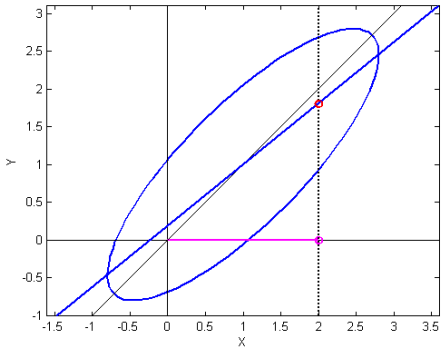
Machine Learning for Signal Processing
Predicting and Estimation from Time Series

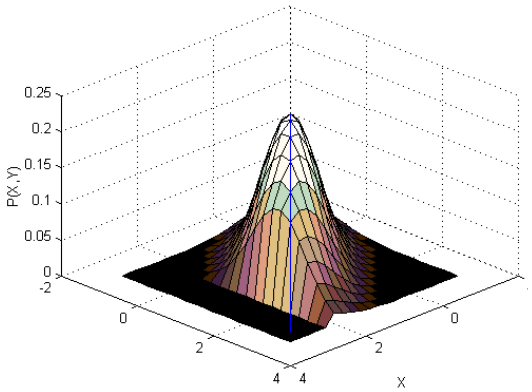
Bhiksha Raj

Preliminaries : P(y|x) for Gaussian

- If P(x,y) is Gaussian:

$$P(\mathbf{x},\mathbf{y}) = N(\begin{bmatrix}\mu_x\\ \mu_y\end{bmatrix}, \begin{bmatrix}C_{xx} & C_{xy}\\ C_{yx} & C_{yy}\end{bmatrix})$$





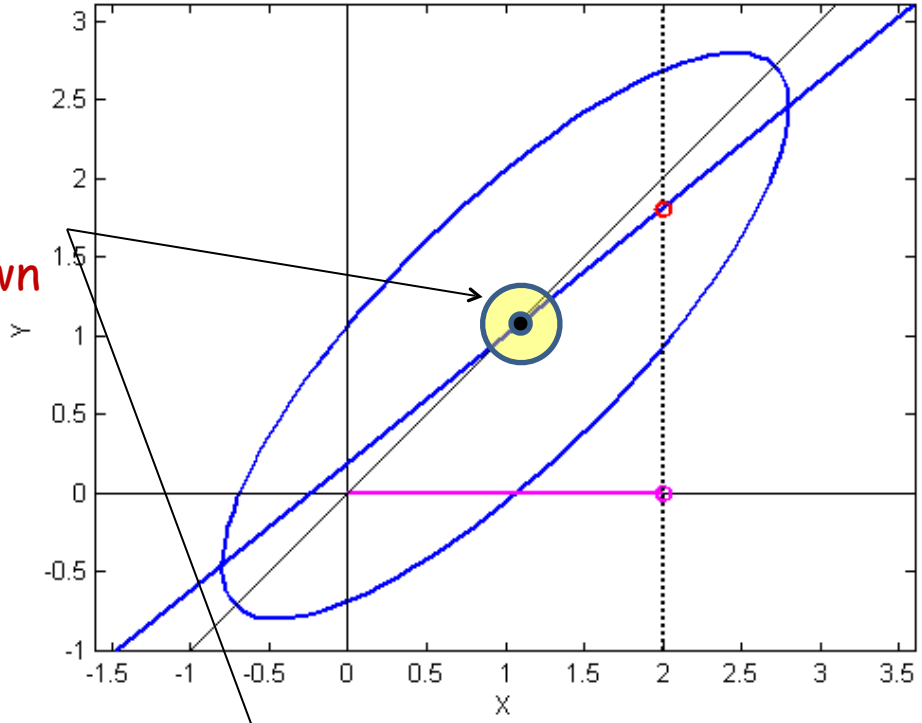
- The conditional probability of y given x is also Gaussian
  - The slice in the figure is Gaussian

$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x - \mu_x), C_{yy} - C_{yx}C_{xx}^{-1}C_{xy})$$

- The mean of this Gaussian is a function of x
- The variance of y reduces if x is known
  - Uncertainty is reduced

Preliminaries : P(y|x) for Gaussian

Best guess for Y
when X is not known



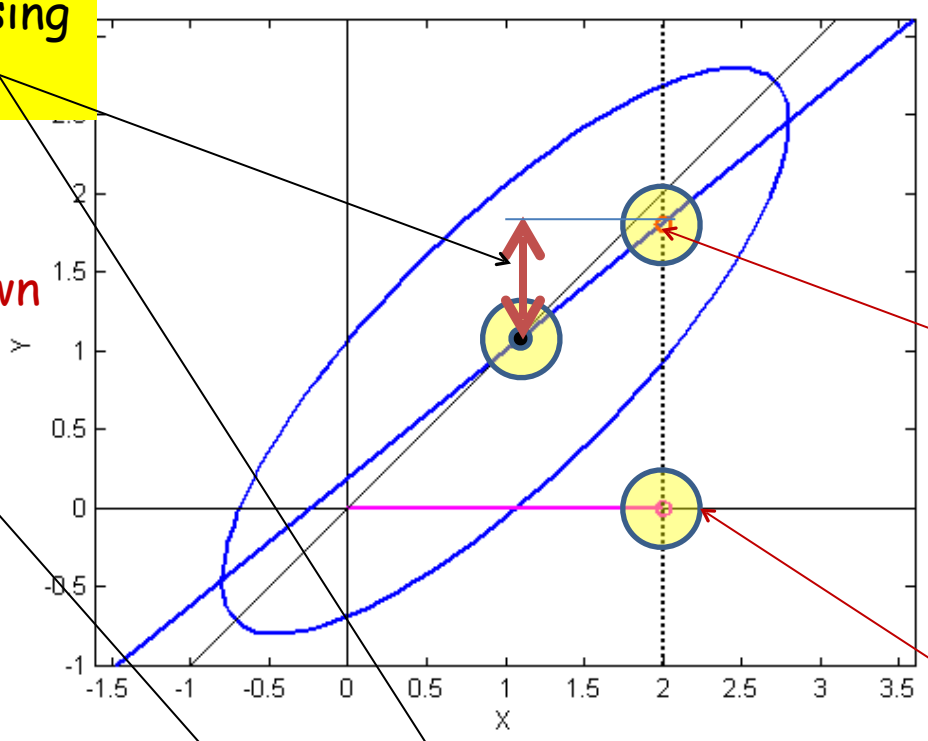
$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x - \mu_x), C_{yy} - C_{yx}C_{xx}^{-1}C_{xy})$$

Preliminaries : P(y|x) for Gaussian

Update guess of Y based on information in X
Correction is 0 if X and Y are uncorrelated, i.e $C_{yx} = 0$

Correction of Y using information in X

Best guess for Y when X is not known



Mean of Y given X

Given X value

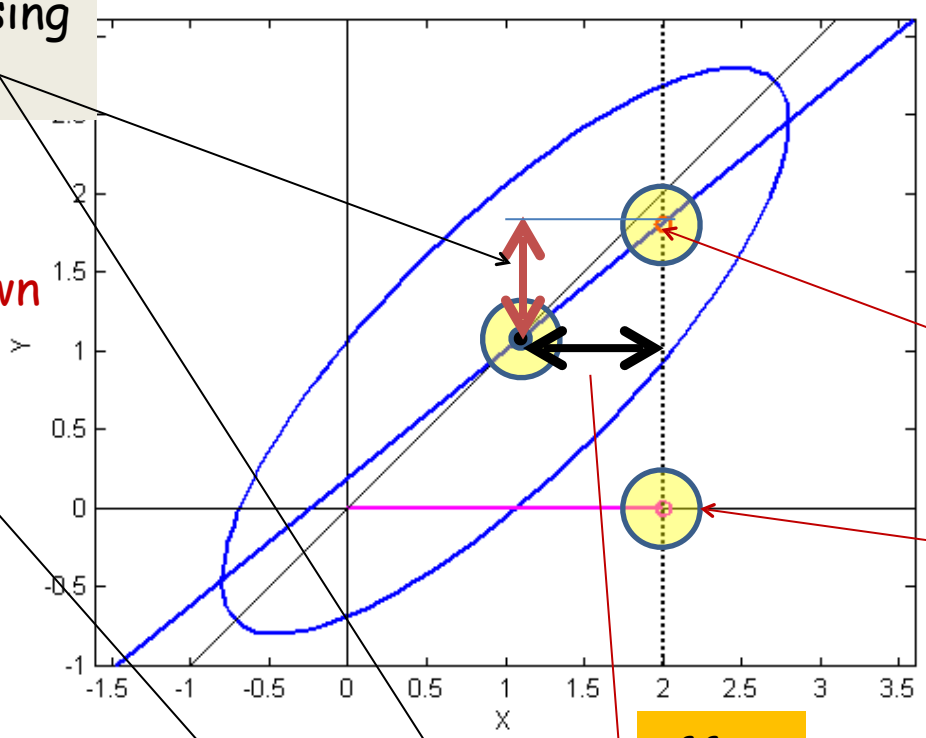
$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x - \mu_x), C_{yy} - C_{yx}C_{xx}^{-1}C_{xy})$$

