Machine Learning for Signal Processing

Hidden Markov Models

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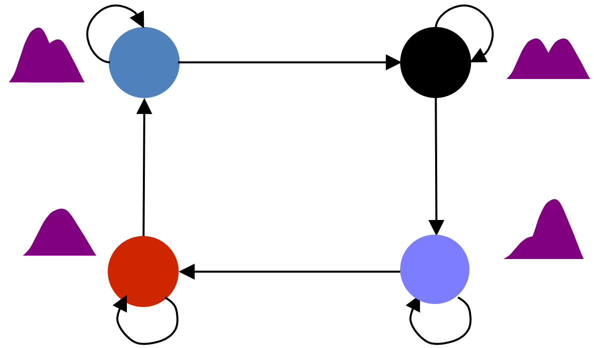
Problem 2: State segmentation

- Given only a sequence of observations, how do we determine which sequence of states was followed in producing it?

The HMM as a generator

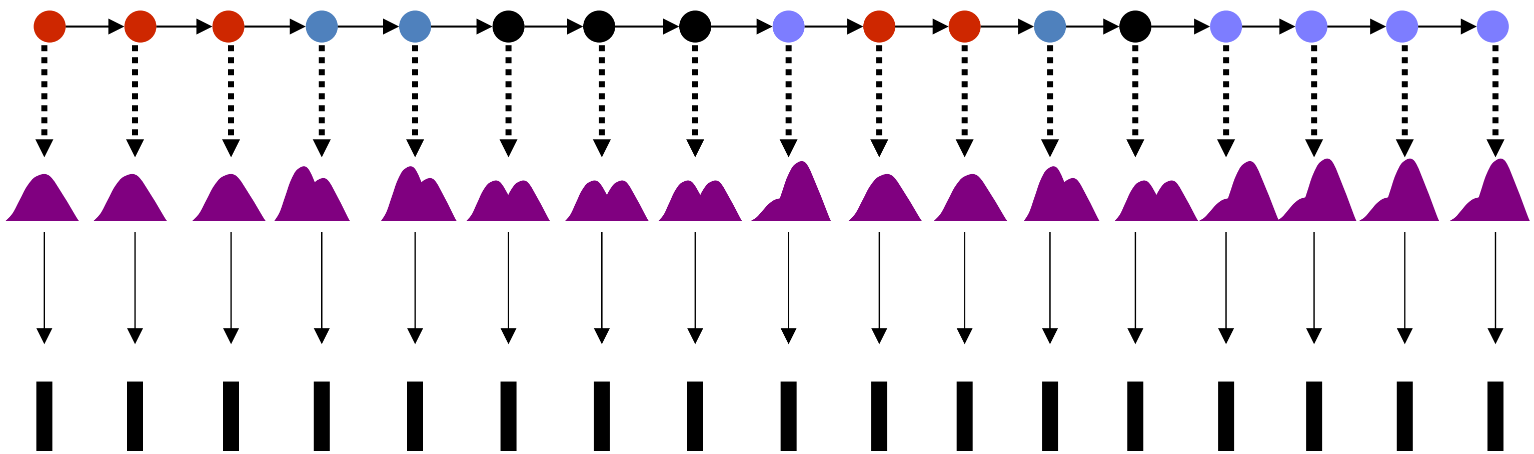
HMM assumed to be generating data



state sequence

state distributions

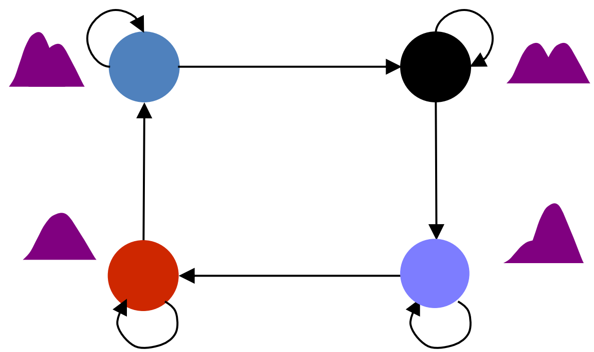
observation sequence



- The process goes through a series of states and produces observations from them

States are hidden

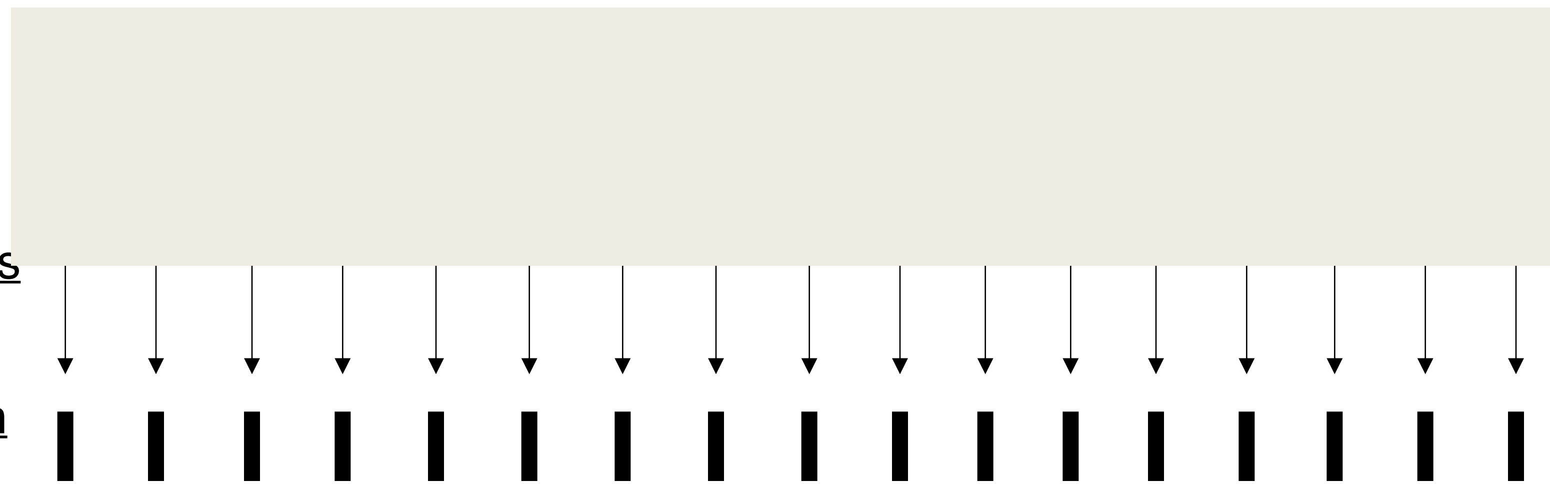
HMM assumed to be generating data



state sequence

state distributions

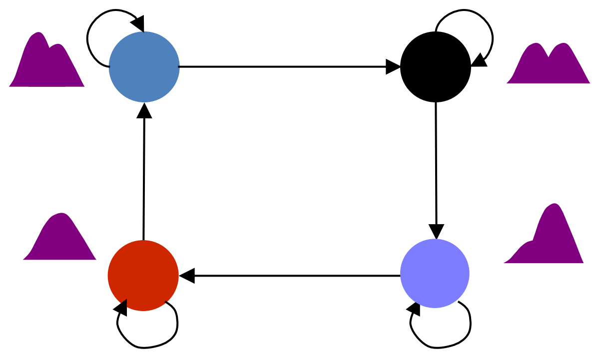
observation sequence



- The observations do not reveal the underlying state

The state segmentation problem

HMM assumed to be generating data



state sequence

state distributions

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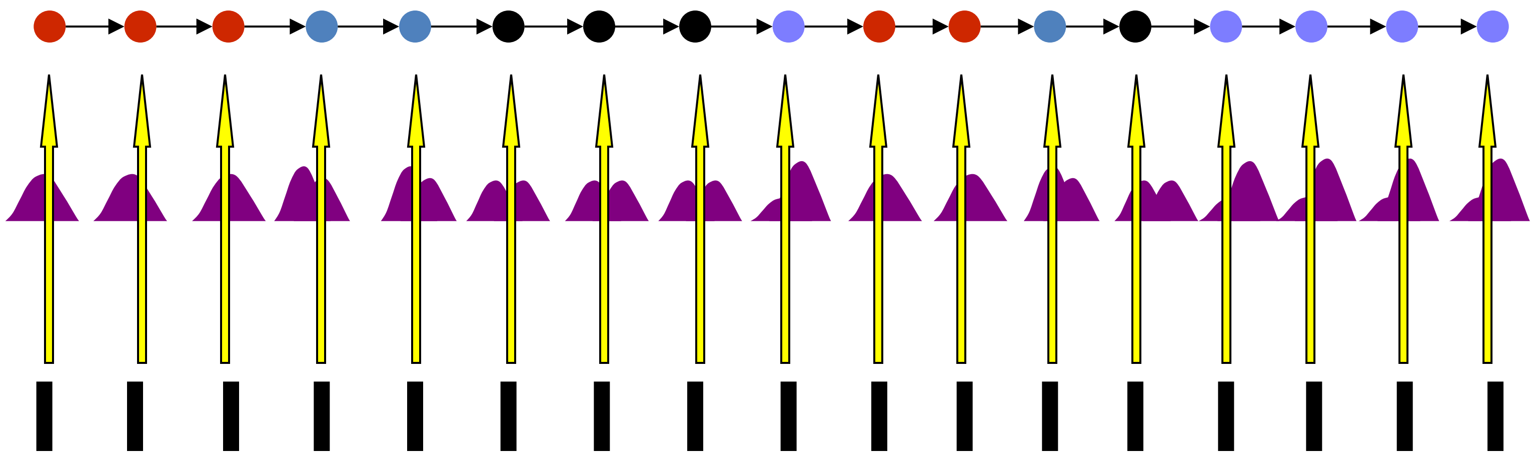

- State segmentation: Estimate state sequence given observations

Estimating the State Sequence

- Many different state sequences are capable of producing the observation

- Solution: Identify the most *probable* state sequence
  - The state sequence for which the probability of progressing through that sequence and generating the observation sequence is maximum
  - i.e $P(o_1, o_2, o_3, \ldots, s_1, s_2, s_3, \ldots)$ is maximum

Estimating the state sequence

- Once again, exhaustive evaluation is impossibly expensive
- But once again a simple dynamic-programming solution is available

$$P(o_1, o_2, o_3, \ldots, s_1, s_2, s_3, \ldots) =$$

$$\underline{P(o_1|s_1)P(o_2|s_2)P(o_3|s_3)\ldots}\;\underline{P(s_1)P(s_2|s_1)P(s_3|s_2)\ldots}$$

- Needed:

$$\arg\max_{s_1, s_2, s_3, \ldots} P(o_1 \mid s_1)P(s_1)P(o_2 \mid s_2)P(s_2 \mid s_1)P(o_3 \mid s_3)P(s_3 \mid s_2)$$

Estimating the state sequence

- Once again, exhaustive evaluation is impossibly expensive
- But once again a simple dynamic-programming solution is available

$$P(o_1, o_2, o_3, \ldots, s_1, s_2, s_3, \ldots) =$$

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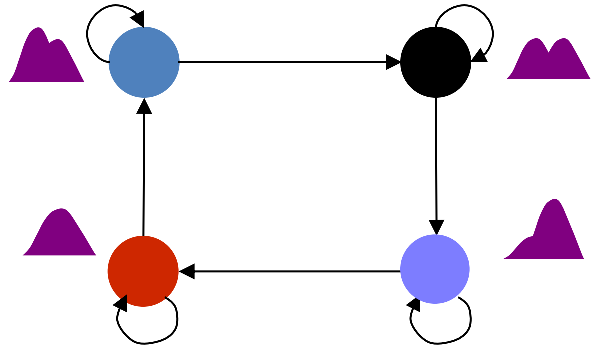
- Needed:

$$\arg\max_{s_1, s_2, s_3, \ldots} P(o_1 | s_1)P(s_1)P(o_2 | s_2)P(s_2 | s_1)P(o_3 | s_3)P(s_3 | s_2)$$

The HMM as a generator

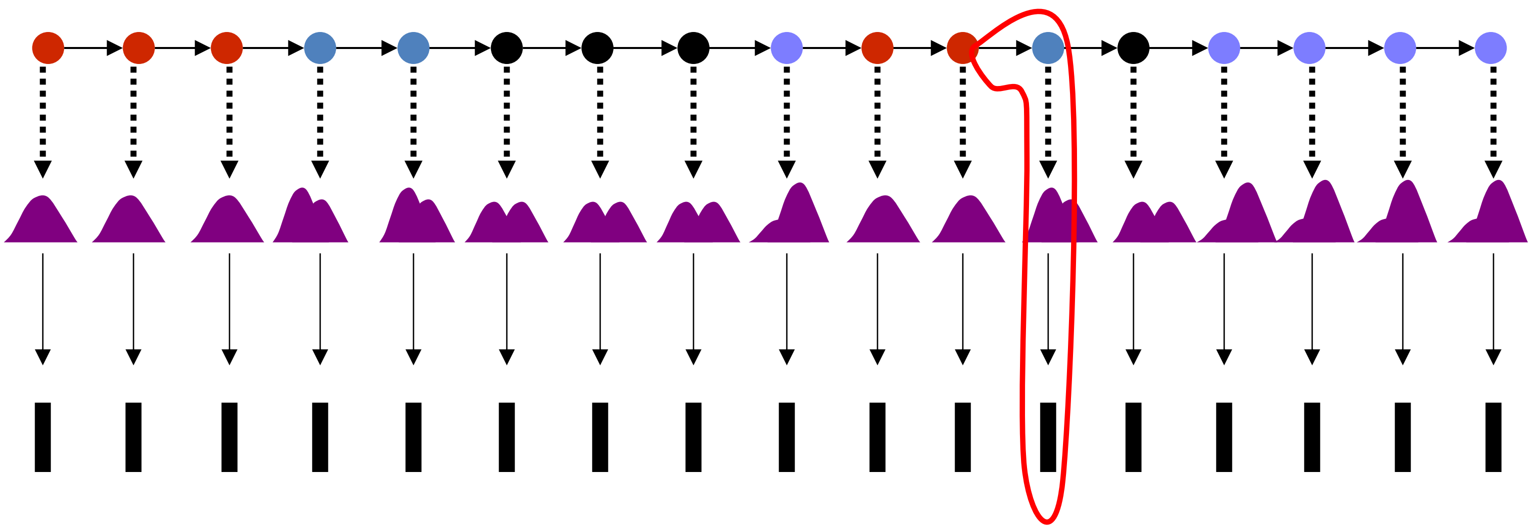
HMM assumed to be generating data



state sequence

state distributions

observation sequence



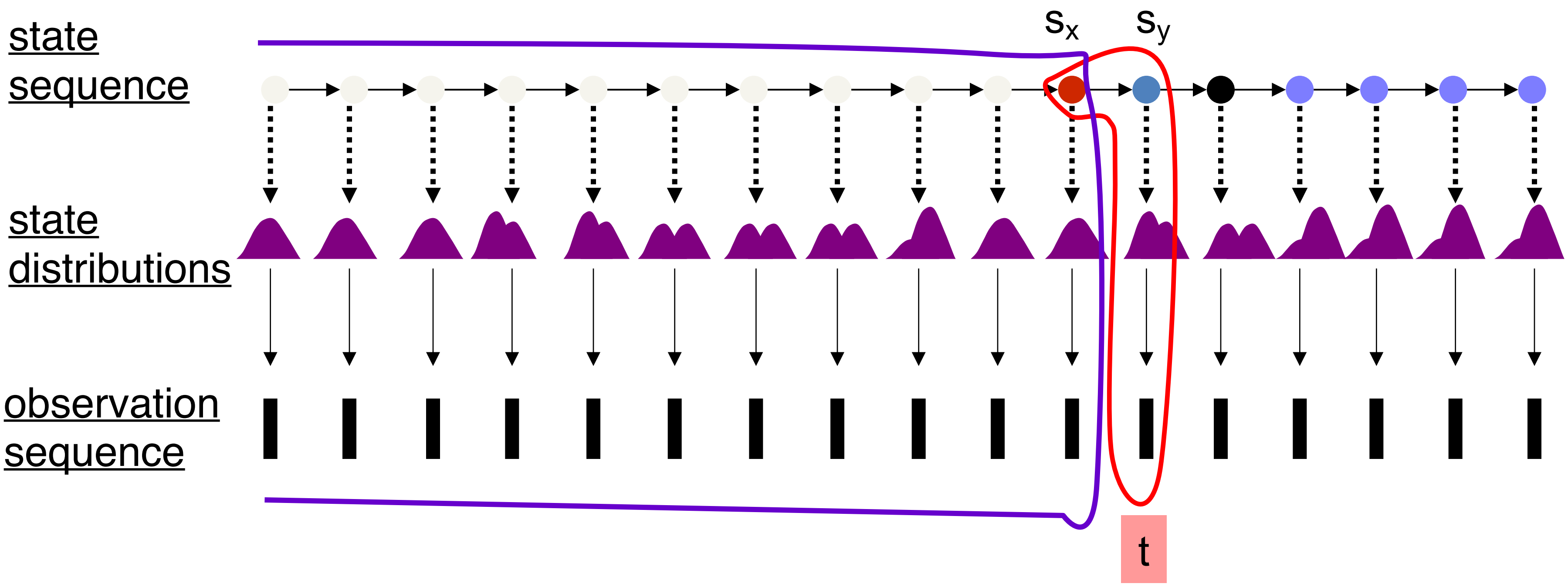
- Each enclosed term represents one forward transition and a subsequent emission

The state sequence

- The probability of a state sequence ?,?,?,?,s_x,s_y ending at time t , and producing all observations until o_t
  - $P(o_{1..t-1}, ?,?,?,?, s_x , o_t, s_y) = P(\underline{o_{1..t-1},?,?,?,?, s_x}) P(o_t|s_y)P(s_y|s_x)$

