MLSP linear algebra refresher


"YOU LEARN SOMETHING NEW EVERYDAY"
FALSE.
YOU LEARN SOMETHING OLD EVERY DAY. JUST BECAUSE YOU'VE JUST LEARNED IT DOESN'T MEAN IT'S NEW, OTHER PEOPLE ALREADY KNEW IT.
quickmeme.com

I learned something old today!

Book

- Fundamentals of Linear Algebra, Gilbert Strang

- Important to be very comfortable with linear algebra
  - Appears repeatedly in the form of Eigen analysis, SVD, Factor analysis
  - Appears through various properties of matrices that are used in machine learning
    - Often used in the processing of data of various kinds
    - Will use sound and images as examples

- Today's lecture: Definitions
  - Very small subset of all that's used
  - Important subset, intended to help you recollect

Incentive to use linear algebra

- Simplified notation!

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{y} \quad \longleftrightarrow \quad \sum_j y_j \sum_i x_i a_{ij}$$

- Easier intuition
  - *Really convenient geometric interpretations*

- Easy code translation!

```
for i=1:n
  for j=1:m
    c(i)=c(i)+y(j)*x(i)*a(i,j)
  end
end
```

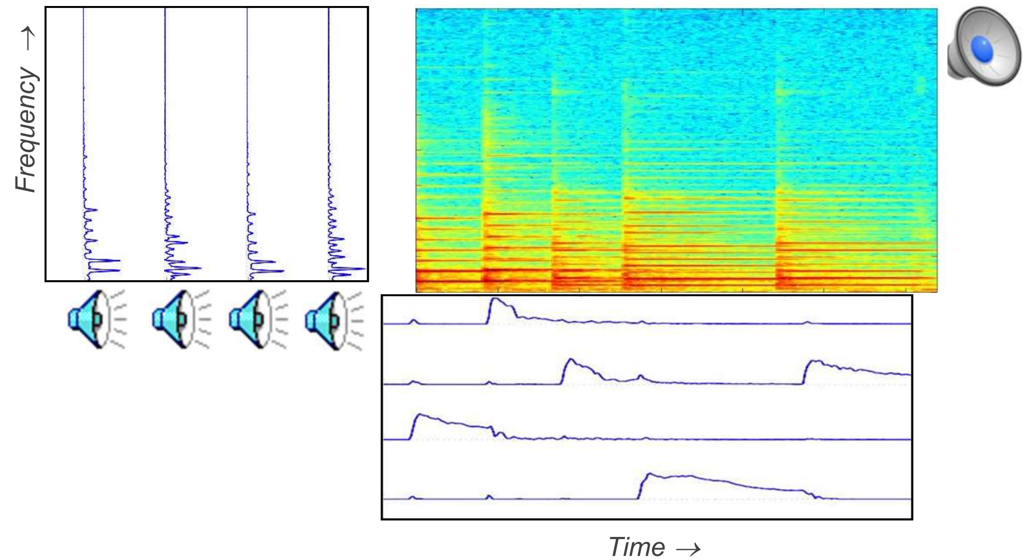
$\longleftrightarrow$

```
C=x*A*y
```

And other things you can do

Rotation + Projection +
Scaling + Perspective

From Bach's Fugue in Gm



Decomposition (NMF)

- Manipulate Data
- Extract information from data
- Represent data..
- Etc.

Overview

- Vectors and matrices
- Basic vector/matrix operations
- Various matrix types
- Matrix properties
  - Determinant
  - Inverse
  - Rank
- Solving simultaneous equations
- Projections
- Eigen decomposition
- SVD

Overview

- **Vectors and matrices**
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- Solving simultaneous equations
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- SVD

What is a vector

Column vector

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

An Nx1 vector

$$\begin{bmatrix} a & b & c \end{bmatrix}$$

Row vector

A 1xN vector

- A rectangular or horizontal arrangement of numbers

What is a vector

Column vector

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An Nx1 vector

$$\begin{bmatrix} a & b & c \end{bmatrix}$$

Row vector

A 1xN vector

- A rectangular or horizontal arrangement of numbers
- Which, without additional context, is actually a useless and meaningless mathematical object

A *meaningful* vector

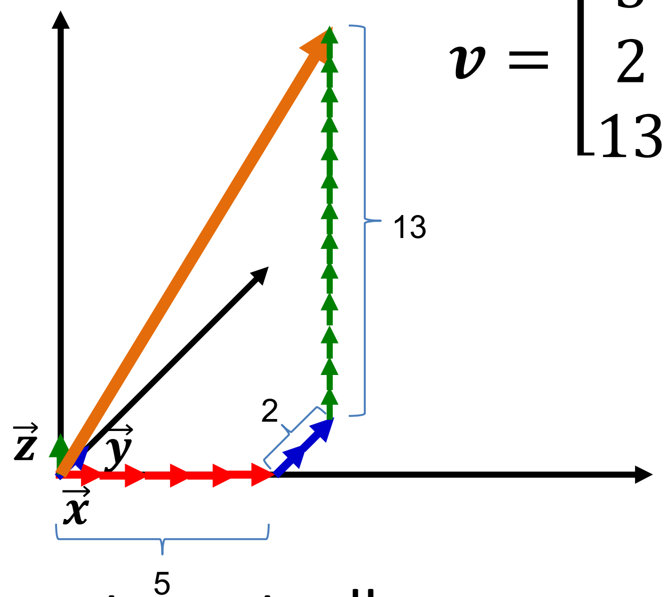
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{array}{l} \leftarrow \text{height} \\ \leftarrow \text{weight} \\ \leftarrow \text{age} \end{array}$$

- A rectangular or horizontal arrangement of numbers
- Where each number refers to a different quantity

What is a vector

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boldsymbol{v} = a\vec{\boldsymbol{x}} + \mathrm{b}\vec{\boldsymbol{y}} + c\vec{\boldsymbol{z}}$$



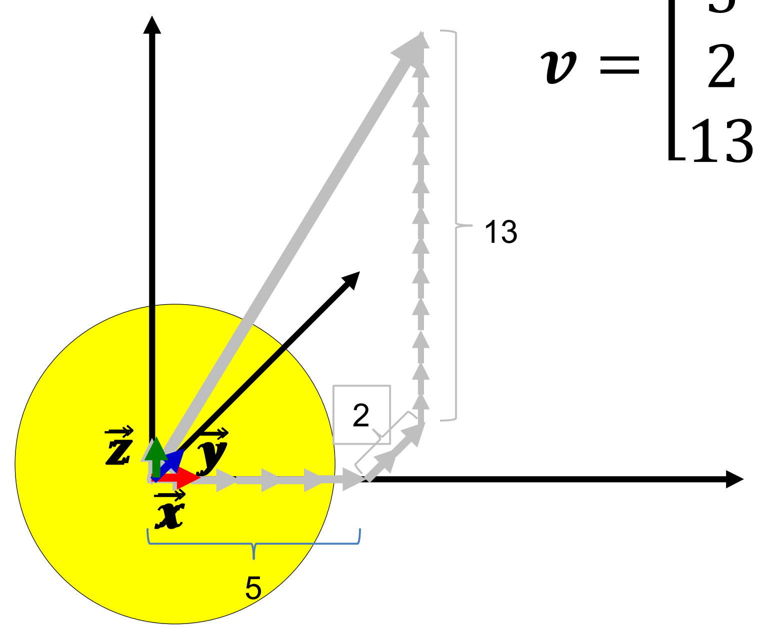
$$\boldsymbol{v} = \begin{bmatrix} 5 \\ 2 \\ 13 \end{bmatrix}$$

- Each component of the vector actually represents the *number of steps* along a set of *basis* directions
  - The vector cannot be interpreted without reference to the bases!!!!!
  - The bases are often *implicit* – we all just agree upon them and don't have to mention them

Standard Bases

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boldsymbol{v} = a\vec{\boldsymbol{x}} + \mathrm{b}\vec{\boldsymbol{y}} + c\vec{\boldsymbol{z}}$$



$$\boldsymbol{v} = \begin{bmatrix} 5 \\ 2 \\ 13 \end{bmatrix}$$

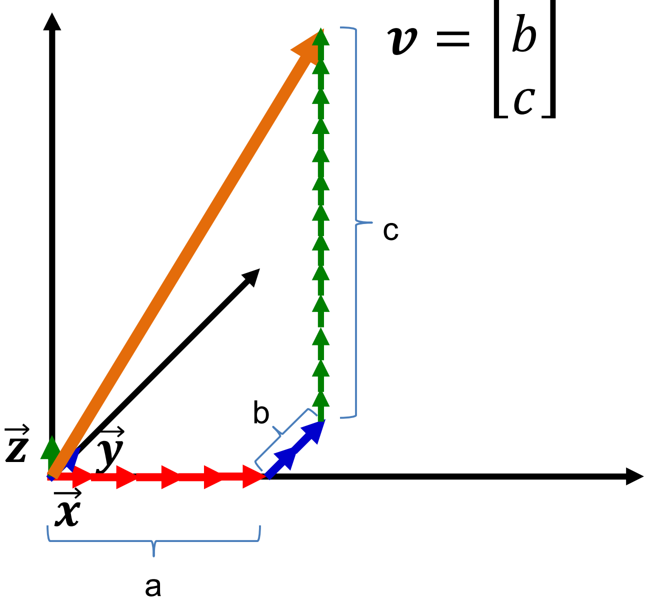
- “Standard” bases are “Orthonormal”
  - Each of the bases is at 90° to every other basis
    - Moving in the direction of one basis results in *no* motion along the directions of other bases
  - All bases are unit length

A vector by another basis..

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{using } \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}}$$

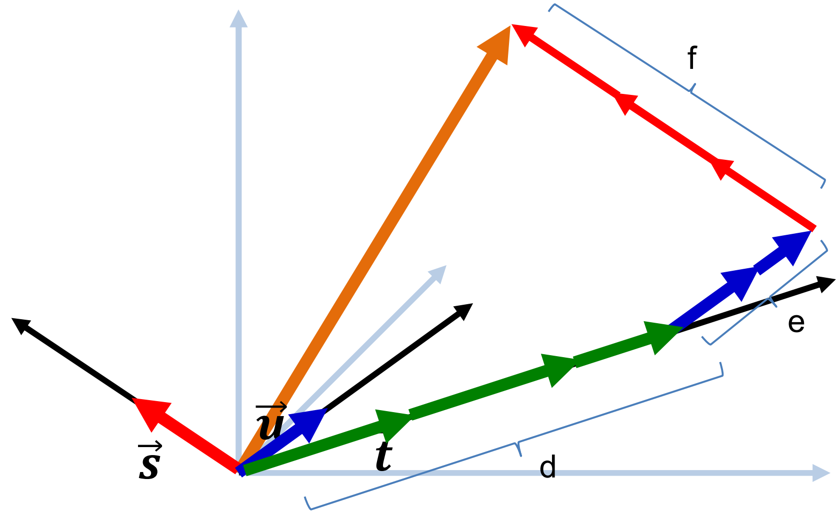
$$\boldsymbol{v} = a\vec{\boldsymbol{x}} + b\vec{\boldsymbol{y}} + c\vec{\boldsymbol{z}}$$

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



$$\boldsymbol{v} = d\vec{\boldsymbol{s}} + e\vec{\boldsymbol{t}} + f\vec{\boldsymbol{u}}$$

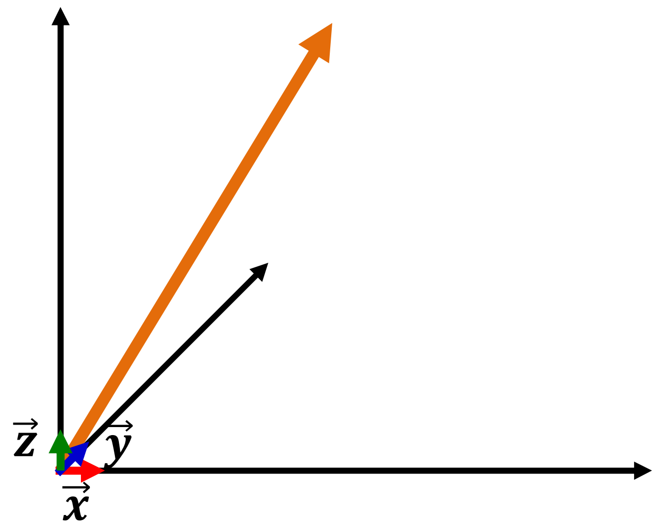
$$\boldsymbol{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$



- For non-standard bases we will generally *have* to specify the bases to be understood

Length of a vector

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



$$|\boldsymbol{v}| = \sqrt{a^2 + b^2 + c^2}$$

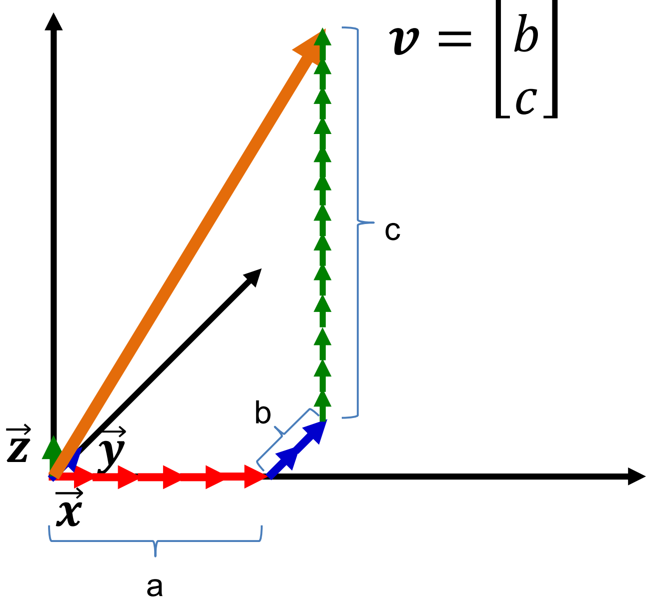
- The Euclidean distance from origin to the location of the vector

Length of a vector..

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{using } \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}, \vec{\boldsymbol{z}}$$

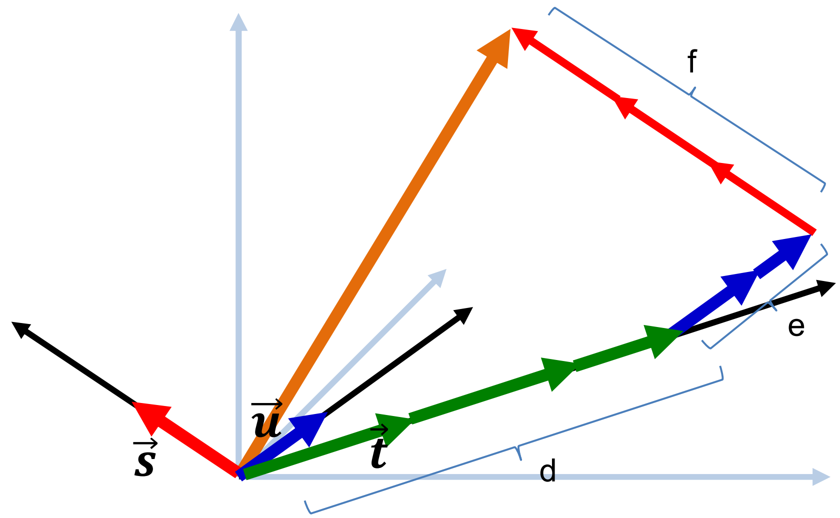
$$\boldsymbol{v} = a\vec{\boldsymbol{x}} + b\vec{\boldsymbol{y}} + c\vec{\boldsymbol{z}}$$

$$\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



$$\boldsymbol{v} = d\vec{\boldsymbol{s}} + e\vec{\boldsymbol{t}} + f\vec{\boldsymbol{u}}$$

$$\boldsymbol{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

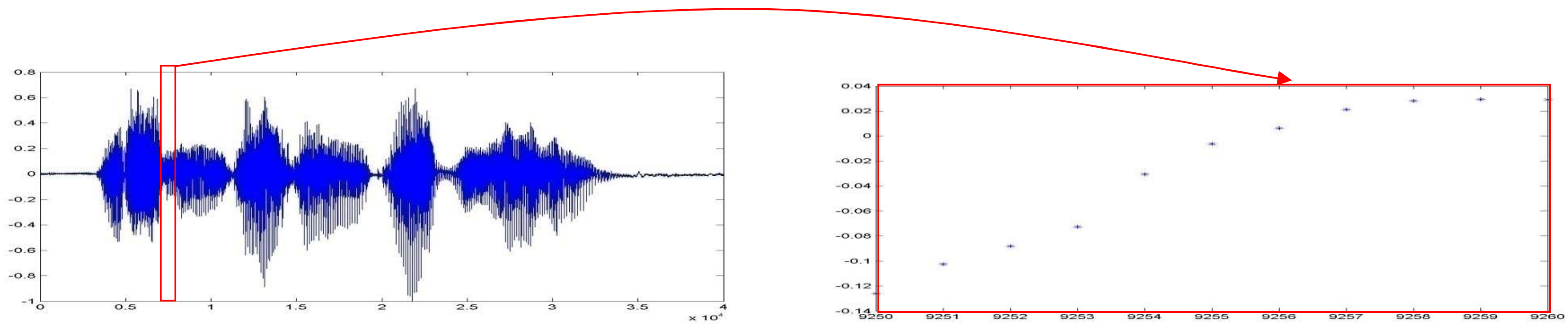


The norm of a vector depends on the bases used to specify it

$$|\boldsymbol{v}| = \sqrt{a^2 + b^2 + c^2} \quad \textbf{OR} \quad |\boldsymbol{v}| = \sqrt{d^2 + e^2 + f^2}$$

Representing signals as vectors

- Signals are frequently represented as vectors for manipulation
- E.g. A segment of an audio signal

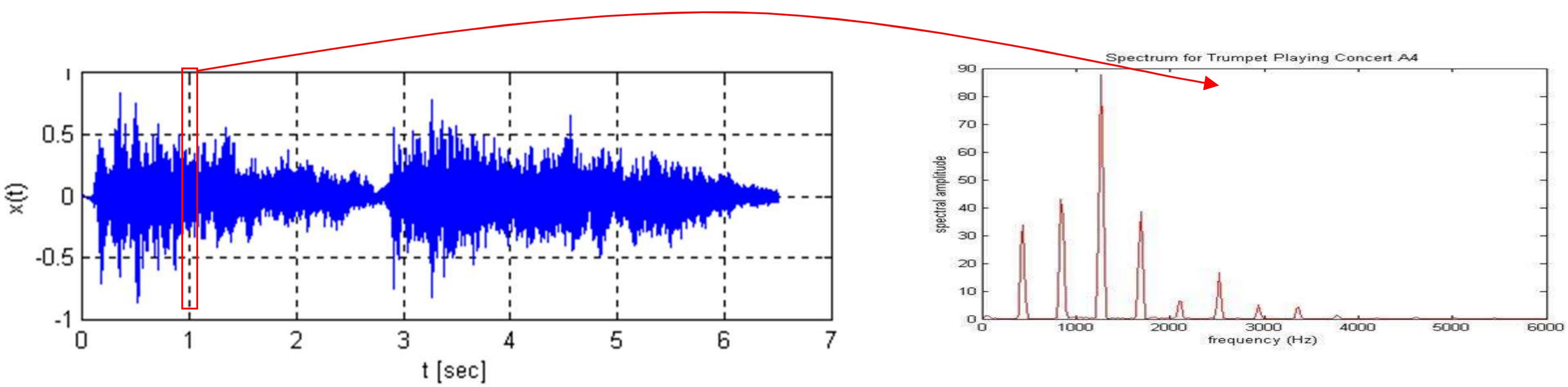


- Represented as a vector of sample values

$$[s_1 \;\; s_2 \;\; s_3 \;\; s_4 \;\ldots\; s_N]$$

Representing signals as vectors

- Signals are frequently represented as vectors for manipulation
- E.g. The ***spectrum*** segment of an audio signal

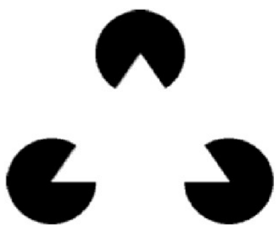


- Represented as a vector of sample values

$$[S_1 \;\; S_2 \;\; S_3 \;\; S_4 \; \ldots \; S_M]$$

  – Each component of the vector represents a frequency component of the spectrum

Representing an image as a vector



- 3 pacmen
- A 321 x 399 grid of pixel values
  - Row and Column = position
- A 1 x 128079 vector
  - “Unraveling” the image