Preliminaries : P(y|x) for Gaussian

Correction to Y = slope * (offset of X from mean)

Correction of Y using information in X

Best guess for Y when X is not known



Mean of Y given X

Given X value

offset

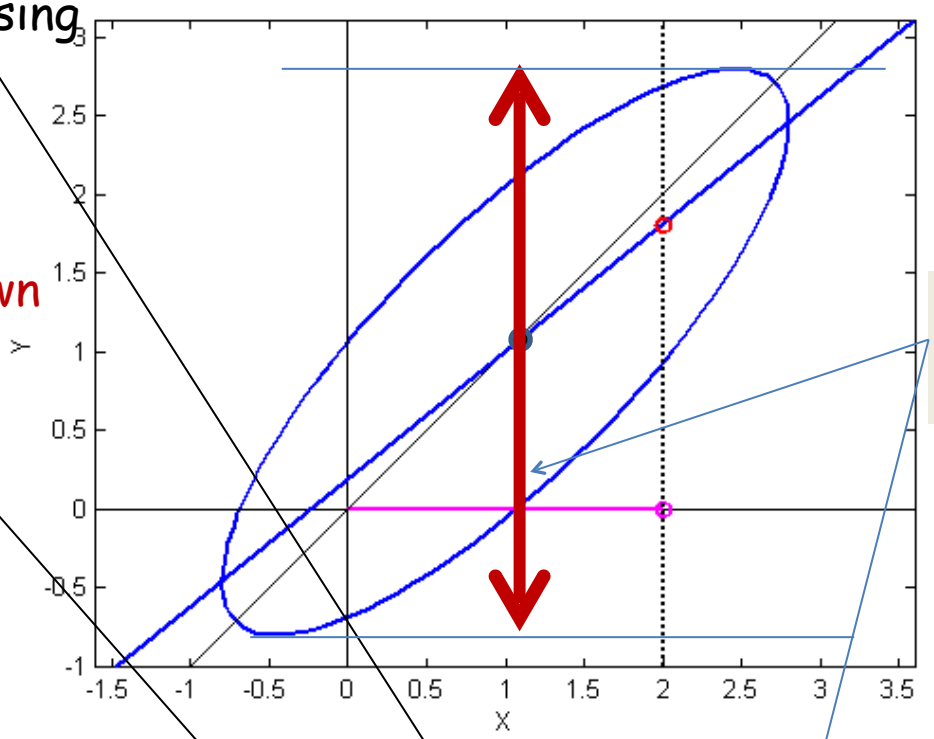
$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x - \mu_x), C_{yy} - C_{yx}C_{xx}^{-1}C_{xy})$$

Slope

Preliminaries : P(y|x) for Gaussian

Correction of Y using information in X

Best guess for Y when X is not known



Uncertainty in Y when X is not known

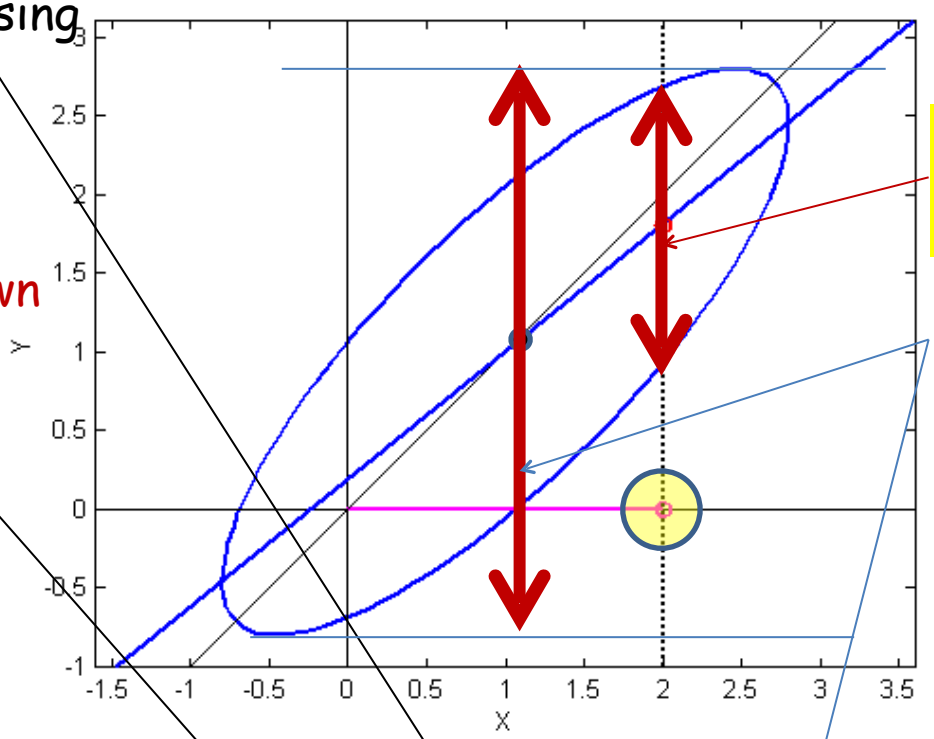
$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x - \mu_x), C_{yy} - C_{yx}C_{xx}^{-1}C_{xy})$$

Preliminaries : P(y|x) for Gaussian

Shrinkage of variance is 0 if X and Y are uncorrelated, i.e $C_{yx} = 0$

Correction of Y using information in X

Best guess for Y when X is not known



Reduced uncertainty from knowing X

Uncertainty in Y when X is not known

Shrinkage of uncertainty from knowing X

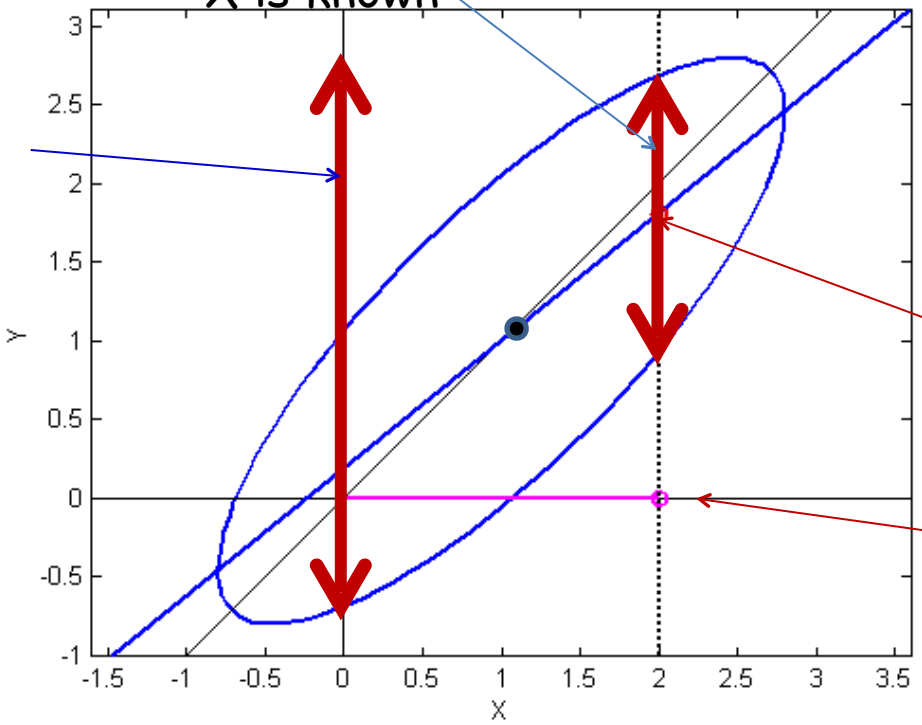
$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x - \mu_x), C_{yy} - C_{yx}C_{xx}^{-1}C_{xy})$$

Preliminaries : P(y|x) for Gaussian

Knowing X modifies the mean of Y and shrinks its variance

Variance of Y when X is known

Overall variance of Y when X is unknown



Mean of Y given X (MAP estimate of Y)

Given X value

$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x - \mu_x), C_{yy} - C_{yx}C_{xx}^{-1}C_{xy})$$

Background: Sum of Gaussian RVs

$$O = AS + \varepsilon$$

$$S \sim N(\mu_s, \Theta_s) \qquad \varepsilon \sim N(\mu_\varepsilon, \Theta_\varepsilon)$$