- The *best* state sequence that ends with s_x,s_y at t will have a probability equal to the probability of the best state sequence ending at t-1 at s_x times $P(o_t|s_y)P(s_y|s_x)$

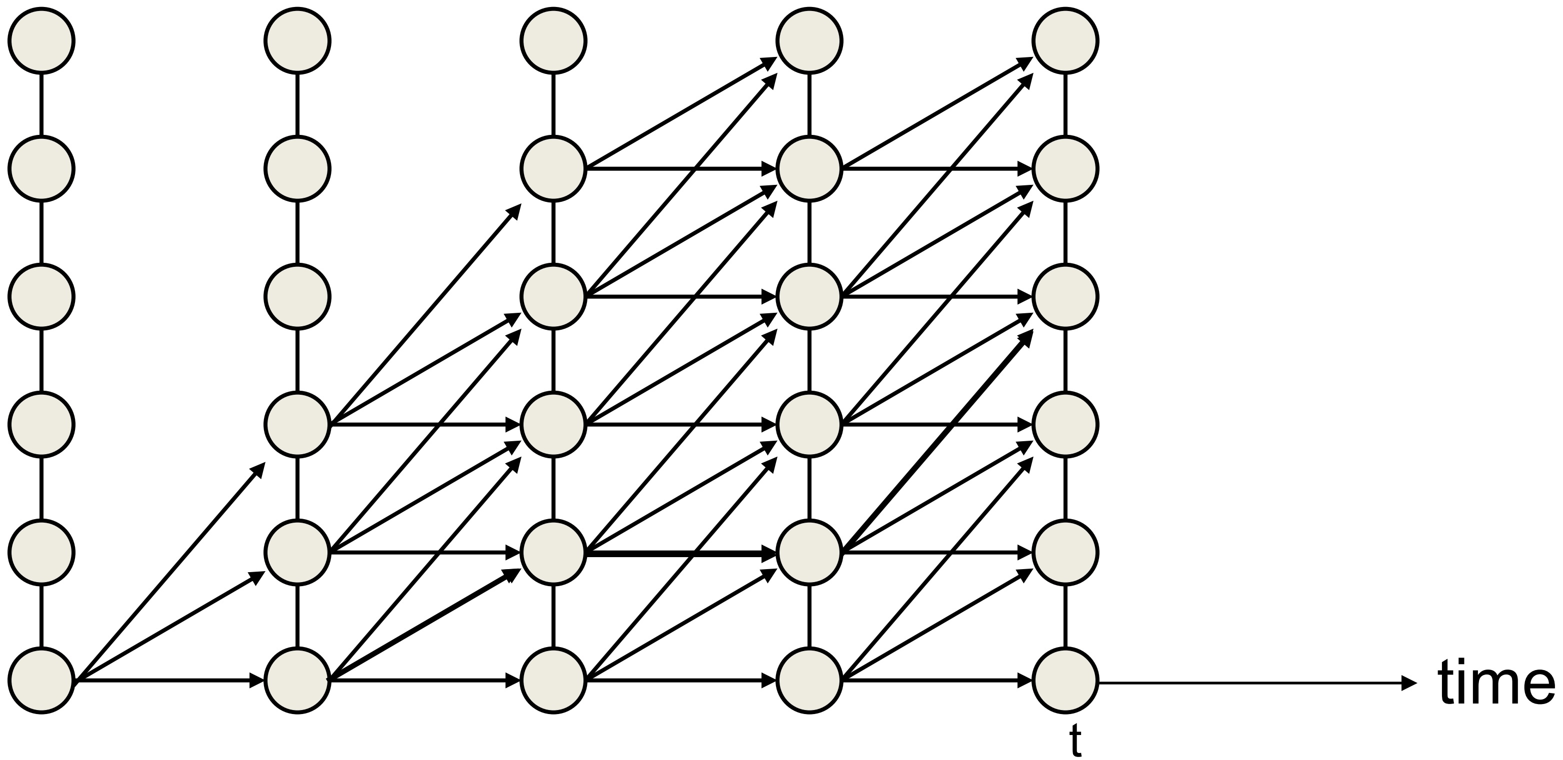
Extending the state sequence



- The probability of a state sequence ?,?,?,?,s_x,s_y ending at time *t* and producing observations until o_t
  - $P(o_{1..t-1}, o_t, ?,?,?,?, s_x, s_y) = P(o_{1..t-1},?,?,?,?, s_x)P(o_t|s_y)P(s_y|s_x)$

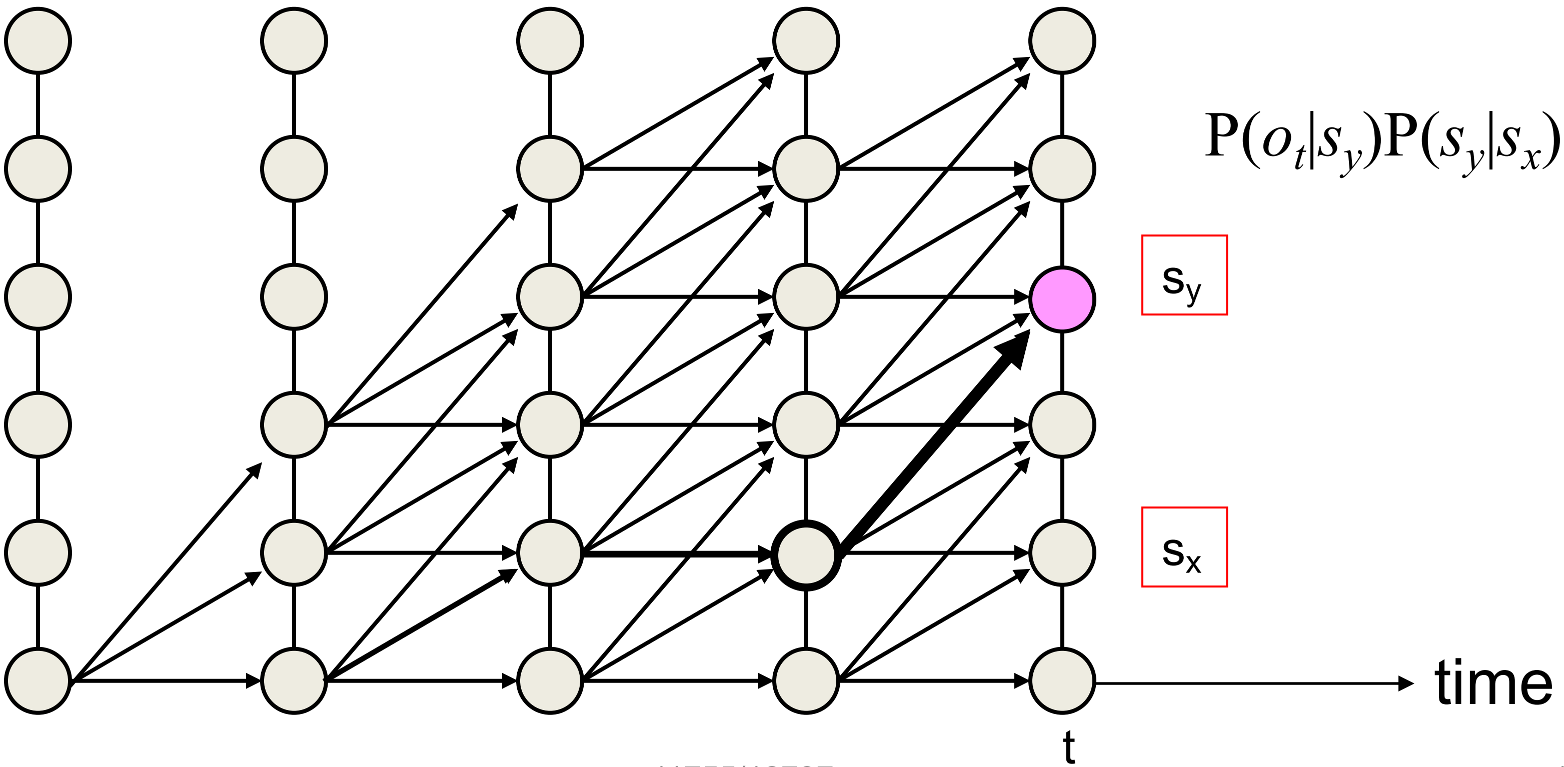
Trellis

- The graph below shows the set of all possible state sequences through this HMM in five time instants



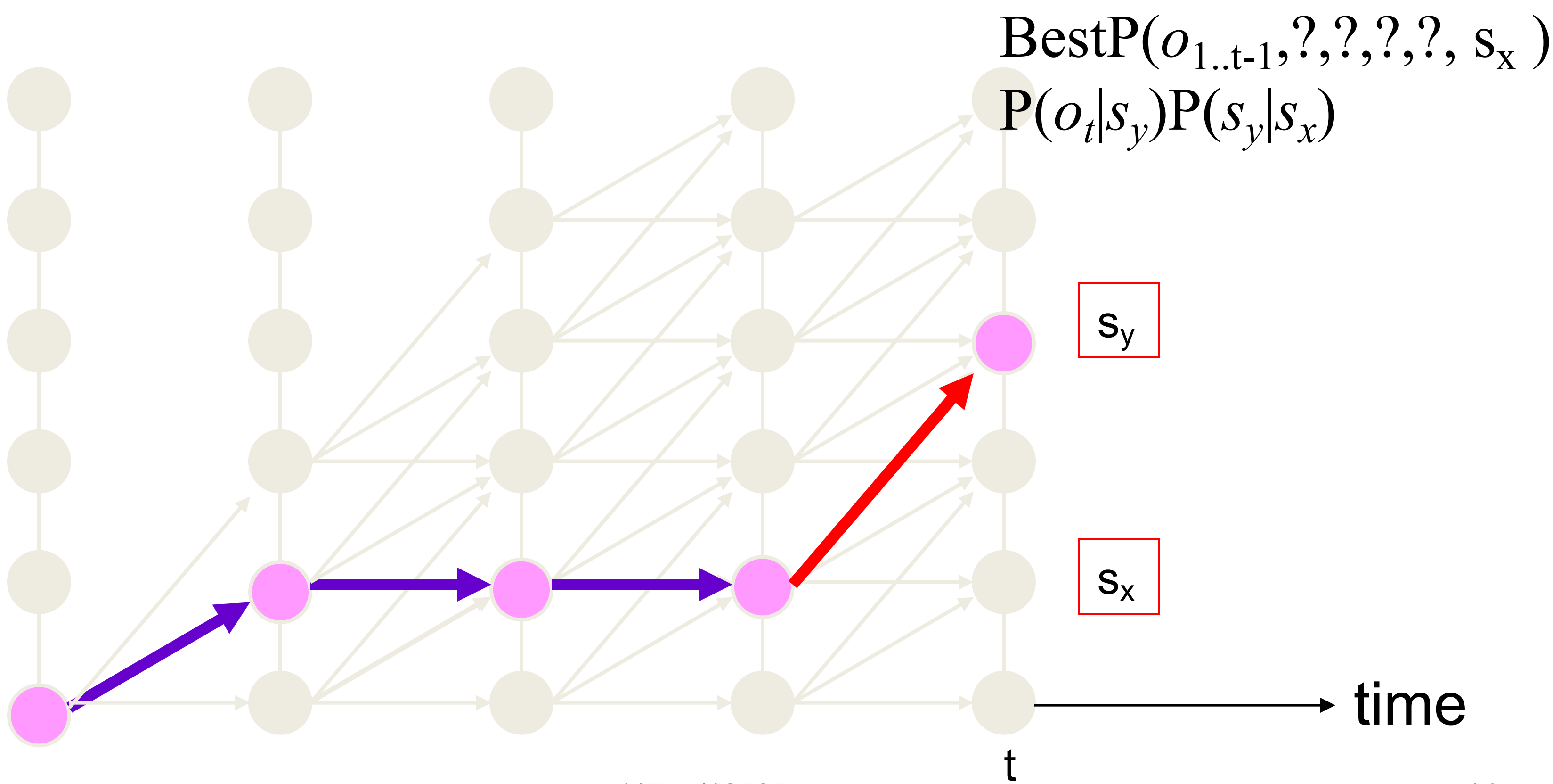
The cost of extending a state sequence

- The cost of *extending* a state sequence ending at s_x is only dependent on the transition from s_x to s_y, and the observation probability at s_y



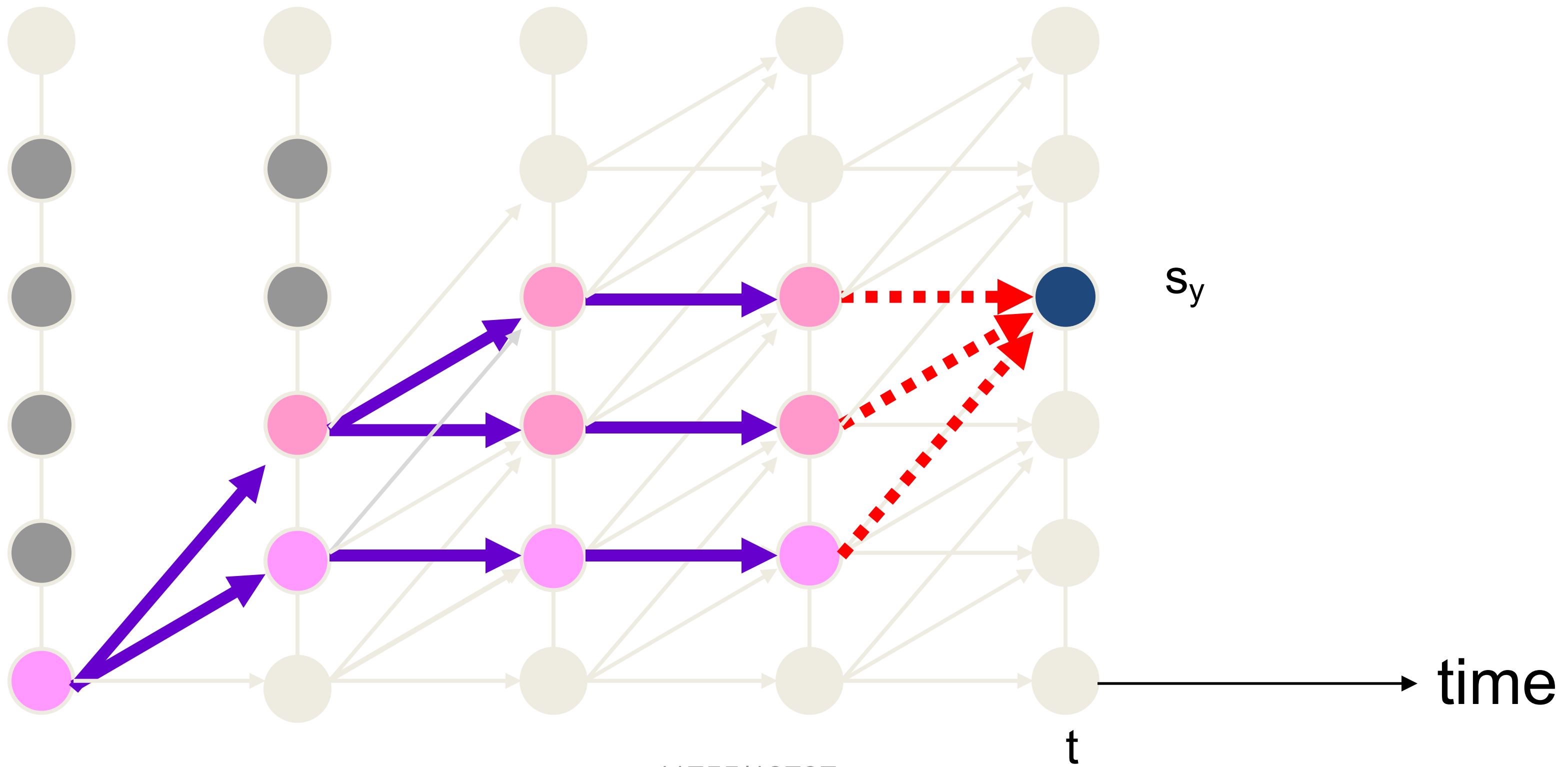
The cost of extending a state sequence

- The best path to s_y through s_x is simply an extension of the best path to s_x