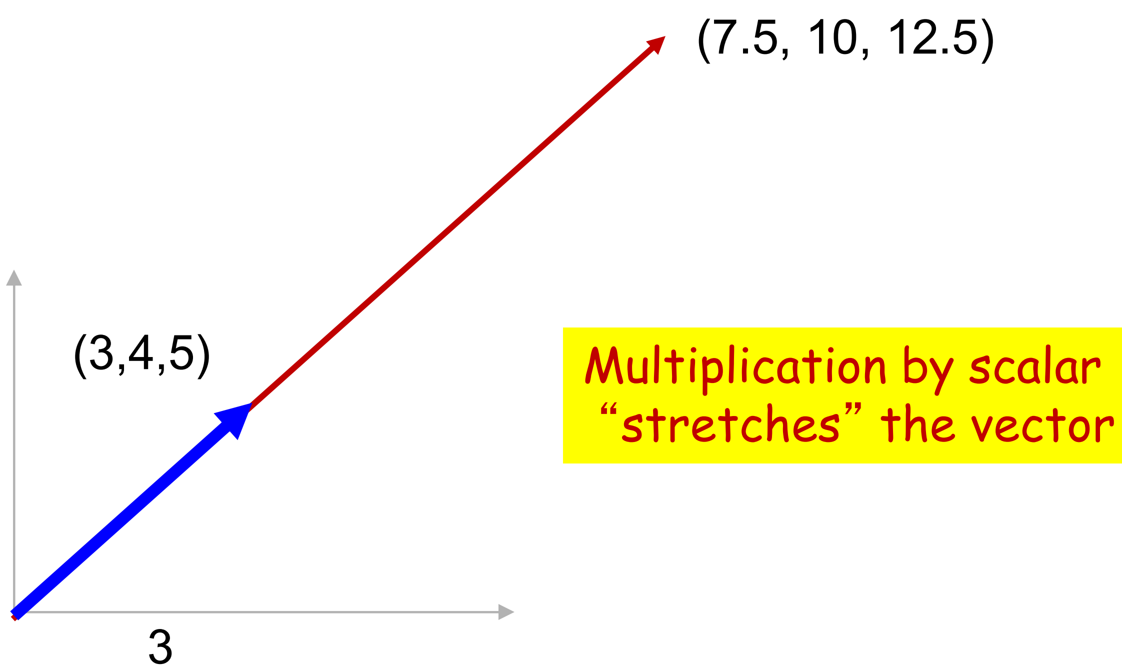
$$\begin{bmatrix}1 & 1 & . & 1 & 1 & . & 0 & 0 & 0 & . & . & 1\end{bmatrix}$$

  - Note: This can be recast as the grid that forms the image

Vector operations

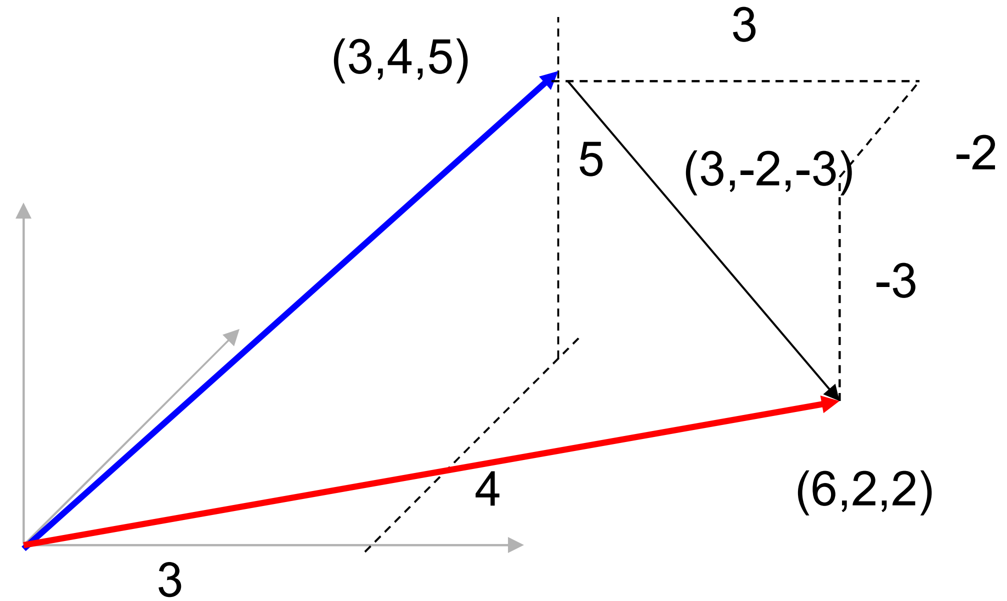
- Addition
- Multiplication
- Inner product
- Outer product

Vector Operations: Multiplication by scalar



- Vector multiplication by scalar: each component multiplied by scalar
  – 2.5 x[3,4,5] = [7.5, 10, 12.5]
- Note: as a result, vector norm is also multiplied by the scalar
  – ||2.5 x [3,4,5]|| = 2.5x|| [3, 4, 5]||

Vector Operations: Addition



- Vector addition: individual components add
  - [3,4,5] + [3,-2,-3] = [6,2,2]

Vector operation: Inner product

- Multiplication of a row vector by a column vector to result in a scalar
  - Note order of operation
  - The *inner* product between two row vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is the product of $\boldsymbol{u}^T$ and $\boldsymbol{v}$
  - Also called the “dot” product

$$\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$\boldsymbol{u}.\boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{v} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = a.d + b.e + c.f$$

Vector operation: Inner product

- The inner product of a vector with itself is its squared norm
  - This will be the squared length

$$\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\boldsymbol{u}.\boldsymbol{u} = \boldsymbol{u}^T\boldsymbol{u} = a^2 + b^2 + c^2 = \|\boldsymbol{u}\|^2$$

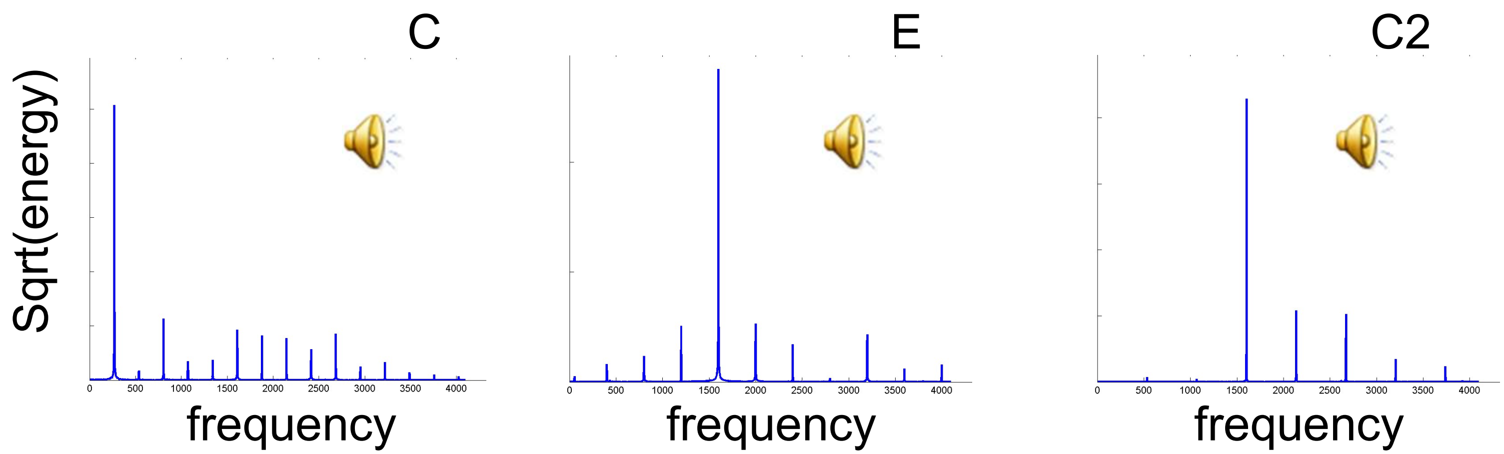
Vector dot product

- Example:
  - Coordinates are yards, not ave/st
  - $\mathbf{a} = [200\ 1600]$,
    $\mathbf{b} = [770\ 300]$

- The dot product of the two vectors relates to the length of a *projection*
  - How much of the first vector have we covered by following the second one?
  - Must normalize by the length of the "target" vector

$$\frac{\mathbf{a}\cdot\mathbf{b}^T}{\|\mathbf{a}\|} = \frac{\begin{bmatrix}200 & 1600\end{bmatrix}\cdot\begin{bmatrix}770\\300\end{bmatrix}}{\left\|\begin{bmatrix}200 & 1600\end{bmatrix}\right\|} \approx 393\text{yd}$$

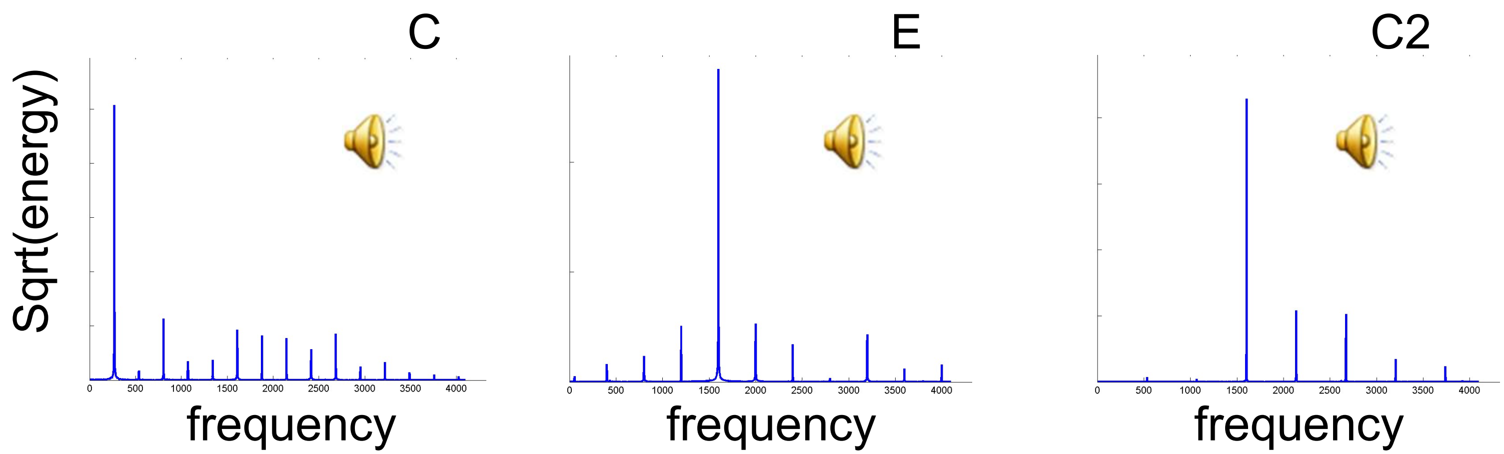
Vector dot product



$[11 \;\; 9 \;\; . \;\; 0 \;\; 54 \;\; 1 \;\; . \;\; . \;\; . \;\; 1]$ $[3 \;\; . \;\; 24 \;\; . \;\; . \;\; 16 \;\; . \;\; 14 \;\; . \;\; 1]$ $[0 \;\; . \;\; 0 \;\; . \;\; 3 \;\; 0 \;\; . \;\; 13 \;\; . \;\; 0]$

- Vectors are spectra
  - Energy at a discrete set of frequencies
  - Actually 1 x 4096
  - X axis is the *index* of the number in the vector
    - Represents frequency
  - Y axis is the value of the number in the vector
    - Represents magnitude

Vector dot product

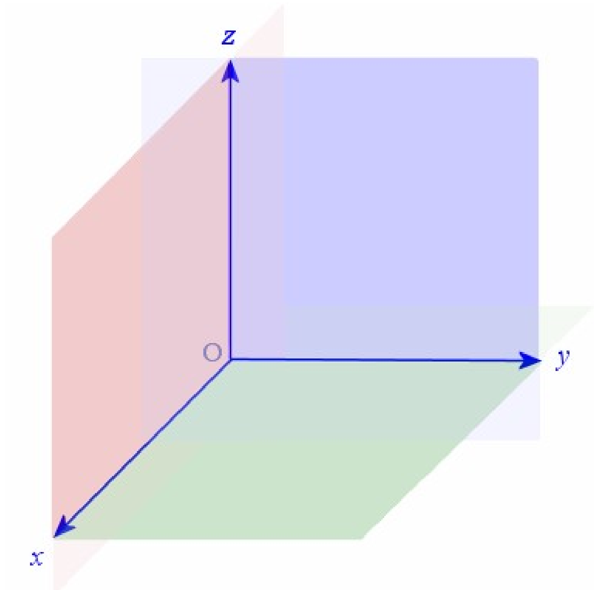


$[11 \quad 9 \quad . \quad 0 \quad 54 \quad 1 \quad . \quad . \quad . \quad 1]$ $[3 \quad . \quad 24 \quad . \quad . \quad 16 \quad . \quad 14 \quad . \quad 1]$ $[0 \quad . \quad 0 \quad . \quad 3 \quad 0 \quad . \quad 13 \quad . \quad 0]$

- How much of C is also in E
  - How much can you fake a C by playing an E
  - C.E / |C||E| = 0.1
  - Not very much
- How much of C is in C2?
  - C.C2 / |C| /|C2| = 0.5
  - Not bad, you can fake it

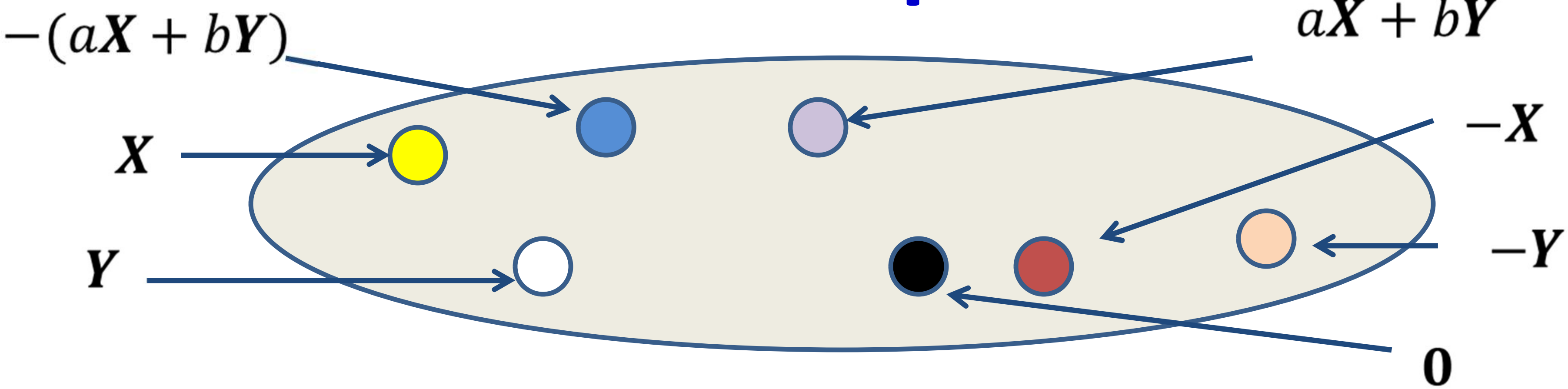
The notion of a “Vector Space”

An introduction to *spaces*



- Conventional notion of "space": a geometric construct of a certain number of "dimensions"
  - E.g. the 3-D space that this room and every object in it lives in

A *vector* space



- A *vector space* is an infinitely large set of vectors with the following properties
  - The set includes the zero vector (of all zeros)
  - The set is "closed" under addition
    - If X and Y are in the set, aX + bY is also in the set for any two scalars a and b
  - For every X in the set, the set also includes the additive inverse Y = -X, such that X + Y = 0

Additional Properties

- Additional requirements:
  - Scalar multiplicative identity element exists: $1X = X$
  - Addition is associative: $X + Y = Y + X$
  - Addition is commutative: $(X+Y)+Z = X+(Y+Z)$
  - Scalar multiplication is commutative: $a(bX) = (ab)\,X$
  - Scalar multiplication is distributive: $(a+b)X = aX + bX$
    $a(X+Y) = aX + aY$

Example of vector space

$$\mathbf{S} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ for\ all\ x, y, z \in \mathcal{R} \right\}$$

- Set of *all* three-component column vectors
  - Note we used the term three-*component,* rather than three-*dimensional*
- The set includes the zero vector
- For every $\mathbf{X}$ in the set $\alpha \in \mathcal{R}$, every $\alpha\mathbf{X}$ is in the set
- For every $\mathbf{X}, \mathbf{Y}$ in the set, $\alpha\mathbf{X} + \beta\mathbf{Y}$ is in the set
- $-\mathbf{X}$ is in the set
- Etc.

Example: a function space

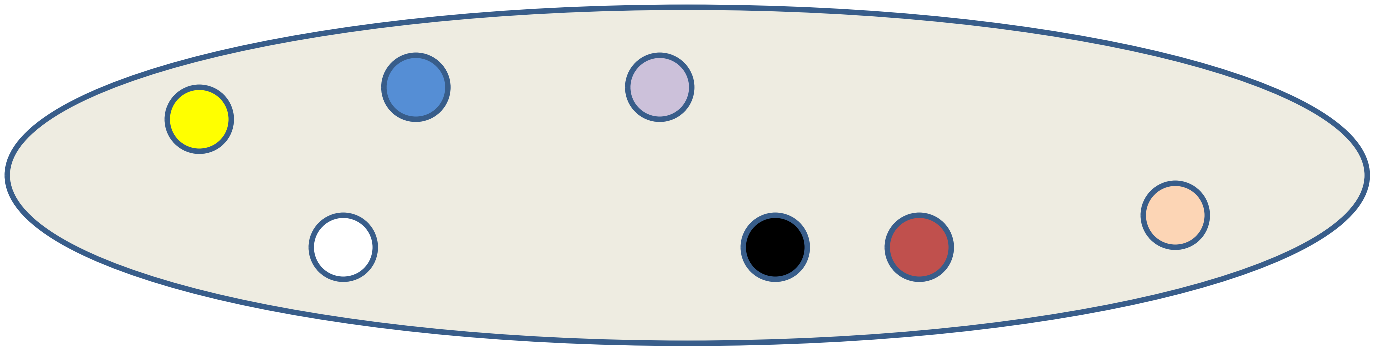
$$\mathbf{S} = \left\{ \begin{matrix} a\cos(\mathrm{x}) + b\sin(3\mathrm{x}) \ for \ all \ a, b, \in \ \mathcal{R}\}, \\ \mathrm{x} \in [-\pi, \pi] \end{matrix} \right\}$$

- Entries are *functions* from $[-\pi, \pi]$ to $[-1,1]$

$$f: [-\pi, \pi] \longrightarrow [-1,1]$$

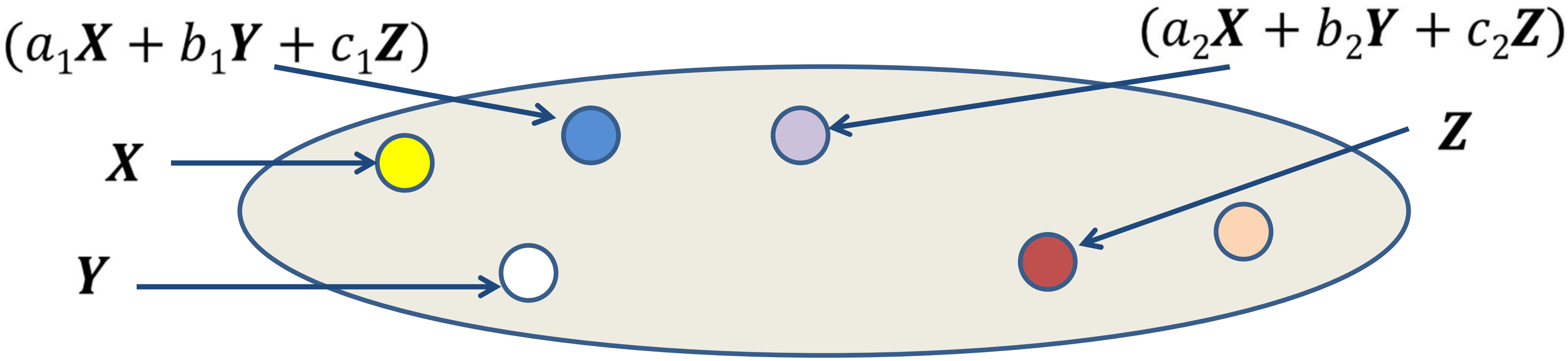
- Define $(f+g)(\mathrm{x}) = f(\mathrm{x}) + g(\mathrm{x})$ for any f and g in the set
- Verify that this is a space!

Dimension of a space



- Every element in the space can be composed of linear combinations of some other elements in the space
  - For any $\mathbf{X}$ in $\mathbf{S}$ we can write $\mathbf{X} = a\mathbf{Y}_1 + b\mathbf{Y}_2 + c\mathbf{Y}_3..$ for some other $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ .. in $\mathbf{S}$
    - Trivial to prove..

