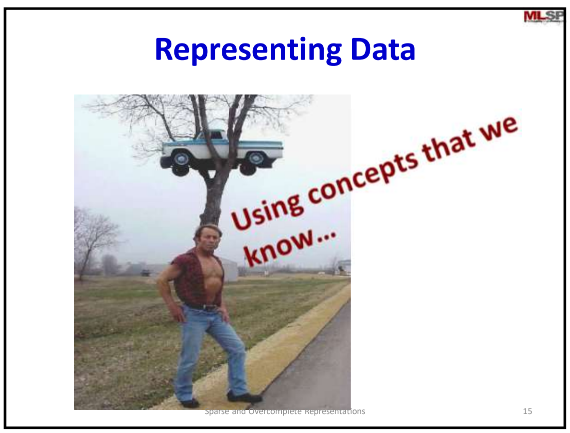

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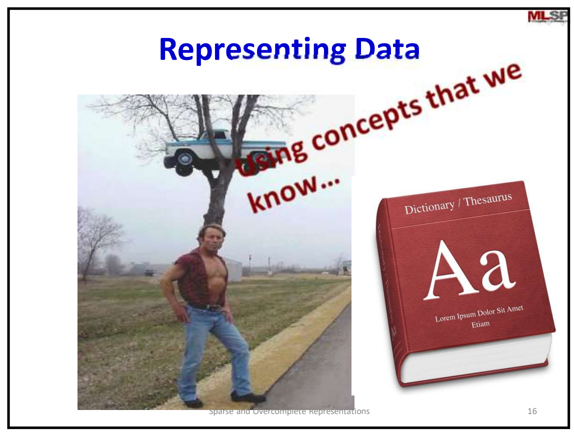

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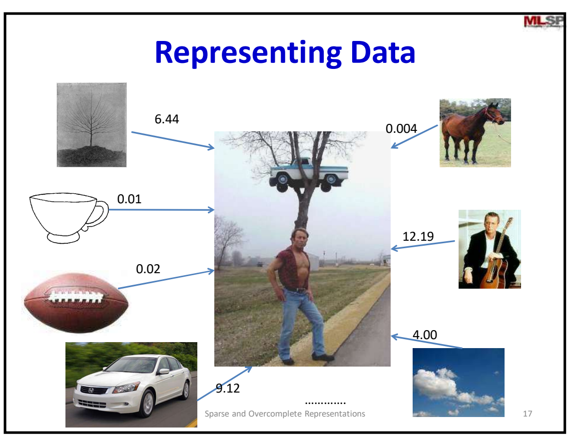

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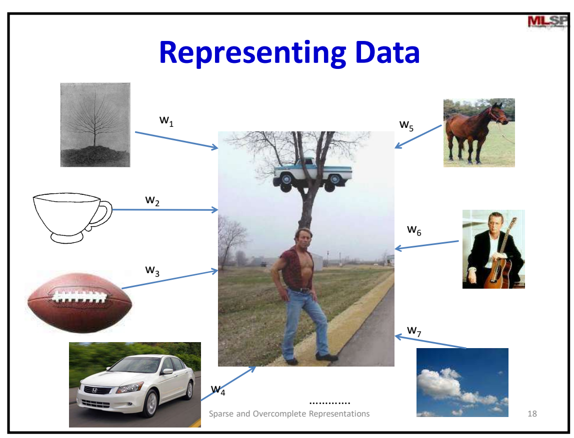

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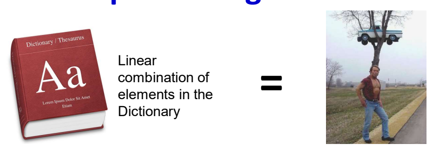
Representing Data



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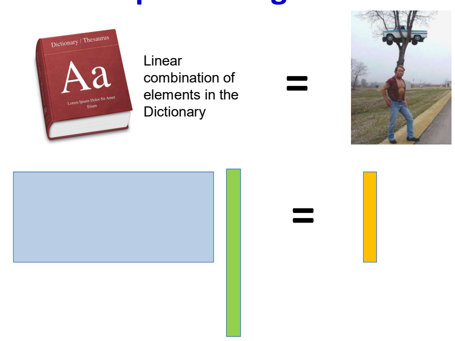
Representing Data



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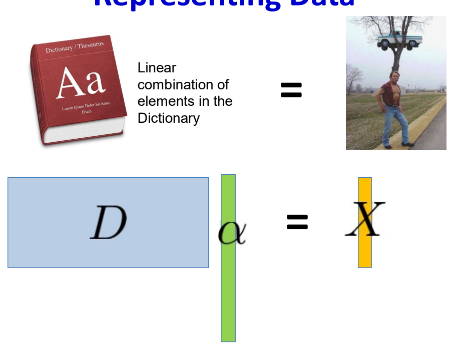
Representing Data



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Quick Linear Algebra Refresher

- Remember, #(Basis Vectors)= #unknowns

$$D \cdot \alpha = X$$

Basis Vectors ***(from Dictionary)***

Weights

Input data

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Overcomplete Representations

- What is the dimensionality of the input image? (say 64x64 image)
  - **4096**
- What is the dimensionality of the dictionary? (each image = 64x64 pixels)
  - **4096 x N**

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Overcomplete Representations

- What is the dimensionality of the input image? (say 64x64 image)
  - **4096**
- What is the dimensionality of the dictionary? (each image = 64x64 pixels)
  - **4096 x N** ???

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Overcomplete Representations

- What is the dimensionality of the input image? (say 64x64 image)
  - ➢ **4096**
- What is the dimensionality of the dictionary? (each image = 64x64 pixels)
  - ➢ **4096 x N**

VERY LARGE!!!

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Overcomplete Representations

- What is the dimensionality of the input image? (say 64x64 image)

If N > 4096 (as it likely is)
we have an **overcomplete** representation

- What is the dimensionality of the dictionary? (each image = 64x64 pixels)
  - ➢ **4096 x N**

VERY LARGE!!!

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Overcomplete Representations

- What is the dimensionality of the input image? (say 64x64 image)

More generally:

If #(dictionary units) > dimensions of input

we have an **overcomplete** representation

- ➢ **4096 x N**

VERY LARGE!!!

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Quick Linear Algebra Refresher

- Remember, #(Basis Vectors)= #unknowns

$$D \cdot \alpha = X$$

Dictionary Units

Weights

Input data

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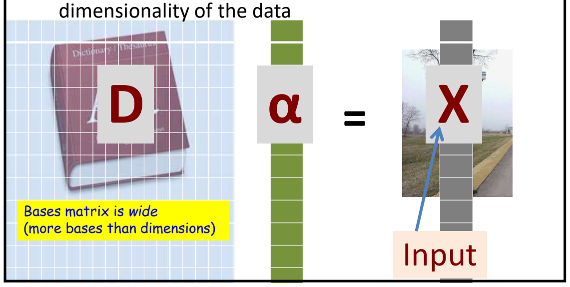
Dictionary based Representations

- Overcomplete "dictionary"-based representations are linear-composition-based representations with more "atomic building blocks" than the data dimensionality of the data



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Why Dictionary-based Representations?