- Consider a random variable *O* obtained as above
- The expected value of *O* is given by

$$E[O] = E[AS + \varepsilon] = A\mu_s + \mu_\varepsilon$$

- Notation:

$$E[O] = \mu_O$$

Background: Sum of Gaussian RVs

$$O = AS + \varepsilon$$

$$S \sim N(\mu_S, \Theta_S) \qquad \varepsilon \sim N(\mu_\varepsilon, \Theta_\varepsilon)$$

- The variance of *O* is given by

$$Var(O) = \Theta_O = E\left[(O - \mu_O)(O - \mu_O)^T\right]$$

- This is just the sum of the variance of AS and the variance of ε

$$\Theta_O = A\Theta_S A^T + \Theta_\varepsilon$$

Background: Sum of Gaussian RVs

$$O = AS + \varepsilon$$

$$S \sim N(\mu_S, \Theta_S) \qquad \varepsilon \sim N(\mu_\varepsilon, \Theta_\varepsilon)$$

- The conditional probability of *O*:

$$P(O \mid S) = N(AS + \mu_\varepsilon, \Theta_\varepsilon)$$

- The overall probability of *O*:

$$P(O) = N(A\mu_S + \mu_\varepsilon, A\Theta_S A^T + \Theta_\varepsilon)$$

Background: Sum of Gaussian RVs

$$O = AS + \varepsilon$$

$$S \sim N(\mu_s, \Theta_s) \qquad \varepsilon \sim N(\mu_\varepsilon, \Theta_\varepsilon)$$

- **The *cross-correlation* between *O* and *S***

$$\Theta_{OS} = E\left[(O-\mu_O)(S-\mu_s)^T\right] = E\left[(A(S-\mu_s)+(\varepsilon-\mu_\varepsilon))(S-\mu_s)^T\right] = E\left[A(S-\mu_s)(S-\mu_s)^T + (\varepsilon-\mu_\varepsilon)(S-\mu_s)^T\right] = AE\left[(S-\mu_s)(S-\mu_s)^T\right] + E\left[(\varepsilon-\mu_\varepsilon)(S-\mu_s)^T\right] = AE\left[(S-\mu_s)(S-\mu_s)^T\right]$$

- $= A\,\Theta_s$
- The cross-correlation between *O* and *S* is

$$\Theta_{OS} = A\Theta_S$$
$$\Theta_{SO} = \Theta_S A T$$

Background: Joint Prob. of O and S

$$O = AS + \varepsilon$$

$$\begin{bmatrix} O \\ S \end{bmatrix}$$

- The joint probability of *O* and *S* (i.e. P(*Z*)) is also Gaussian

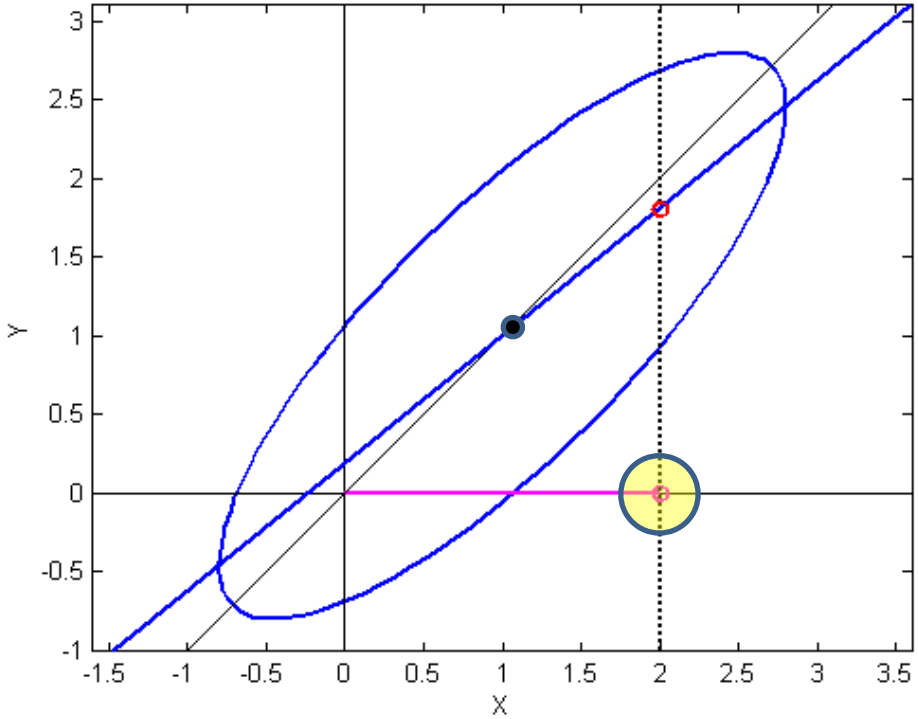
$$P(Z) = P(O, S) = N(\mu_Z, \Theta_Z)$$

- Where

$$\mu_Z = \begin{bmatrix} \mu_O \\ \mu_S \end{bmatrix} = \begin{bmatrix} A\mu_s + \mu_\varepsilon \\ \mu_S \end{bmatrix}$$

- $$\Theta_Z = \begin{bmatrix} \Theta_O & \Theta_{OS} \\ \Theta_{SO} & \Theta_S \end{bmatrix} = \begin{bmatrix} A\Theta_S A^{\mathrm{T}} + \Theta_\varepsilon & A\Theta_S \\ \Theta_S A^{\mathrm{T}} & \Theta_S \end{bmatrix}$$

Preliminaries : Conditional of S given O: P(S|O)



$$O = AS + \varepsilon$$

$$P(S|O) = N(\mu_S + \Theta_{SO}\Theta_O^{-1}(O - \mu_O),\quad \Theta_S - \Theta_{SO}\Theta_O^{-1}\Theta_{OS})$$

$$P(S|O) = N(\mu_S + \Theta_S A^{\mathrm{T}}\left(A\Theta_S A^{\mathrm{T}} + \Theta_\varepsilon\right)^{-1}(O - A\mu_s - \mu_\varepsilon),$$
$$\Theta_S - \Theta_S A^{\mathrm{T}}\left(A\Theta_S A^{\mathrm{T}} + \Theta_\varepsilon\right)^{-1} A\Theta_S)$$

The little parable

You've been kidnapped

And blindfolded

You can only hear the car

You must find your way back home from wherever they drop you off

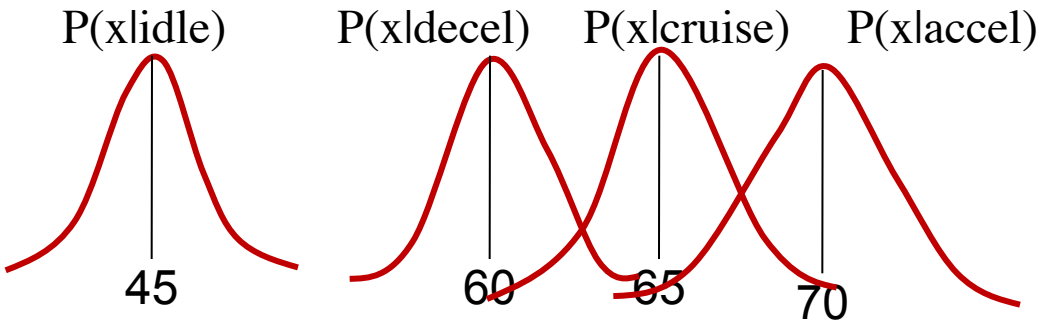
Kidnapped!