Dimension of a space



- What is the smallest subset of elements that can compose the entire set?
  - There may be multiple such sets
- The elements in this subset are called "bases"
  - The subset is a "basis" set
- The number of elements in the subset is the "dimensionality" of the space

Dimensions: Example

$$\mathbf{S} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; for\, all\; x, y, z \;\in\; \mathcal{R} \right\}$$

- What is the dimensionality of this vector space

Dimensions: Example

$$\mathbf{Z} = \left\{ a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, for\ all\ a, b\ \in\ \mathcal{R} \right\}$$

- What is the dimensionality of this vector space?
  - First confirm this is a proper vector space
- Note: all elements in $\mathbf{Z}$ are also in $\mathbf{S}$ (slide 36)
  - $\mathbf{Z}$ is a *subspace* of $\mathbf{S}$

Poll 1

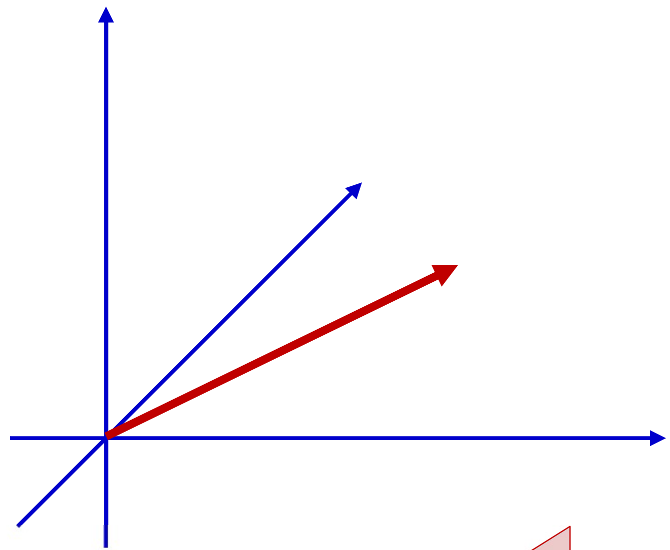
$$\mathbf{S} = \left\{ \begin{matrix} a\cos(\mathrm{x}) + b\sin(3\mathrm{x}) \ for\ all\ a, b, \in\ \mathcal{R}\}, \\ \mathrm{x} \in [-\pi, \pi] \end{matrix} \right\}$$

- What is the dimensionality of this space?

- Return to reality..

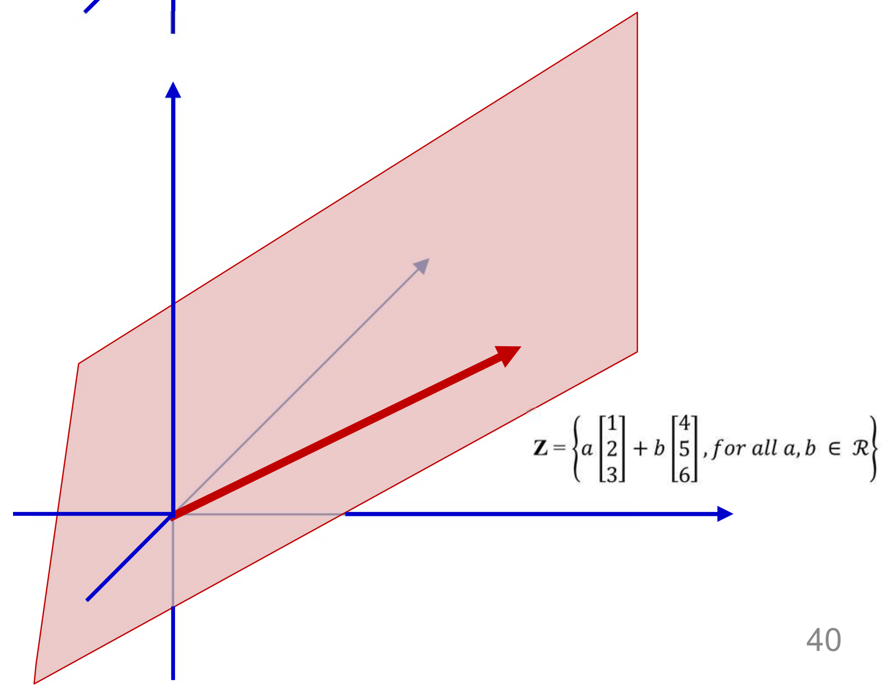
Returning to dimensions..

- Two interpretations of "dimension"
- The *spatial* dimension of a vector:
  - The number of components in the vector
  - An N-component vector "lives" in an N-dimensional space
  - Essentially a "stand-alone" definition of a vector against "standard" bases



- The *embedding* dimension of the vector
  - The dimensionality of the *subspace* the vector actually lives in
  - Only makes sense in the context where the vector is one element of a restricted set, e.g. a subspace or hyperplane

- Much of machine learning and signal processing is aimed at finding the latter from collections of vectors



Matrices..

What is a *matrix*

A 2x3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2.2 & 6 \\ 3.1 & 1 & 5 \end{bmatrix}$$

A 3x2 matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- Rectangular (or square) arrangement of numbers

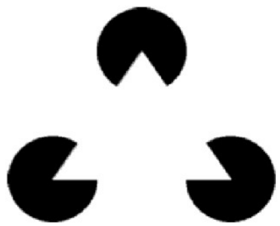
Dimensions of a matrix

- The matrix size is specified by the number of rows and columns

$$\mathbf{c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ \mathbf{r} = \begin{bmatrix} a & b & c \end{bmatrix}$$

  - c = 3x1 matrix: 3 rows and 1 column (vectors are matrices too)
  - r = 1x3 matrix: 1 row and 3 columns

$$\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$



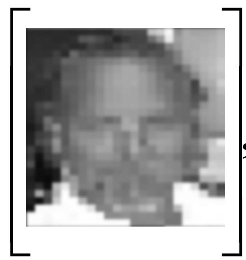
  - S = 2 x 2 matrix
  - R = 2 x 3 matrix
  - Pacman = 321 x 399 matrix

Dimensionality and Transposition

- A transposed matrix gets all its row (or column) vectors transposed in order
  - An NxM matrix becomes an MxN matrix

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \mathbf{x}^T = \begin{bmatrix} a & b & c \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} a & b & c \end{bmatrix}, \quad \mathbf{y}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad \mathbf{X}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

$\mathbf{M} =$ , $\mathbf{M}^T =$ 

What is a *matrix*

A 2x3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2.2 & 6 \\ 3.1 & 1 & 5 \end{bmatrix}$$

A 3x2 matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- A matrix by itself is uninformative, except through its relationship to vectors

Interpreting matrices

- Matrices as transforms
- Matrices as data containers
- Matrices as compositional building blocks for vector spaces

Interpreting matrices

- Matrices as transforms
- Matrices as data containers
- Matrices as compositional building blocks for vector spaces

Matrices as transforms

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

- Multiplying a vector by a matrix *transforms* the vector
  - $\boldsymbol{Ab} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{12}b_3 + a_{14}b_4 \\ a_{21}b_1 + a_{22}b_2 + a_{32}b_3 + a_{44}b_4 \\ a_{31}b_1 + a_{32}b_2 + a_{32}b_3 + a_{44}b_4 \end{bmatrix}$
- A matrix is a *transform* that *transforms* a vector
  - Above example: *left multiplication*. Matrix transforms a column vector
  - Dimensions must match!!
    - No. of columns of matrix = size of vector
    - Result inherits the number of rows from the matrix

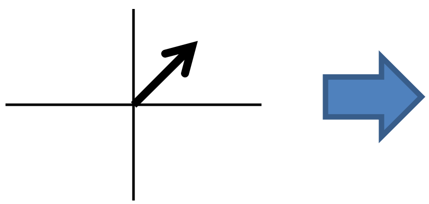
Matrices as transforms

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

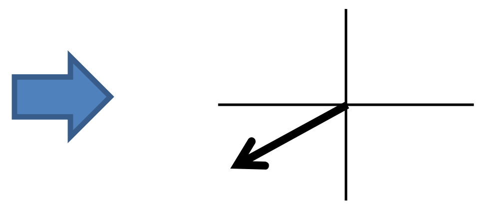
- Multiplying a vector by a matrix *transforms* the vector
  - $\boldsymbol{bA} = [b_1 \quad b_2 \quad b_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{21}b_2 + a_{31}b_3 \\ a_{12}b_1 + a_{22}b_2 + a_{32}b_3 \\ a_{13}b_1 + a_{23}b_2 + a_{33}b_3 \\ a_{14}b_1 + a_{24}b_2 + a_{34}b_3 \end{bmatrix}^T$
- A matrix is a *transform* that *transforms* a vector
  - Example: *right multiplication.* Matrix transforms a row vector
  - Dimensions must match!!
    - No. of rows of matrix = size of vector
    - Result inherits the number of columns from the matrix

Matrices transform a space

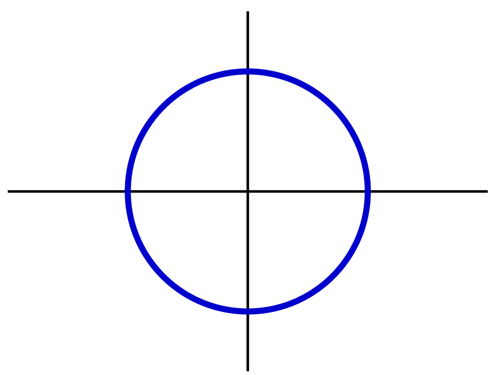
- A matrix is a ***transform*** that modifies vectors and vector spaces



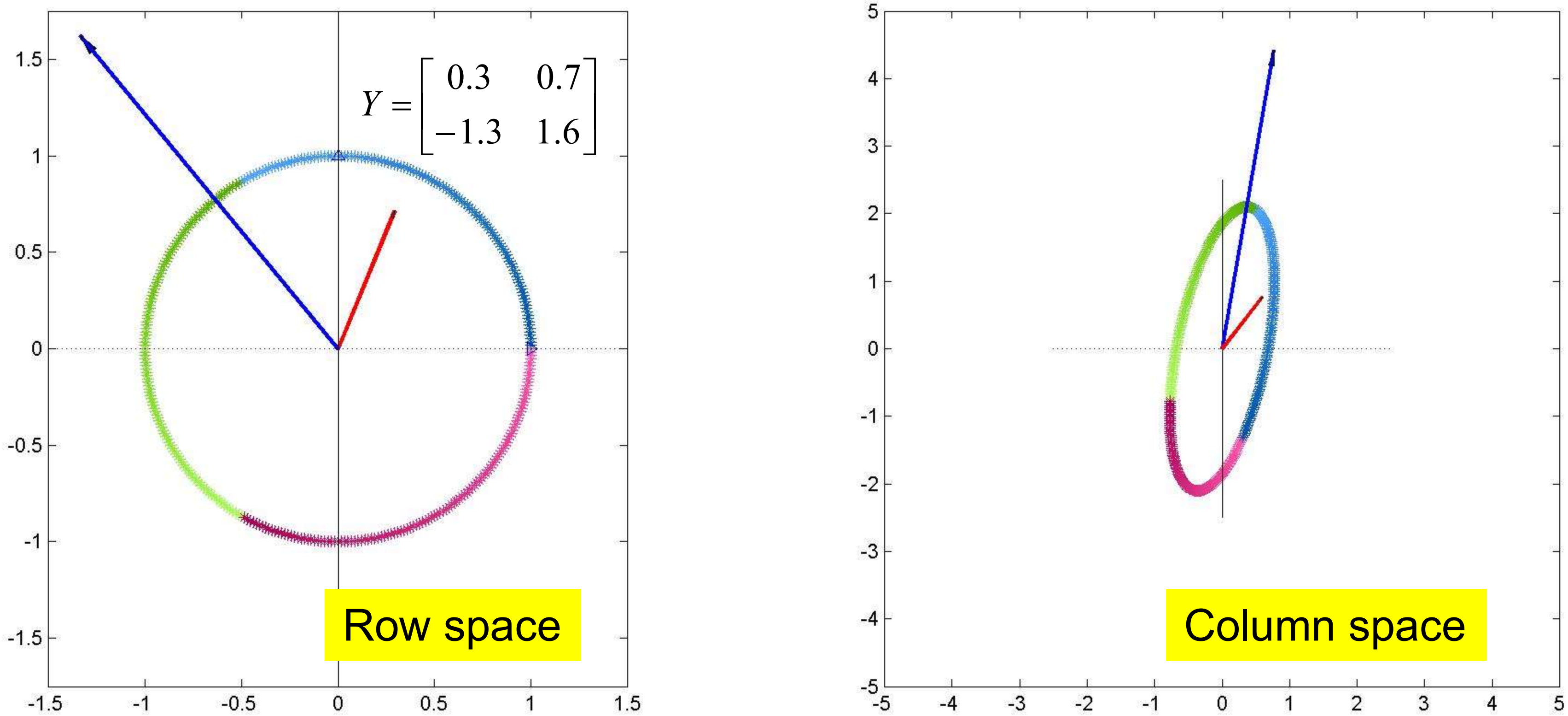
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



- So how does it transform the *entire space*?
- E.g. how will it transform the following figure?

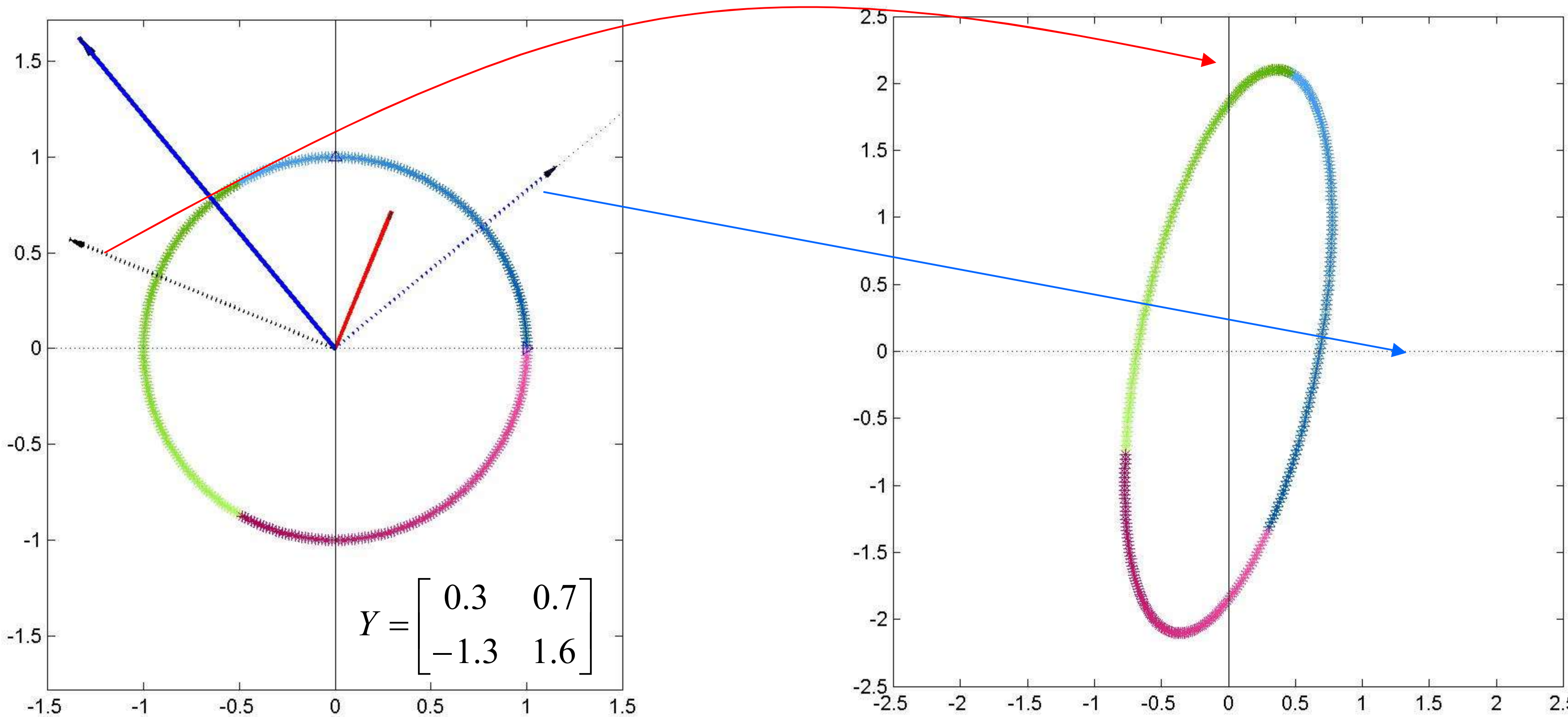


Multiplication of vector space by matrix



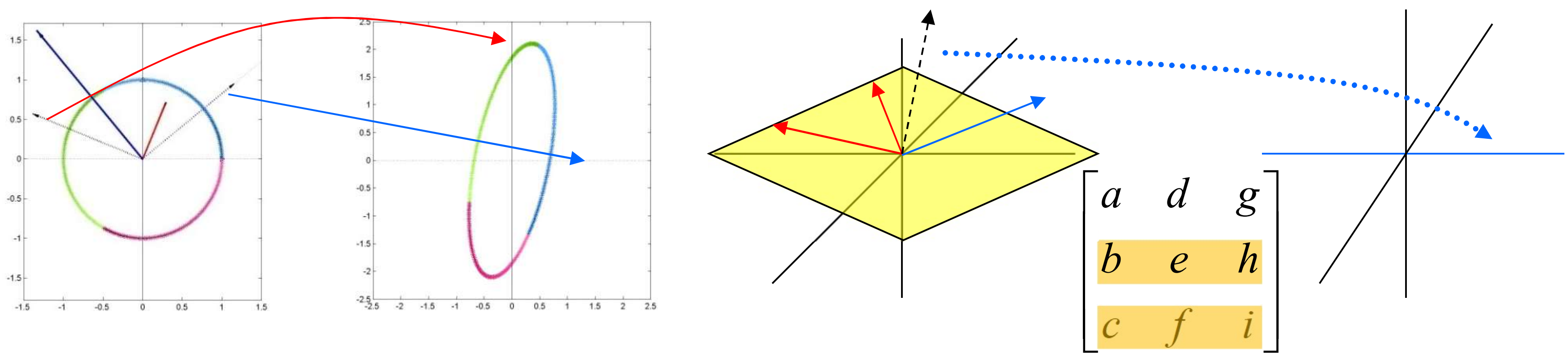
- The matrix rotates and scales the space
  - Including its own row vectors

Multiplication of vector space by matrix



- The *normals* to the row vectors in the matrix become the new axes
  - X axis = normal to the *second* row vector
    - Scaled by the inverse of the length of the *first* row vector

Matrix Multiplication



- The k-th axis corresponds to the normal to the hyperplane represented by the 1..k-1,k+1..N-th row vectors in the matrix
  - Any set of K-1 vectors represent a hyperplane of dimension K-1 or less

- The distance along the new axis equals the length of the projection on the k-th row vector
  - Expressed in inverse-lengths of the vector

Interpreting matrices

- Matrices as transforms
- Matrices as data containers
- Matrices as compositional building blocks for vector spaces

Matrices as data containers

- A matrix can be vertical stacking of row vectors

$$\mathbf{R} = \begin{bmatrix} \boxed{a \quad b \quad c} \\ \boxed{d \quad e \quad f} \end{bmatrix}$$

  - The space of all vectors that can be composed from the rows of the matrix is the *row space* of the matrix

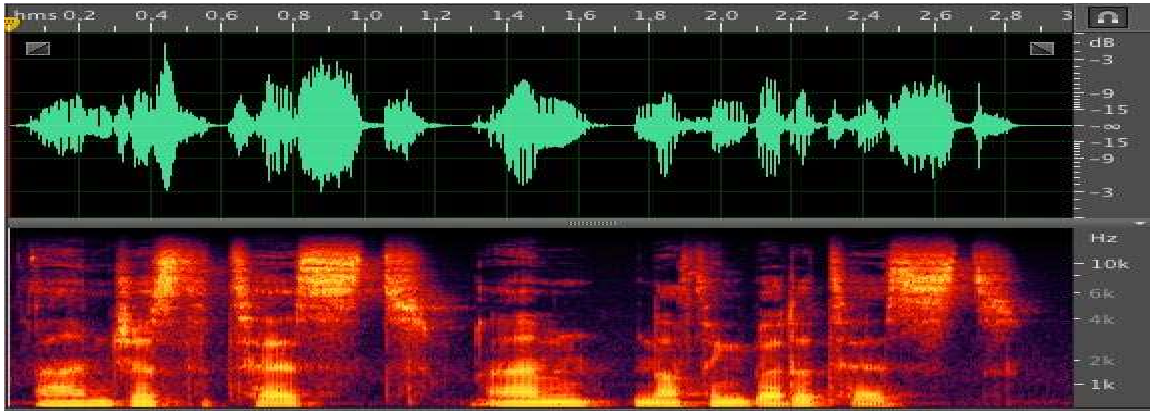
- Or a horizontal arrangement of column vectors

$$\mathbf{R} = \begin{bmatrix} \boxed{\begin{matrix} a \\ d \end{matrix}} & \boxed{\begin{matrix} b \\ e \end{matrix}} & \boxed{\begin{matrix} c \\ f \end{matrix}} \end{bmatrix}$$

  - The space of all vectors that can be composed from the columns of the matrix is the *column space* of the matrix