- Dictionary based representations are semantically more meaningful
- Enable content-based description
  - Bases can capture entire structures in data
  - E.g. notes in music
  - E.g. image structures (such as faces) in images
- Enable content-based processing
  - Reconstructing, separating, denoising, manipulating speech/music signals
  - Coding, compression, etc.
- Statistical reasons: We will get to that shortly..

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Poll 2

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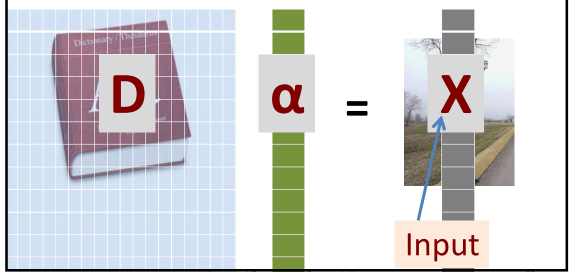
Problems

- How to obtain the dictionary
  - Which will give us meaningful representations
- How to compute the weights?



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Problems

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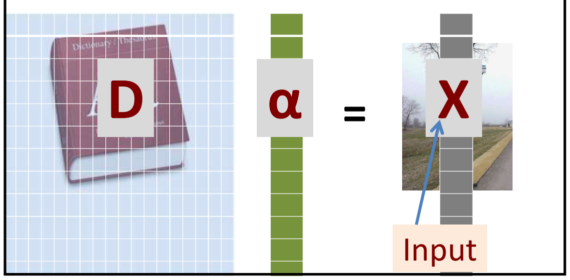

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Quick Linear Algebra Refresher

- Remember, #(Basis Vectors)= #unknowns

$$D \cdot \alpha = X$$

Dictionary entries

Weights

Input data

When can we solve for α?

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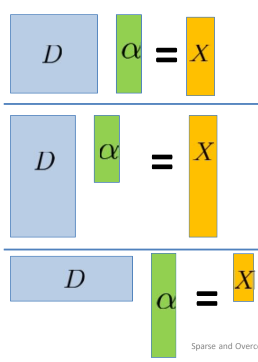
Quick Linear Algebra Refresher

$$D \cdot \alpha = X$$

D: full rank



Unique solution

We may have no exact solution

Infinite Solutions

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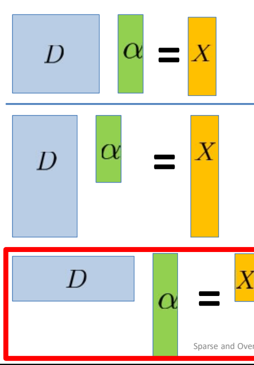
Quick Linear Algebra Refresher

$$D \cdot \alpha = X$$

D: full rank



Unique solution

We may have no exact solution

Our Case

Infinite Solutions

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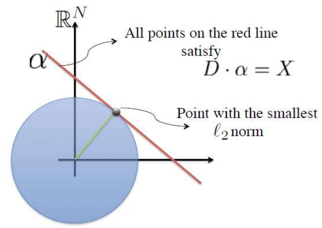
Using Pseudo-Inverse?



This is equivalent to

minimize $\|\alpha\|_2$ subject to $D\alpha = X$

α will generally be "dense"

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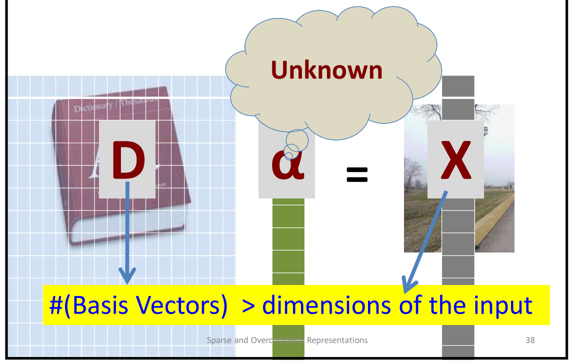
Overcomplete Representation



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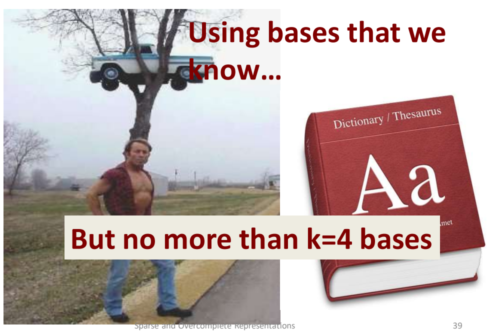
Representing Data



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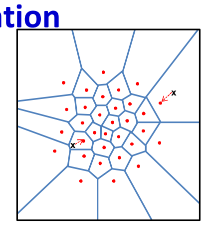
Alternate view: Recall quantization

$$V = \sum_i w_i d_i$$

$$V = \mathbf{D}\mathbf{w} \qquad |\mathbf{w}| = 1$$
$$|\mathbf{w}|_0 = 1$$



- d_i are the "representative" vectors of each cluster
- Restriction: only one of the w_i is 1, the rest are 0
  - $\sum_i w_i = 0$
  - $\mathbf{w}$ is unit length and one-sparse
- What if we let *more* than one entry of $\mathbf{w}$ to be non zero?

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Overcompleteness and Sparsity

- To solve an overcomplete system of the type:

  D.α = X

- Make assumptions about the data.
- Suppose, we say that **X** is composed of no more than a fixed number (**k**) of "bases" from **D** (k ≤ dim(**X**))
  - The term "bases" is an abuse of terminology..
- Now, we can find the set of **k** bases that best fit the data point, **X**.