- Determine by only *listening* to a running automobile, if it is:
  - Idling; or
  - Travelling at constant velocity; or
  - Accelerating; or
  - Decelerating
- You only record energy level (SPL) in the sound
  - The SPL is measured once per second

What we know

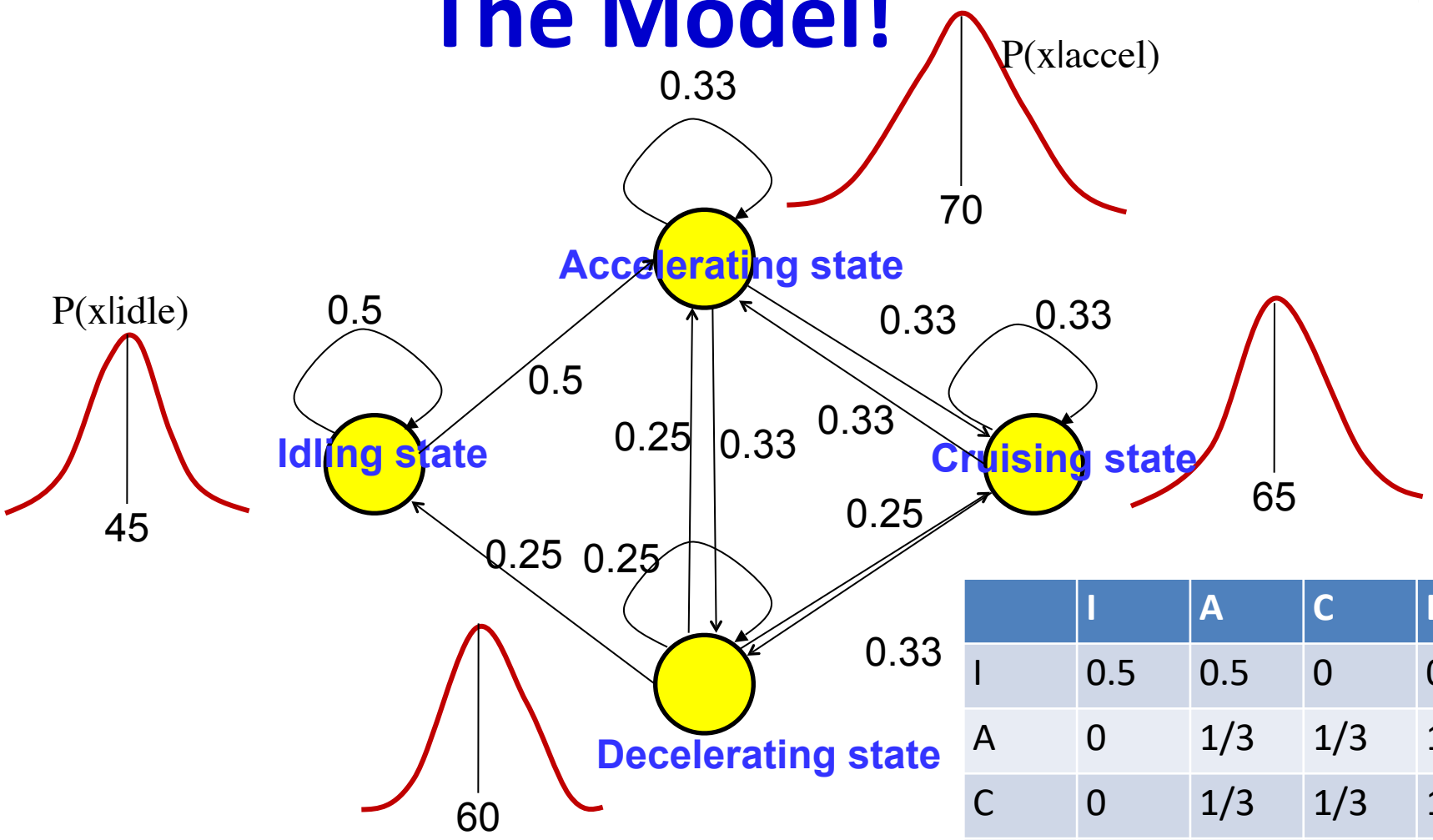
- An automobile that is at rest can accelerate, or continue to stay at rest
- An accelerating automobile can hit a steady-state velocity, continue to accelerate, or decelerate
- A decelerating automobile can continue to decelerate, come to rest, cruise, or accelerate
- A automobile at a steady-state velocity can stay in steady state, accelerate or decelerate

What else we know



- The probability distribution of the SPL of the sound is different in the various conditions
  - As shown in figure
    - In reality, depends on the car
- The distributions for the different conditions overlap
  - Simply knowing the current sound level is not enough to know the state of the car

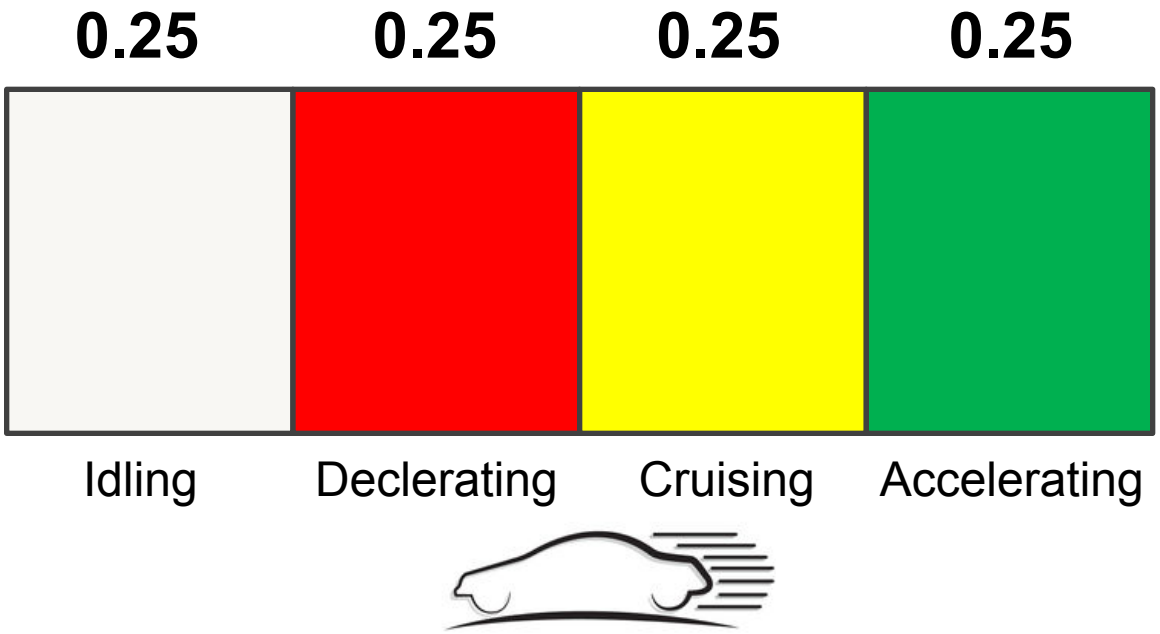
The Model!



	I	A	C	D
I	0.5	0.5	0	0
A	0	1/3	1/3	1/3
C	0	1/3	1/3	1/3
D	0.25	0.25	0.25	0.25

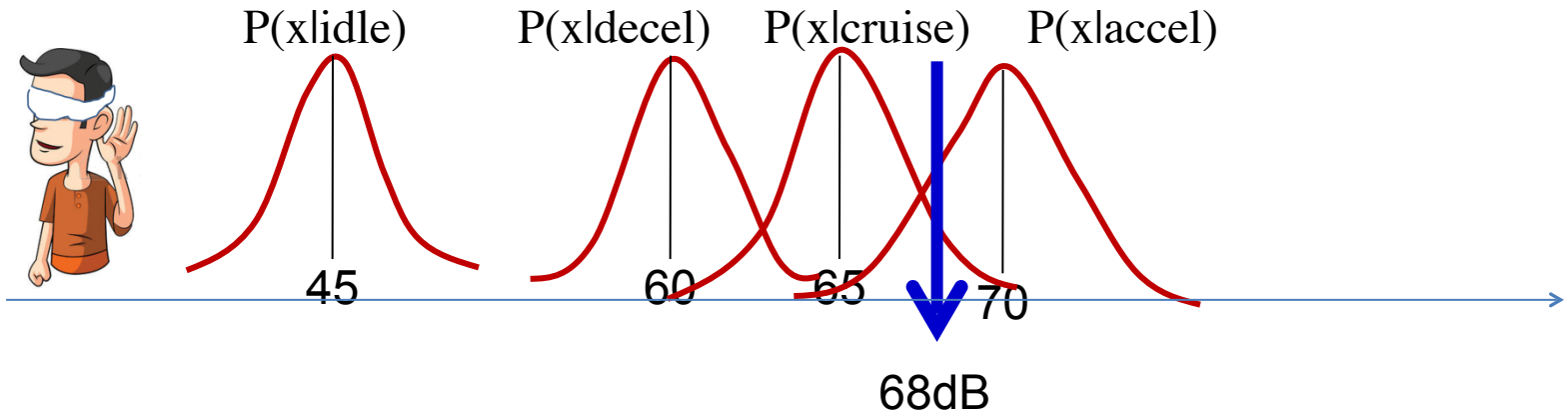
- The state-space model
  - Assuming all transitions from a state are equally probable
  - This is a Hidden Markov Model!