Representing a signal as a matrix

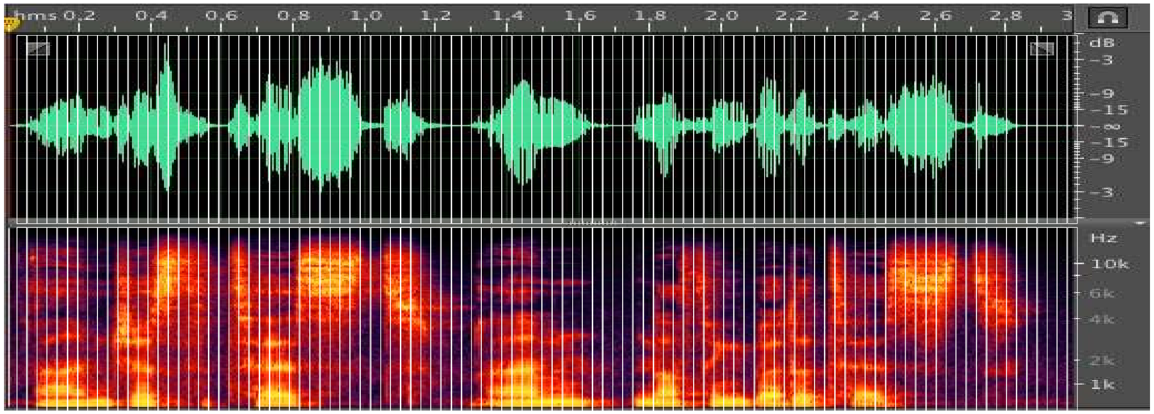
- Time series data like audio signals are often represented as spectrographic matrices



- Each column is the spectrum of a short segment of the audio signal

Representing a signal as a matrix

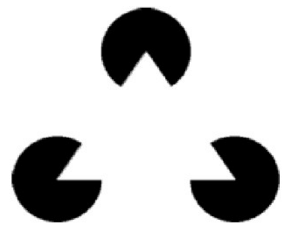
- Time series data like audio signals are often represented as spectrographic matrices



- Each column is the spectrum of a short segment of the audio signal

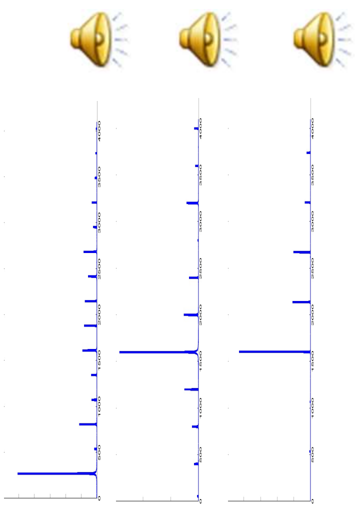
Representing a signal as a matrix

- Images are often just represented as matrices



```
>> X(1:32:end,1:40:end)

ans =

     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     0     0     0     1     1     1
     1     1     1     1     0     0     0     1     1     1
     1     1     1     1     0     1     0     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     0     1     1     1     1     1     0     1
     1     0     0     1     1     1     1     1     0     0
     1     0     0     0     1     1     1     0     0     0
     1     0     0     0     1     1     1     0     0     0
     1     1     1     1     1     1     1     1     1     1
```

Storing collections of data



X

$$\begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix}$$

$$\begin{bmatrix} f_{11} & \cdots & f_{K1} \\ \vdots & \ddots & \vdots \\ f_{1N} & \cdots & f_{KN} \end{bmatrix}$$

- Individual data instances can be packed into columns (or rows) of a matrix
  - A “data container” matrix

Interpreting matrices

- Matrices as transforms
- Matrices as data containers
- Matrices as compositional building blocks for vector spaces

Matrices as space constructors

- Right multiplying a matrix by a column vector mixes the columns of the matrix according to the numbers in the vector

  - $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{32} & a_{32} & a_{33} & a_{34} \end{bmatrix}$ $\quad\quad$ $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

$$\boldsymbol{Ab} = b_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + b_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + b_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} + b_4 \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$$

- “Mixes” the columns
  - “Transforms” row space to column space
- “Generates” the space of vectors that can be formed by mixing its own columns

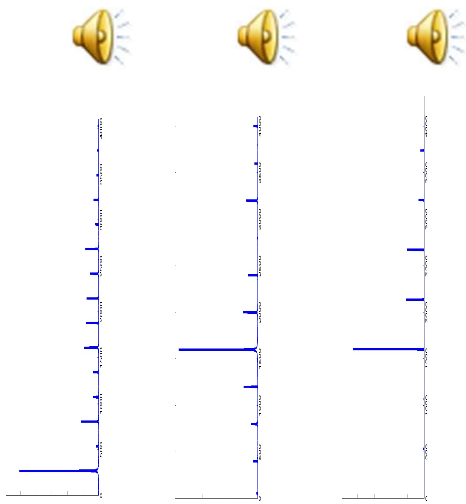
Multiplying a vector by a matrix

- Left multiplying a matrix by a row vector mixes the rows of the matrix according to the numbers in the vector
  - $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{32} & a_{32} & a_{33} & a_{34} \end{bmatrix}$ $\quad\quad \boldsymbol{b} = [b_1 \quad b_2 \quad b_3]$

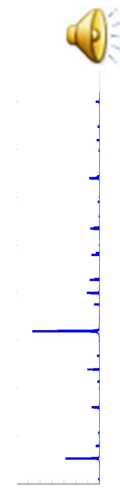
$$\boldsymbol{bA} = b_1[a_{11} \quad a_{12} \quad a_{13} \quad a_{14}] + b_2[a_{21} \quad a_{22} \quad a_{23} \quad a_{24}]$$
$$+b_3[a_{31} \quad a_{32} \quad a_{33} \quad a_{34}]$$

- "Mixes" the rows
  - "Transforms" column space to row space
- "Generates" the space of vectors that can be formed by mixing its own rows

Matrix multiplication: Mixing vectors

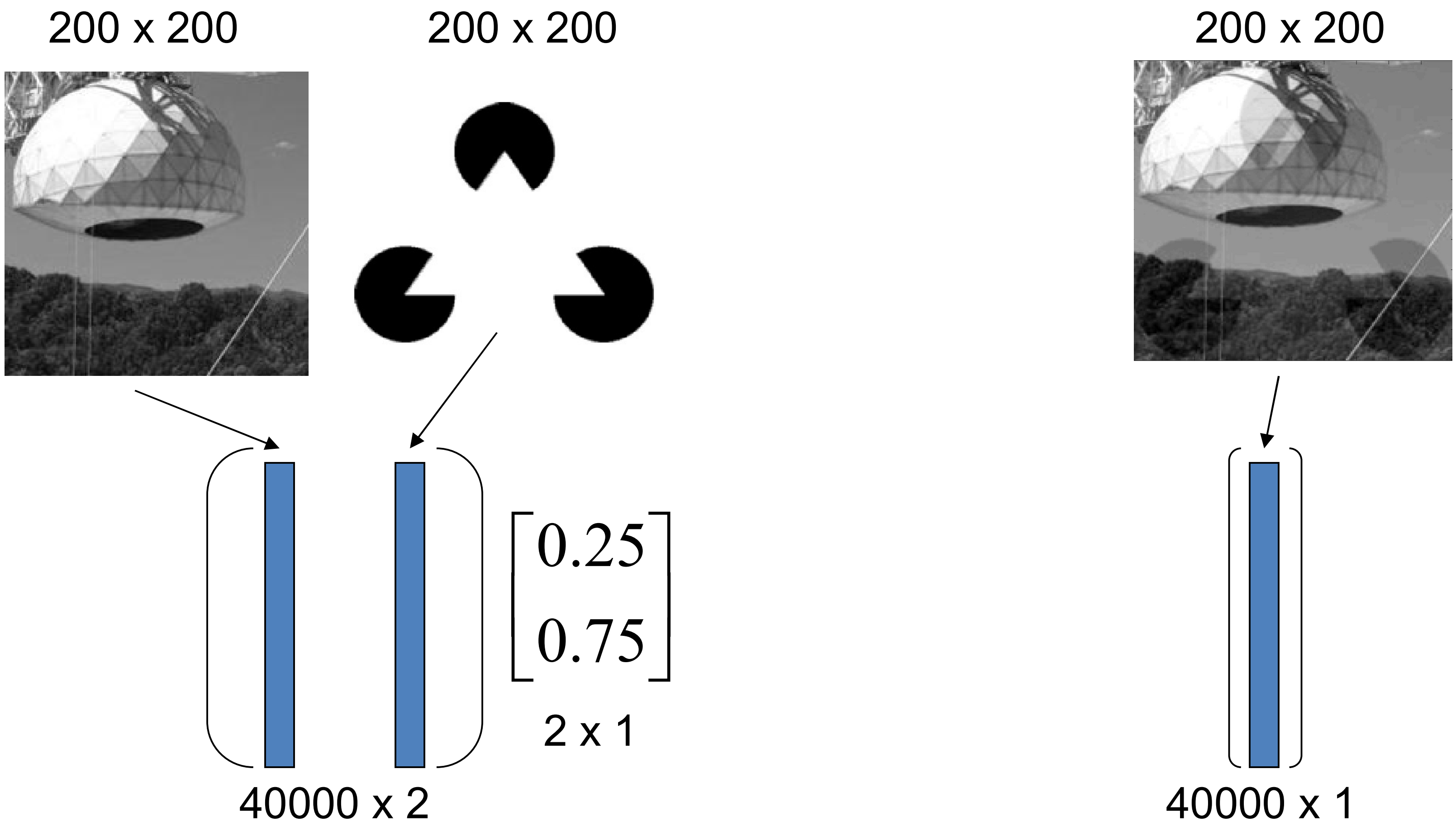


$$\begin{array}{c} X \\ \begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix} \end{array} \begin{array}{c} Y \\ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{array} = \begin{bmatrix} 7 \\ . \\ . \\ 2 \end{bmatrix}$$



- A physical example
  - The three column vectors of the matrix X are the spectra of three notes
  - The multiplying column vector Y is just a mixing vector
  - The result is a sound that is the mixture of the three notes

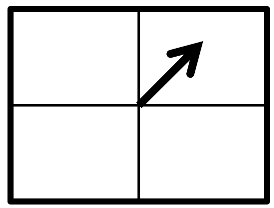
Matrix multiplication: Mixing vectors



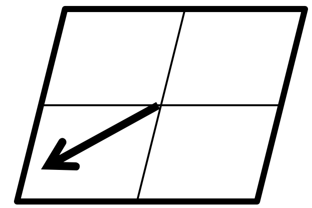
- Mixing two images
  - The images are arranged as columns
    - position value not included
  - The result of the multiplication is rearranged as an image

Interpretations of a matrix

- As a ***transform*** that modifies vectors and vector spaces



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



- As a ***container*** for data (vectors)

$$\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \end{bmatrix}$$

- As a ***generator*** of vector spaces..

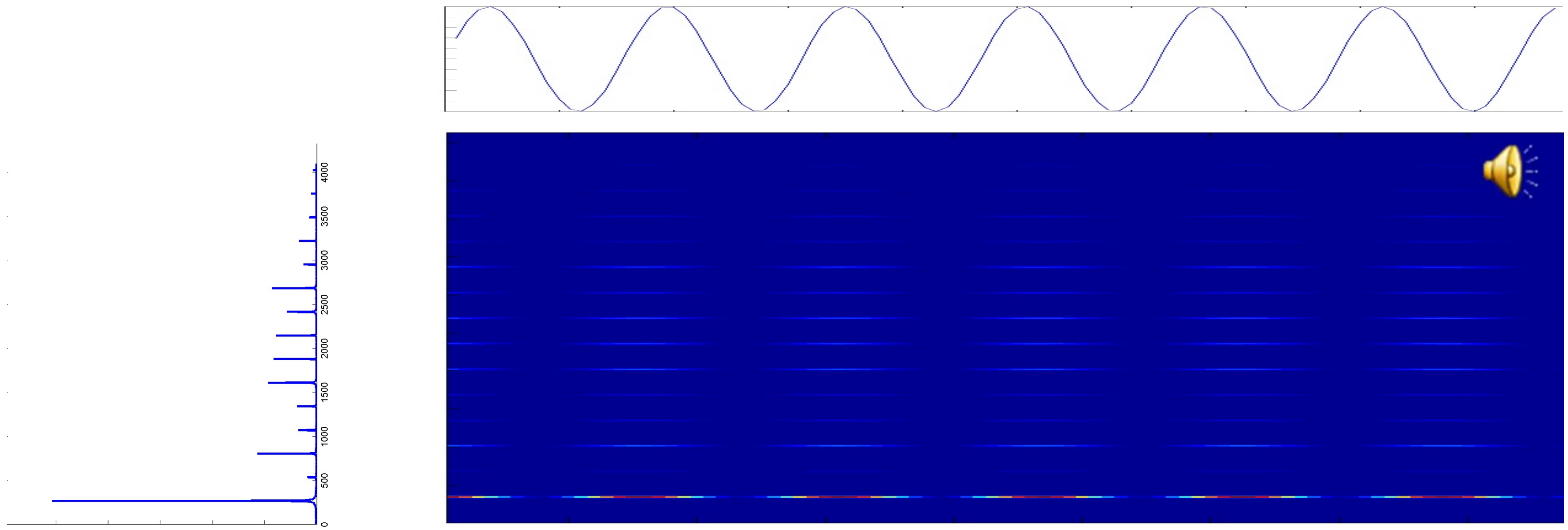
Matrix ops..

Vector multiplication: Outer product

- Product of a column vector by a row vector
- Also called vector *direct* product
- Results in a *matrix*
- *Transform or collection of vectors?*

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d & e & f \end{bmatrix} = \begin{bmatrix} a \cdot d & a \cdot e & a \cdot f \\ b \cdot d & b \cdot e & b \cdot f \\ c \cdot d & c \cdot e & c \cdot f \end{bmatrix}$$

Vector outer product



- The column vector is the spectrum
- The row vector is an amplitude modulation
- The outer product is a spectrogram
  – Shows how the energy in each frequency varies with time
  – The pattern in each column is a scaled version of the spectrum
  – Each row is a scaled version of the modulation

Matrix multiplication

$$\begin{bmatrix} a_{11} & . & . & a_{1N} \\ a_{21} & . & . & a_{2N} \\ . & . & . & . \\ a_{M1} & . & . & a_{MN} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & . & b_{1K} \\ . & . & . \\ b_{N1} & . & b_{NK} \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j} b_{j1} & . & . & \sum_j a_{1j} b_{jK} \\ . & . & . & . \\ . & . & . & . \\ \sum_j a_{Mj} b_{j1} & . & . & \sum_j a_{Mj} b_{jK} \end{bmatrix}$$

- Standard formula for matrix multiplication

Matrix multiplication

$$\begin{bmatrix} a_{11} & . & . & a_{1N} \\ a_{21} & . & . & a_{2N} \\ . & . & . & . \\ a_{M1} & . & . & a_{MN} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & . & b_{1K} \\ . & . & . \\ b_{N1} & . & b_{NK} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & . & . & \mathbf{a}_1 \cdot \mathbf{b}_K \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & . & . & \mathbf{a}_2 \cdot \mathbf{b}_K \\ . & . & . & . \\ \mathbf{a}_M \cdot \mathbf{b}_1 & . & . & \mathbf{a}_M \cdot \mathbf{b}_K \end{bmatrix}$$

- Matrix $\boldsymbol{A}$: A column of row vectors
- Matrix $\boldsymbol{B}$: A row of column vectors
- $\boldsymbol{AB}$: A matrix of inner products
  - Mimics the vector outer product rule

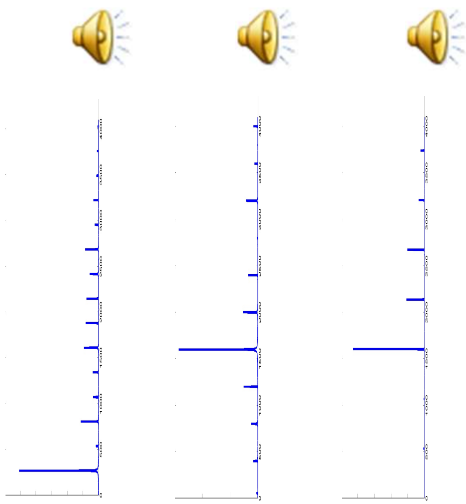
Matrix multiplication: another view

$$\begin{bmatrix} a_{11} & . & . & a_{1N} \\ a_{21} & . & . & a_{2N} \\ . & . & . & . \\ a_{M1} & . & . & a_{MN} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & . & b_{NK} \\ . & . & . \\ b_{N1} & . & b_{NK} \end{bmatrix} = \begin{bmatrix} a_{11} \\ . \\ . \\ a_{M1} \end{bmatrix} \begin{bmatrix} b_{11} & . & b_{1K} \end{bmatrix} + \begin{bmatrix} a_{12} \\ . \\ . \\ a_{M2} \end{bmatrix} \begin{bmatrix} b_{21} & . & b_{2K} \end{bmatrix} + \ldots + \begin{bmatrix} a_{1N} \\ . \\ . \\ a_{MN} \end{bmatrix} \begin{bmatrix} b_{N1} & . & b_{NK} \end{bmatrix}$$

- The outer product of the first column of A and the first row of B + outer product of the second column of A and the second row of B + ….

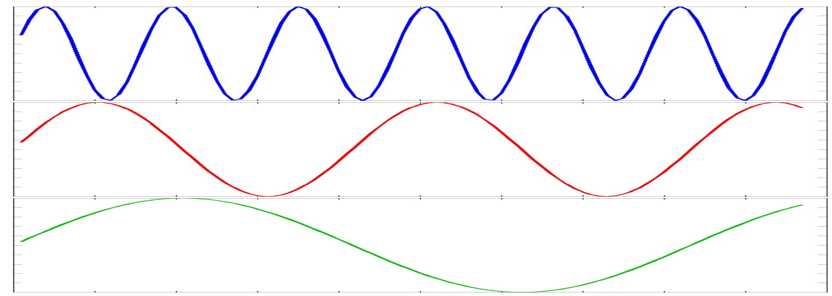
- *Sum of outer products*

Why is that useful?



$$\begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix}$$

X

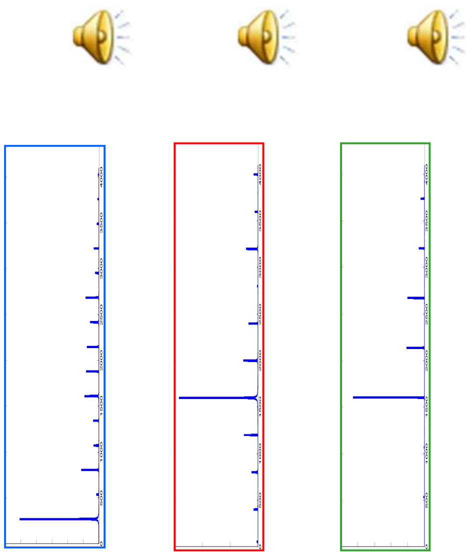


$$\begin{bmatrix} 0 & 0.5 & 0.75 & 1 & 0.75 & 0.5 & 0 & . & . & . & . & . \\ 1 & 0.9 & 0.7 & 0.5 & 0 & 0.5 & . & . & . & . & . & . \\ 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 0.95 & 1 & . & . & . & . & . \end{bmatrix}$$

Y

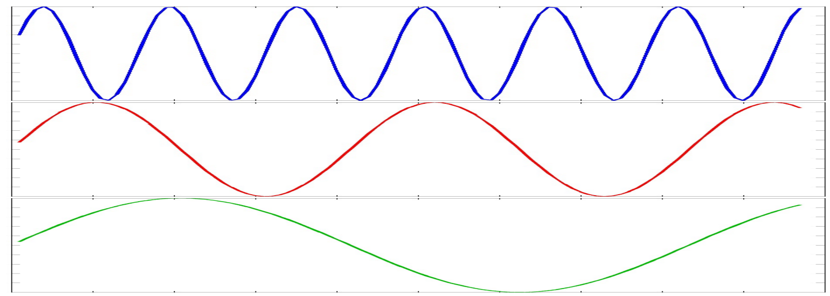
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



$$\begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix}$$

X

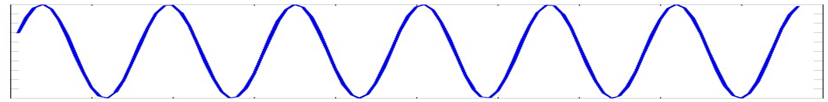


$$\begin{bmatrix} 0 & 0.5 & 0.75 & 1 & 0.75 & 0.5 & 0 & . & . & . & . & . \\ 1 & 0.9 & 0.7 & 0.5 & 0 & 0.5 & . & . & . & . & . & . \\ 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 0.95 & 1 & . & . & . & . & . \end{bmatrix}$$

Y

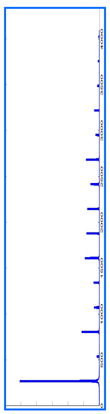
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