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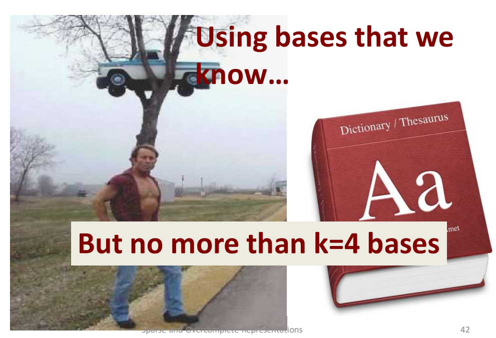
Representing Data



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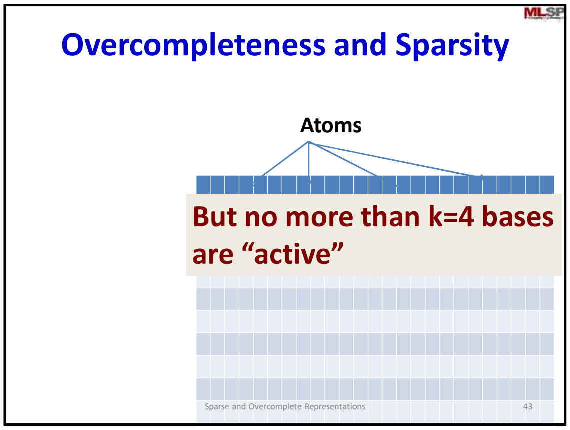

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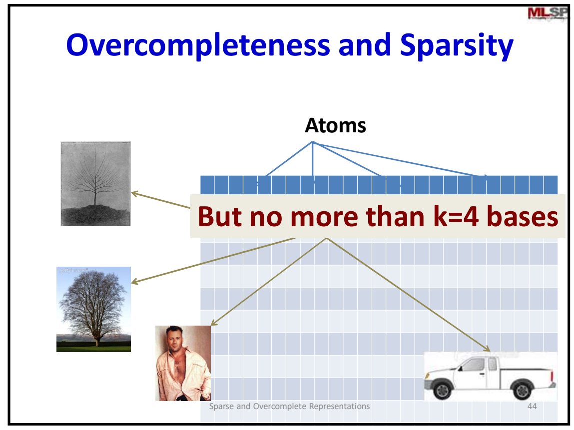

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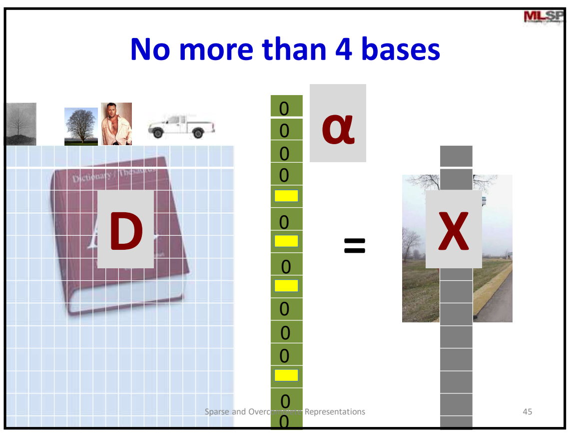

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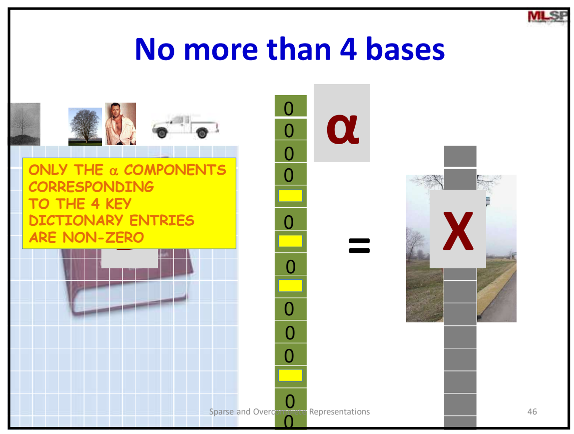

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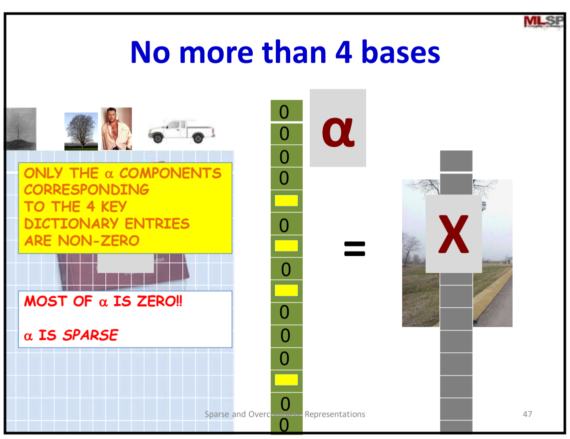

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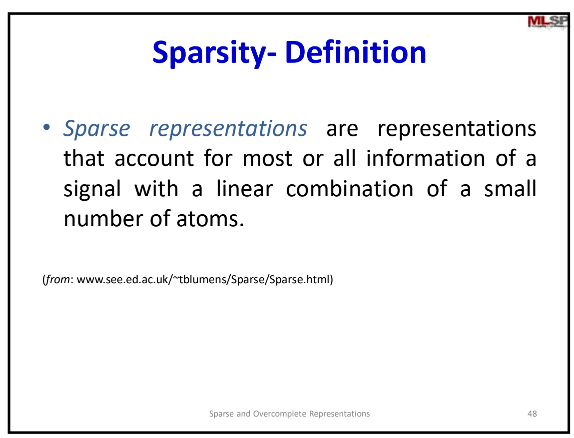

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The Sparsity Problem

- We don't really know **k**
- You are given a signal **X**
- Assuming **X** was generated using the dictionary, can we find α that generated it?

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The Sparsity Problem

- We want to use as few dictionary entries as possible to do this.

$$\boxed{\begin{aligned} &\underset{\underline{\alpha}}{Min}\ \|\underline{\alpha}\|_0 \\ &s.t.\ \ \underline{X} = \mathbf{D}\underline{\alpha} \end{aligned}}$$

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The Sparsity Problem

- We want to use as few dictionary entries as possible to do this.

$$\boxed{\begin{aligned} &\underset{\underline{\alpha}}{Min}\ \|\underline{\alpha}\|_0 \\ &s.t.\ \ \underline{X} = \mathbf{D}\underline{\alpha} \end{aligned}}$$

Counts the number of non-zero elements in α

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The Sparsity Problem

- We want to use **as few dictionary entries** as possible to do this
  - Ockham's razor: Choose the simplest explanation invoking the fewest variables

$$\boxed{\begin{aligned} &\underset{\underline{\alpha}}{Min}\ \|\underline{\alpha}\|_0 \\ &s.t.\ \ \underline{X} = \mathbf{D}\underline{\alpha} \end{aligned}}$$

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Poll 3

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The Sparsity Problem

- We want to use as few dictionary entries as possible to do this.

$$\boxed{\begin{aligned} &\underset{\underline{\alpha}}{Min}\ \|\underline{\alpha}\|_0 \\ &s.t.\ \ \underline{X} = \mathbf{D}\underline{\alpha} \end{aligned}}$$

How can we solve the above?

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Obtaining Sparse Solutions

- We will look at 2 algorithms:
  - Matching Pursuit (MP)
  - Basis Pursuit (BP)

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Matching Pursuit (MP)

- Greedy algorithm
- Finds an atom in the dictionary that best matches the input signal
- Remove the weighted value of this atom from the signal
- Again, find an atom in the dictionary that best matches the remaining signal.
- Continue till a defined stop condition is satisfied.