Estimating the state at T = 0-



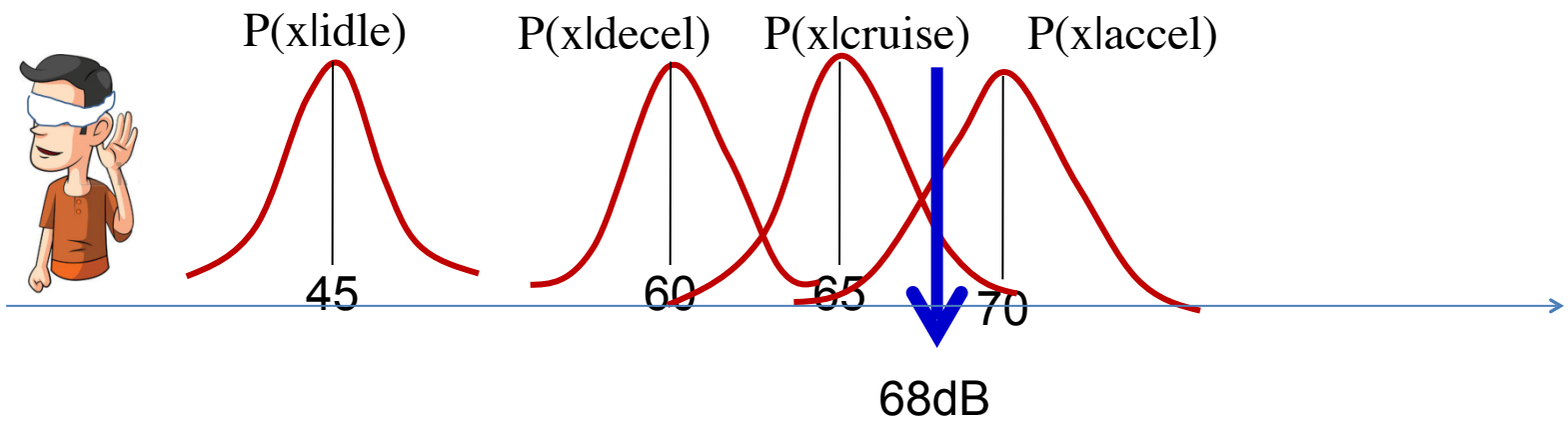
- A T=0, before the first observation, we know nothing of the state
  - Assume all states are equally likely

The first observation: T=0



- At T=0 you observe the sound level x_0 = 68dB SPL
  - The observation modifies our belief in the state of the system

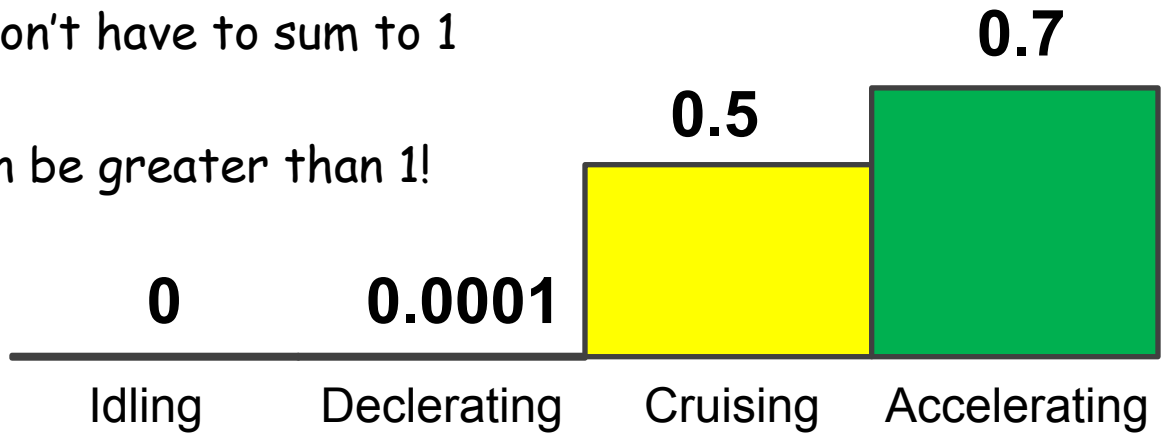
The first observation: T=0



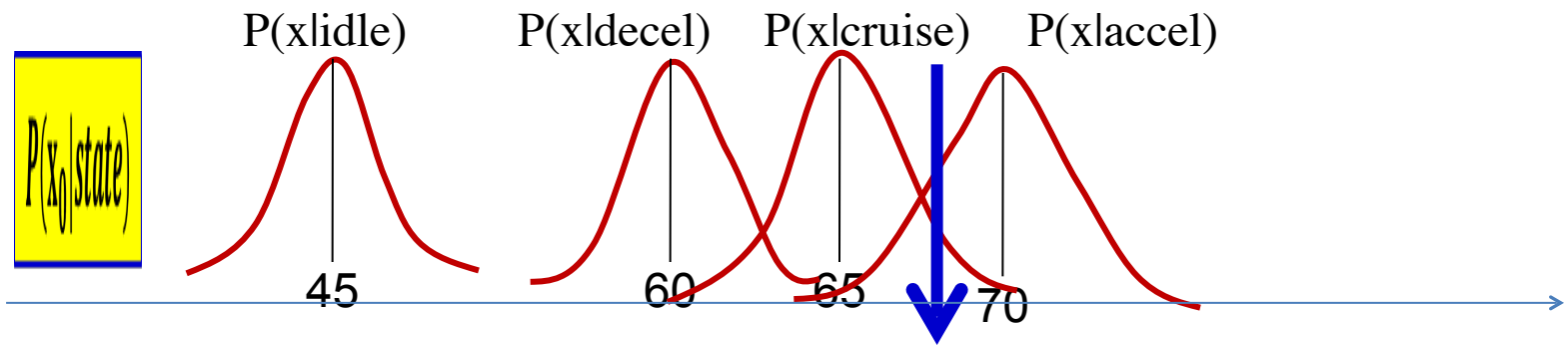
P(x|idle)	P(x|deceleration)	P(x|cruising)	P(x|acceleration)
0	0.0001	0.5	0.7

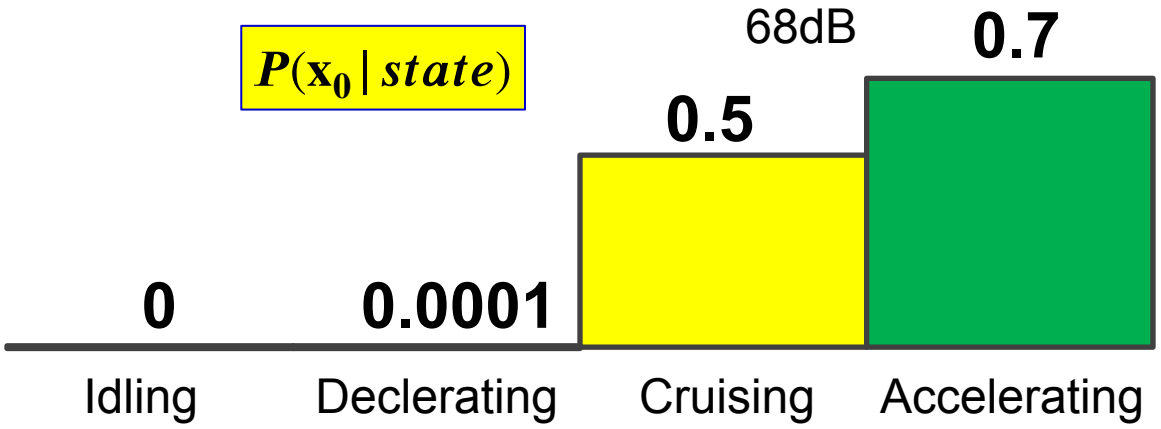
These don't have to sum to 1

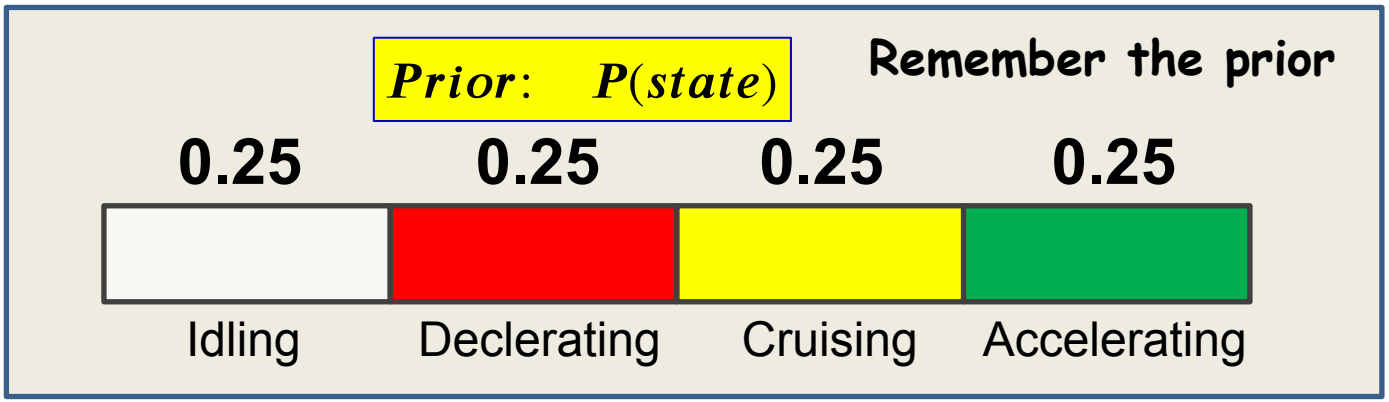
Can even be greater than 1!