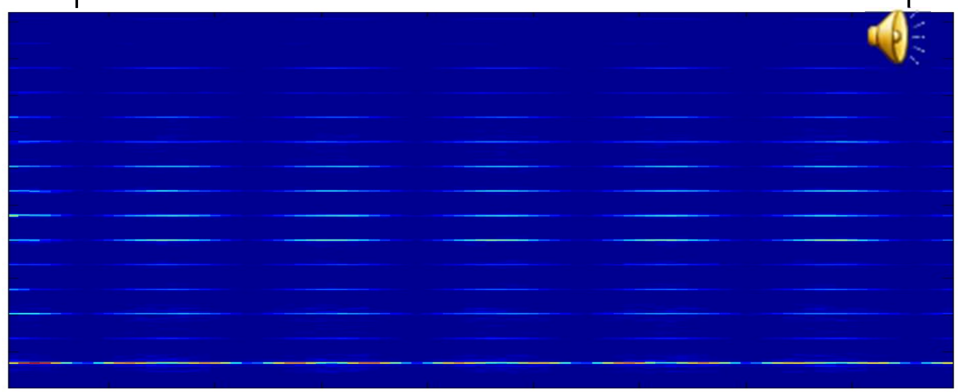
$$\begin{bmatrix} 0 & 0.5 & 0.75 & 1 & 0.75 & 0.5 & 0 & . & . & . & . & . \end{bmatrix}$$

Y



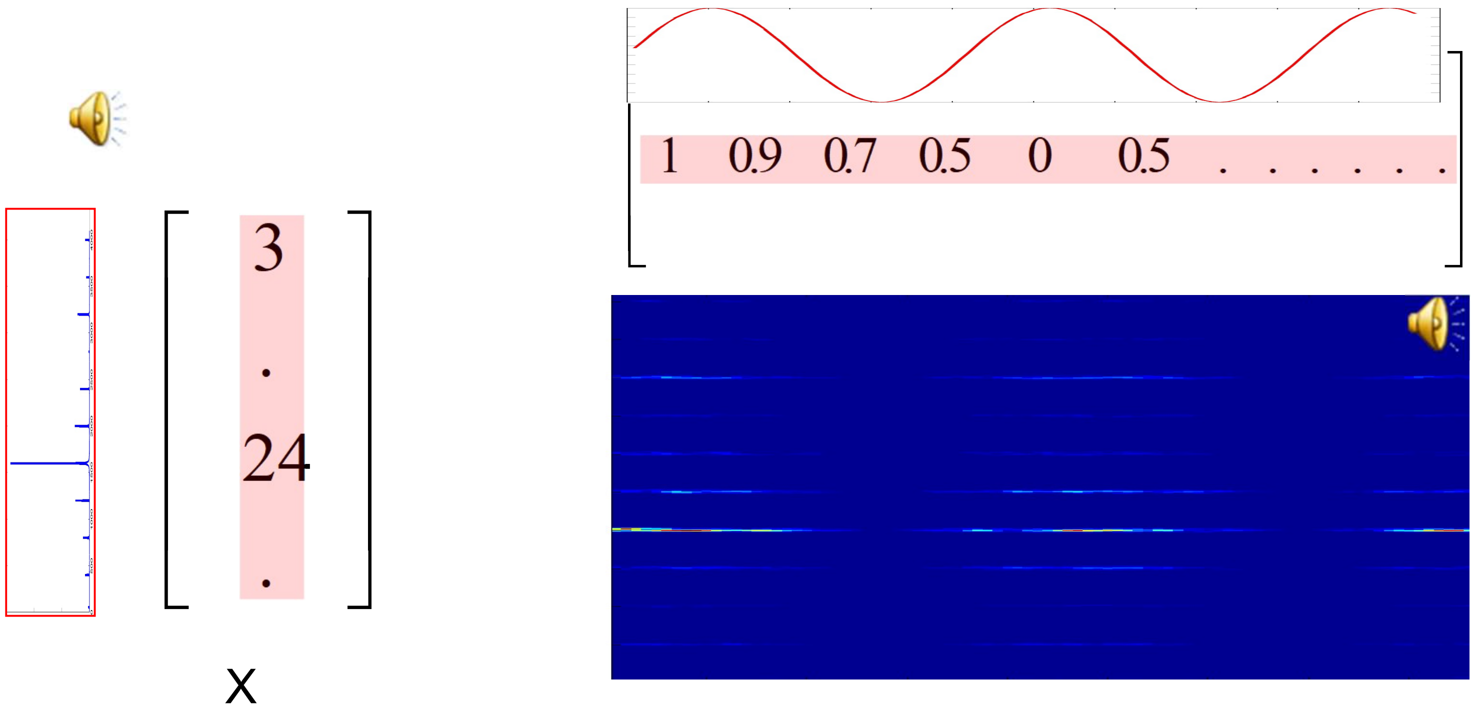
$$\begin{bmatrix} 1 & \\ . & \\ 9 & . \\ . & \end{bmatrix}$$

X



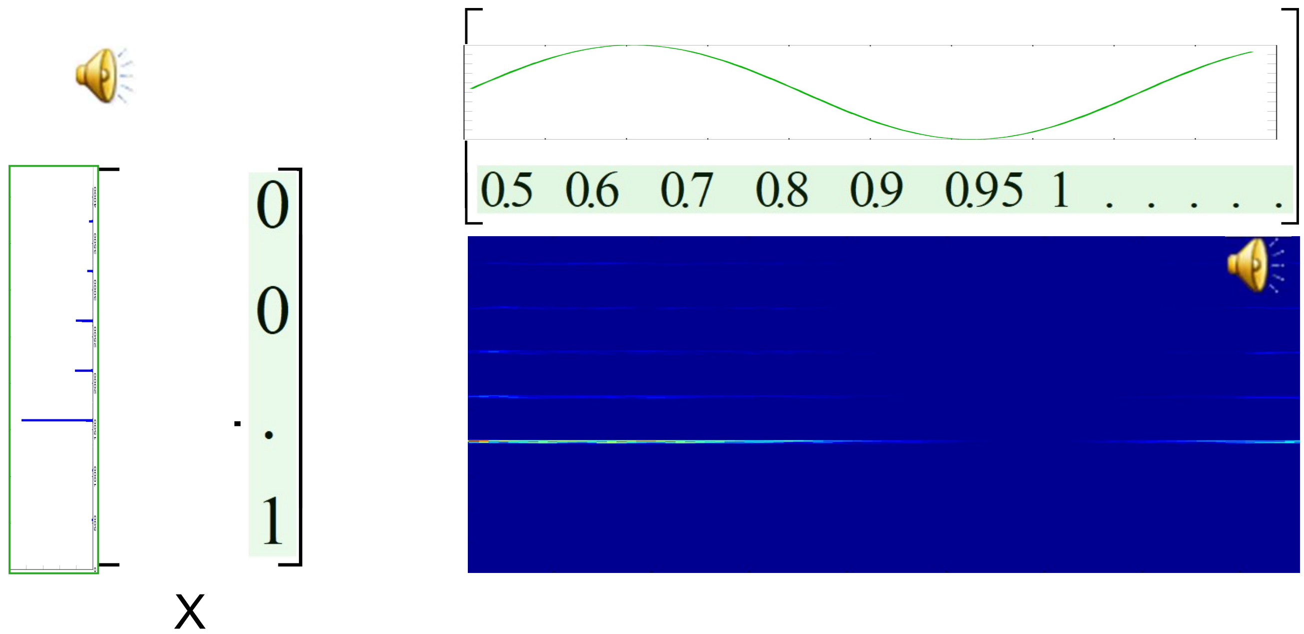
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



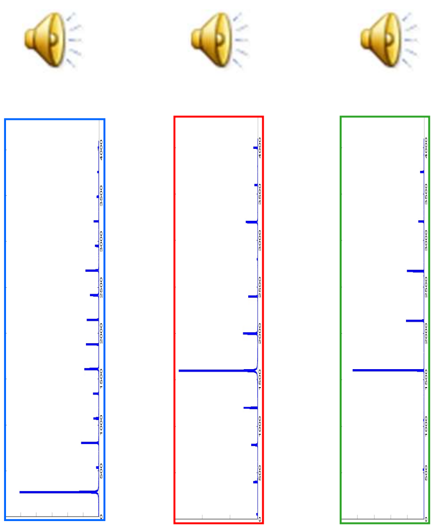
- Sounds: Three notes modulated independently

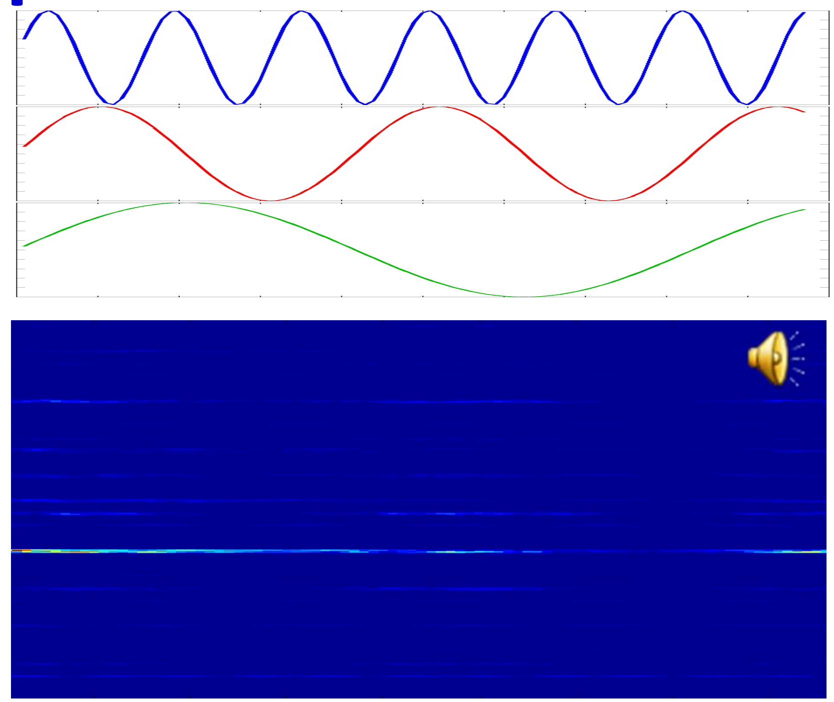
Matrix multiplication: Mixing modulated spectra



- Sounds: Three notes modulated independently

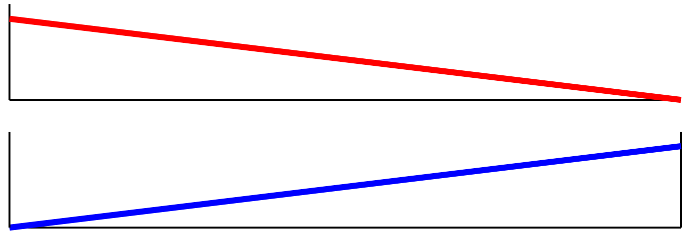
Matrix multiplication: Mixing modulated spectra



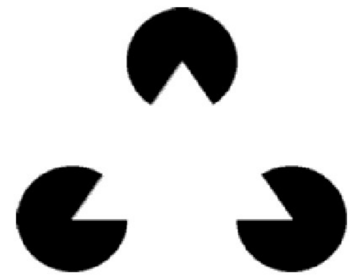


- Sounds: Three notes modulated independently

Matrix multiplication: Image transition

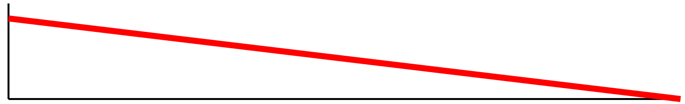


$$\begin{bmatrix} i_1 & j_1 \\ i_2 & j_2 \\ . & . \\ . & . \\ . & . \end{bmatrix} \begin{bmatrix} 1 & .9 & .8 & .7 & .6 & .5 & .4 & .3 & .2 & .1 & 0 \\ 0 & .1 & .2 & .3 & .4 & .5 & .6 & .7 & .8 & .9 & 1 \end{bmatrix}$$



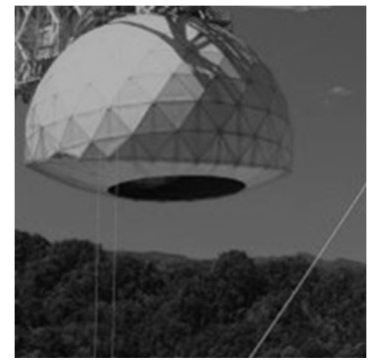
- Image1 fades out linearly
- Image 2 fades in linearly

Matrix multiplication: Image transition



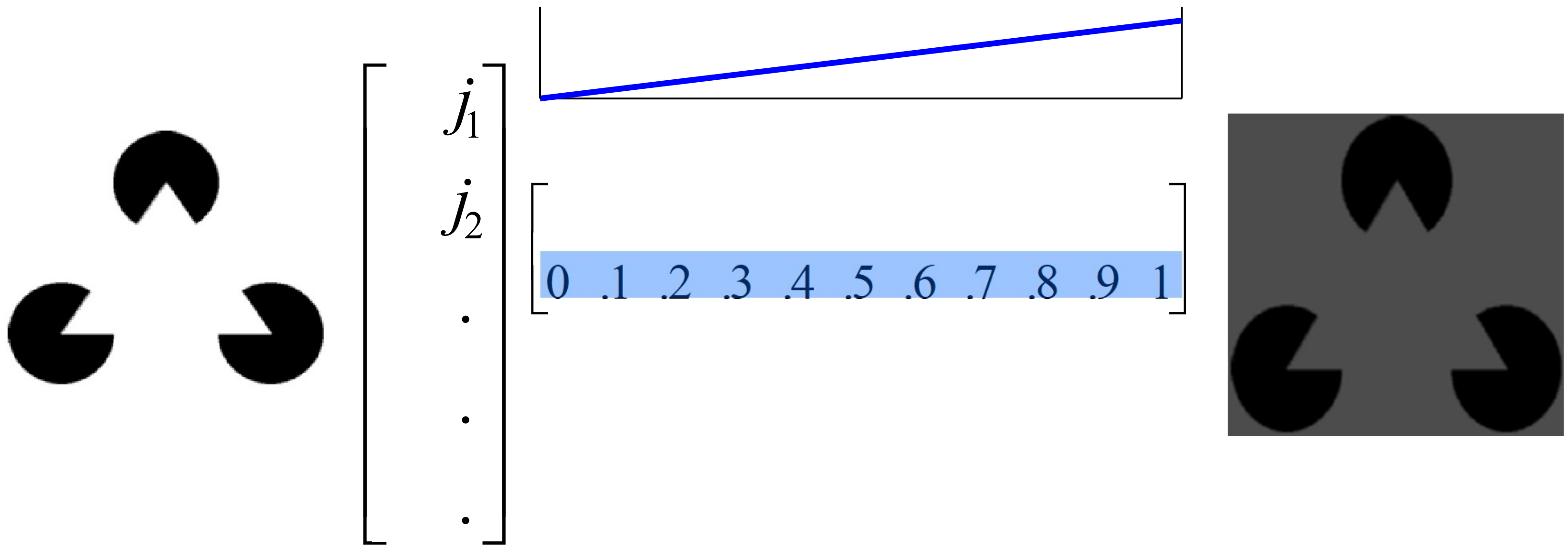
$$\begin{bmatrix} i_1 \\ i_2 \\ . \\ . \\ . \end{bmatrix} \begin{bmatrix} 1 & .9 & .8 & .7 & .6 & .5 & .4 & .3 & .2 & .1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} i_1 & 0.9i_1 & 0.8i_1 & . & . & . & . & . & . & 0 \\ i_2 & 0.9i_2 & 0.8i_2 & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & 0. \\ i_N & 0.9i_N & 0.8i_N & . & . & . & . & . & . & 0 \end{bmatrix}$$



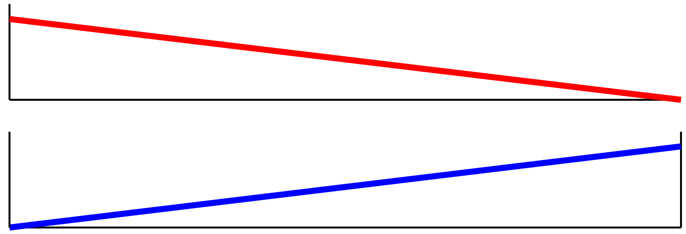
- Each column is one image
  - The columns represent a sequence of images of decreasing intensity
- Image1 fades out linearly

Matrix multiplication: Image transition



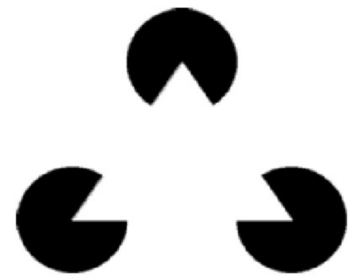
- Image 2 fades in linearly

Matrix multiplication: Image transition



$$\begin{bmatrix} i_1 & j_1 \\ i_2 & j_2 \\ . & . \\ . & . \\ . & . \end{bmatrix}$$

$$\begin{bmatrix} 1 & .9 & .8 & .7 & .6 & .5 & .4 & .3 & .2 & .1 & 0 \\ 0 & .1 & .2 & .3 & .4 & .5 & .6 & .7 & .8 & .9 & 1 \end{bmatrix}$$



- Image1 fades out linearly
- Image 2 fades in linearly

Matrix Operations: Properties

- $\mathbf{A} + \mathbf{B} \;=\; \mathbf{B} + \mathbf{A}$
  - Actual interpretation: for any vector $\mathbf{x}$
    - $(\mathbf{A} + \mathbf{B})\mathbf{x} \;=\; (\mathbf{B} + \mathbf{A})\mathbf{x}$ (column vector $\mathbf{x}$ of the right size)
    - $\mathbf{x}(\mathbf{A} + \mathbf{B}) \;=\; \mathbf{x}(\mathbf{B} + \mathbf{A})$ (row vector $\mathbf{x}$ of the appropriate size)

- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Multiplication properties

- Properties of vector/matrix products
  - Associative

$$\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$$

  - Distributive

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

  - NOT commutative!!!

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

    - *left multiplications ≠ right multiplications*
  - Transposition

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$$

Poll 2

Poll 2

- What properties are true for matrix multiplication?
  - **Transposition property**
  - **Distributive property**
  - **Associative property**
  - Commutative property

- True or false: $(A+B)x = (B+A)x$ for appropriate dimensions of A, B and x
  - **T**
  - F

The Space of Matrices

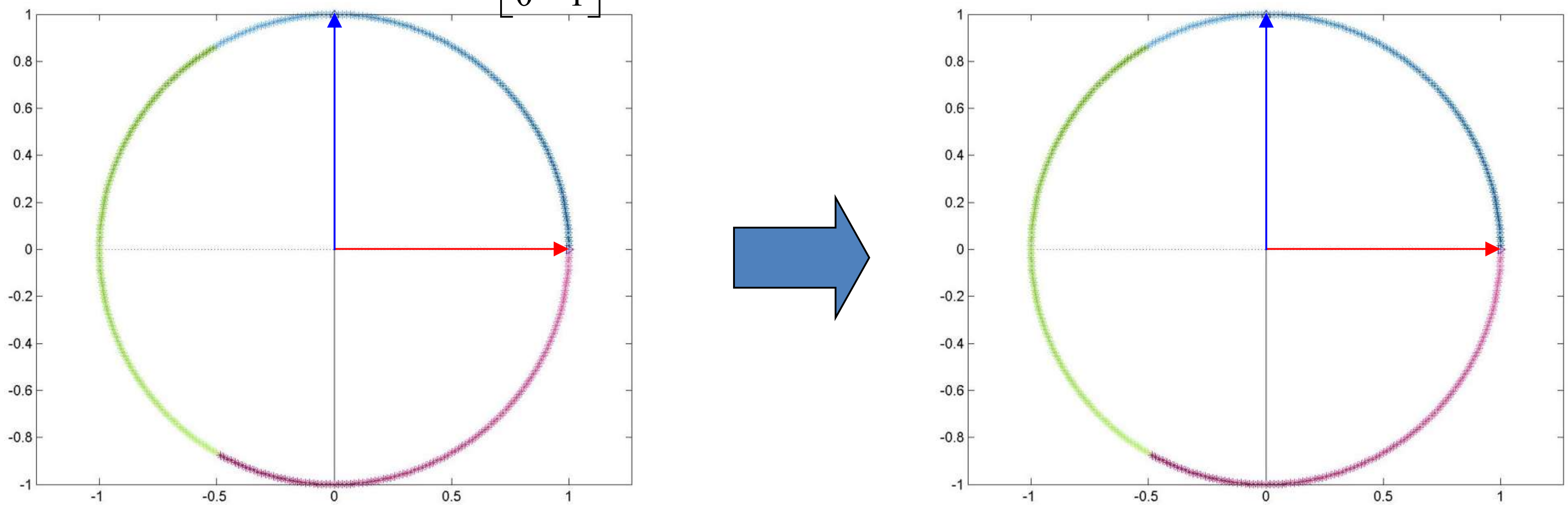
- The set of all matrices of a given size (e.g. all 3x4 matrices) is a space!
  - Addition is closed
  - Scalar multiplication is closed
  - Zero matrix exists
  - Matrices have additive inverses
  - Associativity and commutativity rules apply!