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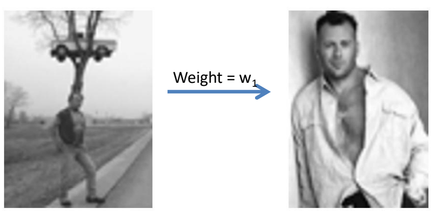
Matching Pursuit

- Find the dictionary atom that best matches the given signal.



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Matching Pursuit

- Remove weighted image to obtain updated signal

Find best match for this signal from the dictionary

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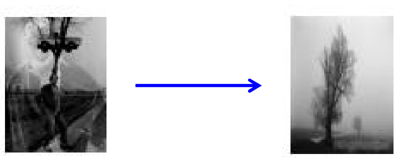
Matching Pursuit

- Find best match for updated signal



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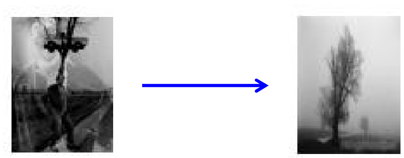
Matching Pursuit

- Find best match for updated signal



Iterate till you reach a stopping condition,
norm(ResidualInputSignal) < threshold

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Matching Pursuit

```
Algorithm Matching Pursuit
 Input: Signal: f(t).
 Output: List of coefficients: (a_n, g_γn).
 Initialization:
   Rf_1 ← f(t);
 Repeat
   find g_γn ∈ D with maximum inner product < Rf_n, g_γn >;
   a_n ← < Rf_n, g_γn >;
   Rf_{n+1} ← Rf_n − a_n g_γn;
   n ← n + 1;
 Until stop condition (for example: || Rf_n || < threshold)
```

From http://en.wikipedia.org/wiki/Matching_pursuit

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Matching Pursuit

- Problems ???

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Matching Pursuit

- Main Problem
  - Computational complexity
  - The entire dictionary has to be searched at every iteration

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Comparing MP and BP

Matching Pursuit	Basis Pursuit
Hard thresholding (remember the equations)	
Greedy optimization at each step	
Weights obtained using greedy rules	

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Basis Pursuit (BP)

- Remember,

$$\boxed{\begin{array}{l} \underset{\underline{\alpha}}{Min} \ \|\underline{\alpha}\|_0 \\ s.t. \ \ \underline{X} = \mathbf{D}\underline{\alpha} \end{array}}$$

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Basis Pursuit

- Remember,

$$\boxed{\begin{array}{l} \underset{\underline{\alpha}}{Min} \ \|\underline{\alpha}\|_0 \\ s.t. \ \ \underline{X} = \mathbf{D}\underline{\alpha} \end{array}}$$

In the general case, this is intractable

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Basis Pursuit

- Remember,

$$\min_{\alpha} \|\alpha\|_0 \quad s.t. \; X = \mathbf{D}\alpha$$

In the general case, this is intractable

Requires combinatorial optimization

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Basis Pursuit

- Replace the intractable expression by an expression that is solvable

$$\min_{\alpha} \|\alpha\|_1 \quad s.t. \; X = \mathbf{D}\alpha$$

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Basis Pursuit

- Replace the intractable expression by an expression that is solvable

$$\min_{\alpha} \|\alpha\|_1 \quad s.t. \; X = \mathbf{D}\alpha$$

This will provide identical solutions when $\mathbf{D}$ obeys the ***Restricted Isometry Property***.

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Basis Pursuit

- Replace the intractable expression by an expression that is solvable

$$\min_{\alpha} \|\alpha\|_1 \quad s.t. \; X = \mathbf{D}\alpha$$

Objective

Constraint

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Basis Pursuit

- We can formulate the optimization term as:

$$\min_{\alpha} \{\|X - \mathbf{D}\alpha\|^2 + \lambda\|\alpha\|_1\}$$

Constraint Objective

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Basis Pursuit

- We can formulate the optimization term as:

$$\min_{\alpha} \{\|X - \mathbf{D}\alpha\|^2 + \lambda\|\alpha\|_1\}$$

λ is a penalty term on the non-zero elements and promotes sparsity

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Basis Pursuit

Equivalent to *LASSO*; for more details, see [this paper by Tibshirani](http://www-stat.stanford.edu/~tibs/ftp/lasso.ps)
http://www-stat.stanford.edu/~tibs/ftp/lasso.ps

$$\underset{\alpha}{Min}\ \{\|X - \mathbf{D}\alpha\|^2 + \lambda\|\alpha\|_1\}$$

λ is a penalty term on the non-zero elements and promotes sparsity

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Basis Pursuit

$$\underset{\alpha}{Min}\{\|X - \mathbf{D}\alpha\|^2 + \lambda\|\alpha\|_1\}$$

$$\frac{\partial\|\alpha\|_1}{\partial\alpha_j} = \begin{cases} +1 & \text{at } \alpha_j > 0 \\ [-1,1] & \text{at } \alpha_j = 0 \\ -1 & \text{at } \alpha_j < 0 \end{cases}$$

- $\|\alpha\|_1$ is not differentiable at $\alpha_j = 0$
- Gradient of $\|\alpha\|_1$ for gradient descent update
- At optimum, following conditions hold

$$\nabla_j\|X - \mathbf{D}\alpha\|^2 + \lambda sign(\alpha_j) = 0, \quad \text{if } |\alpha_j| > 0$$
$$\nabla_j\|X - \mathbf{D}\alpha\|^2 \le \lambda, \quad \text{if } \alpha_j = 0$$

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Basis Pursuit

- There are efficient ways to solve the LASSO formulation.
  - http://web.stanford.edu/~hastie/glmnet_matlab/
- Simplest solution: Coordinate descent algorithms
  - On webpage..