The first observation: T=0





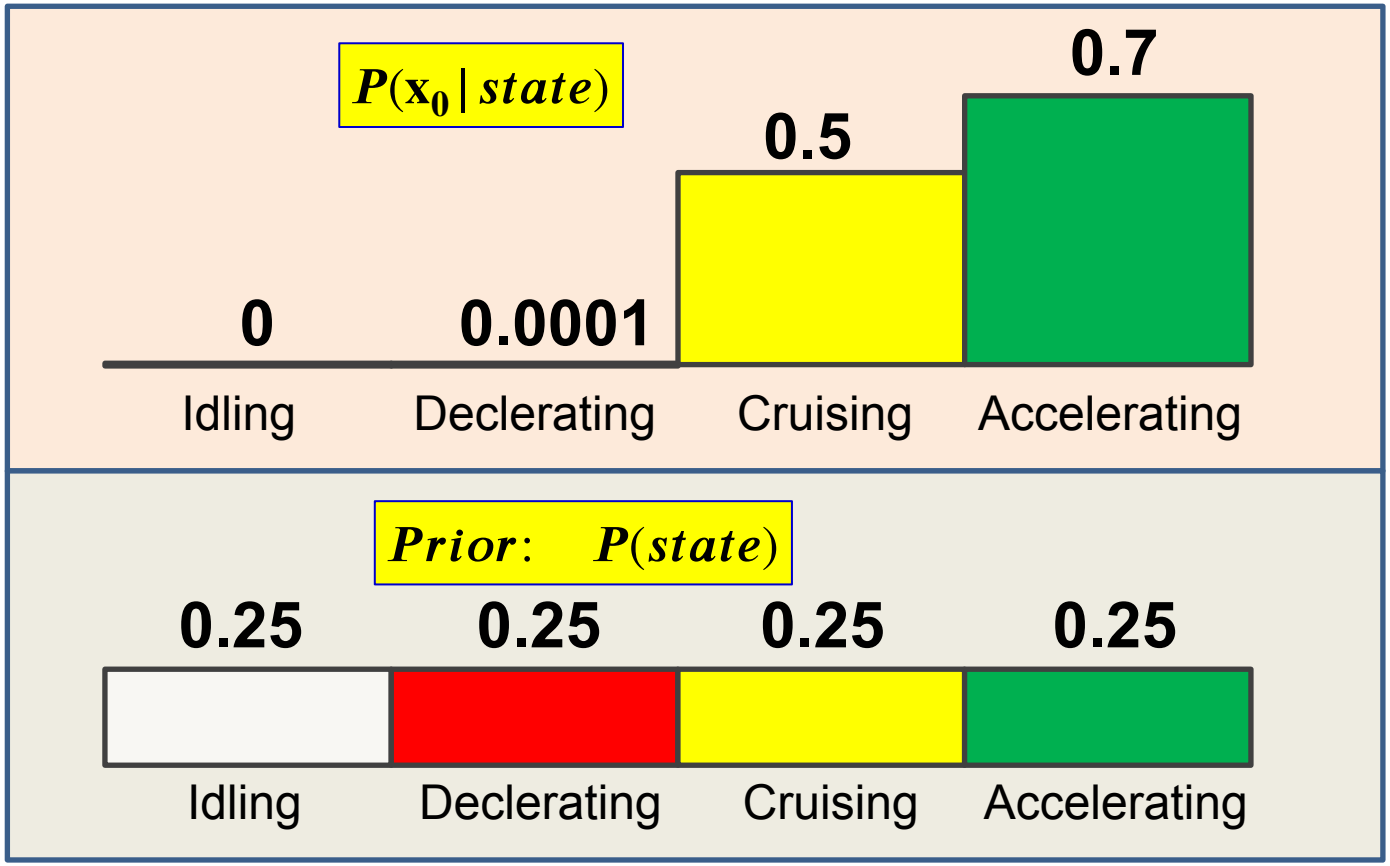


Estimating state *after* at observing x_0

- Combine prior information about state and evidence from observation
- We want $P(state \mid \mathbf{x}_0)$
- We can compute it using Bayes rule as

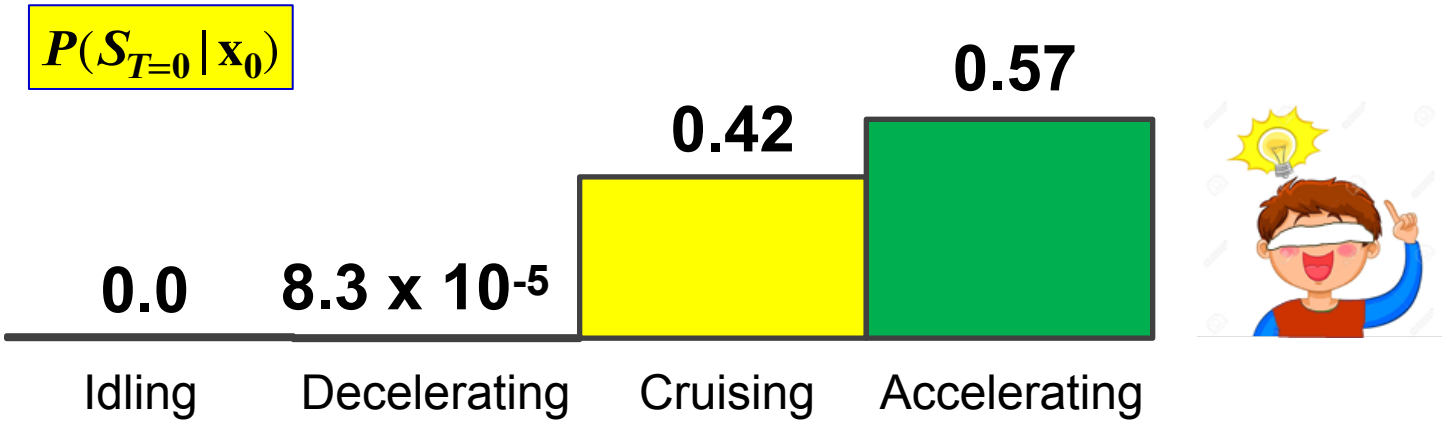
$$P(state \mid x_0) = \frac{P(state)P(\mathrm{x}_0 \mid state)}{\sum_{state'} P(state')P(\mathrm{x}_0 \mid state')}$$

The Posterior



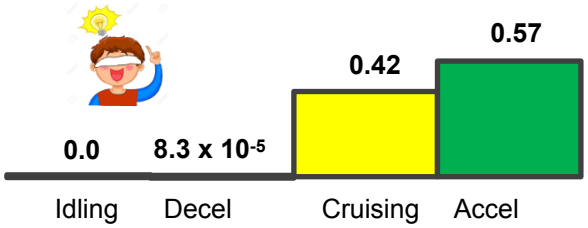
- Multiply the two, term by term, and normalize them so that they sum to 1.0

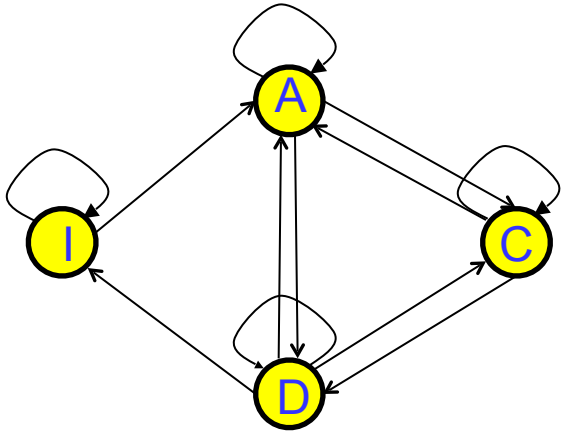
Estimating the state at T = 0+



- At T=0, after the first observation x_0, we update our belief about the states
  - The first observation provided some evidence about the state of the system
  - It modifies our belief in the state of the system