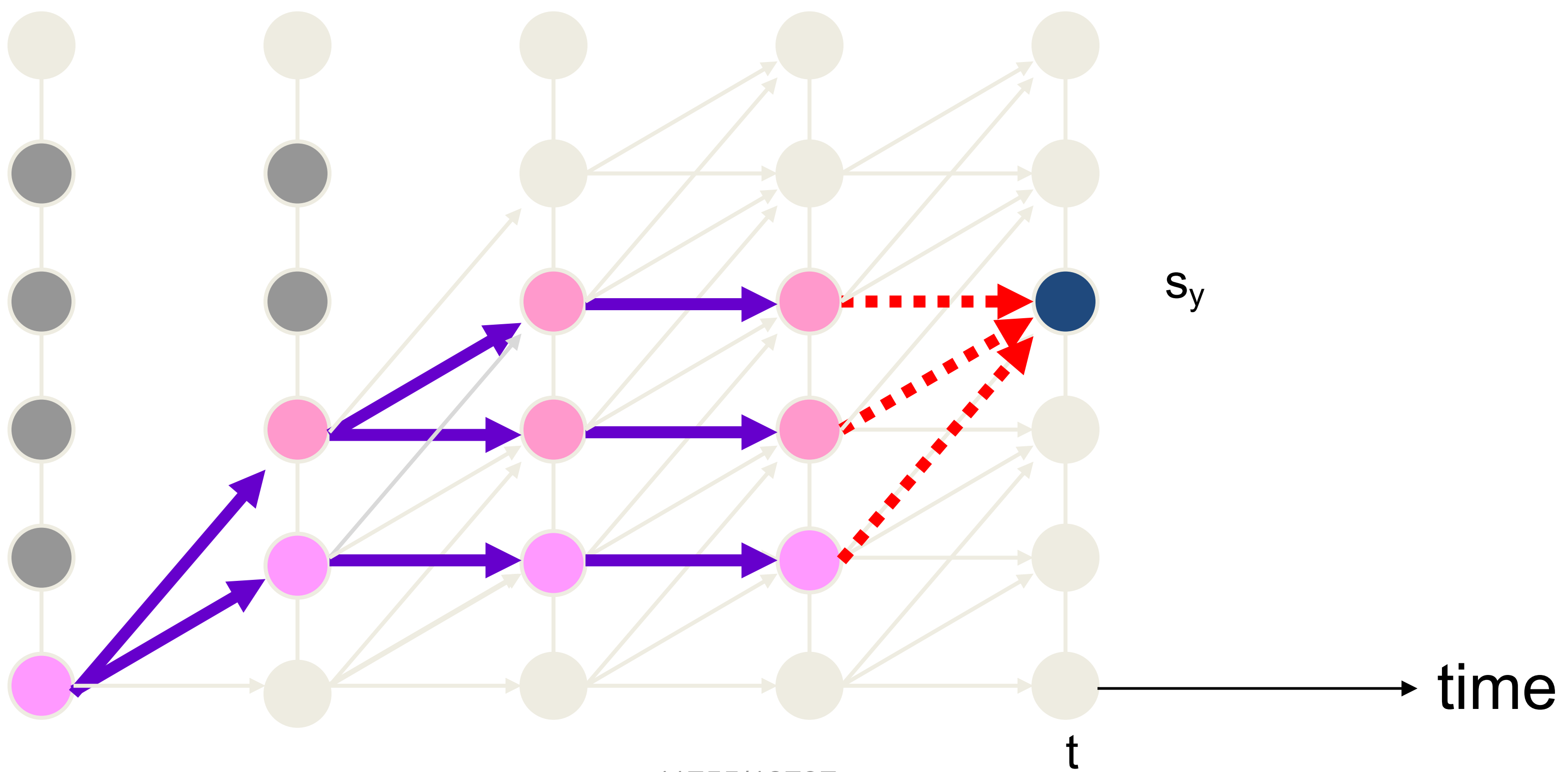
The Recursion

- The overall best path to s_y is an extension of the best path to one of the states at the previous time



The Recursion

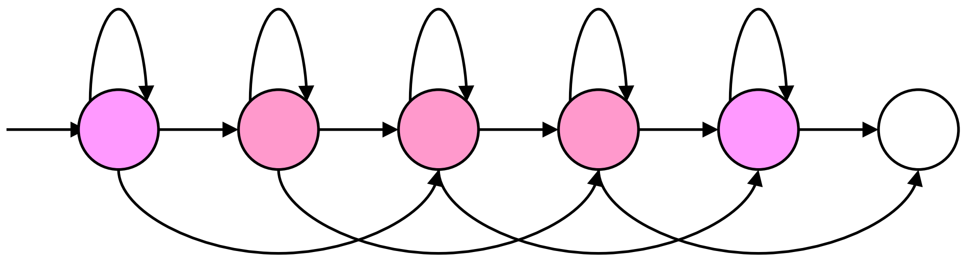
- Prob. of best path to s_y =
  Max_{s_x} $\text{BestP}(o_{1..t-1},?,?,?,?,\ s_x\ )\ P(o_t|s_y)P(s_y|s_x)$

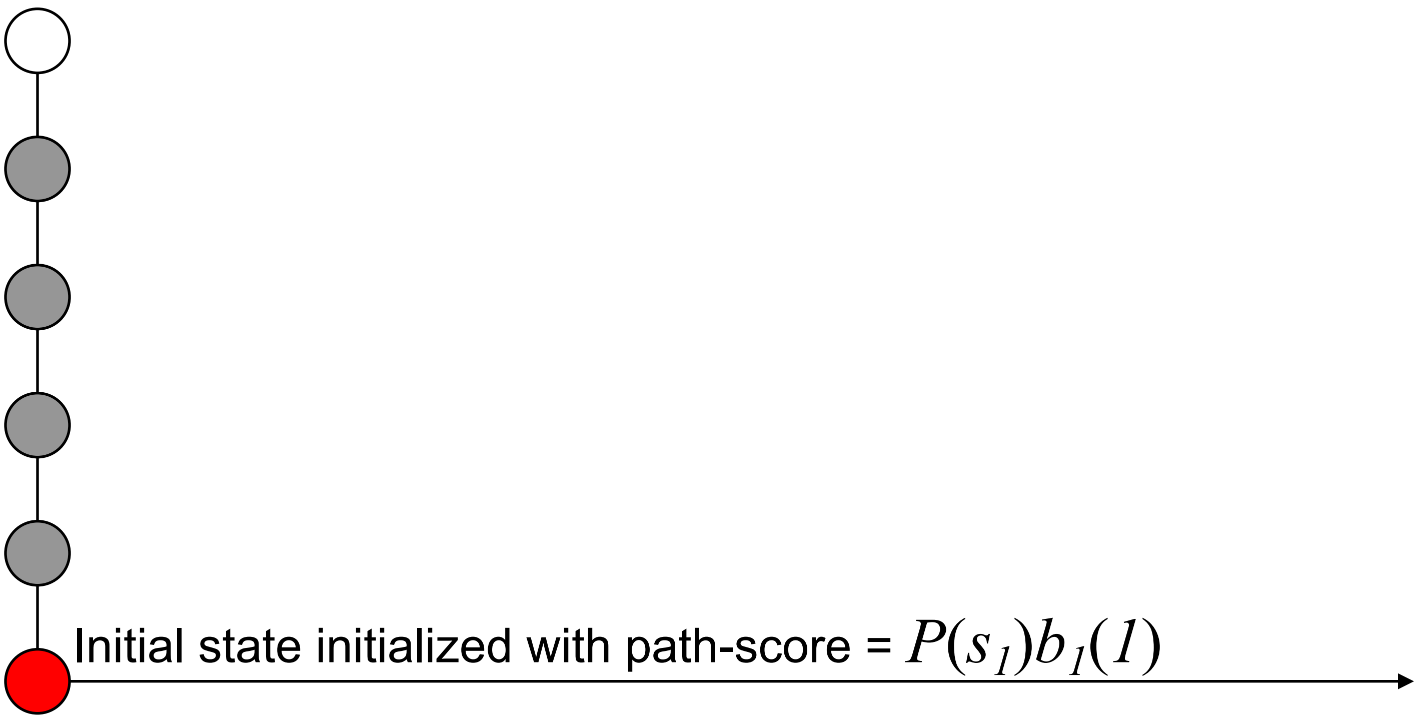


Finding the best state sequence

- The simple algorithm just presented is called the VITERBI algorithm in the literature
  - After A.J.Viterbi, who invented this dynamic programming algorithm for a completely different purpose: decoding error correction codes!

Viterbi Search (contd.)



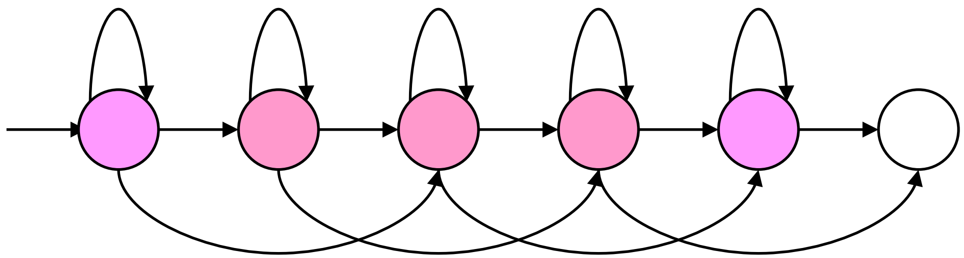


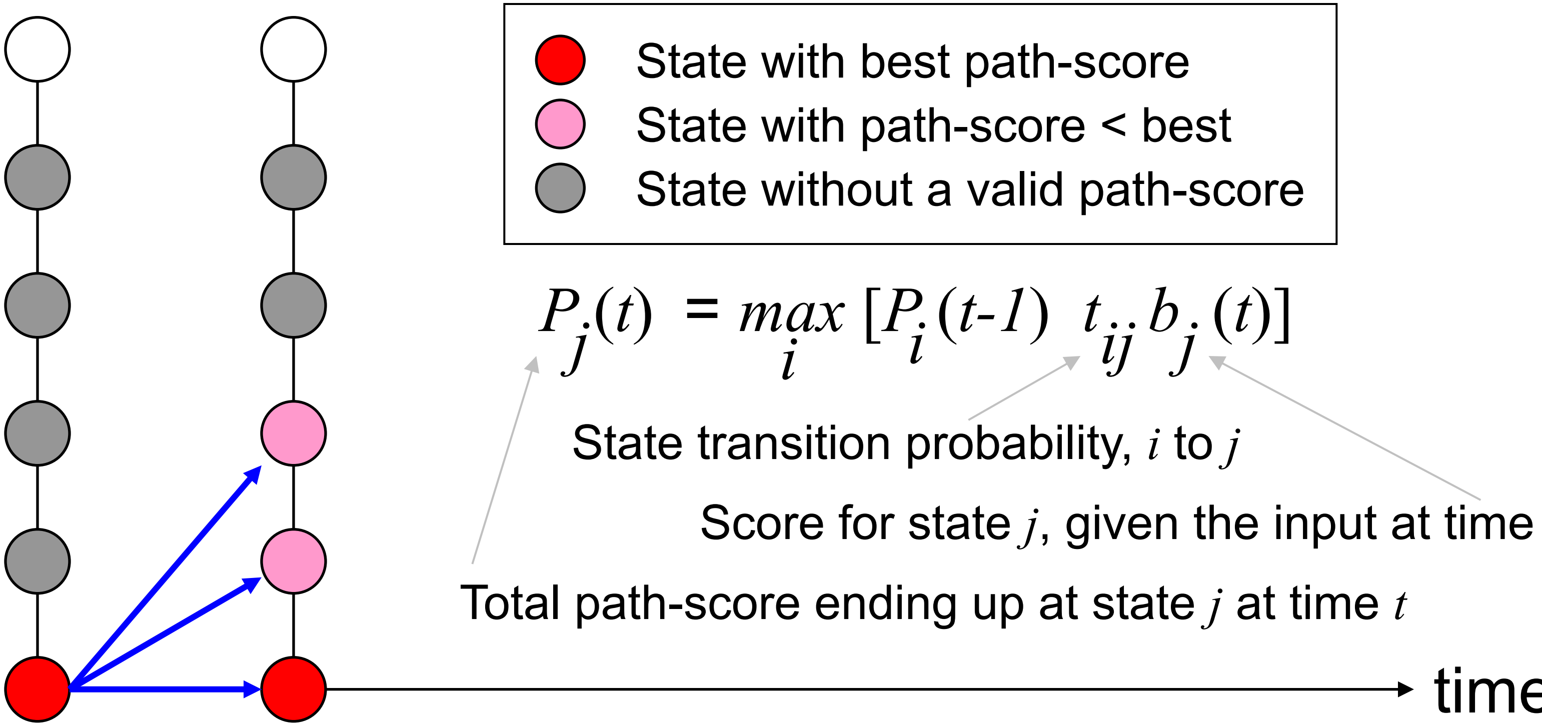
Initial state initialized with path-score = $P(s_1)b_1(1)$

time

In this example all other states have score 0 since $P(s_i) = 0$ for them

Viterbi Search (contd.)





- State with best path-score
- State with path-score < best
- State without a valid path-score

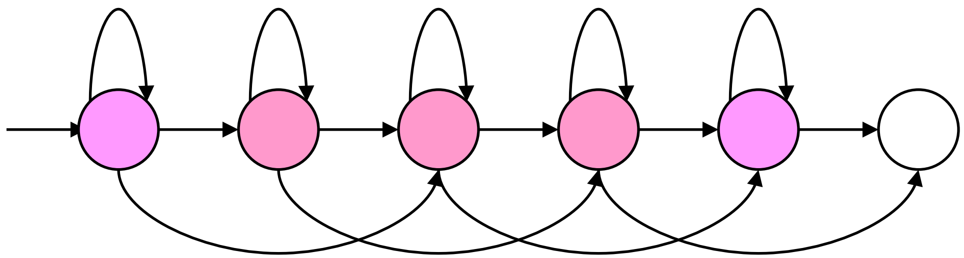
$$P_j(t) = \max_i \left[P_i(t-1)\; t_{ij}\; b_j(t) \right]$$

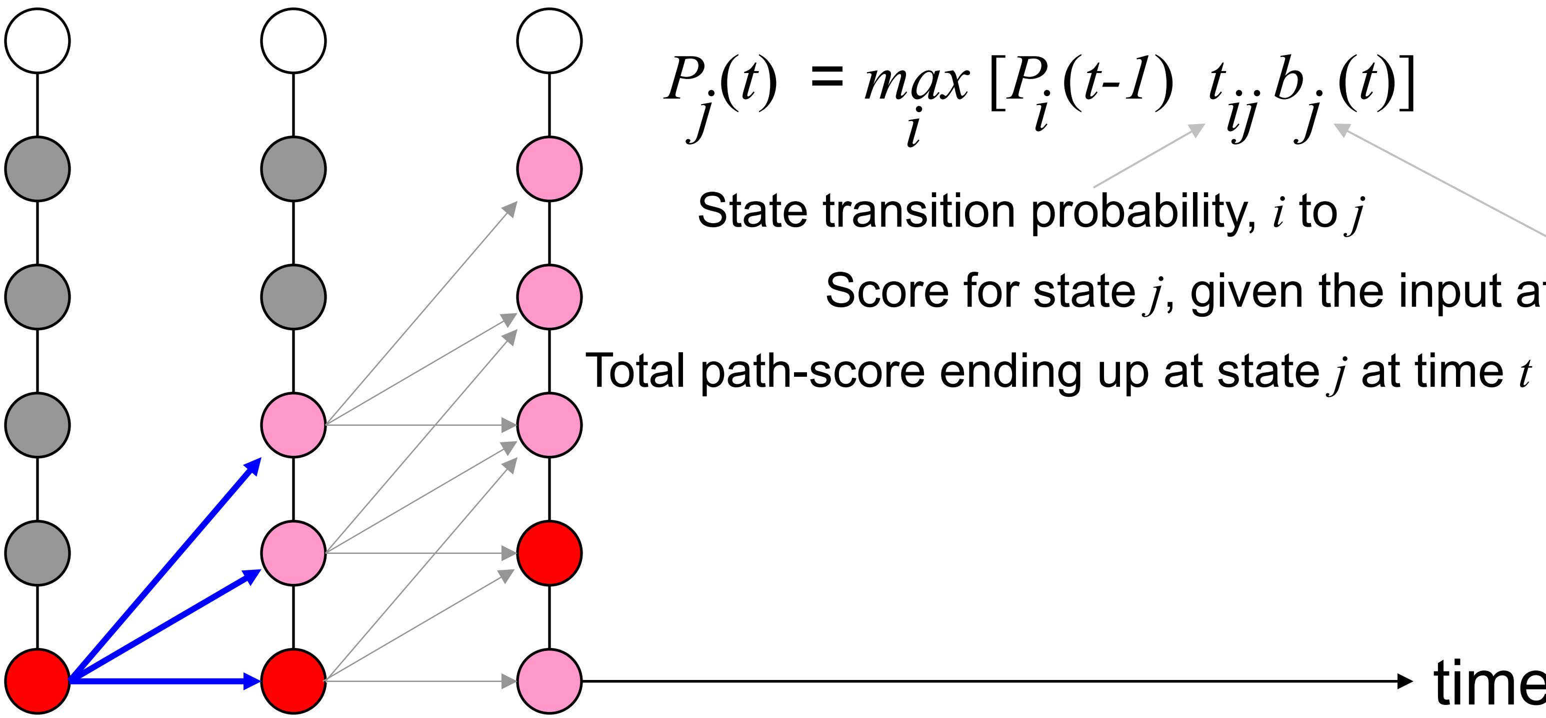
State transition probability, i to j

Score for state j, given the input at time t

Total path-score ending up at state j at time t

Viterbi Search (contd.)