Overview

- Vectors and matrices
- Basic vector/matrix operations
- **Various matrix types**
- Matrix properties
  - Determinant
  - Inverse
  - Rank
- Projections
- Eigen decomposition
- SVD

The Identity Matrix

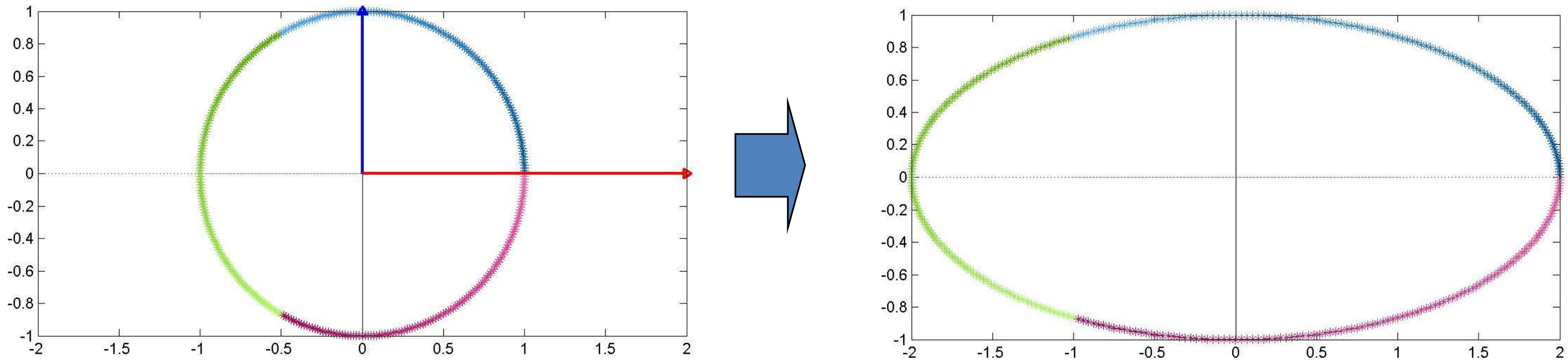
$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



- An identity matrix is a square matrix where
  - All diagonal elements are 1.0
  - All off-diagonal elements are 0.0
- Multiplication by an identity matrix does not change vectors

Diagonal Matrix

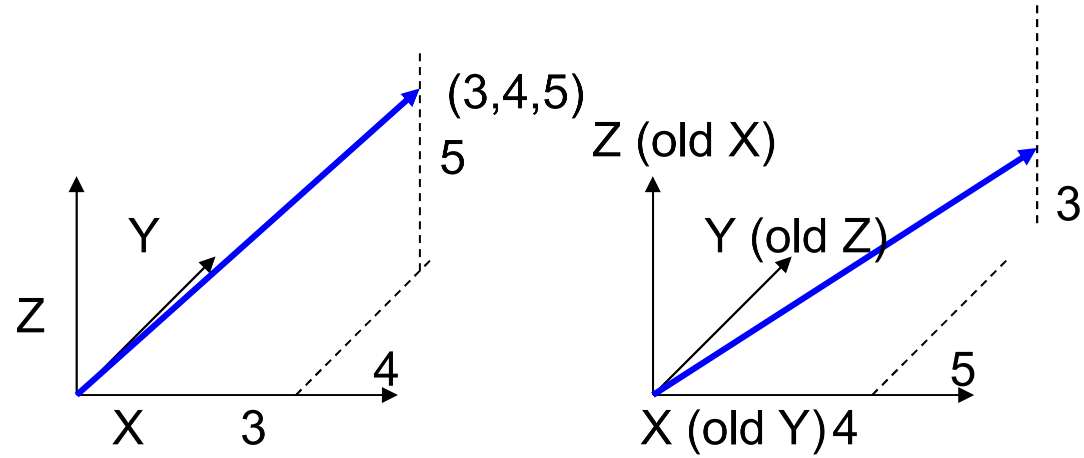
$$Y = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



- All off-diagonal elements are zero
- Diagonal elements are non-zero
- Scales the axes
  - May flip axes

Permutation Matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$



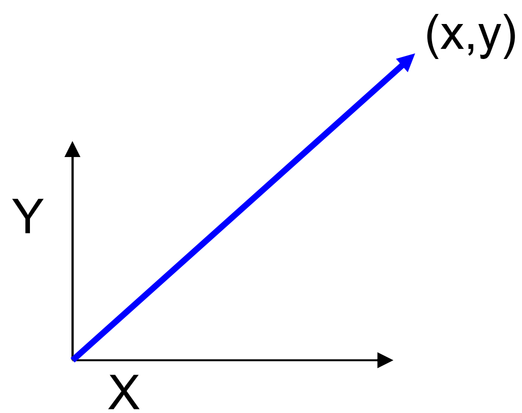
- A permutation matrix simply rearranges the axes
  - The row entries are axis vectors in a different order
  - The result is a combination of rotations and reflections
- The permutation matrix effectively *permutes* the arrangement of the elements in a vector

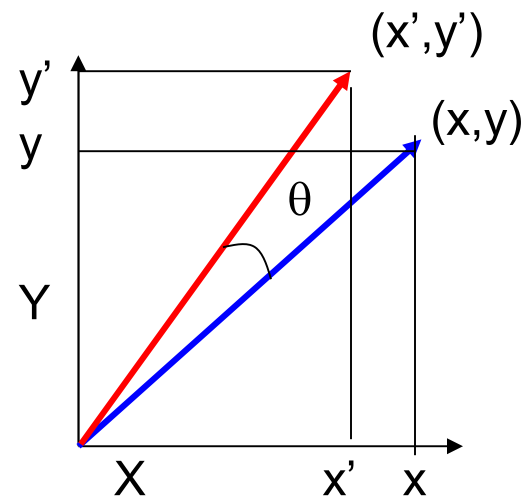
Rotation Matrix

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$X_{new} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

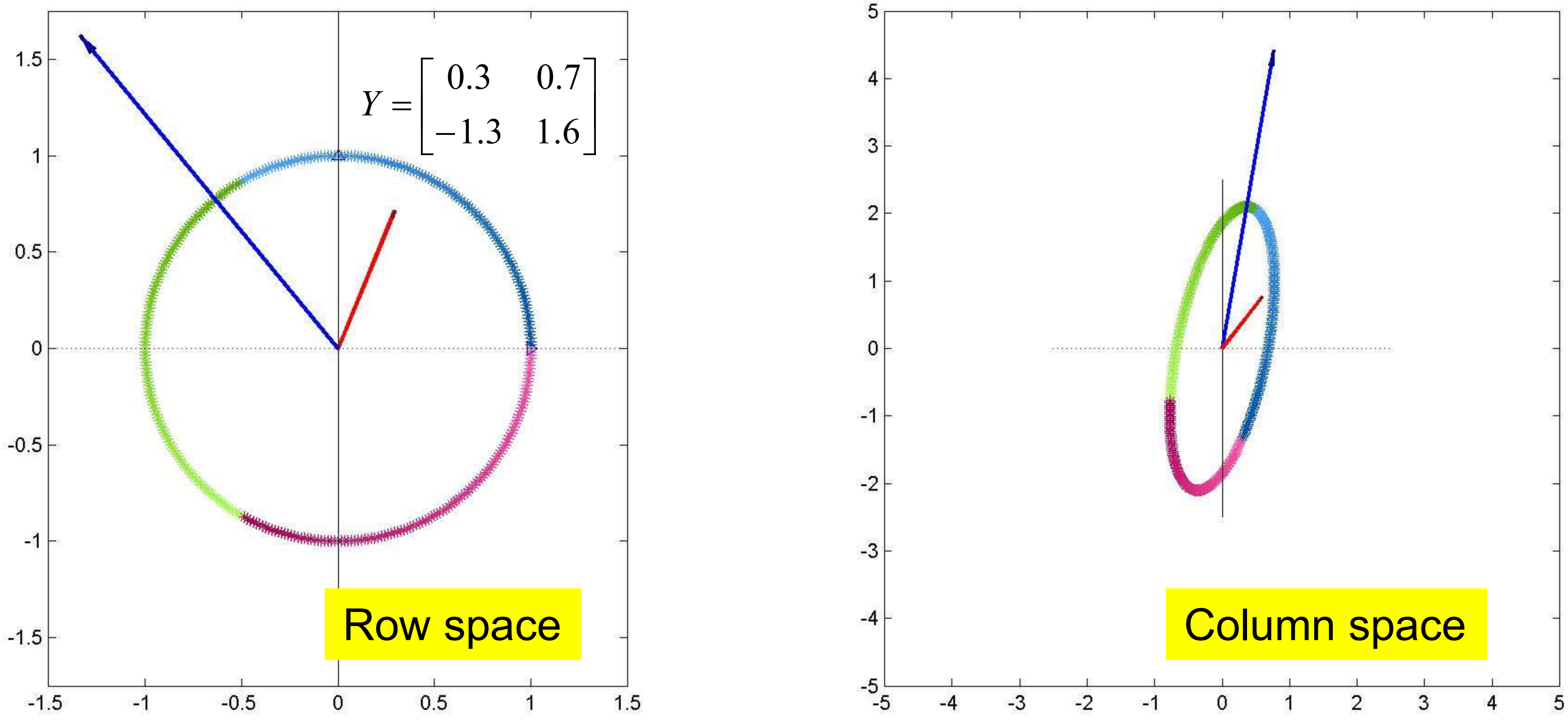
$$R_\theta X = X_{new}$$





- A rotation matrix *rotates* the vector by some angle θ
- Alternately viewed, it rotates the axes
  - The new axes are at an angle θ to the old one

More generally

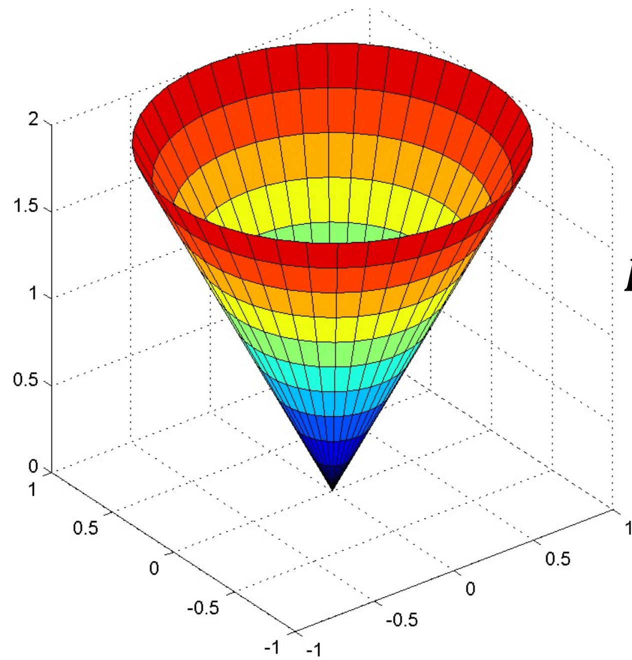


- Matrix operations are combinations of rotations, permutations and stretching

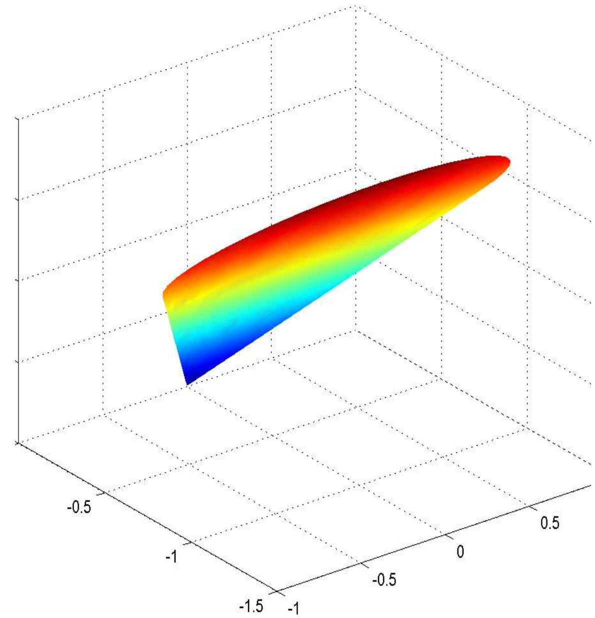
Overview

- Vectors and matrices
- Basic vector/matrix operations
- Various matrix types
- **Matrix properties**
  - Rank
  - Determinant
  - Inverse
- Solving simultaneous equations
- Projections
- Eigen decomposition
- SVD

Matrix Rank and Rank-Deficient Matrices



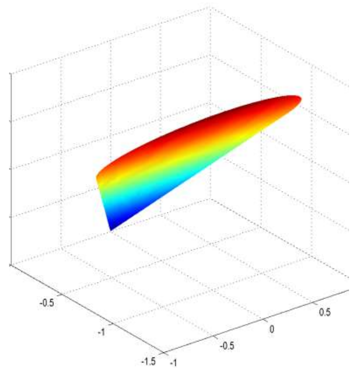
$$\boldsymbol{P} \times Cone =$$



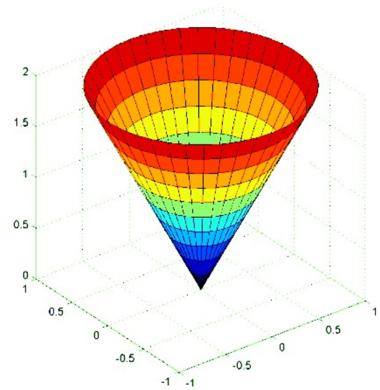
- Some matrices will eliminate one or more dimensions during transformation
  - These are *rank deficient* matrices
  - The **rank** of the matrix is the dimensionality of the transformed version of a **full-dimensional** object

Matrix Rank and Rank-Deficient Matrices

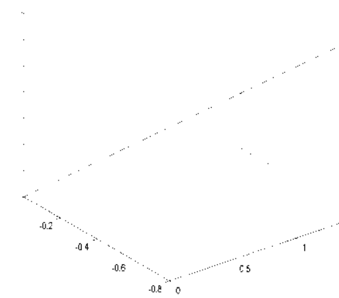
```
P =

    1.0000         0         0
         0    0.2500   -0.4330
         0   -0.4330    0.7500
```



Rank = 2



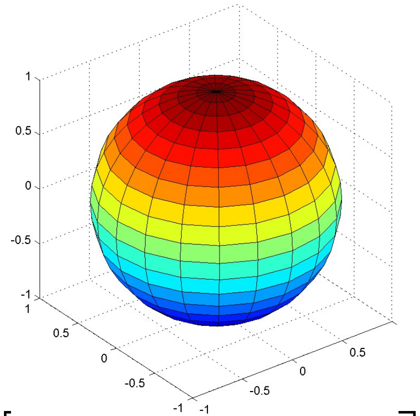
```
P2 =

    0.5000   -0.2500    0.4330
   -0.2500    0.1250   -0.2165
    0.4330   -0.2165    0.3750
```



Rank = 1

- Some matrices will eliminate one or more dimensions during transformation
  - These are *rank deficient* matrices
  - The rank of the matrix is the dimensionality of the transformed version of a full-dimensional object

Non-square Matrices

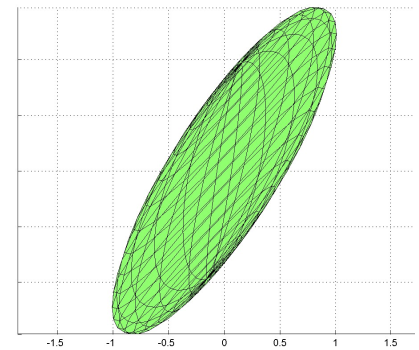


$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \\ z_1 & z_2 & . & . & z_N \end{bmatrix}$$

X = 3D data, rank 3

$$\begin{bmatrix} .3 & 1 & .2 \\ .5 & 1 & 1 \end{bmatrix}$$

P = transform

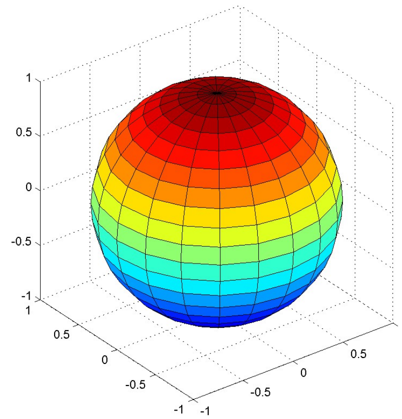


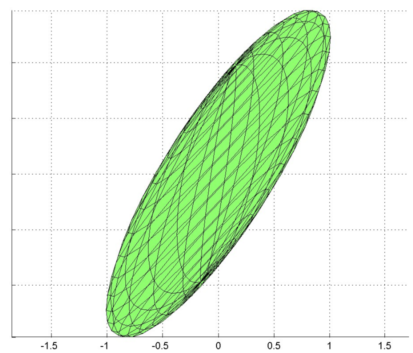
$$\begin{bmatrix} \hat{x}_1 & \hat{x}_2 & . & . & \hat{x}_N \\ \hat{y}_1 & \hat{y}_2 & . & . & \hat{y}_N \end{bmatrix}$$

PX = 2D, rank 2

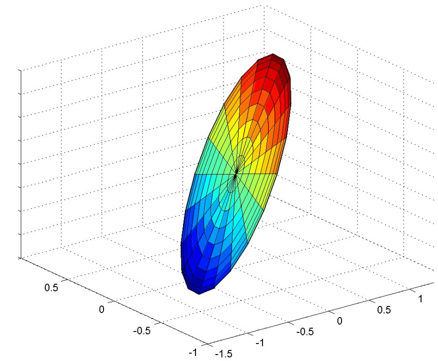
- Non-square matrices add or subtract axes
  - More rows than columns → add axes
    - But does not increase the dimensionality of the data
  - Fewer rows than columns → reduce axes
    - May reduce dimensionality of the data

The Rank of a Matrix





$$\begin{bmatrix} .3 & 1 & .2 \\ .5 & 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} .8 & .9 \\ .1 & .9 \\ .6 & 0 \end{bmatrix}$$

- The matrix rank is the dimensionality of the transformation of a full-dimensioned object in the original space

- The matrix can never *increase* dimensions
  - Cannot convert a circle to a sphere or a line to a circle

- The rank of a matrix can never be greater than the lower of its two dimensions

Rank – an alternate definition

- In terms of bases..
- Will get back to this shortly..

Poll 3

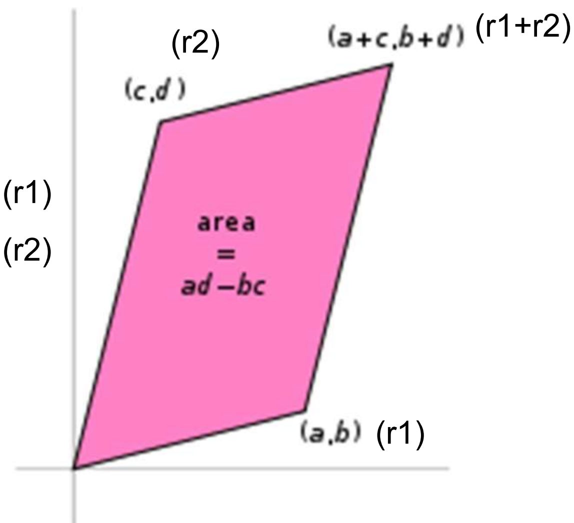
Poll 3

- True or false: Given a circle in R^3, there will always exist a matrix, M, such that the image of the circle under the transformation of M will be a sphere.
  - T
  - **F**

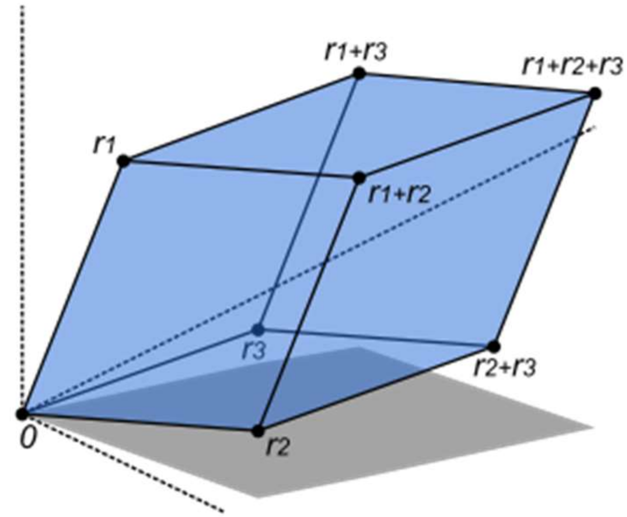
- True or false: A square matrix with N rows and N columns will always have rank N.
  - T
  - **F**

Matrix Determinant

$$A = \begin{array}{cc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \begin{array}{c} (r1) \\ (r2) \end{array} \end{array}$$



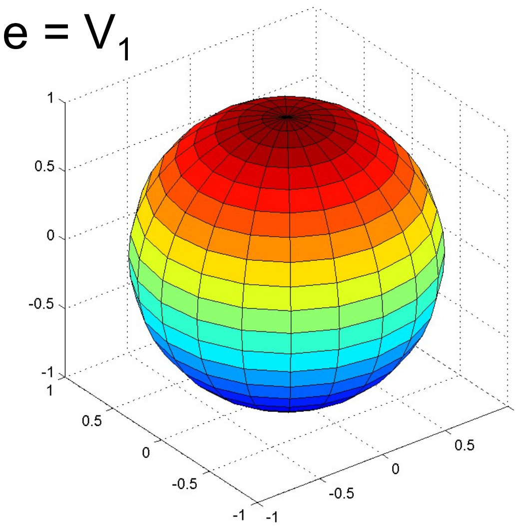
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$