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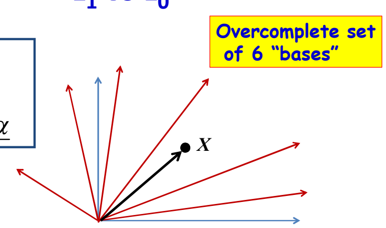
L_1 vs L_0

$$\underset{\alpha}{Min}\|\alpha\|_0$$
$$s.t.\ X = \mathbf{D}\alpha$$



- **L_0 minimization**
  - Two-sparse solution
  - ANY pair of bases can explain X with 0 error

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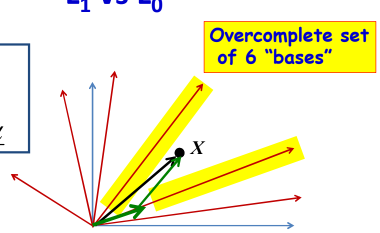
L_1 vs L_0

$$\underset{\alpha}{Min}\|\alpha\|_1$$
$$s.t.\ X = \mathbf{D}\alpha$$



- **L_1 minimization**
  - Two-sparse solution
  - All else being equal, the two closest bases are chosen

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Comparing MP and BP

Matching Pursuit	Basis Pursuit
Hard thresholding	Soft thresholding
(remember the equations)	
Greedy optimization at each step	Global optimization
Weights obtained using greedy rules	Can force N-sparsity with appropriately chosen weights

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General Formalisms

- L_0 minimization $\begin{array}{l} \underset{\alpha}{Min} \|\alpha\|_0 \\ s.t.\ X = \mathbf{D}\alpha \end{array}$
- L_0 constrained optimization $\begin{array}{l} \underset{\alpha}{Min} \|X - \mathbf{D}\alpha\|_2^2 \\ s.t. \|\alpha\|_0 < C \end{array}$
- L_1 minimization $\begin{array}{l} \underset{\alpha}{Min} \|\alpha\|_1 \\ s.t.\ X = \mathbf{D}\alpha \end{array}$
- L_1 constrained optimization $\begin{array}{l} \underset{\alpha}{Min} \|X - \mathbf{D}\alpha\|_2^2 \\ s.t. \|\alpha\|_1 < C \end{array}$

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Many Other Methods..

- Iterative Hard Thresholding (IHT)
- CoSAMP
- OMP
- …

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Poll 4

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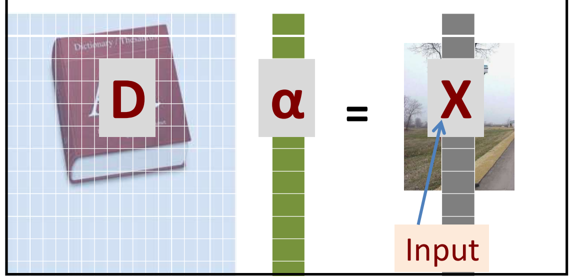
Problems

- How to obtain the dictionary
  - Which will give us meaningful representations
- How to compute the weights?



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Trivial Solution

- D = Training data
- Impractical in most situations
  - Popular approach: sample random vectors from training data

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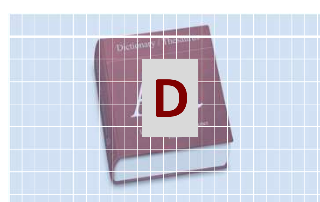
Dictionaries: Compressive Sensing

- Just random vectors!



84

More Structured ways of Constructing Dictionaries

- Dictionary entries must be structurally "meaningful"
  – Represent true compositional units of data
- Have already encountered two ways of building dictionaries
  – NMF for non-negative data
  – K-means ..

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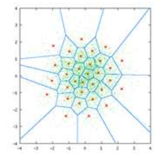
K-Means for Composing Dictionaries



Train the codebook from training data using K-means

- Every vector is approximated by the centroid of the cluster it falls into
- Cluster means are "codebook" entries
  – Dictionary entries
  – Also compositional units the compose the data

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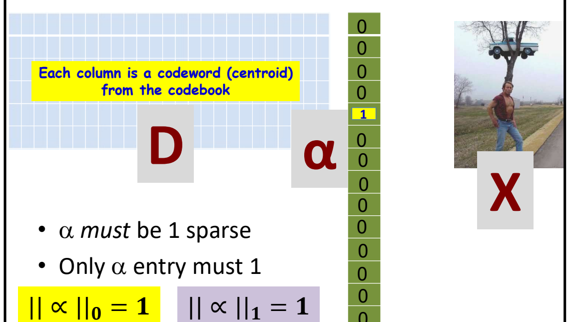
K-Means for Dictionaries



- α *must* be 1 sparse
- Only α entry must 1

$||\propto||_0 = 1$ $||\propto||_1 = 1$

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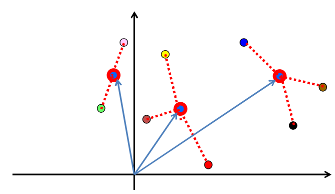
K-Means



- Learn Codewords to minimize the total squared length of the training vectors from the closest codeword

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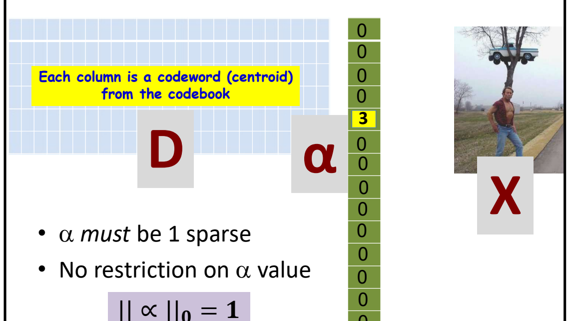
Length-unconstrained K-Means for Dictionaries

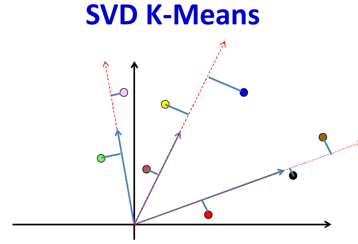


- α *must* be 1 sparse
- No restriction on α value

$||\propto||_0 = 1$

89

SVD K-Means

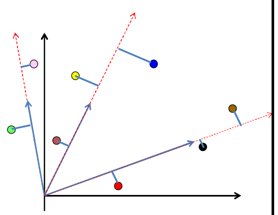


- Learn Codewords to minimize the total squared *projection error* of the training vectors from the closest codeword

90

SVD K–means

1. Initialize a set of centroids randomly
2. For each data point x, find the projection from the centroid for each cluster
   - $p_{cluster} = \left| x^T m_{cluster} \right|$
3. Put data point in the cluster of the closest centroid
   - Cluster for which $p_{cluster}$ is *maximum*
4. When all data points are clustered, recompute centroids



$$m_{cluster} = \text{Principal Eigenvector}(\{x \mid x \in cluster\})$$

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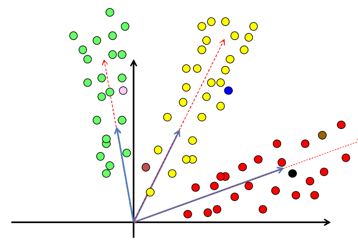
Problem



- Only represents *Radial* patterns

Sparse and Overcomplete Representations 92

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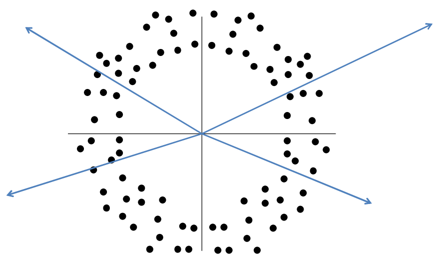
What about this pattern?