$$P_j(t) = \max_i \left[P_i(t\text{-}1)\ t_{ij}\ b_j(t)\right]$$

State transition probability, i to j

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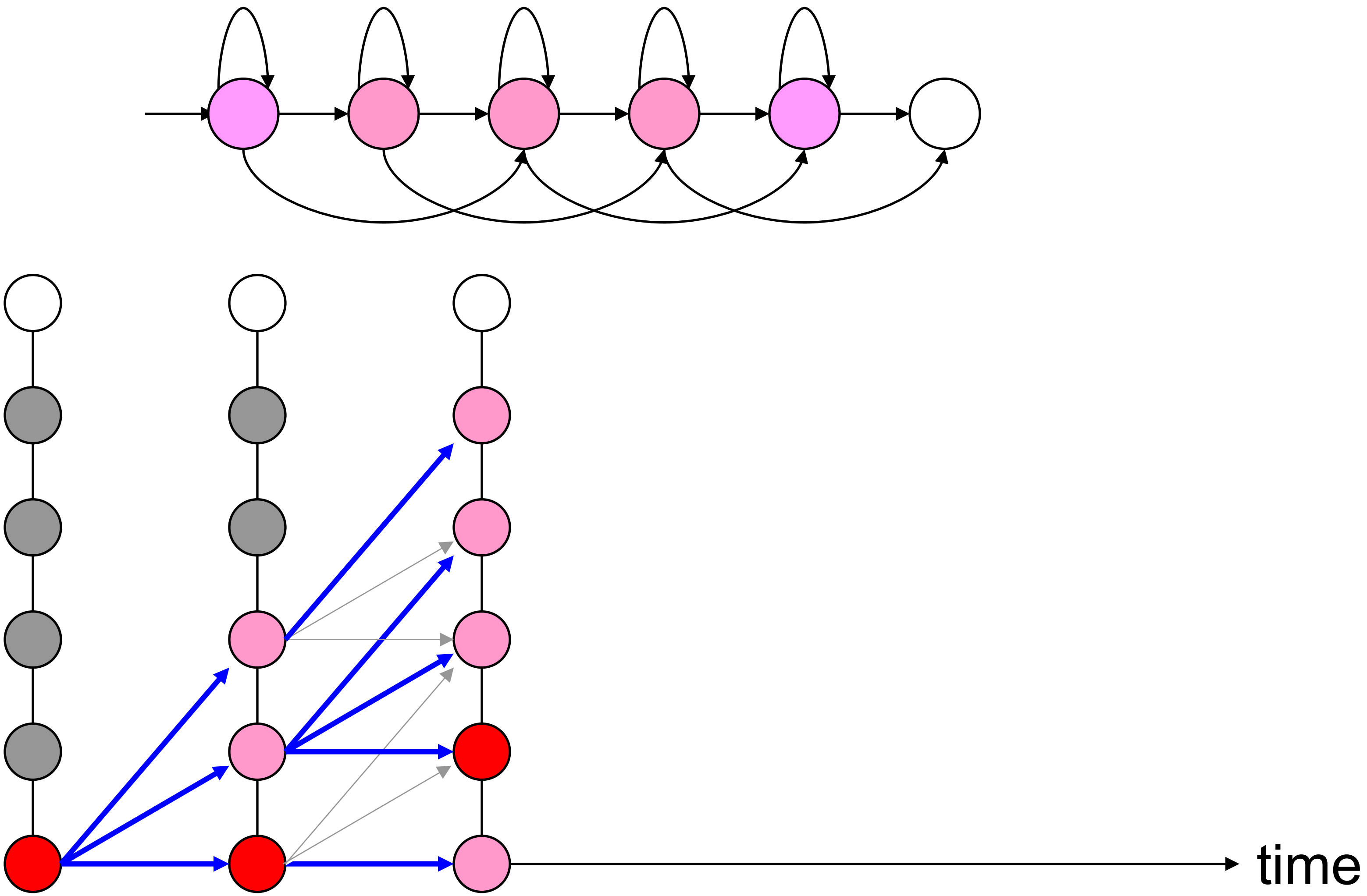
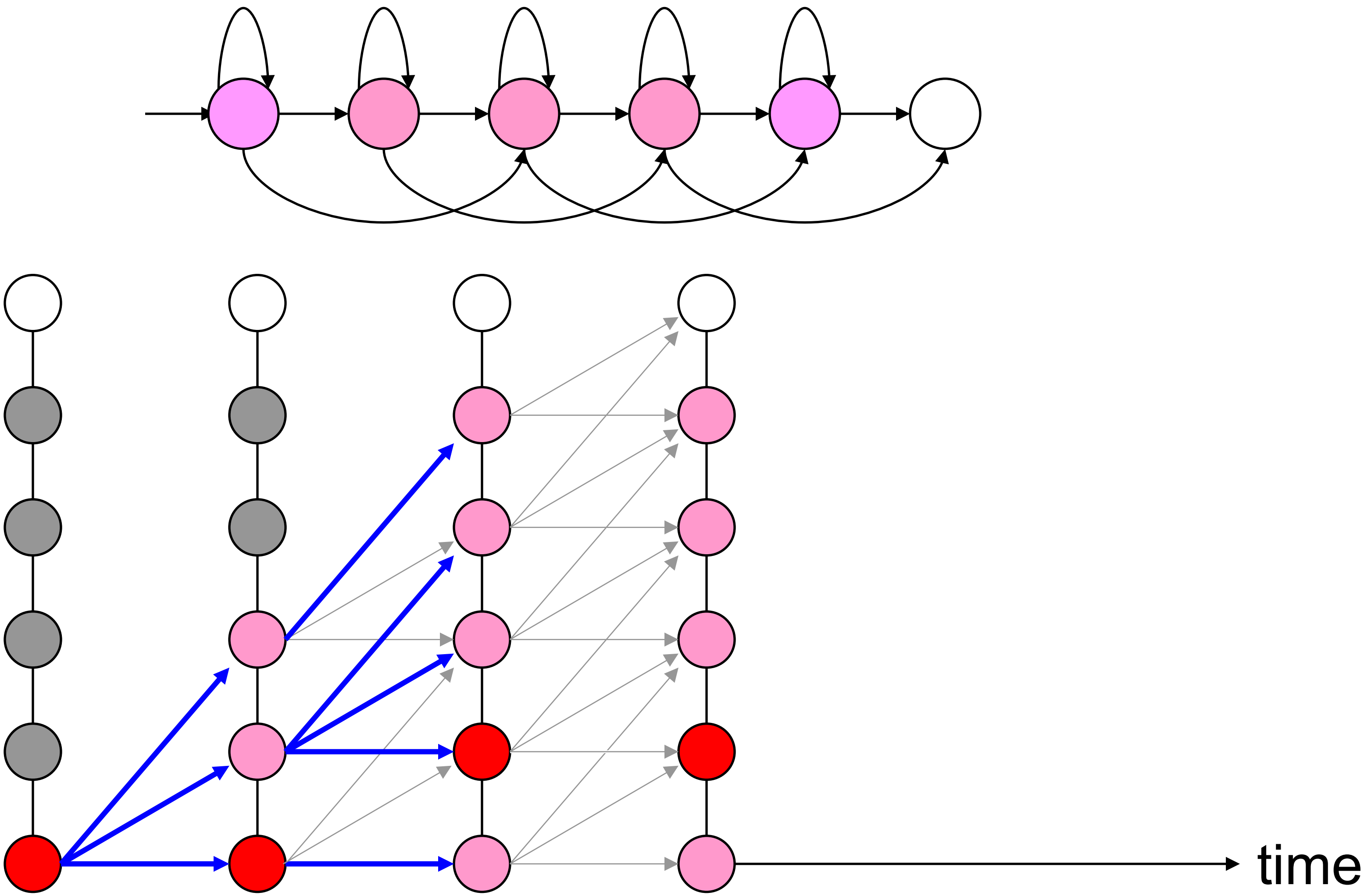
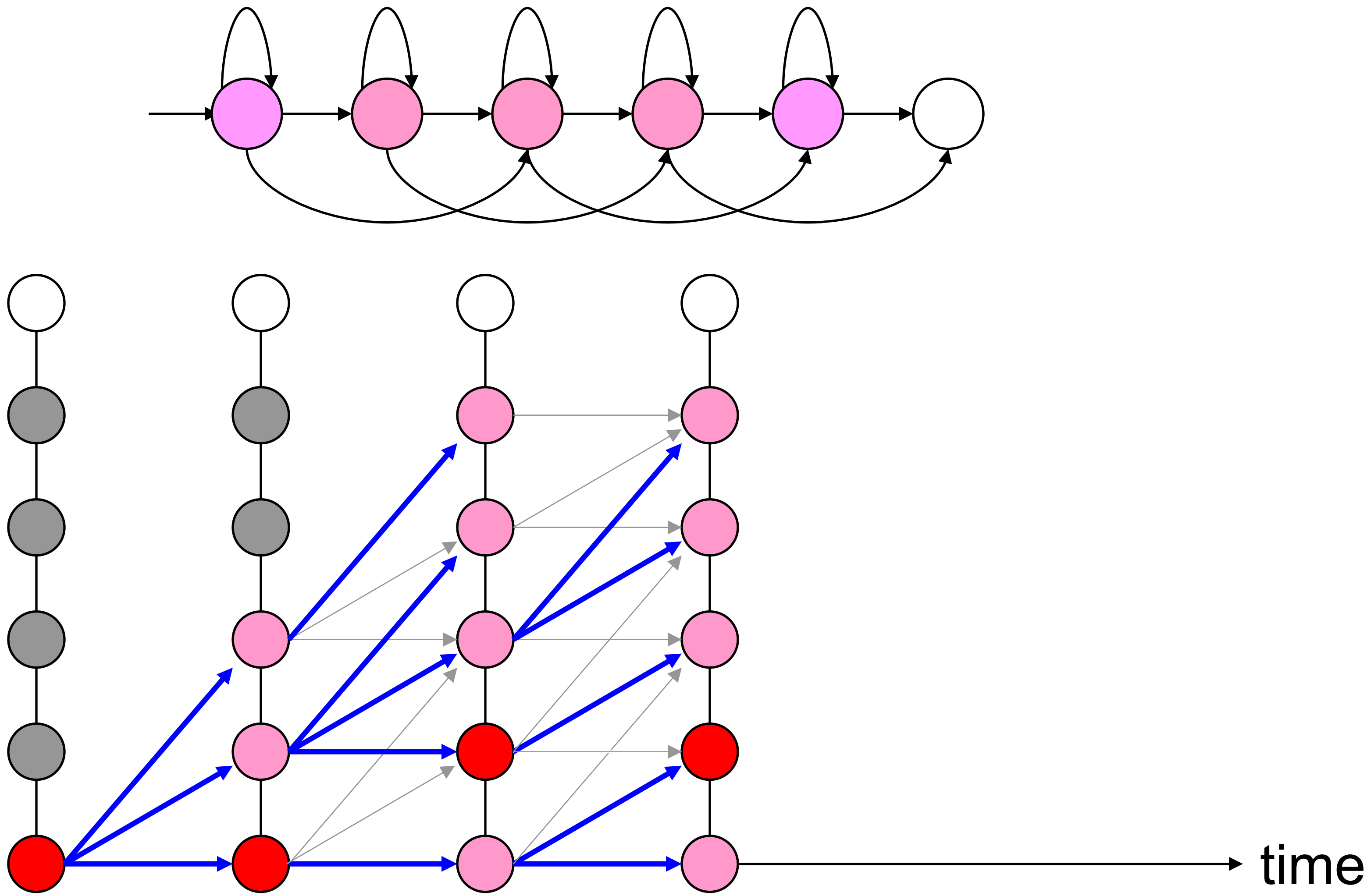


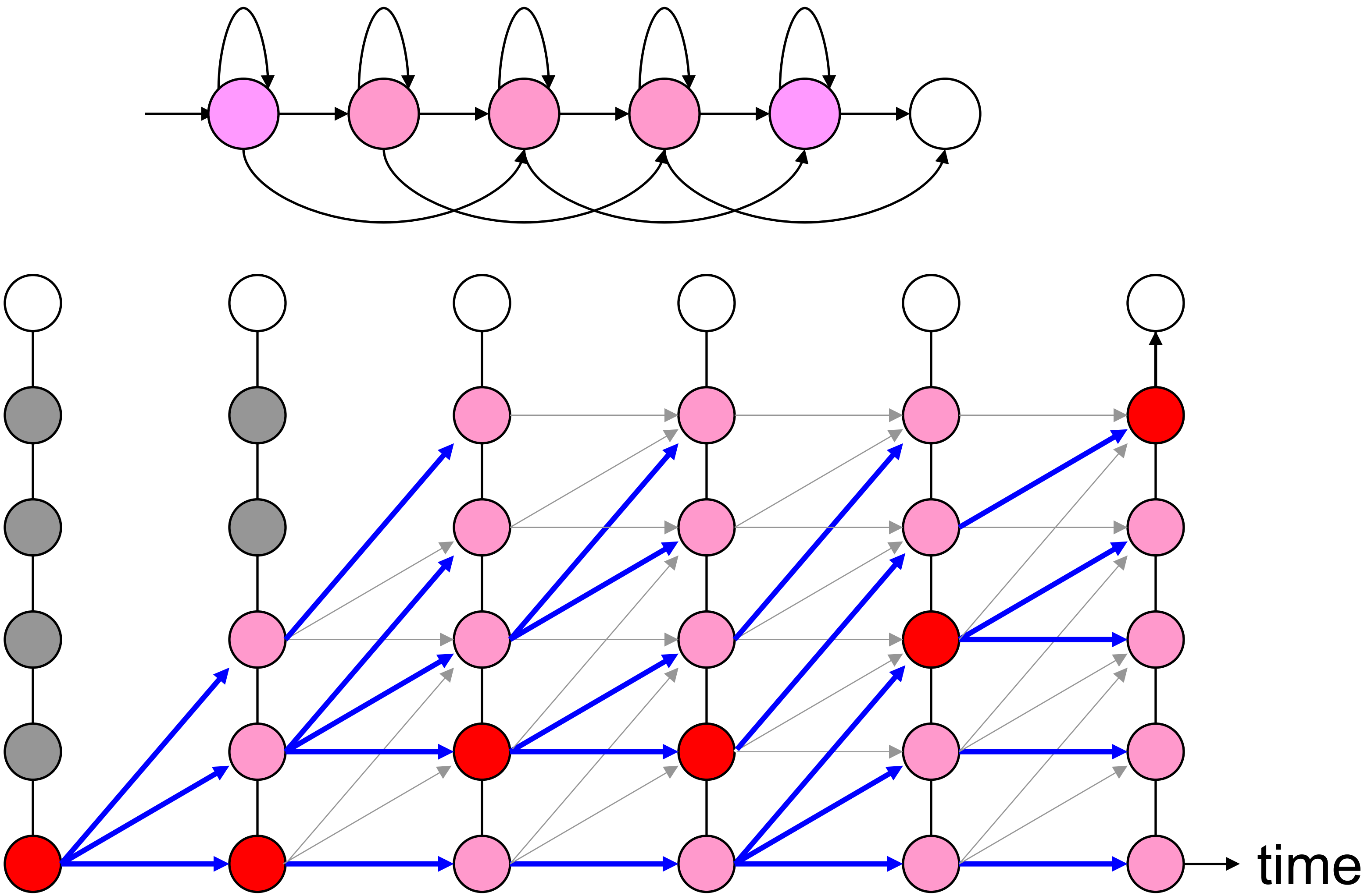
Viterbi Search (contd.)



Viterbi Search (contd.)



Viterbi Search (contd.)