- The determinant is the “volume” of a matrix
- Actually, the volume of a parallelepiped formed from its row vectors
  - Also, the volume of the parallelepiped formed from its column vectors
- Standard formula for determinant in textbooks

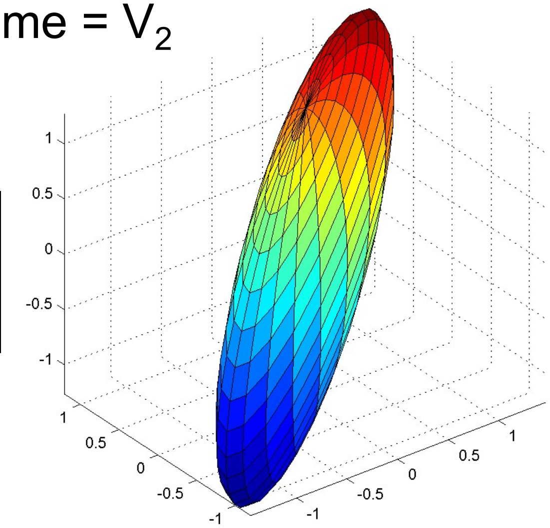
Matrix Determinant: Another Perspective

Volume = V_1



$$\begin{bmatrix} 0.8 & 0 & 0.7 \\ 1.0 & 0.8 & 0.8 \\ 0.7 & 0.9 & 0.7 \end{bmatrix}$$

Volume = V_2



- The (magnitude of the) determinant is the ratio of N-volumes
  - If V_1 is the volume of an N-dimensional sphere "O" in N-dimensional space
    - O is the complete set of points or vertices that specify the object
  - If V_2 is the volume of the N-dimensional ellipsoid specified by A*O, where A is a matrix that transforms the space
  - $|A| = V_2 / V_1$

Matrix Determinants

- Matrix determinants are *only defined for square matrices*
  - They characterize volumes in linearly transformed space of the same dimensionality as the vectors

- Rank deficient matrices have determinant 0
  - Since they compress full-volumed N-dimensional objects into zero-volume N-dimensional objects
    - E.g. a 3-D sphere into a 2-D ellipse: The ellipse has 0 volume (although it does have area)

- Conversely, all matrices of determinant 0 are rank deficient
  - Since they compress full-volumed N-dimensional objects into zero-volume objects

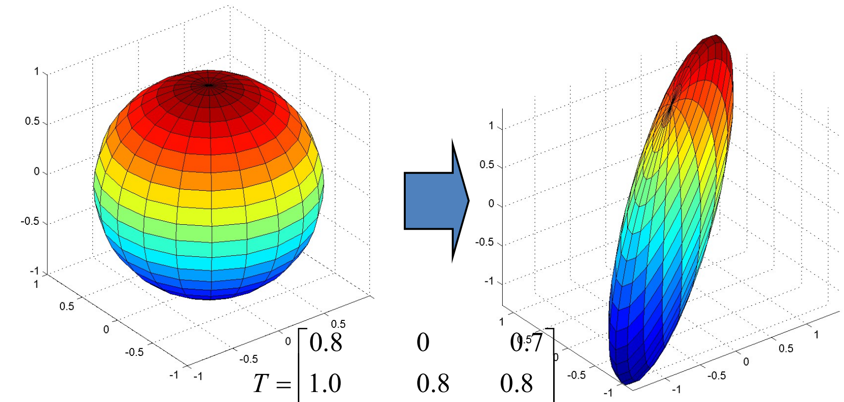
Determinant properties

- Associative for square matrices $$|\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}| = |\mathbf{A}|\cdot|\mathbf{B}|\cdot|\mathbf{C}|$$
  - Scaling volume sequentially by several matrices is equal to scaling once by the product of the matrices
- Volume of sum != sum of Volumes $$|(\mathbf{B}+\mathbf{C})| \neq |\mathbf{B}| + |\mathbf{C}|$$
- Commutative
  - The order in which you scale the volume of an object is irrelevant

$$|\mathbf{A}\cdot\mathbf{B}| = |\mathbf{B}\cdot\mathbf{A}| = |\mathbf{A}|\cdot|\mathbf{B}|$$

Matrix Inversion

- A matrix transforms an N-dimensional object to a different N-dimensional object

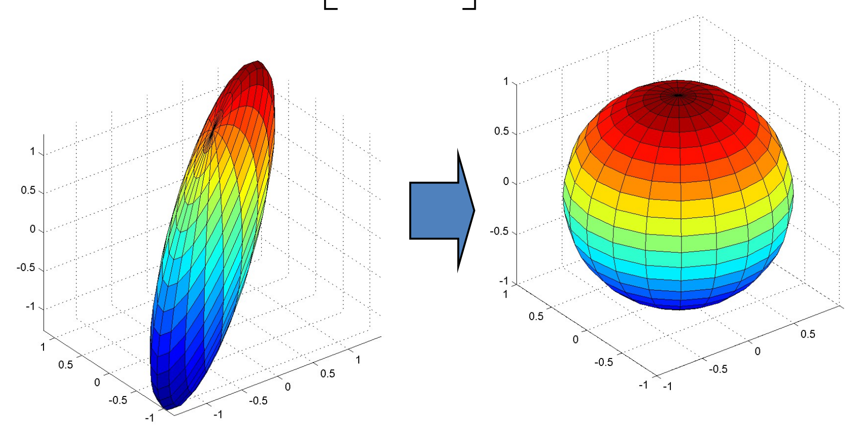


$$T = \begin{bmatrix} 0.8 & 0 & 0.7 \\ 1.0 & 0.8 & 0.8 \\ 0.7 & 0.9 & 0.7 \end{bmatrix}$$

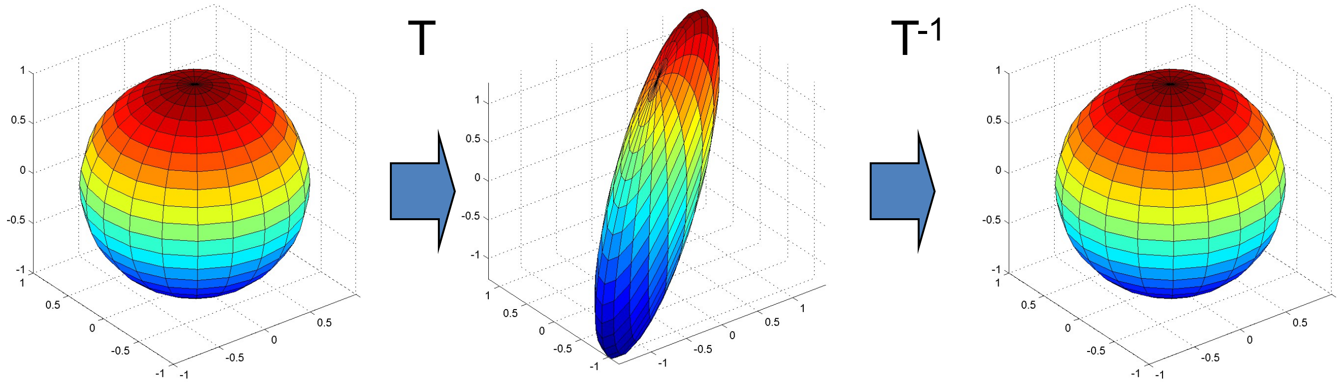
- What transforms the new object back to the original?
  - The *inverse transformation*

$$Q = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} = T^{-1}$$

- The inverse transformation is called the matrix inverse



Matrix Inversion

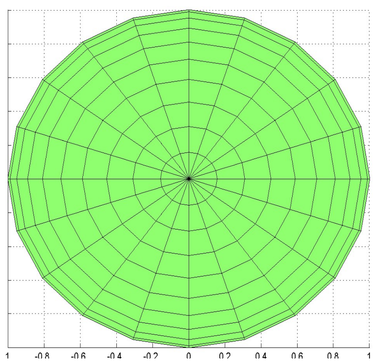


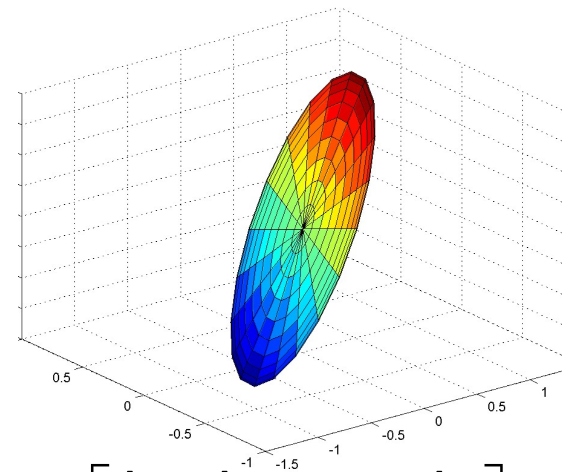
$$\mathbf{T}^{-1}\mathbf{T}\mathbf{D} = \mathbf{D} \Rightarrow \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$

- The product of a matrix and its inverse is the identity matrix
  - Transforming an object, and then inverse transforming it gives us back the original object

$$\mathbf{T}\mathbf{T}^{-1}\mathbf{D} = \mathbf{D} \Rightarrow \mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$$

Non-square Matrices





$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \end{bmatrix}$$

X = 2D data

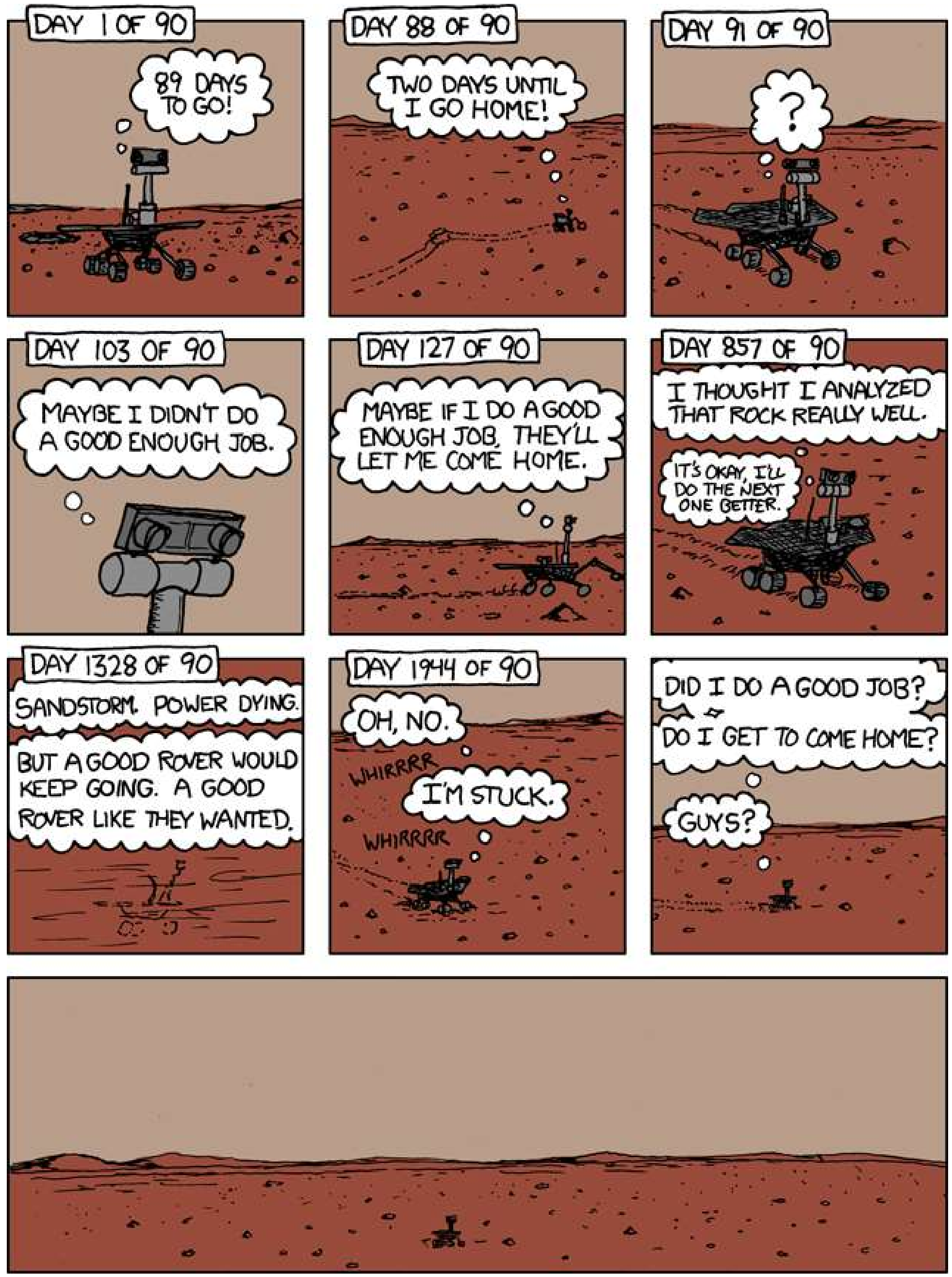
$$\begin{bmatrix} .8 & .9 \\ .1 & .9 \\ .6 & 0 \end{bmatrix}$$

P = transform

$$\begin{bmatrix} \hat{x}_1 & \hat{x}_2 & . & . & \hat{x}_N \\ \hat{y}_1 & \hat{y}_2 & . & . & \hat{y}_N \\ \hat{z}_1 & \hat{z}_2 & . & . & \hat{z}_N \end{bmatrix}$$

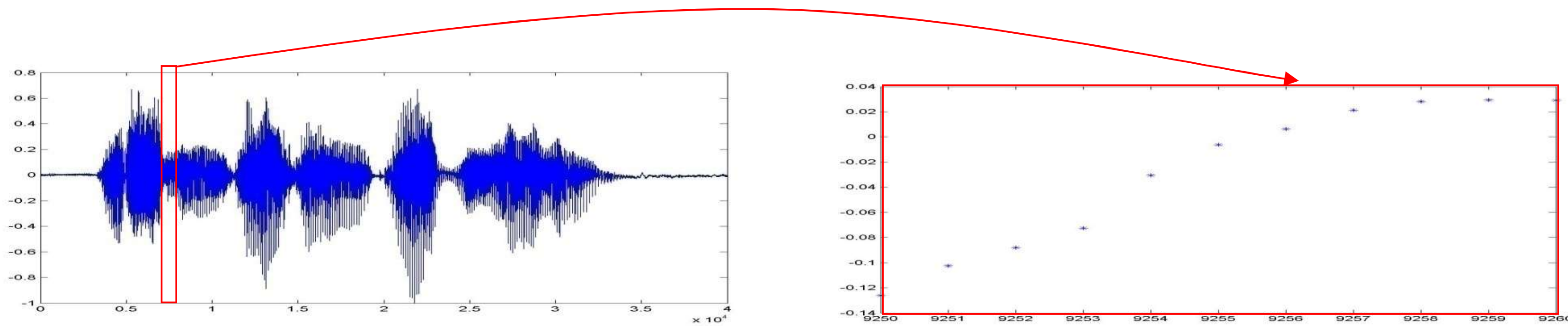
PX = 3D, rank 2

- Non-square matrices add or subtract axes
  - More rows than columns → add axes
    - But does not increase the dimensionality of the data


DAY 1 OF 90
89 DAYS TO GO!
DAY 88 OF 90
TWO DAYS UNTIL I GO HOME!
DAY 91 OF 90
?
DAY 103 OF 90
MAYBE I DIDN'T DO A GOOD ENOUGH JOB.
DAY 127 OF 90
MAYBE IF I DO A GOOD ENOUGH JOB, THEY'LL LET ME COME HOME.
DAY 857 OF 90
I THOUGHT I ANALYZED THAT ROCK REALLY WELL.
IT'S OKAY, I'LL DO THE NEXT ONE BETTER.
DAY 1328 OF 90
SANDSTORM. POWER DYING.
BUT A GOOD ROVER WOULD KEEP GOING. A GOOD ROVER LIKE THEY WANTED.
DAY 1944 OF 90
OH, NO.
WHIRRRR
I'M STUCK.
WHIRRRR
DID I DO A GOOD JOB?
DO I GET TO COME HOME?
GUYS?

Recap: Representing signals as vectors

- Signals are frequently represented as vectors for manipulation
- E.g. A segment of an audio signal

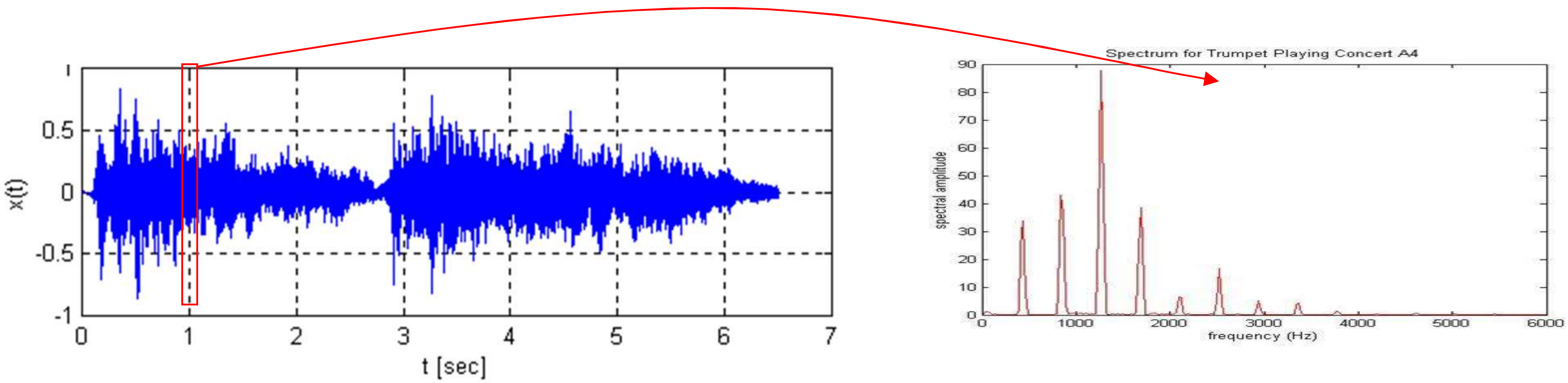


- Represented as a vector of sample values

$$[s_1 \ \ s_2 \ \ s_3 \ \ s_4 \ \ldots \ \ s_N]$$

Representing signals as vectors

- Signals are frequently represented as vectors for manipulation
- E.g. The ***spectrum*** segment of an audio signal



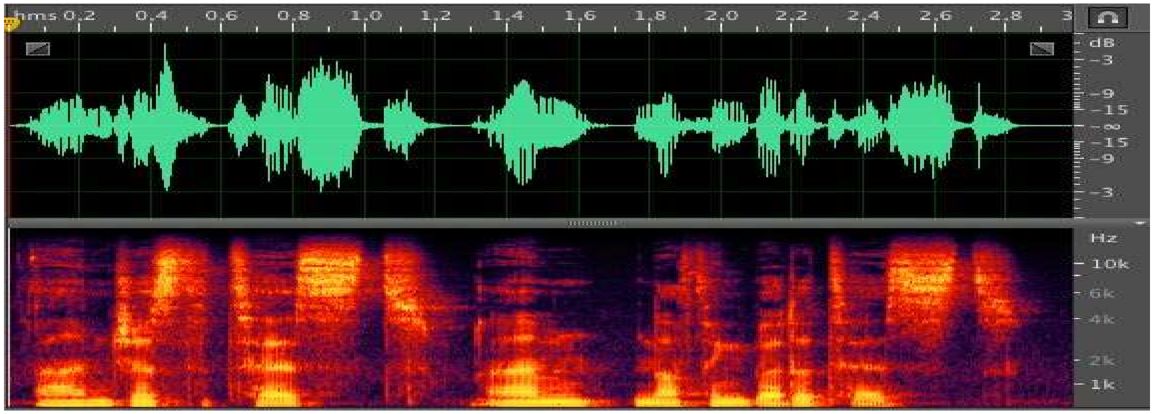
- Represented as a vector of sample values

$$[S_1 \;\; S_2 \;\; S_3 \;\; S_4 \; \dots \; S_M]$$

  – Each component of the vector represents a frequency component of the spectrum

Representing a signal as a matrix

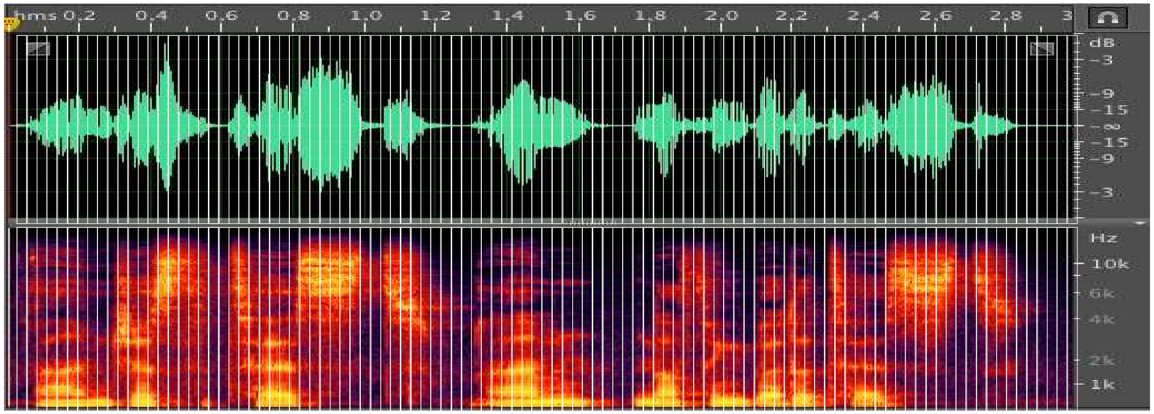
- Time series data like audio signals are often represented as spectrographic matrices