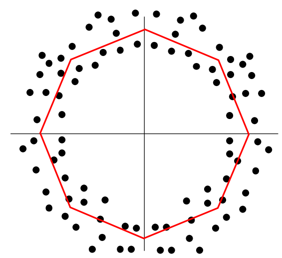


- Dictionary entries that represent radial patterns will not capture this structure
  - 1-sparse representations will not do

93

93

What about this pattern?



- We need ***AFFINE*** *patterns*

Sparse and Overcomplete Representations 94

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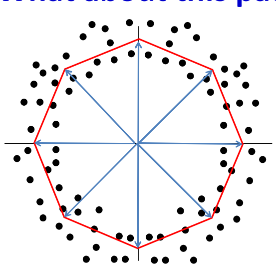
What about this pattern?

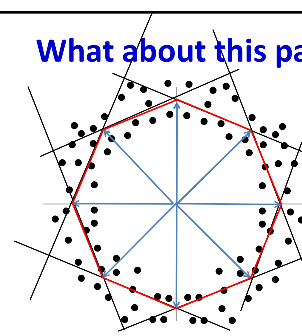


- We need ***AFFINE*** *patterns*
- *Each vector is modeled by a linear combination of K (here 2) bases*

95

95

What about this pattern?



Every line is a (constrained) combination of two bases

2-sparse

Constraint:
Line = a.b_1 + (1-a)b_2

- We need ***AFFINE*** *patterns*
- *Each vector is modeled by a linear combination of K (here 2) bases*

96

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Codebooks for K sparsity?

Each column is a codeword (centroid) from the codebook

D

α

X

$$\alpha = [0, 0, 0, 0, 3, 0, 1, 0, 0, 4, 0, 0, 0, 0]^T$$

- α *must* be *k* sparse
- No restriction on α value

$|| \propto ||_0 = k$

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Formalizing

Given training data

$$\{X_1, X_2, ..., X_T\}$$

We want to find a dictionary D, such that

$$D\alpha_i = X_i$$

With α_i sparse

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Formalizing

Two objectives:

- Approximation $\|D\alpha_i - X_i\|$
- Sparsity in coefficients $\|\alpha_i\|_1$

$$\min_{D,\alpha_i} \sum_{i=1}^{T} \|X_i - D\alpha_i\|^2 + \|\alpha_i\|_1$$

NON-Convex!!!

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An iterative method

- Given D, estimate α_i to get sparse solution
  - We can use any method

$$\min_{\alpha_i} \sum_{i=1}^{T} \|X_i - D\alpha_i\|^2 + \|\alpha_i\|_1$$

- Given α_i, estimate D

$$\min_{D} \sum_{i=1}^{T} \|X_i - D\alpha_i\|^2$$

Difficult!

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K SVD

- Initialize Codebook

D =

1. For every vector, compute K-sparse alphas
   - Using any pursuit algorithm

α =

0	0	2	0	0
1	0	0	0	0
0	0	0	1	0
0	7	0	0	0
0	0	0	1	0
3	0	0	0	0
0	5	0	0	1
0	0	1	0	2

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K-SVD

D = (D_j j != 1)

2. For each codeword (k):
- For each vector x that used k
  - Subtract the contribution of all other codewords to obtain $e_k(x)$
    - Codeword-specific residual
- Compute the principal Eigen vector of $\{e_k(x)\}$

3. Return to step 1

α = ($\alpha_j(x)$ j != 1)

0	0	2	0	0
1	0	0	0	0
0	0	0	1	0
0	7	0	0	0
0	0	0	1	0
3	0	0	0	0
0	5	0	0	1
0	0	1	0	2

$$e_k(x) = x - \sum_{j \neq k} \propto_j D_j$$

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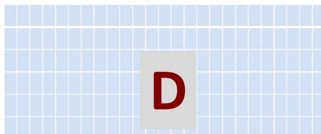
K-SVD



- Termination of each iteration: Updated dictionary
- Conclusion: A dictionary where any data vector can be composed of at most K dictionary entries
  - More generally, sparse composition

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K-SVD algorithm (skip)

Initialization : Set the random normalized dictionary matrix $\mathbf{D}^{(0)} \in \mathbb{R}^{n\times K}$. Set $J = 1$.
Repeat until convergence,
Sparse Coding Stage: Use any pursuit algorithm to compute $\mathbf{x}_i$ for $i = 1, 2, \ldots, N$

$$\min_{\mathbf{x}} \{\|\mathbf{y}_i - \mathbf{D}\mathbf{x}\|_2^2\} \quad \text{subject to} \quad \|\mathbf{x}\|_0 \le T_0.$$

Codebook Update Stage: For $k = 1, 2, \ldots, K$

- Define the group of examples that use $\mathbf{d}_k$, $\omega_k = \{i|\ 1 \le i \le N,\ \mathbf{x}_i(k) \ne 0\}$.
- Compute

$$\mathbf{E}_k = \mathbf{Y} - \sum_{j\ne k} \mathbf{d}_j \mathbf{x}^j,$$

- Restrict $\mathbf{E}_k$ by choosing only the columns corresponding to those elements that initially used $\mathbf{d}_k$ in their representation, and obtain $\mathbf{E}_k^R$.
- Apply SVD decomposition $\mathbf{E}_k^R = \mathbf{U}\mathbf{\Delta}\mathbf{V}^T$. Update: $\mathbf{d}_k = \mathbf{u}_1, \mathbf{x}_R^k = \mathbf{\Delta}(1,1)\cdot \mathbf{v}_1$

Set $J = J + 1$.

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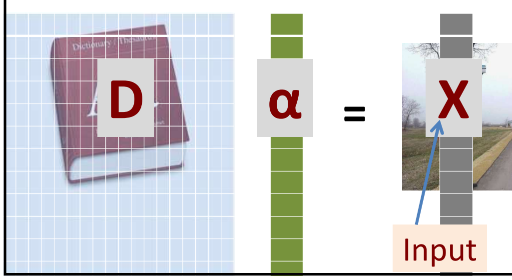
Problems

- How to obtain the dictionary
  - Which will give us meaningful representations
- How to compute the weights?