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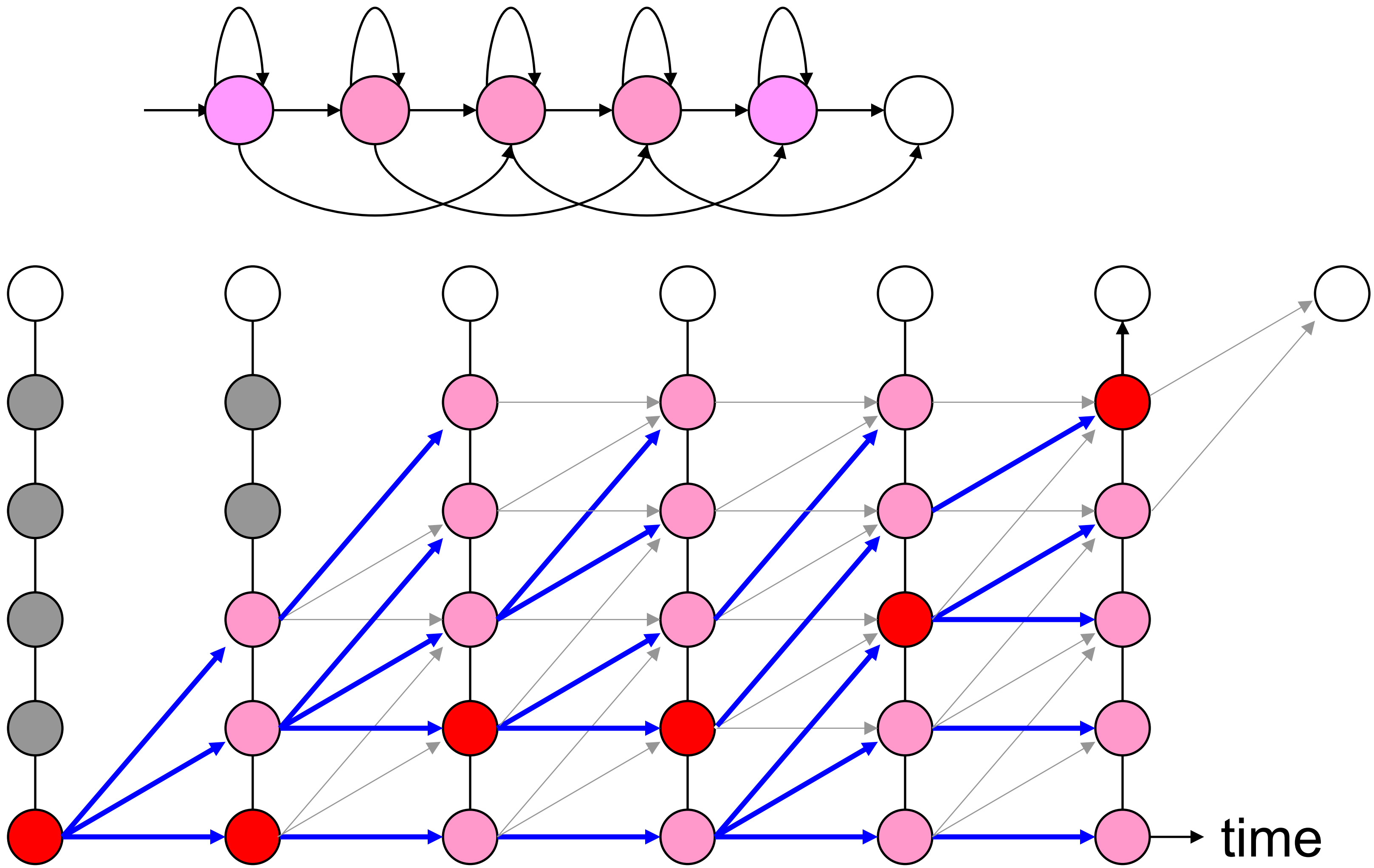
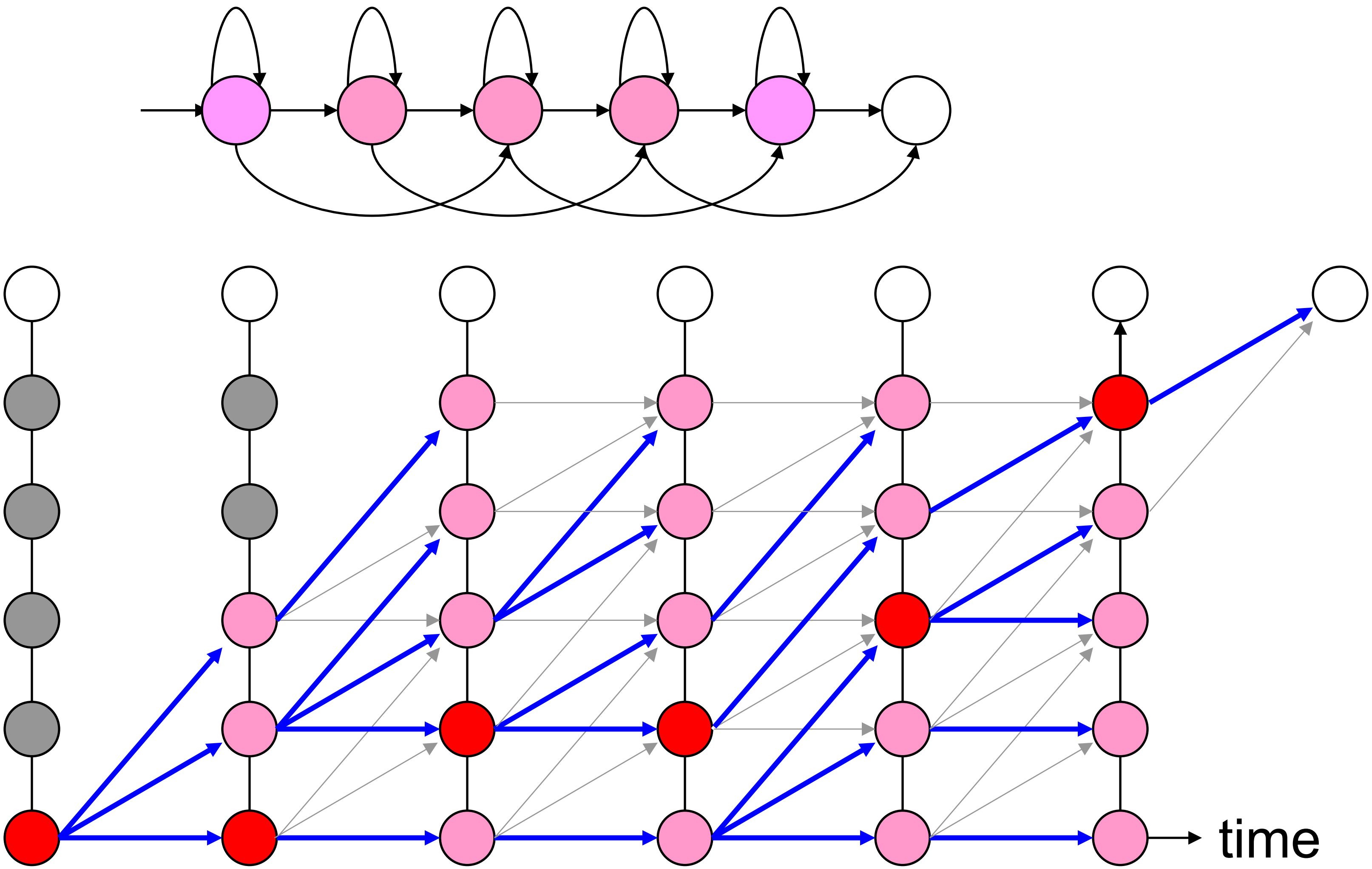


Viterbi Search (contd.)



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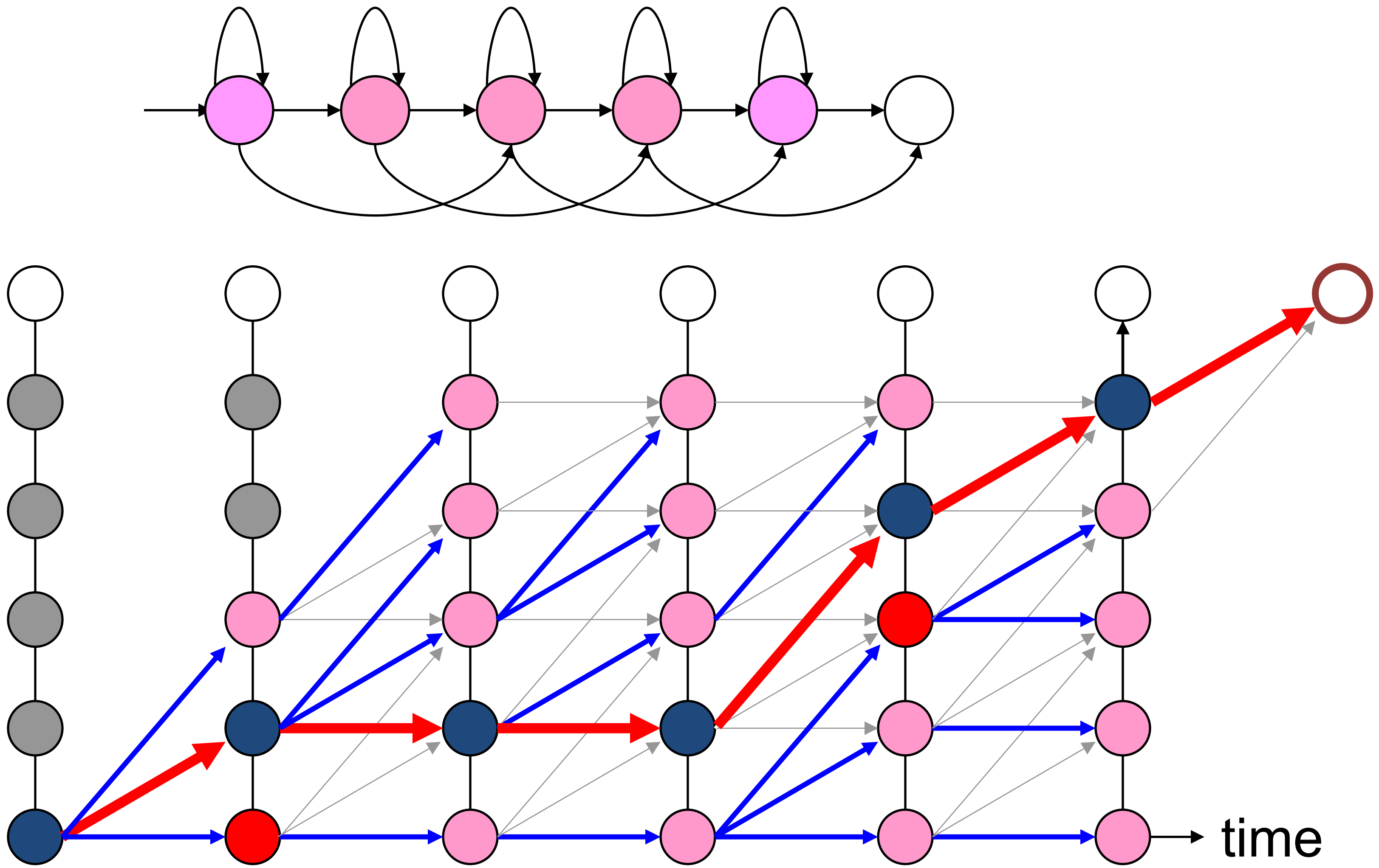
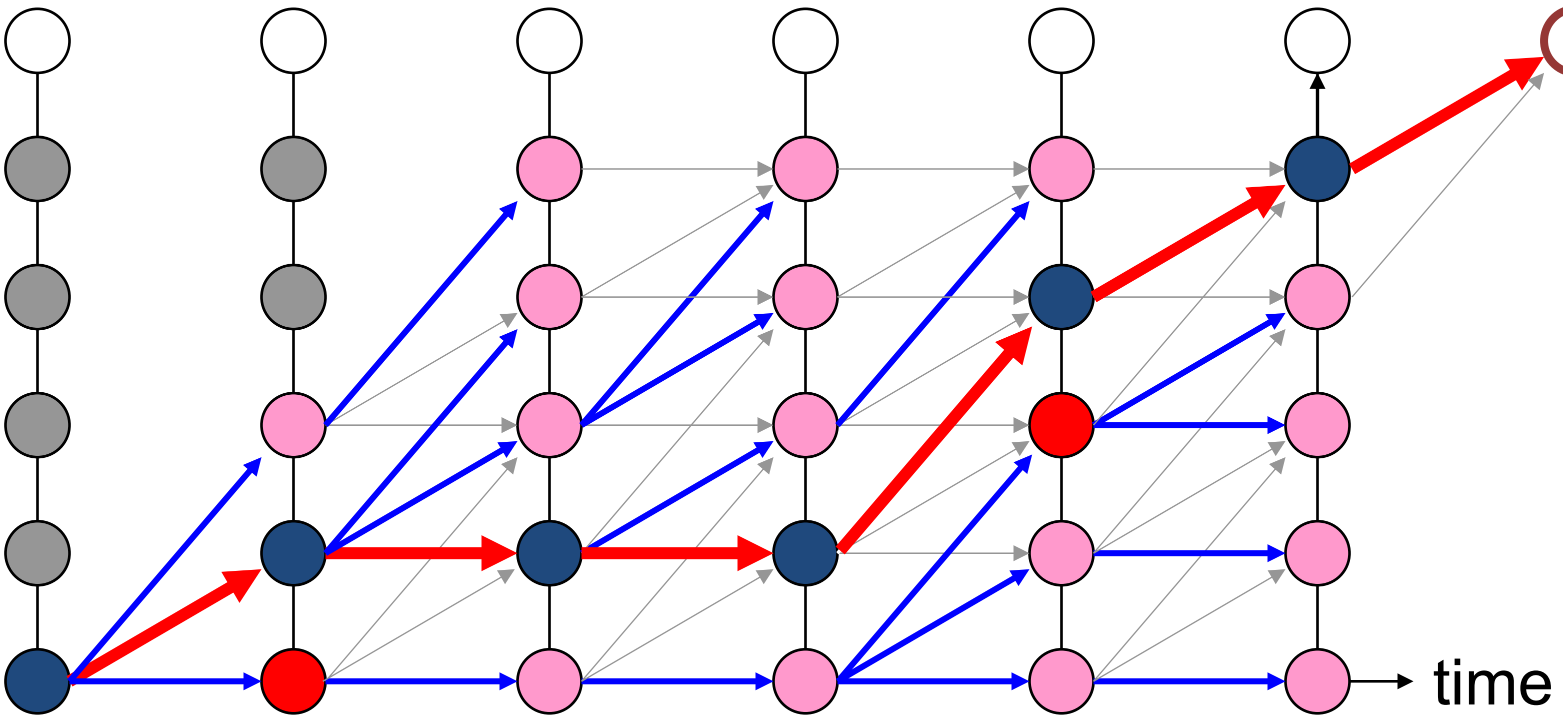


Viterbi Search (contd.)

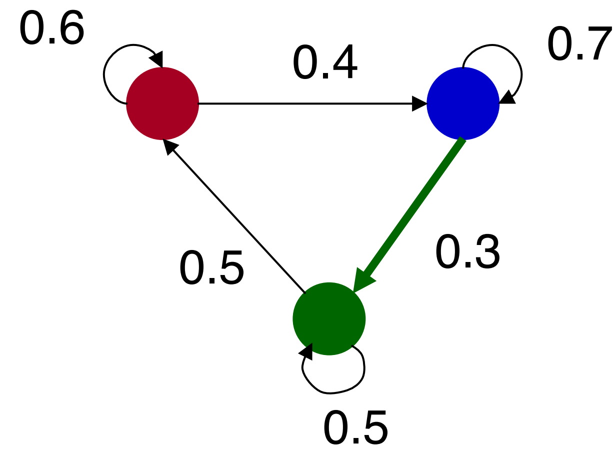
THE BEST STATE SEQUENCE IS THE ESTIMATE OF THE STATE SEQUENCE FOLLOWED IN GENERATING THE OBSERVATION



Problem3: Training HMM parameters

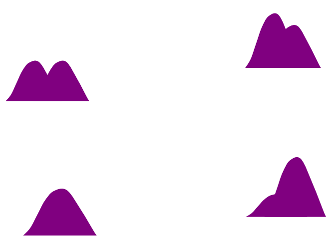
- We can compute the probability of an observation, and the best state sequence given an observation, using the HMM's parameters
- But where do the HMM parameters come from?
- They must be learned from a collection of observation sequences

HMM Parameters



- The transition probabilities
  - Often represented as a matrix as here
  - T_{ij} is the probability that when in state i, the process will move to j
- The probability π_i of beginning at any state s_i
  - The complete set is represented as π
- The *state output distributions*
  - Typically histograms, Gaussians, or Gaussian mixtures
  - Assuming Gaussian
    - Parameters are mean and variance

$$T = \begin{pmatrix} .6 & .4 & 0 \\ 0 & .7 & .3 \\ .5 & 0 & .5 \end{pmatrix}$$



Learning HMM parameters: Simple procedure – counting

- Given a set of training instances
- Iteratively:

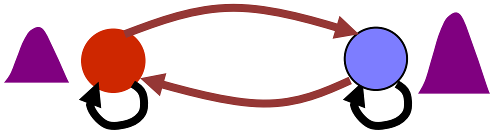
1. Initialize HMM parameters
2. Segment all training instances
3. Estimate transition probabilities and state output probability parameters by counting

Learning by counting example

- Explanation by example in next few slides
- 2-state HMM, Gaussian PDF at states, 3 observation sequences
- Example shows ONE iteration
  - How to count after state sequences are obtained

Example: Learning HMM Parameters

- We have an HMM with two states s1 and s2.
- Observations are vectors x_{ij}
  - i-th sequence, j-th vector
- We are given the following three observation sequences
  - And have already estimated state sequences



Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

Observation 2

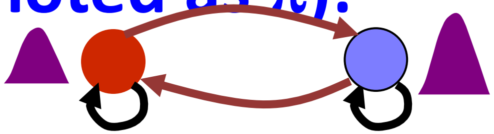
Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Example: Learning HMM Parameters

- **Initial state probabilities (usually denoted as π):**
  - We have 3 observations
  - 2 of these begin with S1, and one with S2
  - $\pi(S1) = 2/3$, $\pi(S2) = 1/3$



Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

Observation 2

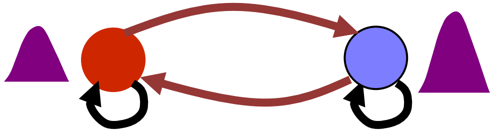
Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Example: Learning HMM Parameters

- **Transition probabilities:**
  - State S1 occurs 11 times in non-terminal locations



Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

Observation 2

Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
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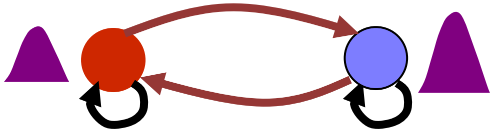
Example: Learning HMM Parameters



- **Transition probabilities:**
  - State S1 occurs 11 times in non-terminal locations
  - Of these, it is followed immediately by S1 6 times

Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

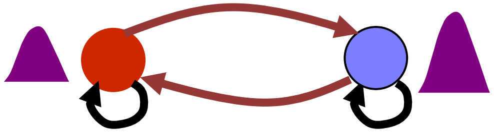
Observation 2

Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Example: Learning HMM Parameters



- **Transition probabilities:**
  - State S1 occurs 11 times in non-terminal locations
  - Of these, it is followed immediately by S1 6 times
  - It is followed immediately by S2 5 times

Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

Observation 2

Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
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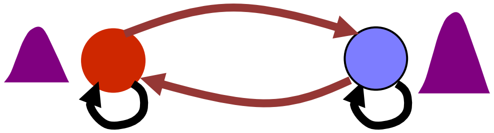
Example: Learning HMM Parameters



- **Transition probabilities:**
  - State S1 occurs 11 times in non-terminal locations
  - Of these, it is followed immediately by S1 6 times
  - It is followed immediately by S2 5 times
  - P(S1 | **S1**) = 6/ 11;   P(S2 | **S1**) = 5 / 11

Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

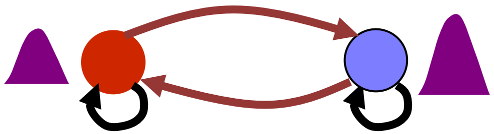
Observation 2

Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Example: Learning HMM Parameters



- **Transition probabilities:**
  - State S2 occurs 13 times in non-terminal locations

Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs.	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

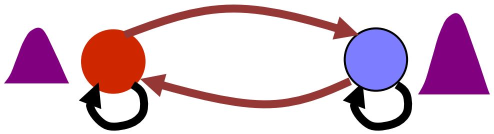
Observation 2

Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Example: Learning HMM Parameters



- **Transition probabilities:**
  - State S2 occurs 13 times in non-terminal locations
  - Of these, it is followed immediately by S1 5 times

Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

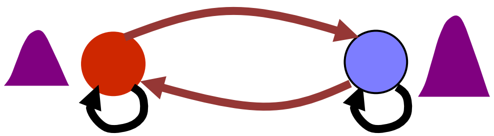
Observation 2

Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Example: Learning HMM Parameters



- **Transition probabilities:**
  - State S2 occurs 13 times in non-terminal locations
  - Of these, it is followed immediately by S1 5 times
  - It is followed immediately by S2 8 times

Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

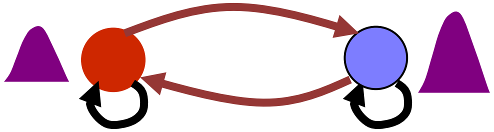
Observation 2

Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Example: Learning HMM Parameters



- **Transition probabilities:**
  - State S2 occurs 13 times in non-terminal locations
  - Of these, it is followed immediately by S1 5 times
  - It is followed immediately by S2 8 times
  - P(S1 | **S2**) = 5 / 13;   P(S2 | **S2**) = 8 / 13

Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

Observation 2

Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Parameters learnt so far

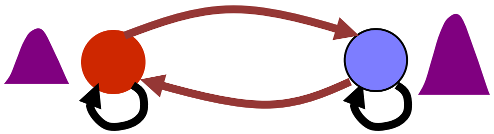
- State initial probabilities, often denoted as π
  - $\pi(S1) = 2/3 = 0.66$
  - $\pi(S2) = 1/3 = 0.33$
- State transition probabilities
  - $P(S1 \mid S1) = 6/11 = 0.545$; $P(S2 \mid S1) = 5/11 = 0.455$
  - $P(S1 \mid S2) = 5/13 = 0.385$; $P(S2 \mid S2) = 8/13 = 0.615$
  - Represented as a transition matrix

$$A = \begin{pmatrix} P(S1 \mid S1) & P(S2 \mid S1) \\ P(S1 \mid S2) & P(S2 \mid S2) \end{pmatrix} = \begin{pmatrix} 0.545 & 0.455 \\ 0.385 & 0.615 \end{pmatrix}$$

Each row of this matrix must sum to 1.0

Example: Learning HMM Parameters

- State output probability for S1
  - There are 13 observations in S1



Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

Observation 2

Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Example: Learning HMM Parameters

- State output probability for S1
  - There are 13 observations in S1
  - Segregate them out and count
    - Compute parameters (mean and variance) of Gaussian output density for state S1

Time	1	2	6	7	9	10
state	S1	S1	S1	S1	S1	S1
Obs	X_{a1}	X_{a2}	X_{a6}	X_{a7}	X_{a9}	X_{a10}

$$P(X \mid S_1) = \frac{1}{\sqrt{(2\pi)^d \, |\Theta_1|}} \exp\left(-0.5 (X-\mu_1)^T \Theta_1^{-1} (X-\mu_1)\right)$$

Time	3	4	9
state	S1	S1	S1
Obs	X_{b3}	X_{b4}	X_{b9}

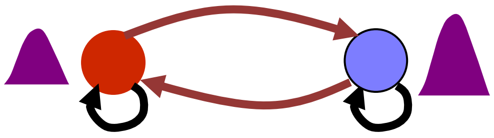
$$\mu_1 = \frac{1}{13}\left(\begin{array}{c} X_{a1} + X_{a2} + X_{a6} + X_{a7} + X_{a9} + X_{a10} + X_{b3} + \\ X_{b4} + X_{b9} + X_{c1} + X_{c2} + X_{c4} + X_{c5} \end{array}\right)$$

Time	1	3	4	5
state	S1	S1	S1	S1
Obs	X_{c1}	X_{c2}	X_{c4}	X_{c5}

$$\Theta_1 = \frac{1}{13}\left(\begin{array}{c} (X_{a1}-\mu_1)(X_{a1}-\mu_1)^T + (X_{a2}-\mu_1)(X_{a2}-\mu_1)^T + \ldots \\ (X_{b3}-\mu_1)(X_{b3}-\mu_1)^T + (X_{b4}-\mu_1)(X_{b4}-\mu_1)^T + \ldots \\ (X_{c1}-\mu_1)(X_{c1}-\mu_1)^T + (X_{c2}-\mu_1)(X_{c2}-\mu_1)^T + \ldots \end{array}\right)$$

Example: Learning HMM Parameters

- State output probability for S2
  - There are 14 observations in S2



Observation 1

Time	1	2	3	4	5	6	7	8	9	10
state	S1	S1	S2	S2	S2	S1	S1	S2	S1	S1
Obs	X_{a1}	X_{a2}	X_{a3}	X_{a4}	X_{a5}	X_{a6}	X_{a7}	X_{a8}	X_{a9}	X_{a10}

Observation 2

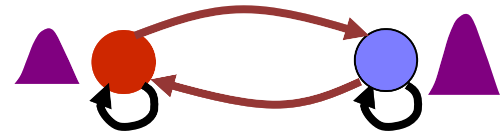
Time	1	2	3	4	5	6	7	8	9
state	S2	S2	S1	S1	S2	S2	S2	S2	S1
Obs	X_{b1}	X_{b2}	X_{b3}	X_{b4}	X_{b5}	X_{b6}	X_{b7}	X_{b8}	X_{b9}

Observation 3

Time	1	2	3	4	5	6	7	8
state	S1	S2	S1	S1	S1	S2	S2	S2
Obs	X_{c1}	X_{c2}	X_{c3}	X_{c4}	X_{c5}	X_{c6}	X_{c7}	X_{c8}

Example: Learning HMM Parameters

- State output probability for S2
  - There are 14 observations in S2
  - Segregate them out and count
    - Compute parameters (mean and variance) of Gaussian output density for state S2



Time	3	4	5	8
state	S2	S2	S2	S2
Obs	X_{a3}	X_{a4}	X_{a5}	X_{a8}

$$P(X \mid S_2) = \frac{1}{\sqrt{(2\pi)^d \,|\Theta_2|}} \exp\left(-0.5(X-\mu_2)^T \Theta_2^{-1}(X-\mu_2)\right)$$

Time	1	2	5	6	7	8
state	S2	S2	S2	S2	S2	S2
Obs	X_{b1}	X_{b2}	X_{b5}	X_{b6}	X_{b7}	X_{b8}

$$\mu_2 = \frac{1}{14}\begin{pmatrix} X_{a3} + X_{a4} + X_{a5} + X_{a8} + X_{b1} + X_{b2} + X_{b5} + \\ X_{b6} + X_{b7} + X_{b8} + X_{c2} + X_{c6} + X_{c7} + X_{c8} \end{pmatrix}$$

Time	2	6	7	8
state	S2	S2	S2	S2
Obs	X_{c2}	X_{c6}	X_{c7}	X_{c8}

$$\Theta_1 = \frac{1}{14}\left((X_{a3}-\mu_2)(X_{a3}-\mu_2)^T + \ldots\right)$$

We have learnt all the HMM parmeters

- State initial probabilities, often denoted as π
  - $\pi(S1) = 0.66$ $\pi(S2) = 1/3 = 0.33$
- State transition probabilities

$$A = \begin{pmatrix} 0.545 & 0.455 \\ 0.385 & 0.615 \end{pmatrix}$$

- State output probabilities

State output probability for S1

$$P(X \mid S_1) = \frac{1}{\sqrt{(2\pi)^d \mid \Theta_1 \mid}} \exp\left(-0.5(X-\mu_1)^T \Theta_1^{-1} (X-\mu_1)\right)$$

State output probability for S2

$$P(X \mid S_2) = \frac{1}{\sqrt{(2\pi)^d \mid \Theta_2 \mid}} \exp\left(-0.5(X-\mu_2)^T \Theta_2^{-1} (X-\mu_2)\right)$$

Update rules at each iteration

$$\pi(s_i) = \frac{\sum_{obs} \delta_{t=1}(s_i)}{N_{obs}}$$

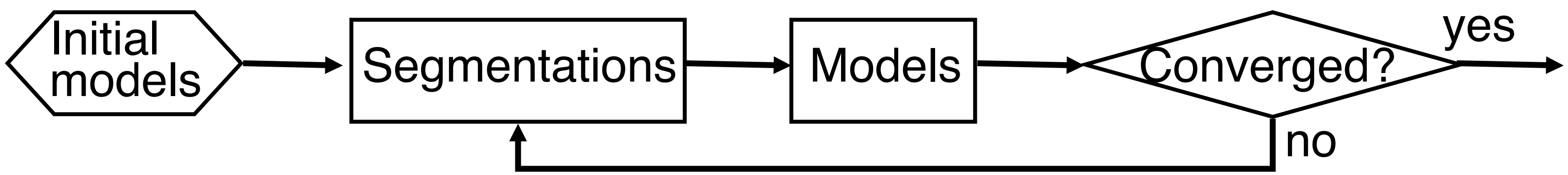
$$P(s_j|s_i) = \frac{\sum_{obs} \sum_{t=1}^{T-1} \delta_{t,t+1}(s_j|s_i)}{\sum_{obs} \sum_{t=1}^{T-1} \delta_t(s_i)}$$

$$\mu_i = \frac{\sum_{obs} \sum_{t=1}^{T} \delta_t(s_i)\, X_{obs}(t)}{\sum_{obs} \sum_{t=1}^{T} \delta_t(s_i)}$$

$$\Theta_i = \frac{\sum_{obs} \sum_{t=1}^{T} \delta_t(s_i)\, (X_{obs}(t) - \mu_i)(X_{obs}(t) - \mu_i)^T}{\sum_{obs} \sum_{t=1}^{T} \delta_t(s_i)}$$

- Assumes state output PDF = Gaussian
  - For GMMs, estimate GMM parameters from collection of observations at any state

Training by segmentation: Viterbi training



- Initialize all HMM parameters

- Segment all training observation sequences into states using the Viterbi algorithm with the current models

- Using estimated state sequences and training observation sequences, reestimate the HMM parameters

- This method is also called a “segmental k-means” learning procedure

Poll 1

Alternative to counting: SOFT counting

- Expectation maximization
- *Every* observation contributes to every state

Update rules at each iteration

$$\pi(s_i) = \frac{\sum_{obs} \delta_{t=1}(s_i)}{N_{obs}}$$

$$P(s_j|s_i) = \frac{\sum_{obs} \sum_{t=1}^{T-1} \delta_{t,t+1}(s_j|s_i)}{\sum_{obs} \sum_{t=1}^{T-1} \delta_t(s_i)}$$

$$\mu_i = \frac{\sum_{obs} \sum_{t=1}^{T} \delta_t(s_i)\, X_{obs}(t)}{\sum_{obs} \sum_{t=1}^{T} \delta_t(s_i)}$$

$$\Theta_i = \frac{\sum_{obs} \sum_{t=1}^{T} \delta_t(s_i)\, (X_{obs}(t) - \mu_i)(X_{obs}(t) - \mu_i)^T}{\sum_{obs} \sum_{t=1}^{T} \delta_t(s_i)}$$

- Assumes state output PDF = Gaussian
  - For GMMs, estimate GMM parameters from collection of observations at any state

Update rules at each iteration

$$\pi(s_i) = \frac{\sum_{Obs} P(state(t=1) = s_i \mid Obs)}{\text{Total no. of observation sequences}}$$

$$P(s_j \mid s_i) = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i, state(t+1) = s_j \mid Obs)}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

$$\mu_i = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs) X_{Obs,t}}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

$$\Theta_i = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)(X_{Obs,t} - \mu_i)(X_{Obs,t} - \mu_i)^T}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

- Every observation contributes to every state

Poll 2

Update rules at each iteration

$$\pi(s_i) = \frac{\sum_{Obs} P(state(t=1) = s_i \mid Obs)}{\text{Total no. of observation sequences}}$$

$$P(s_j \mid s_i) = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i, state(t+1) = s_j \mid Obs)}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

$$\mu_i = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs) X_{Obs,t}}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

$$\Theta_i = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)(X_{Obs,t} - \mu_i)(X_{Obs,t} - \mu_i)^T}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

- Where did these terms come from?

$$P(state(t) = s \mid Obs)$$

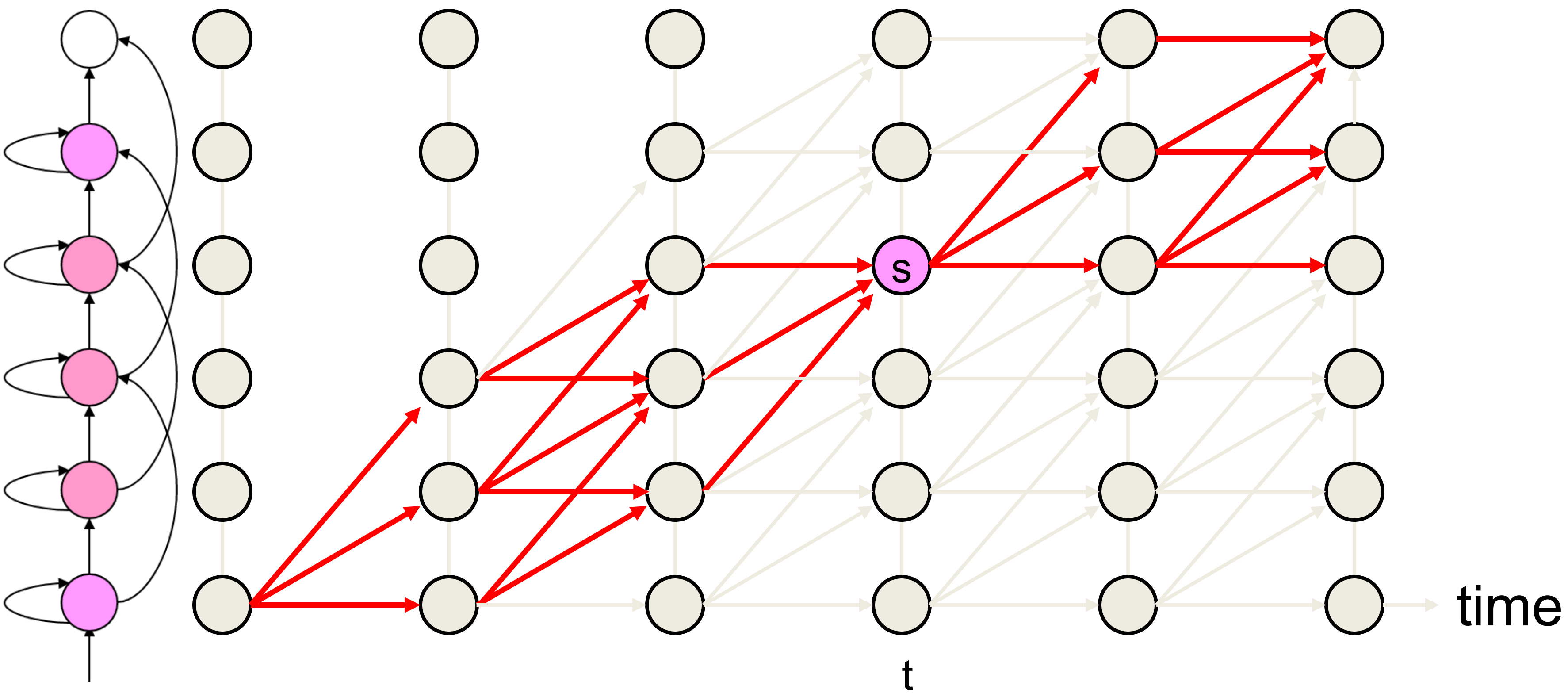
- The probability that the process was at *s* when it generated X_t given the entire observation
  - Dropping the "Obs" subscript for brevity

$$P(state(t) = s \mid X_1, X_2, \ldots, X_T) \propto P(state(t) = s, X_1, X_2, \ldots, X_T)$$

- We will compute $P(state(t) = s_i, x_1, x_2, \ldots, x_T)$ first
  – This is the probability that the process visited *s* at time *t* while producing the entire observation