Predicting the state at T=1





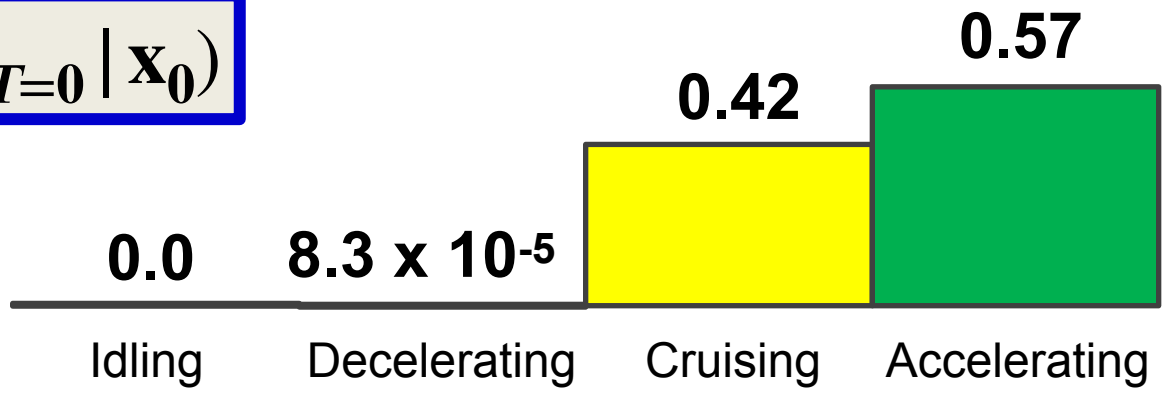
	I	A	C	D
I	0.5	0.5	0	0
A	0	1/3	1/3	1/3
C	0	1/3	1/3	1/3
D	0.25	0.25	0.25	0.25

- Predicting the probability of idling at T=1
  - $P(idling \mid idling) = 0.5;$
  - $P(idling \mid deceleration) = 0.25$
  - $P(idling \text{ at } T=1 \mid x_0) =$
    $P(I_{T=0}|x_0)\, P(I|I) + P(D_{T=0}|x_0)\, P(I|D) = 2.1 \times 10^{-5}$

- In general, for any state S
  - $$P(S_{T=1} \mid \mathbf{x}_0) = \sum_{S_{T=0}} P(S_{T=0} \mid \mathbf{x}_0)\, P(S_{T=1} \mid S_{T=0})$$

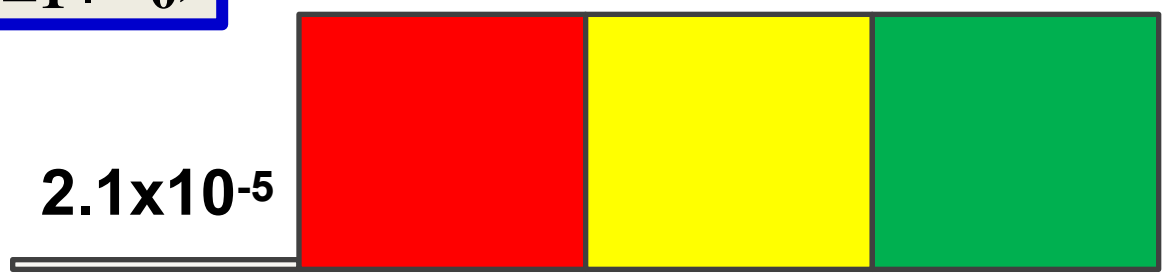
Predicting the state at T = 1

$P(S_{T=0} \mid \mathbf{x}_0)$



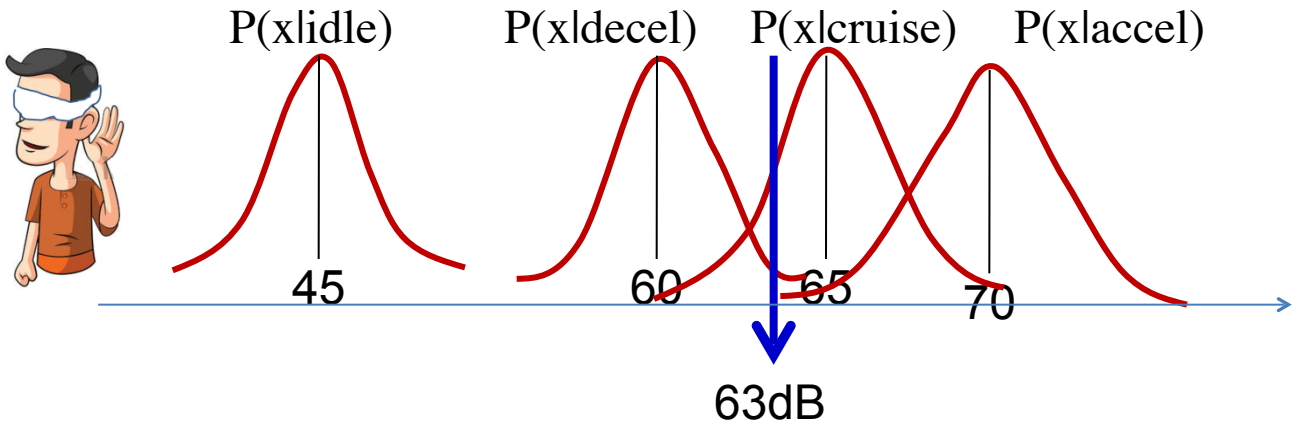
$$P(S_{T=1} \mid \mathbf{x}_0) = \sum_{S_{T=0}} P(S_{T=0} \mid \mathbf{x}_0) P(S_{T=1} \mid S_{T=0})$$