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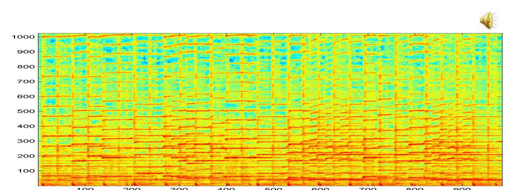
So how does that work



- In case you forgot this music…
- 975 vectors (1025 dimensions)
- N=12, K=5

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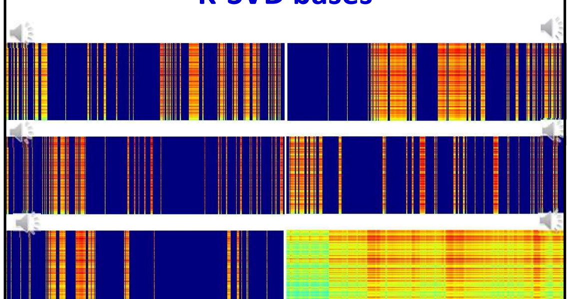
K-SVD bases



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Applications of Sparse Representations

- Many many applications
  - Signal representation
  - Statistical modelling
  - ..
  - We've seen one: Compressive sensing
- Another popular use
  - **Denoising**

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Denoising

- As the name suggests, remove noise!

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Denoising

- As the name suggests, remove noise!
- We will look at image denoising as an example

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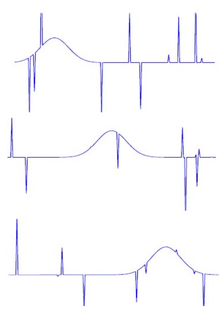
A toy example



111

A toy example

$$D = [I\ G]$$

I Identity matrix

G Translation of a Gaussian pulse

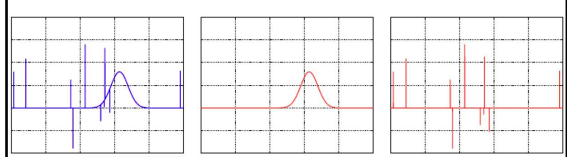


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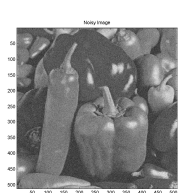
Image Denoising

- Here's what we want



113

Image Denoising

- Here's what we want

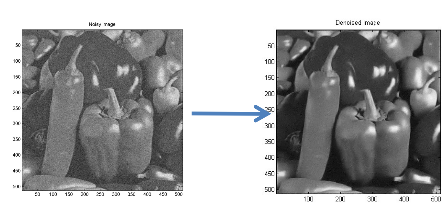


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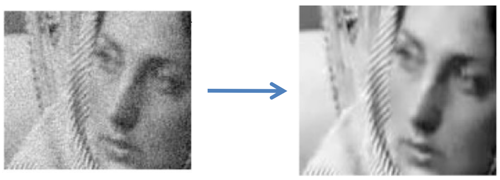
Image Denoising

- Here's what we want



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The Image Denoising Problem

- Given an image
- Remove Gaussian additive noise from it

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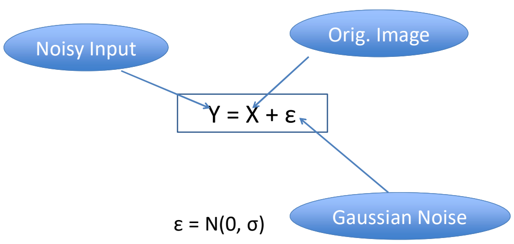
Image Denoising



$\varepsilon = N(0, \sigma)$

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Image Denoising

- Remove the noise from **Y**, to obtain **X** as best as possible.

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Image Denoising

- Remove the noise from **Y**, to obtain **X** as best as possible
- Using sparse representations over learned dictionaries

119

Image Denoising

- Remove the noise from **Y**, to obtain **X** as best as possible
- Using sparse representations over learned dictionaries
- We will *learn* the dictionaries

120

Image Denoising

- Remove the noise from **Y**, to obtain **X** as best as possible
- Using sparse representations over learned dictionaries
- We will *learn* the dictionaries
- What data will we use? *The corrupted image itself!*

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Image Denoising

- We use the data to be denoised to learn the dictionary.
- Training and denoising become an iterated process.
- We use image patches of size $\sqrt{n} \times \sqrt{n}$ pixels (i.e. if the image is 64x64, patches are 8x8)

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Image Denoising

- The data dictionary D
  - Size = n x k (k > n)
  - This is known and fixed, to start with
  - Every image patch can be sparsely represented using D

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Image Denoising

- Recall our equations from before.
- We want to find α so as to minimize the value of the equation below:

$$\underset{\underline{\alpha}}{Min}\ \{\|\underline{X} - \mathbf{D}\underline{\alpha}\|^2 + \lambda\|\underline{\alpha}\|_0\}$$

$$\underset{\underline{\alpha}}{Min}\{\|\underline{X} - \mathbf{D}\underline{\alpha}\|^2 + \lambda\|\underline{\alpha}\|_1\}$$

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Image Denoising

$$\underset{\underline{\alpha}}{Min}\ \{\|\underline{X} - \mathbf{D}\underline{\alpha}\|^2 + \lambda\|\underline{\alpha}\|_1\}$$

- In the above, X is a patch.

125

Image Denoising

$$\underset{\underline{\alpha}}{Min}\ \{\|\underline{X} - \mathbf{D}\underline{\alpha}\|^2 + \lambda\|\underline{\alpha}\|_1\}$$

- In the above, X is a patch.
- If the larger image is fully expressed by the every patch in it, how can we go from patches to the image?

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Image Denoising

$$\underset{\alpha_{ij},X}{Min}\ \{\mu\|X-Y\|_2^2+\sum_{ij}\|R_{ij}X-\mathbf{D}\alpha_{ij}\|_2^2+\sum_{ij}\lambda_{ij}\|\alpha_{ij}\|_0\}$$

Sparse and Overcomplete Representations 127

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Image Denoising

$$\underset{\alpha_{ij},X}{Min}\ \{\mu\|X-Y\|_2^2+\sum_{ij}\|R_{ij}X-\mathbf{D}\alpha_{ij}\|_2^2+\sum_{ij}\lambda_{ij}\|\alpha_{ij}\|_0\}$$

(X – Y) is the error between the input and denoised image. μ is a penalty on the error.

Sparse and Overcomplete Representations 128

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Image Denoising

$$\underset{\alpha_{ij},X}{Min}\ \{\mu\|X-Y\|_2^2+\sum_{ij}\|R_{ij}X-\mathbf{D}\alpha_{ij}\|_2^2+\sum_{ij}\lambda_{ij}\|\alpha_{ij}\|_0\}$$

Error bounding in each patch
- R_{ij} selects the $(ij)^{th}$ patch
- Terms in summation = no. of patches

Sparse and Overcomplete Representations 129

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Image Denoising

$$\underset{\alpha_{ij},X}{Min}\ \{\mu\|X-Y\|_2^2+\sum_{ij}\|R_{ij}X-\mathbf{D}\alpha_{ij}\|_2^2+\sum_{ij}\lambda_{ij}\|\alpha_{ij}\|_0\}$$

λ forces sparsity

Sparse and Overcomplete Representations 130

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Image Denoising

- But, we don't "***know***" our dictionary D.
- We want to estimate D as well.