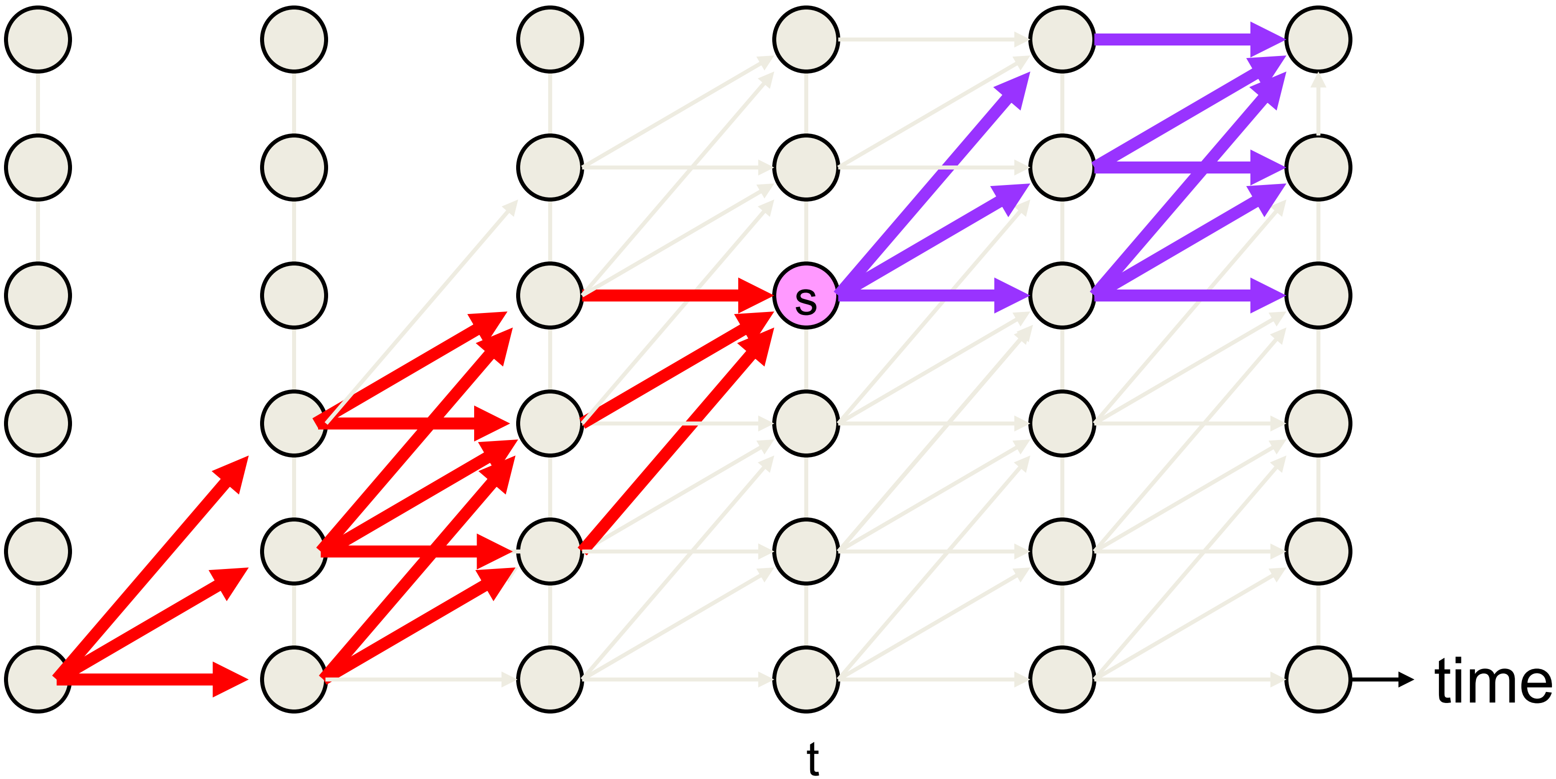
$$P(state(t) = s, x_1, x_2, \ldots, x_T)$$

- The probability that the HMM was in a particular state *s* when generating the observation sequence is the probability that it followed a state sequence that passed through *s* at time *t*



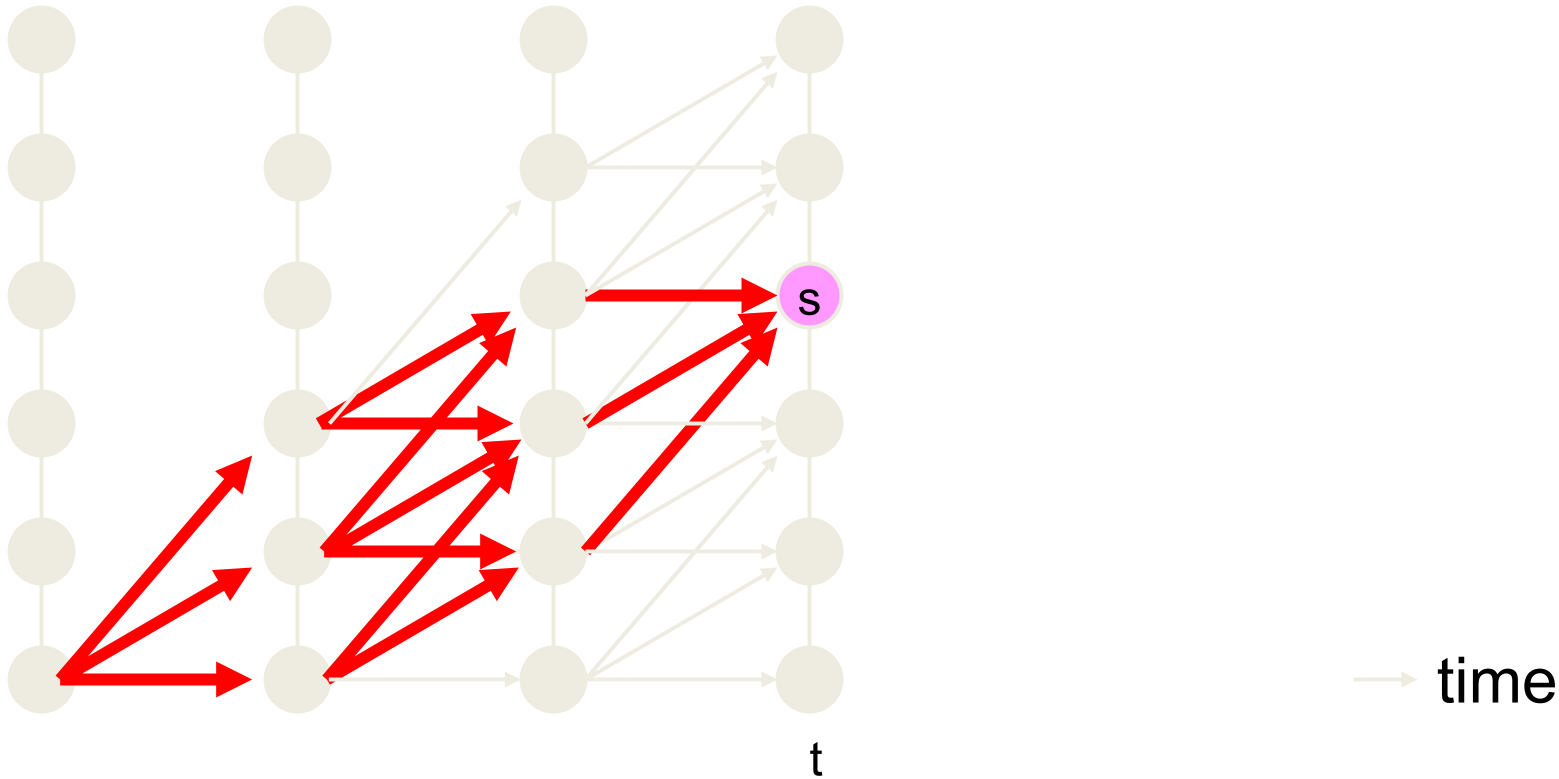
$$P(state(t) = s, x_1, x_2, \ldots, x_T)$$

- This can be decomposed into two multiplicative sections
  - The section of the lattice leading into state *s* at time t and the section leading out of it



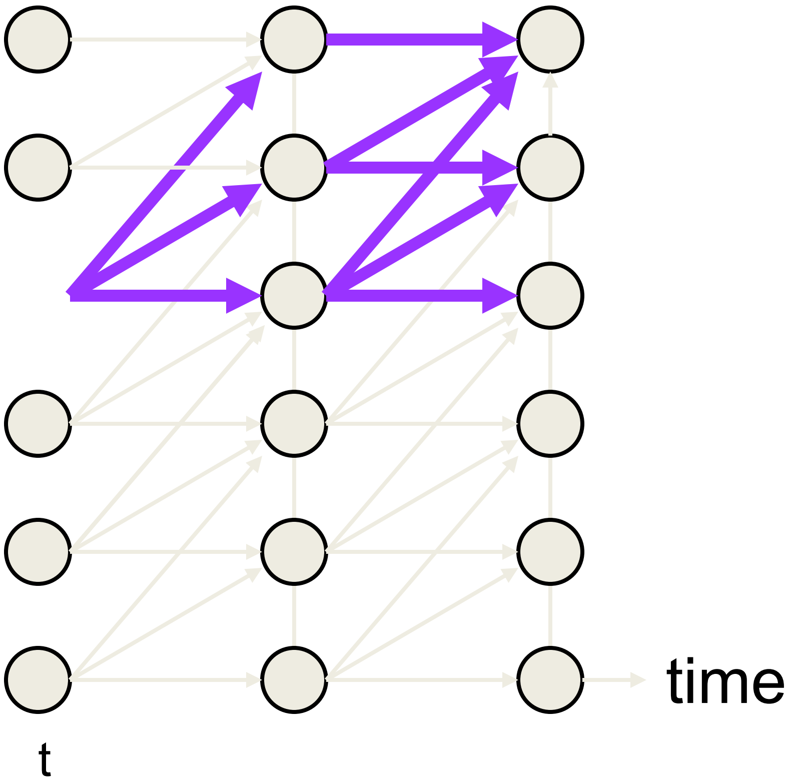
The Forward Paths

- The probability of the red section is the total probability of all state sequences ending at state *s* at time *t*
  - This is simply $\alpha(s,t)$
  - Can be computed using the forward algorithm



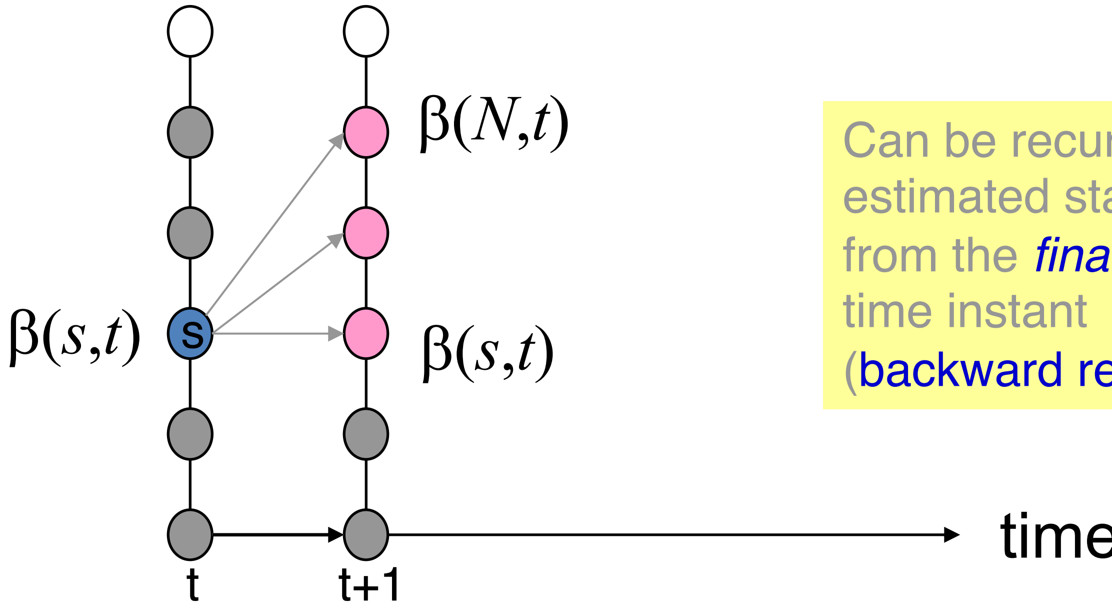
The Backward Paths

- The blue portion represents the probability of all state sequences that began at state *s* at time *t*
  - Like the red portion it can be computed using a *backward recursion*



The Backward Recursion

$$\beta(s,t) = P(x_{t+1}, x_{t+2}, \ldots, x_T \mid state(t) = s)$$



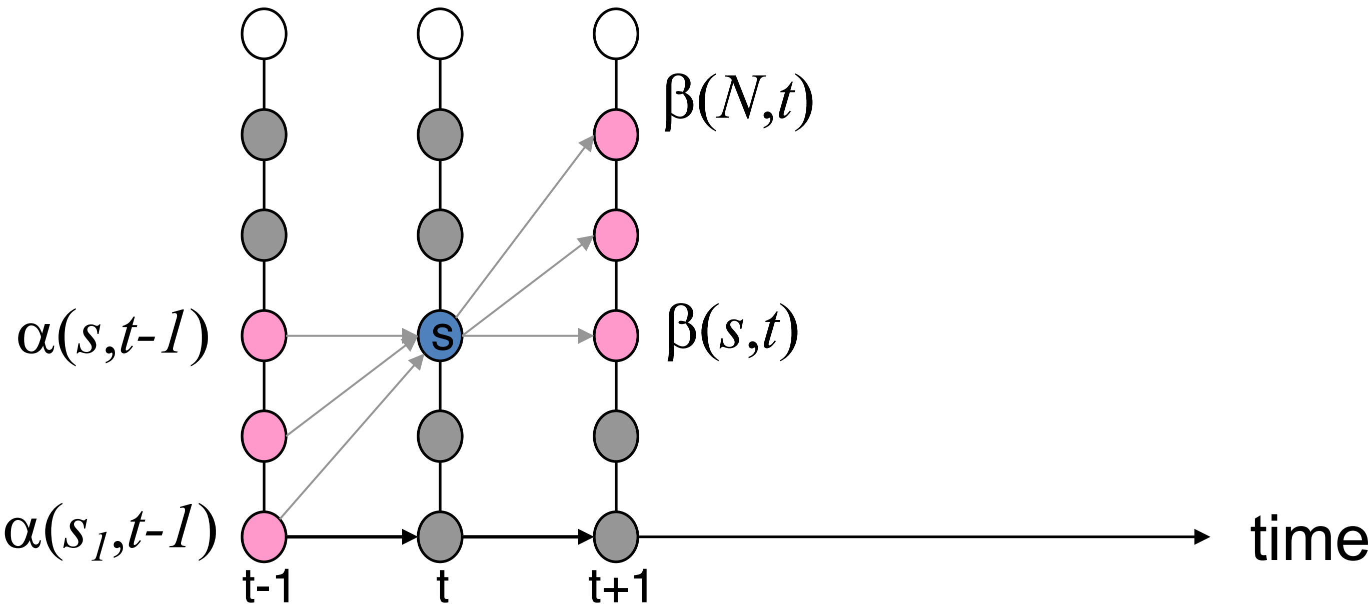
Can be recursively estimated starting from the *final* time time instant (backward recursion)

$$\beta(s,t) = \sum_{s'} \beta(s',t+1) P(s' \mid s) P(x_{t+1} \mid s')$$

- $\beta(s,t)$ is the total probability of ALL state sequences that depart from s at time t, and all observations after x_t
  - $\beta(s,T) = 1$ at the final time instant for all valid final states

The complete probability

$$\alpha(s,t)\beta(s,t) = P(x_{t+1}, x_{t+2}, \ldots, x_T, state(t) = s)$$



Poll 3

Posterior probability of a state

- The probability that the process was in state *s* at time *t*, given that we have observed the data is obtained by simple normalization

$$P(state(t)=s \mid Obs) = \frac{P(state(t)=s, x_1, x_2, \ldots, x_T)}{\sum_{s'} P(state(t)=s, x_1, x_2, \ldots, x_T)} = \frac{\alpha(s,t)\beta(s,t)}{\sum_{s'} \alpha(s',t)\beta(s',t)}$$

- This term is often referred to as the gamma term and denoted by $\gamma_{s,t}$

Update rules at each iteration

$$\pi(s_i) = \frac{\sum_{Obs} P(state(t=1) = s_i \mid Obs)}{\text{Total no. of observation sequences}}$$

$$P(s_j \mid s_i) = \frac{\sum_{Obs} \sum_{t} P(state(t) = s_i, state(t+1) = s_j \mid Obs)}{\sum_{Obs} \sum_{t} P(state(t) = s_i \mid Obs)}$$

$$\mu_i = \frac{\sum_{Obs} \sum_{t} P(state(t) = s_i \mid Obs) X_{Obs,t}}{\sum_{Obs} \sum_{t} P(state(t) = s_i \mid Obs)}$$

$$\Theta_i = \frac{\sum_{Obs} \sum_{t} P(state(t) = s_i \mid Obs)(X_{Obs,t} - \mu_i)(X_{Obs,t} - \mu_i)^T}{\sum_{Obs} \sum_{t} P(state(t) = s_i \mid Obs)}$$

- These have been found

Update rules at each iteration

$$\pi(s_i) = \frac{\sum_{Obs} P(state(t=1) = s_i \mid Obs)}{\text{Total no. of observation sequences}}$$