- Each column is the spectrum of a short segment of the audio signal

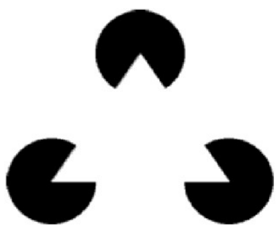
Representing a signal as a matrix

- Time series data like audio signals are often represented as spectrographic matrices



- Each column is the spectrum of a short segment of the audio signal

Representing an image as a vector



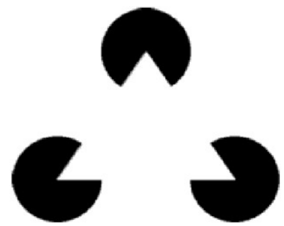
- 3 pacmen
- A 321 x 399 grid of pixel values
  - Row and Column = position
- A 1 x 128079 vector
  - “Unraveling” the matrix

$$\begin{bmatrix}1 & 1 & . & 1 & 1 & . & 0 & 0 & 0 & . & . & 1\end{bmatrix}$$

  - Note: This can be recast as the grid that forms the image

Representing a signal as a matrix

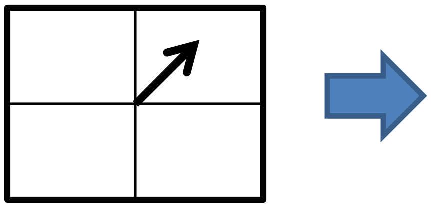
- Images are often just represented as matrices



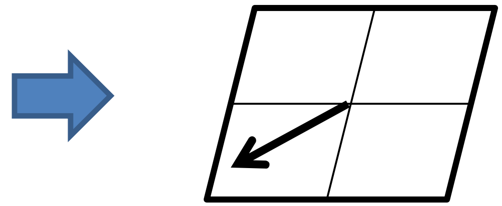
```
>> X(1:32:end,1:40:end)

ans =

     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     0     0     0     1     1     1
     1     1     1     1     0     0     0     1     1     1
     1     1     1     1     0     1     0     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     0     1     1     1     1     1     0     1
     1     0     0     1     1     1     1     1     0     0
     1     0     0     0     1     1     1     0     0     0
     1     0     0     0     1     1     1     0     0     0
     1     1     1     1     1     1     1     1     1     1
```

Interpretations of a matrix

- As a ***transform*** that modifies vectors and vector spaces



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

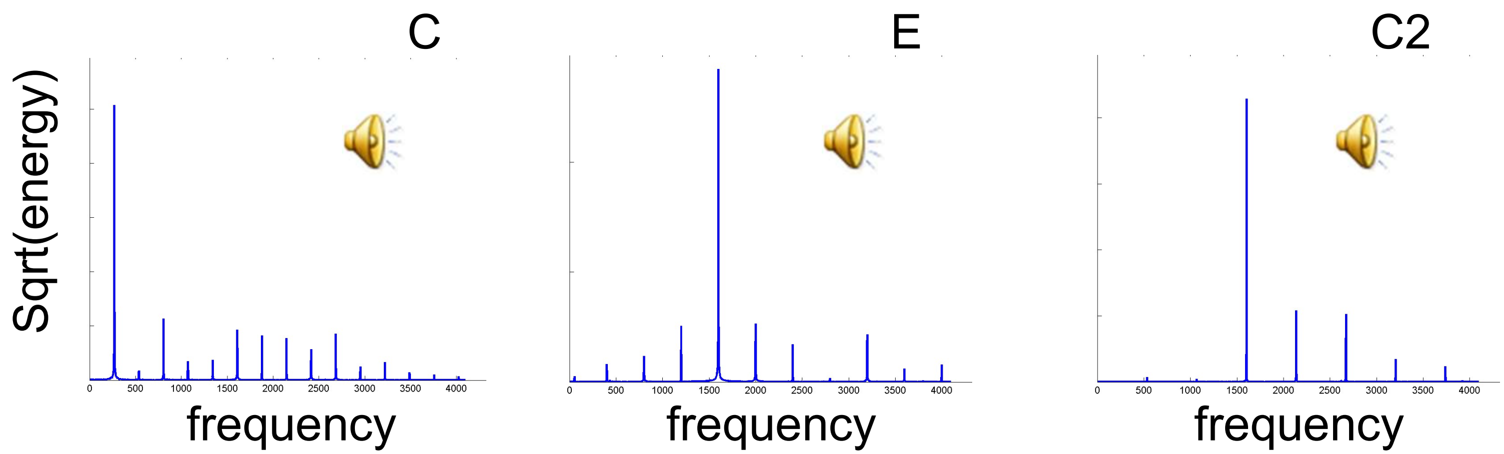


- As a ***container*** for data (vectors)

$$\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \end{bmatrix}$$

- As a generator of vector spaces..

Revise.. Vector dot product



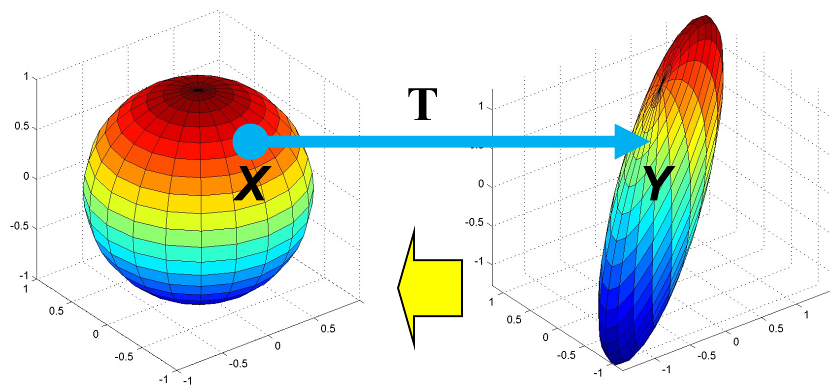
$[11 \;\; 9 \;\; . \;\; 0 \;\; 54 \;\; 1 \;\; . \;\; . \;\; . \;\; 1]$ $[3 \;\; . \;\; 24 \;\; . \;\; . \;\; 16 \;\; . \;\; 14 \;\; . \;\; 1]$ $[0 \;\; . \;\; 0 \;\; . \;\; 3 \;\; 0 \;\; . \;\; 13 \;\; . \;\; 0]$

- How much of C is also in E
  - How much can you fake a C by playing an E
  - C.E / |C||E| = 0.1
  - Not very much
- How much of C is in C2?
  - C.C2 / |C| /|C2| = 0.5
  - Not bad, you can fake it

Overview

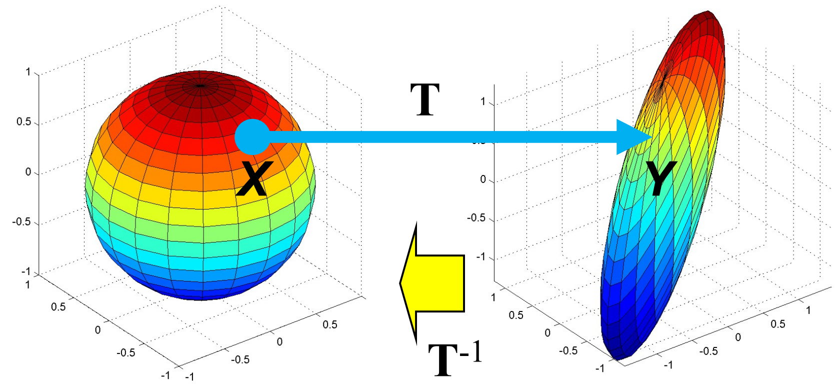
- Vectors and matrices
- Basic vector/matrix operations
- Various matrix types
- Matrix properties
  - Determinant
  - Inverse
  - Rank
- **Solving simultaneous equations**
- Projections
- Eigen decomposition
- SVD

The Inverse Transform and Simultaneous Equations



- Given the Transform $\mathbf{T}$ and transformed vector ***Y***, how do we determine ***X***?

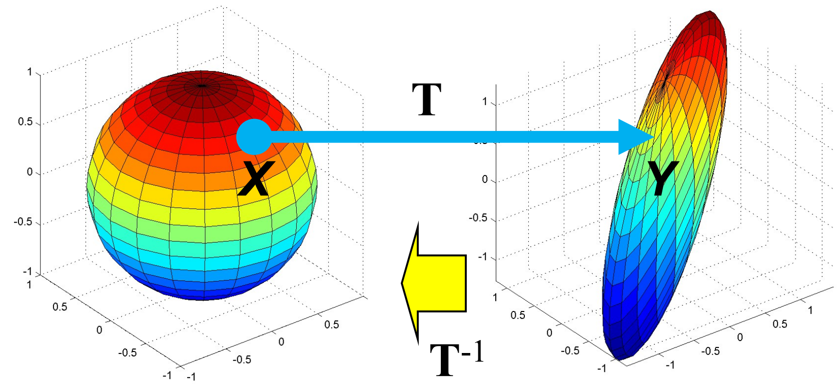
Matrix inversion (division)



- The inverse of matrix multiplication
  - Not element-wise division!!
  - E.g.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

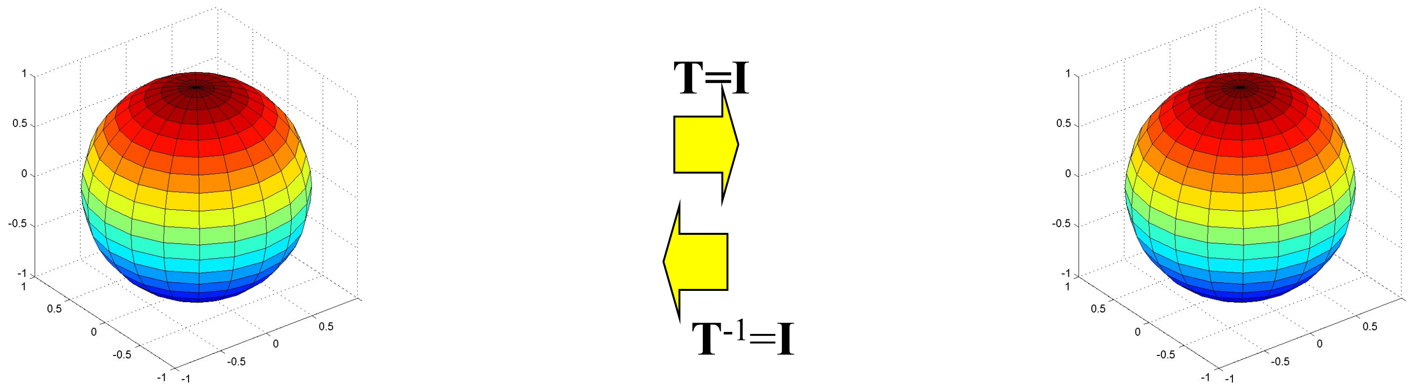
Matrix inversion (division)



- Provides a way to "undo" a linear transform
- Undoing a transform must happen as soon as it is performed
- Effect on matrix inversion: Note order of multiplication

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{C},\quad \mathbf{A}=\mathbf{C}\cdot\mathbf{B}^{-1},\quad \mathbf{B}=\mathbf{A}^{-1}\cdot\mathbf{C}$$

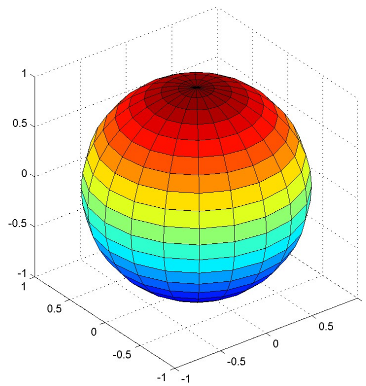
Matrix inversion (division)

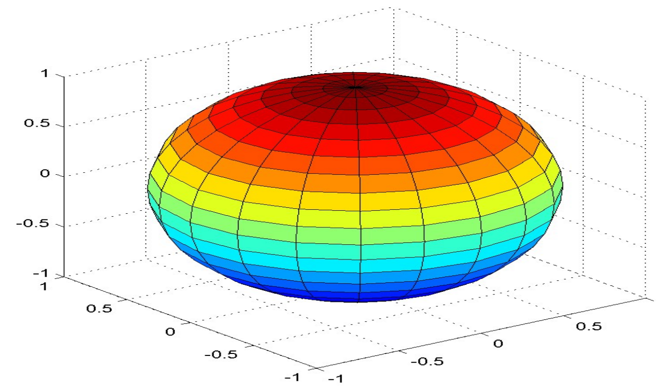


- Inverse of the unit matrix is itself

Matrix inversion (division)

$$\mathbf{T} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



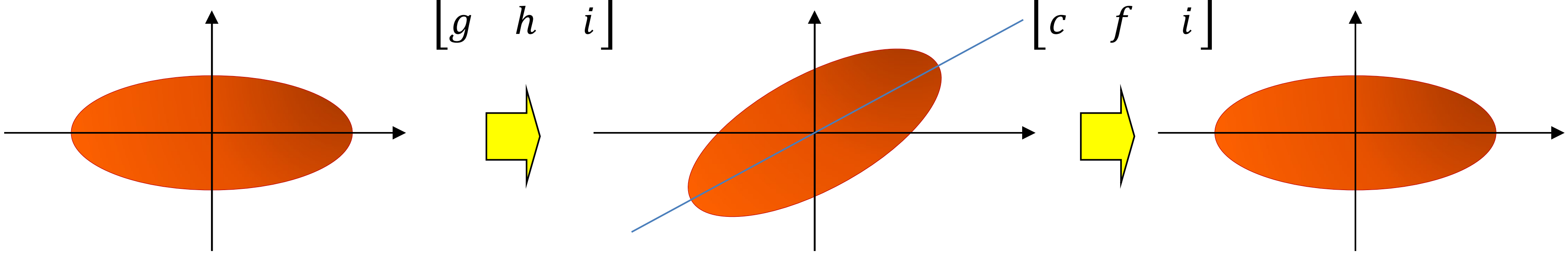


$$\mathbf{T}^{-1} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Inverse of the unit matrix is itself
- Inverse of a diagonal is diagonal

Matrix inversion (division)

$$\mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \mathbf{R}^{-1} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

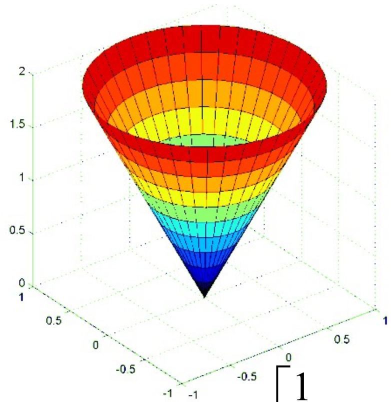


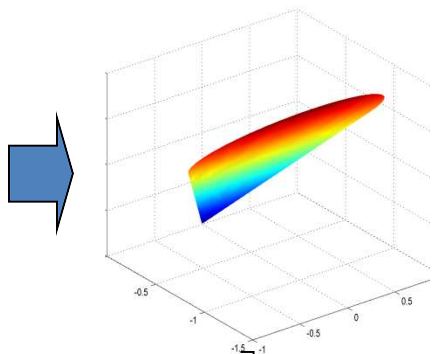
- Inverse of the unit matrix is itself
- Inverse of a diagonal is diagonal
- Inverse of a rotation is a (counter)rotation (its transpose!)
  - In 2D a forward rotation θ by is cancelled by a backward rotation of $-\theta$

$$\mathbf{R} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}, \mathbf{R}^{-1} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

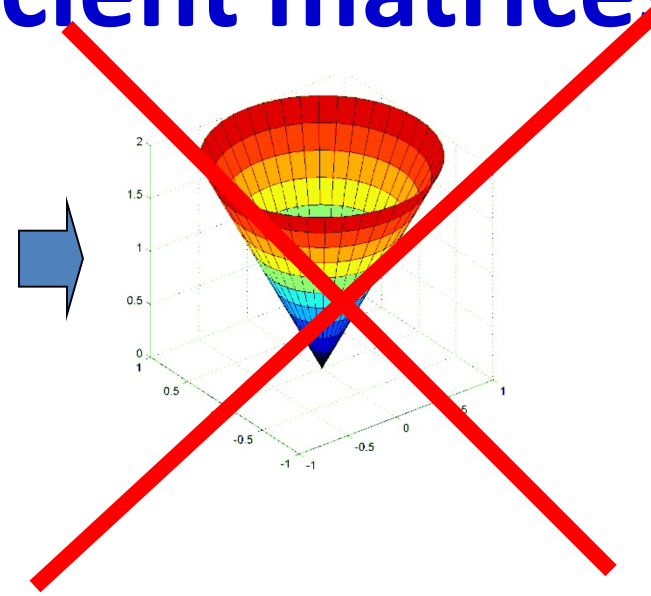
  - More generally, in any number of dimensions: $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$

Inverting rank-deficient matrices





$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & .25 & -0.433 \\ 0 & -0.433 & 0.75 \end{bmatrix}$$

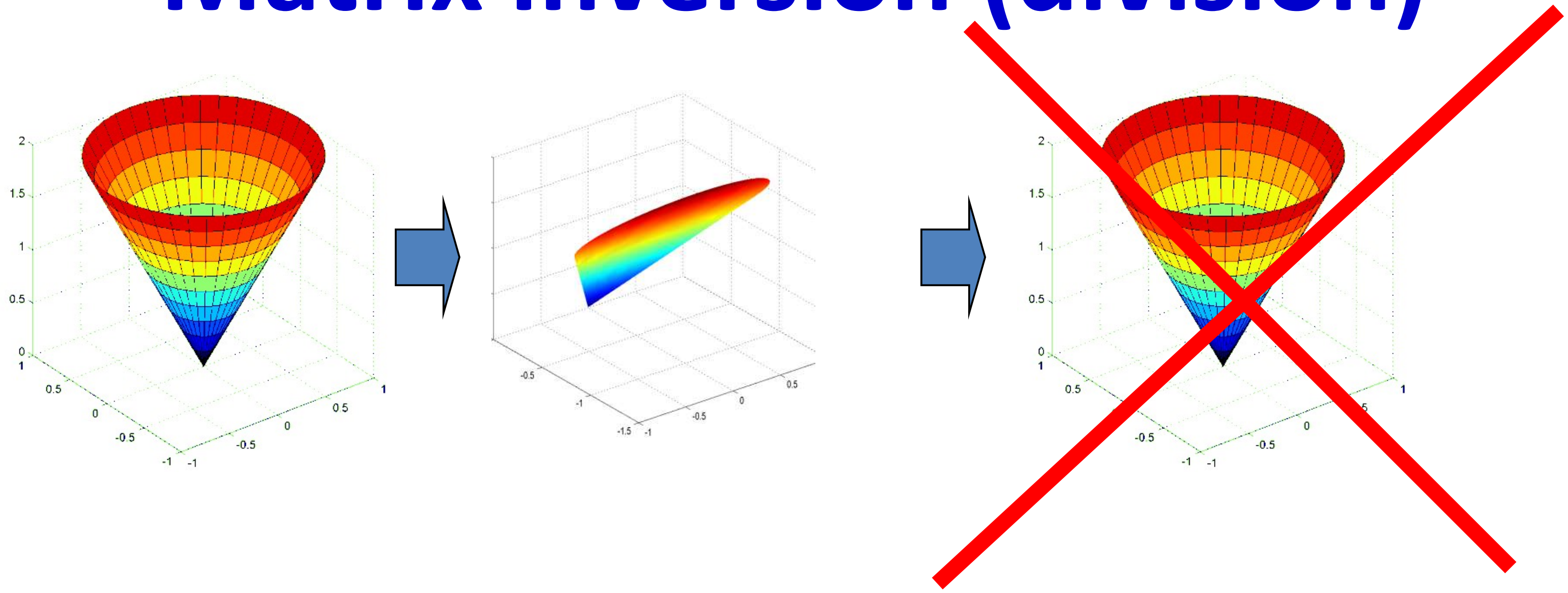


- Rank deficient matrices “flatten” objects
  - In the process, multiple points in the original object get mapped to the same point in the transformed object

- It is not possible to go “back” from the flattened object to the original object
  - Because of the many-to-one forward mapping

- Rank deficient matrices have no inverse

Matrix inversion (division)



- Inverse of the unit matrix is itself
- Inverse of a diagonal is diagonal
- Inverse of a rotation is a (counter)rotation (its transpose!)
- Inverse of a rank deficient matrix does not exist!

Poll 4

Poll 4

- True or false: Using the standard definition of matrix inversion, the inverse of an arbitrary matrix is always the matrix's transpose.
  - T
  - **F**

- True or false: Using the standard definition of matrix inversion, the inverse of a rank deficient matrix does NOT exist.
  - **T**
  - F