Sparse and Overcomplete Representations 131

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Image Denoising

- But, we don't "***know***" our dictionary D.
- We want to estimate D as well.

$$\underset{D,\alpha_{ij},X}{Min}\ \{\mu\|X-Y\|_2^2+\sum_{ij}\|R_{ij}X-\mathbf{D}\alpha_{ij}\|_2^2+\sum_{ij}\lambda_{ij}\|\alpha_{ij}\|_0\}$$

We can use the previous equation itself!!!

Sparse and Overcomplete Representations 132

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Image Denoising

$$\underset{D,\alpha_{ij},X}{Min}\ \{\mu\|\underline{X}-Y\|_2^2+\sum_{ij}\|\underline{R_{ij}X}-\mathbf{D}\underline{\alpha}_{ij}\|_2^2$$
$$+\sum_{ij}\lambda_{ij}\|\underline{\alpha_{ij}}\|_0\}$$

How do we estimate all 3 at once?

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Image Denoising

$$\underset{D,\alpha_{ij},X}{Min}\ \{\mu\|\underline{X}-Y\|_2^2+\sum_{ij}\|\underline{R_{ij}X}-\mathbf{D}\underline{\alpha}_{ij}\|_2^2$$
$$+\sum_{ij}\lambda_{ij}\|\underline{\alpha_{ij}}\|_0\}$$

How do we estimate all 3 at once?

We cannot estimate them at the same time!

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Image Denoising

$$\underset{D,\alpha_{ij},X}{Min}\ \{\mu\|\underline{X}-Y\|_2^2+\sum_{ij}\|\underline{R_{ij}X}-\mathbf{D}\underline{\alpha}_{ij}\|_2^2$$
$$+\sum_{ij}\lambda_{ij}\|\underline{\alpha_{ij}}\|_0\}$$

How do we estimate all 3 at once?

Fix 2, and find the optimal 3rd.

135

Image Denoising

$$\underset{D,\alpha_{ij},X}{Min}\ \{\mu\|\underline{X}-Y\|_2^2+\sum_{ij}\|\underline{R_{ij}X}-\mathbf{D}\underline{\alpha}_{ij}\|_2^2$$
$$+\sum_{ij}\lambda_{ij}\|\underline{\alpha_{ij}}\|_0\}$$

Initialize X = Y

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Image Denoising

$$\underset{\alpha_{ij}}{Min}\ \{\underbrace{\mu\|\underline{X}-Y\|_2^2}_{0}+\sum_{ij}\|\underline{R_{ij}X}-\mathbf{D}\underline{\alpha}_{ij}\|_2^2$$
$$+\sum_{ij}\lambda_{ij}\|\underline{\alpha_{ij}}\|_0\}$$

Initialize X = Y, initialize D

You know how to solve the remaining portion for α – MP, BP!

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Image Denoising

- Now, update the dictionary D.
- Update D one column at a time, following the [K-SVD algorithm]
- K-SVD maintains the sparsity structure

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Image Denoising

- Now, update the dictionary D.
- Update D one column at a time, following the K-SVD algorithm
- K-SVD maintains the sparsity structure
- Iteratively update α and D

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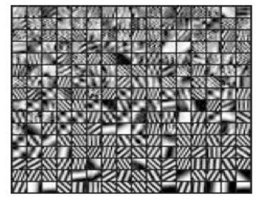
Image Denoising



Learned Dictionary for Face Image denoising

From: M. Elad and M. Aharon, *Image denoising via learned dictionaries and sparse representation*, CVPR, 2006.

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Image Denoising

$$\underset{X}{Min}\ \{\mu\|X - Y\|_2^2 + \sum_{ij}\|R_{ij}X - \mathbf{D}\alpha_{ij}\|_2^2 + \sum_{ij}\lambda_{ij}\|\alpha_{ij}\|_0\}$$

Const. wrt X

We know D and α

The quadratic term above has a closed-form solution

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Image Denoising

$$\underset{X}{Min}\ \{\mu\|X - Y\|_2^2 + \sum_{ij}\|R_{ij}X - \mathbf{D}\alpha_{ij}\|_2^2 + \sum_{ij}\lambda_{ij}\|\alpha_{ij}\|_0\}$$

Const. wrt X

We know D and α

$$X = (\mu I + \sum_{ij} R_{ij}^T R_{ij})^{-1}(\mu Y + \sum_{ij} R_{ij}^T D\alpha_{ij})$$

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Image Denoising

- Summarizing… We wanted to obtain 3 things

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Image Denoising

- Summarizing… We wanted to obtain 3 things

- Weights **α**
- Dictionary **D**
- Denoised Image **X**

144

Image Denoising

- Summarizing… We wanted to obtain 3 things

➢ Weights **α** – Your favorite pursuit algorithm
➢ Dictionary **D** – Using K-SVD
➢ Denoised Image **X**

145

Image Denoising

- Summarizing… We wanted to obtain 3 things

➢ Weights **α** – Your favorite pursuit algorithm
➢ Dictionary **D** – Using K-SVD
➢ Denoised Image **X**

Iterating

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Image Denoising

- Summarizing… We wanted to obtain 3 things

➢ Weights **α**
➢ Dictionary **D**
➢ Denoised Image **X**- Closed form solution

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Image Denoising

- Here's what we want

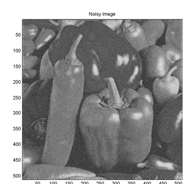
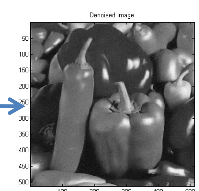

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Image Denoising

- Here's what we want


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Comparing to Other Techniques

Non-Gaussian data


PCA of ICA Which is which?

Images from Lewicki and Sejnowski, *Learning Overcomplete Representations*, 2000.

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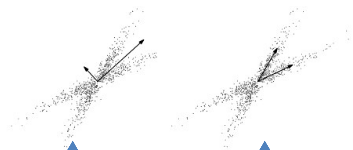
Comparing to Other Techniques

Non-Gaussian data



PCA ICA

Images from Lewicki and Sejnowski, *Learning Overcomplete Representations*, 2000.

Sparse and Overcomplete Representations 151

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Comparing to Other Techniques

Non-Gaussian data

PCA ICA

Images from Lewicki and Sejnowski, *Learning Overcomplete Representations*, 2000.

Sparse and Overcomplete Representations 152

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Comparing to Other Techniques

Data still in 2-D space

ICA Overcomplete

Doesn't capture the underlying representation, which Overcomplete representations can do…

Sparse and Overcomplete Representations 153

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Summary

- Overcomplete representations can be more powerful than component analysis techniques.
- Dictionary can be learned from data.
- Relative advantages and disadvantages of the pursuit algorithms.

Sparse and Overcomplete Representations 154

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