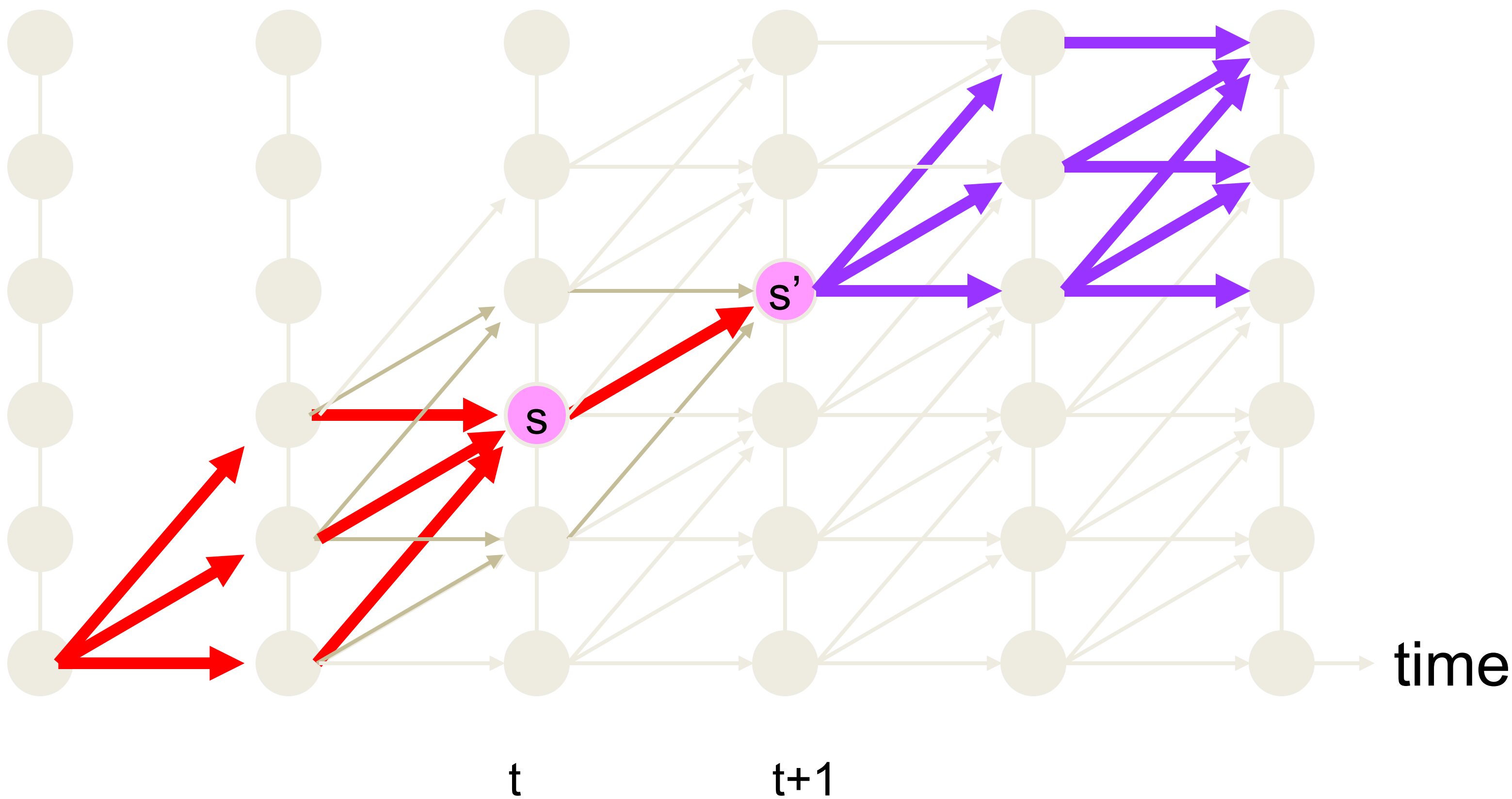
$$P(s_j \mid s_i) = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i, state(t+1) = s_j \mid Obs)}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

$$\mu_i = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs) X_{Obs,t}}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

$$\Theta_i = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)(X_{Obs,t} - \mu_i)(X_{Obs,t} - \mu_i)^T}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

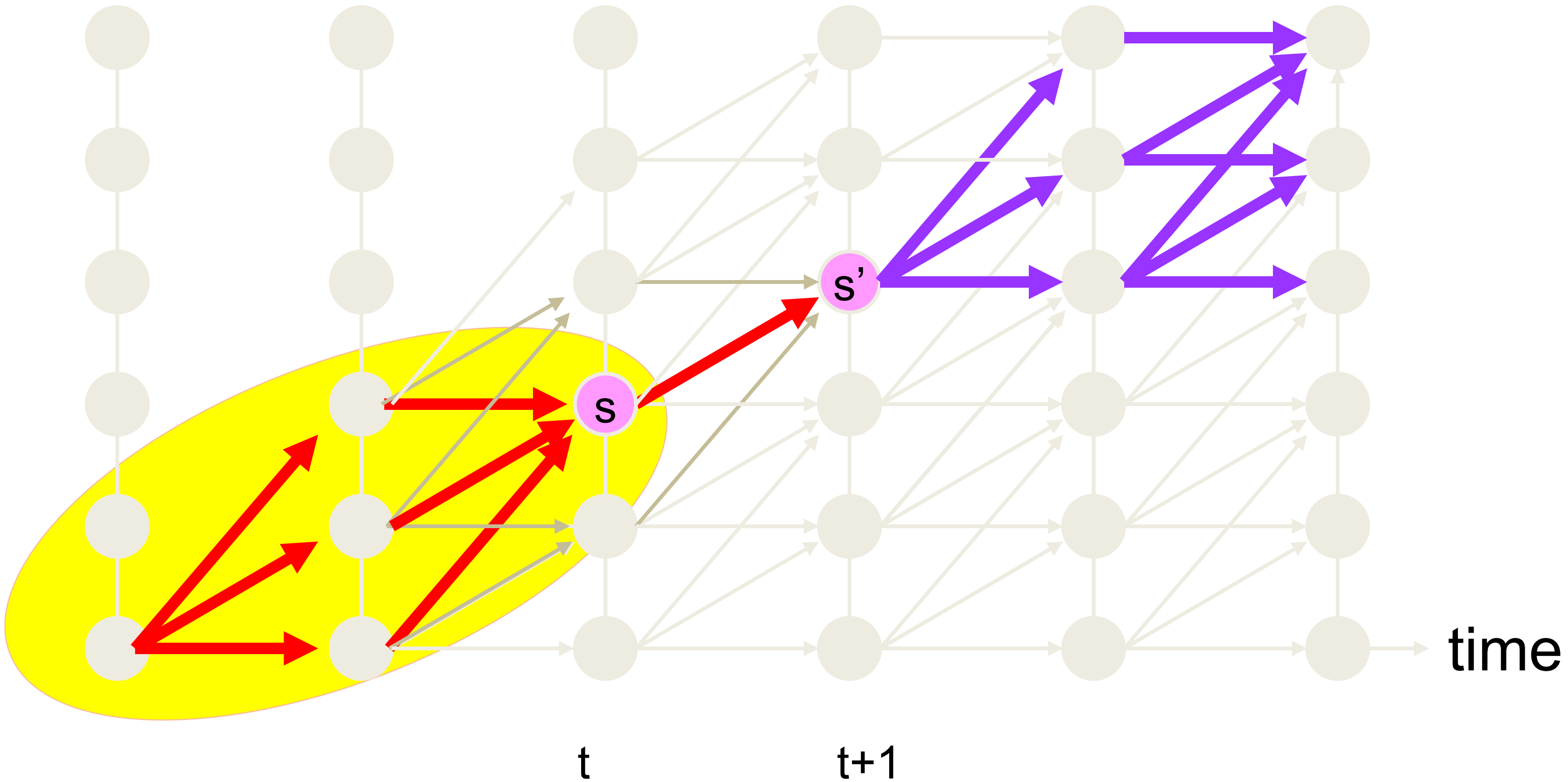
- Where did these terms come from?

$$P(state(t)=s, state(t+1)=s', x_1, x_2, \ldots, x_T)$$



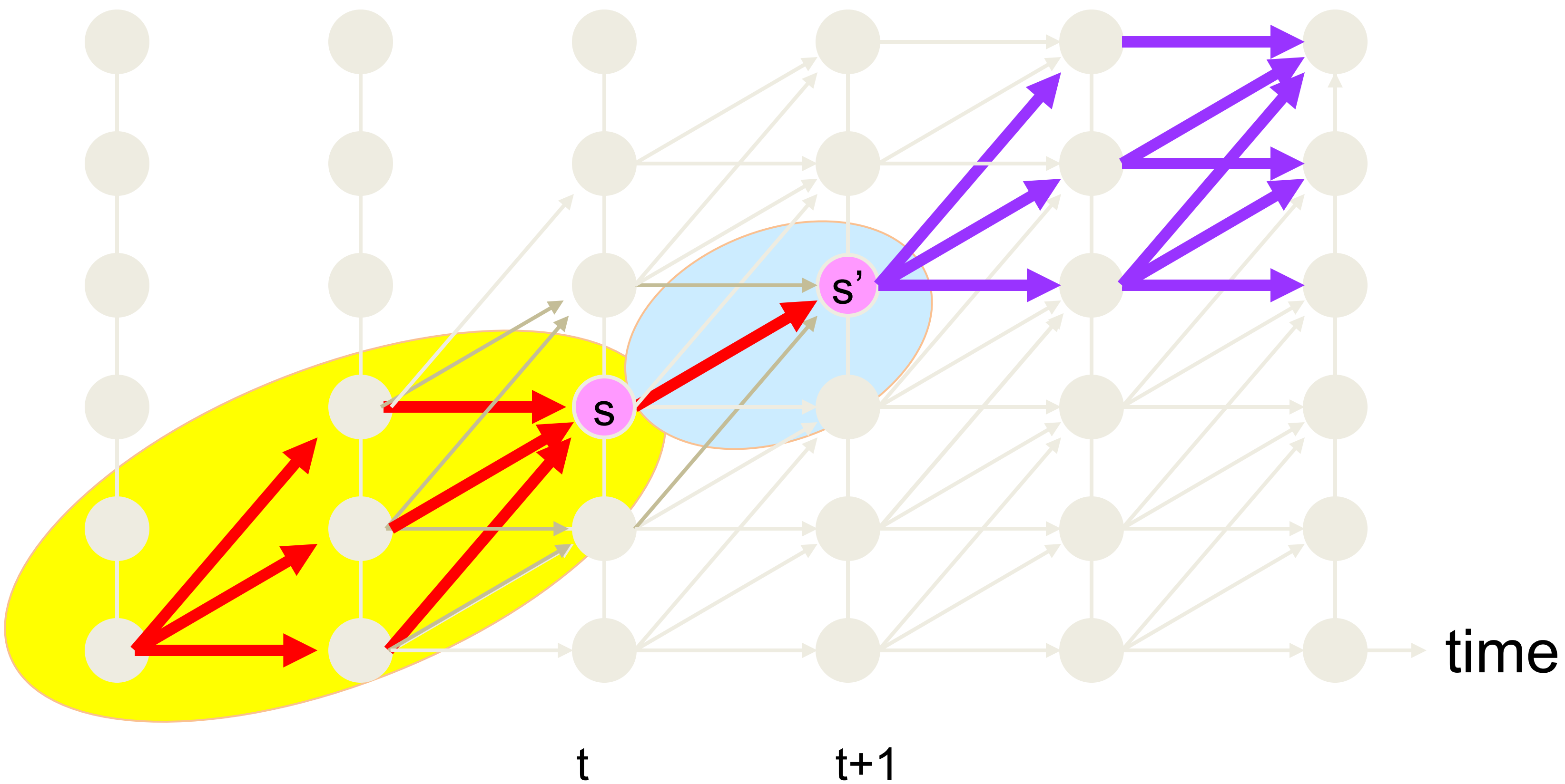
$$P(state(t)=s, state(t+1)=s', x_1, x_2, \ldots, x_T)$$

$$\alpha(s,t)$$



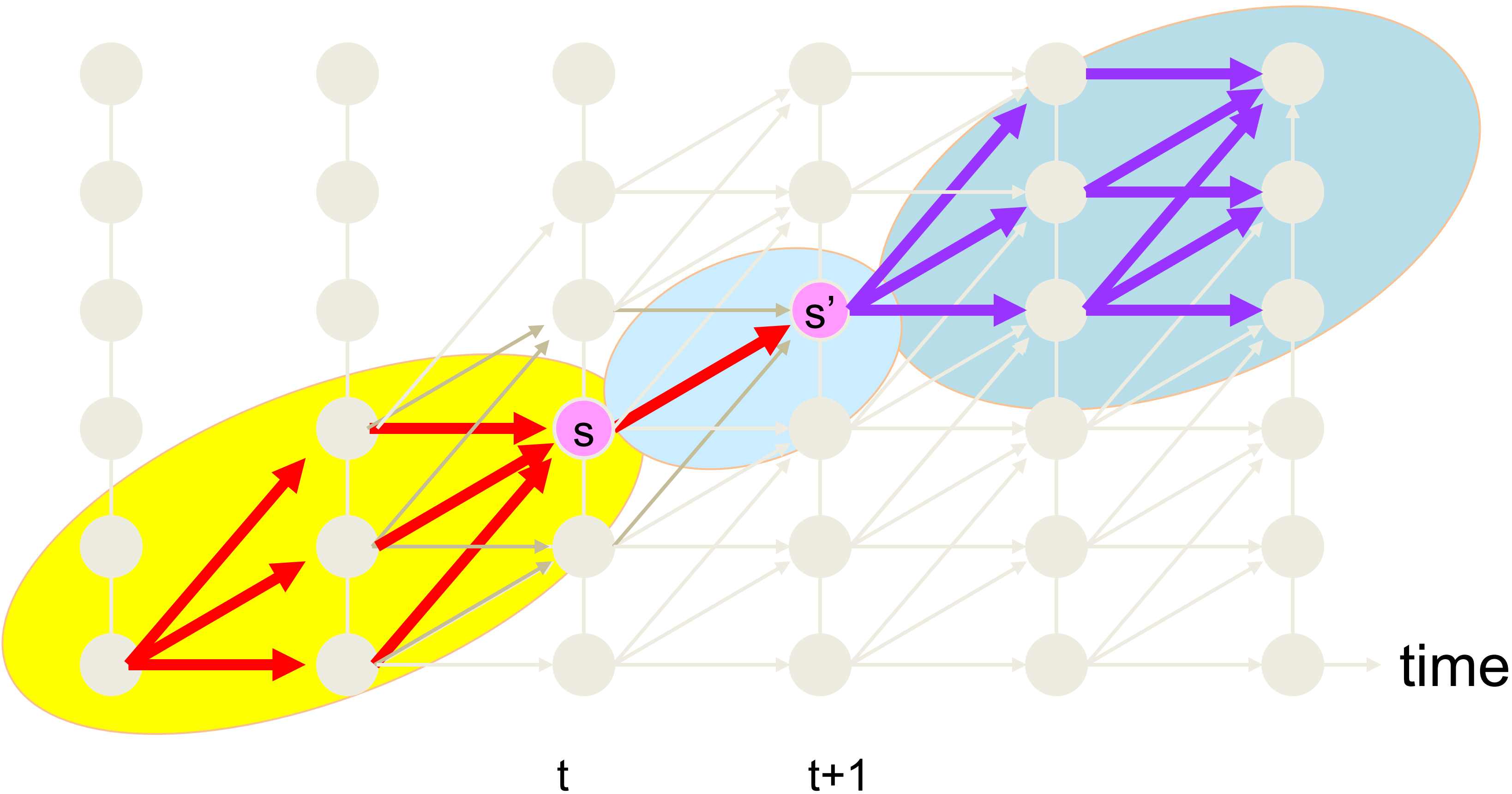
$$P(state(t)=s, state(t+1)=s', x_1, x_2, \ldots, x_T)$$

$$\alpha(s,t)\, P(s'|s) P(x_{t+1}|s')$$



$$P(state(t) = s, state(t+1) = s', x_1, x_2, \ldots, x_T)$$

$$\alpha(s,t) P(s'|s) P(x_{t+1}|s') \beta(s', t+1)$$



The a posteriori probability of transition

$$P(state(t)=s, state(t+1)=s' \mid Obs) = \frac{\alpha(s,t)P(s'\mid s)P(x_{t+1}\mid s')\beta(s',t+1)}{\sum_{s_1}\sum_{s_2}\alpha(s_1,t)P(s_2\mid s_1)P(x_{t+1}\mid s_2)\beta(s_2,t+1)}$$

- The a posteriori probability of a transition given an observation

Update rules at each iteration

$$\pi(s_i) = \frac{\sum_{Obs} P(state(t=1) = s_i \mid Obs)}{\text{Total no. of observation sequences}}$$

$$P(s_j \mid s_i) = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i, state(t+1) = s_j \mid Obs)}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

$$\mu_i = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs) X_{Obs,t}}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

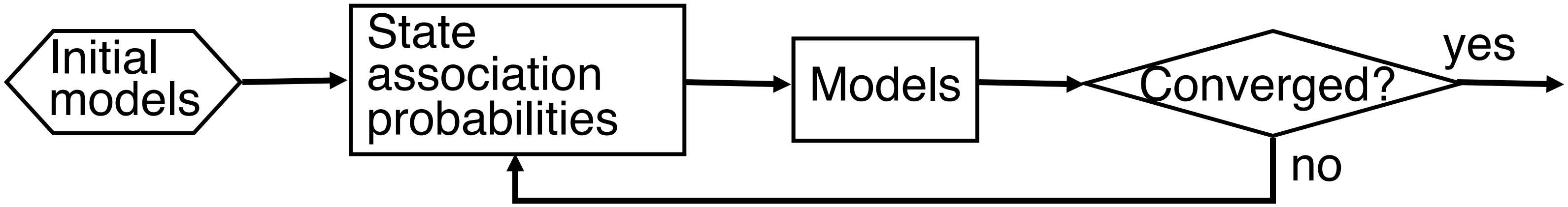
$$\Theta_i = \frac{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)(X_{Obs,t} - \mu_i)(X_{Obs,t} - \mu_i)^T}{\sum_{Obs}\sum_{t} P(state(t) = s_i \mid Obs)}$$

- These have been found

Poll 4

Training without explicit segmentation: Baum-Welch training

- Every feature vector associated with every state of every HMM with a probability



- Probabilities computed using the forward-backward algorithm
- Soft decisions taken at the level of HMM state
- In practice, the segmentation based Viterbi training is much easier to implement and is much faster
- The difference in performance between the two is small, especially if we have lots of training data

HMM Issues

- How to find the best state sequence: Covered
- How to learn HMM parameters: Covered
- How to compute the probability of an observation sequence: Covered

Magic numbers

- How many states:
  - No nice automatic technique to learn this
  - You choose
    - For speech, HMM topology is usually left to right (no backward transitions)
    - For other cyclic processes, topology must reflect nature of process
    - No. of states – 3 per phoneme in speech
    - For other processes, depends on estimated no. of distinct states in process

Applications of HMMs

- Classification:
  - Learn HMMs for the various classes of time series from training data
  - Compute probability of test time series using the HMMs for each class
  - Use in a Bayesian classifier
  - Speech recognition, vision, gene sequencing, character recognition, text mining…
- Prediction
- Tracking

Applications of HMMs

- Segmentation:
  - Given HMMs for various events, find event boundaries
    - Simply find the best state sequence and the locations where state identities change

- Automatic speech segmentation, text segmentation by topic, geneome segmentation, …