$P(S_{T=1} \mid \mathbf{x}_0)$



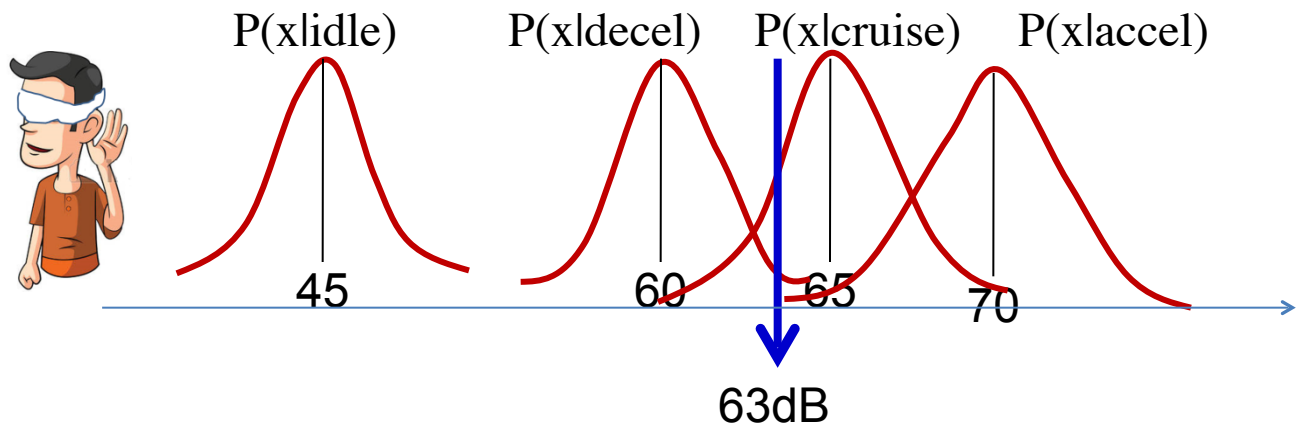
Rounded. In reality, they sum to 1.0

Updating after the observation at T=1

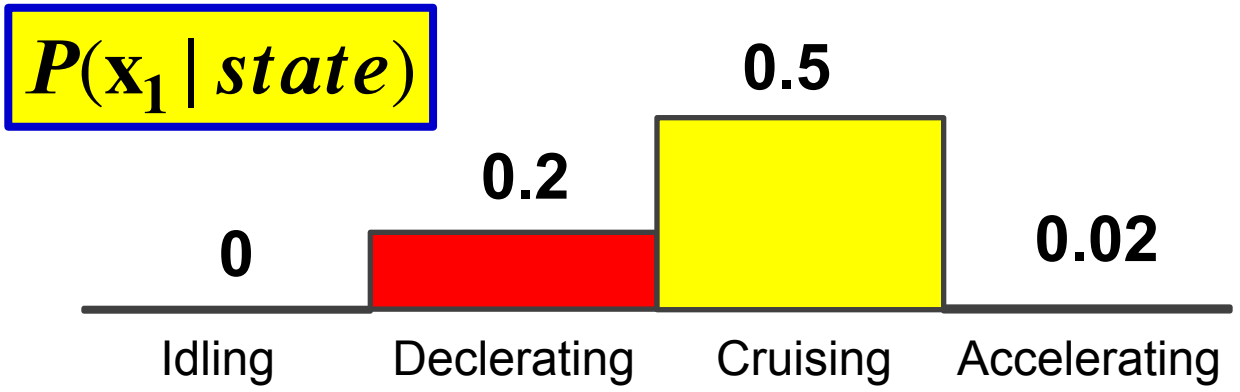


- At T=1 we observe x_1 = 63dB SPL

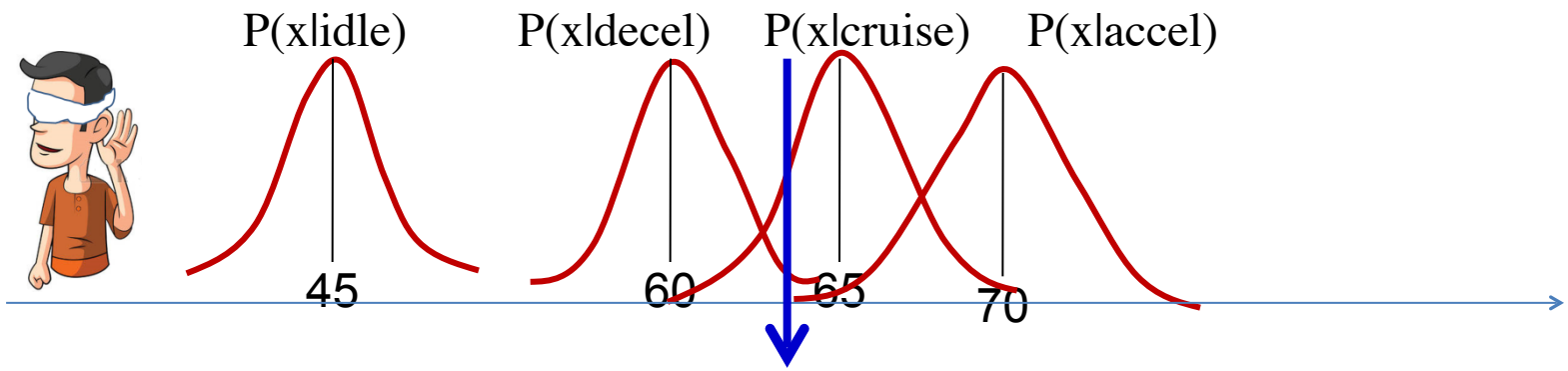
Updating after the observation at T=1



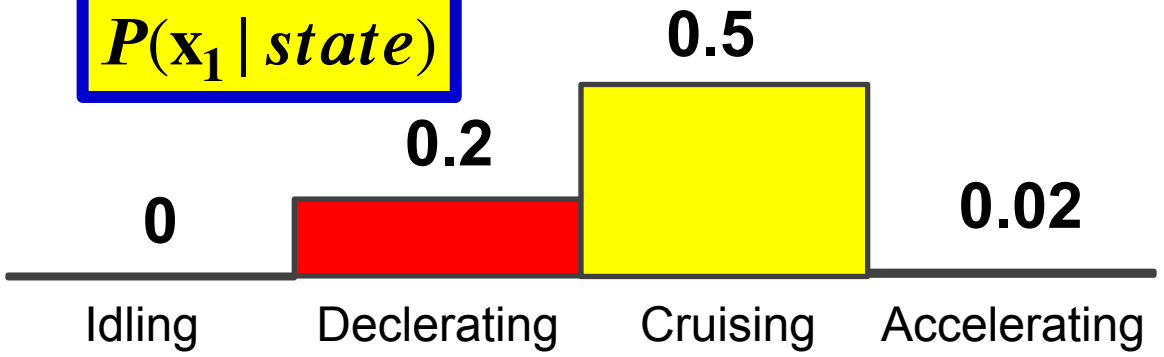
P(x|idle)	P(x|deceleration)	P(x|cruising)	P(x|acceleration)
0	0.2	0.5	0.01



The second observation: T=1

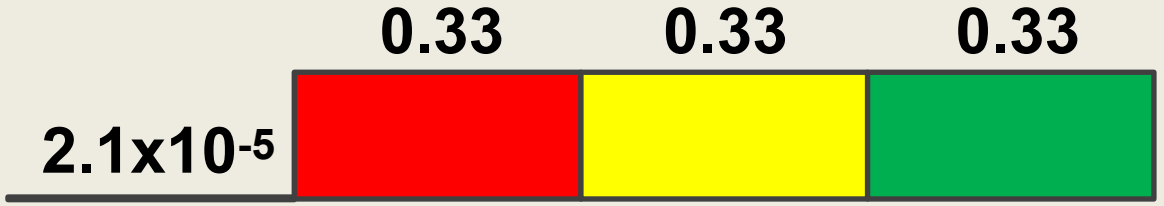


$P(x_1 \mid state)$



$Prior: \quad P(state \mid x0)$

Remember the prior



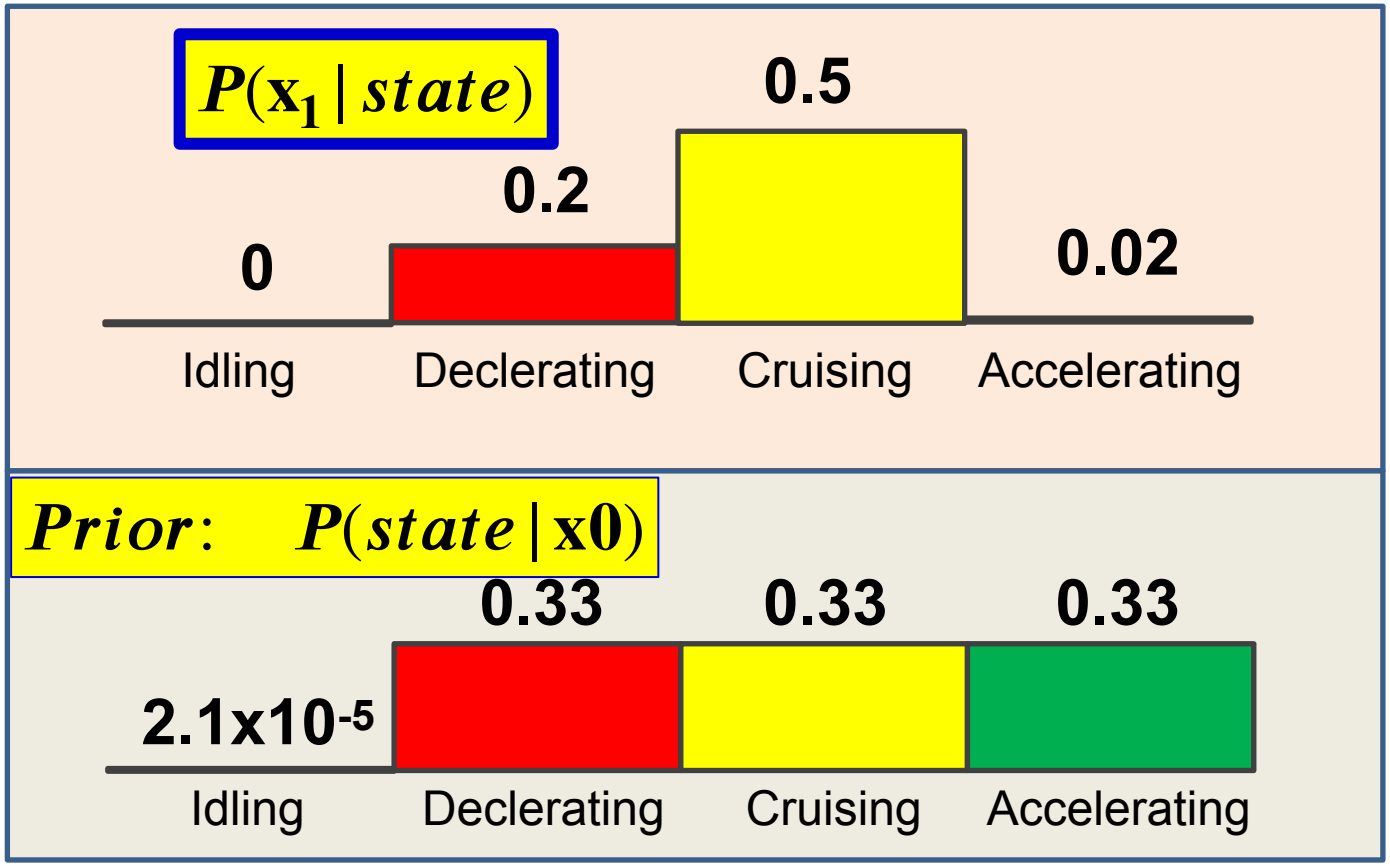
Estimating state *after* at observing x_1

- Combine prior information from the observation at time T=0, AND evidence from observation at T=1 to estimate ***state*** at T=1

- We want $P(state \mid \mathbf{x}_0, \mathbf{x}_1)$
- We can compute it using Bayes rule as

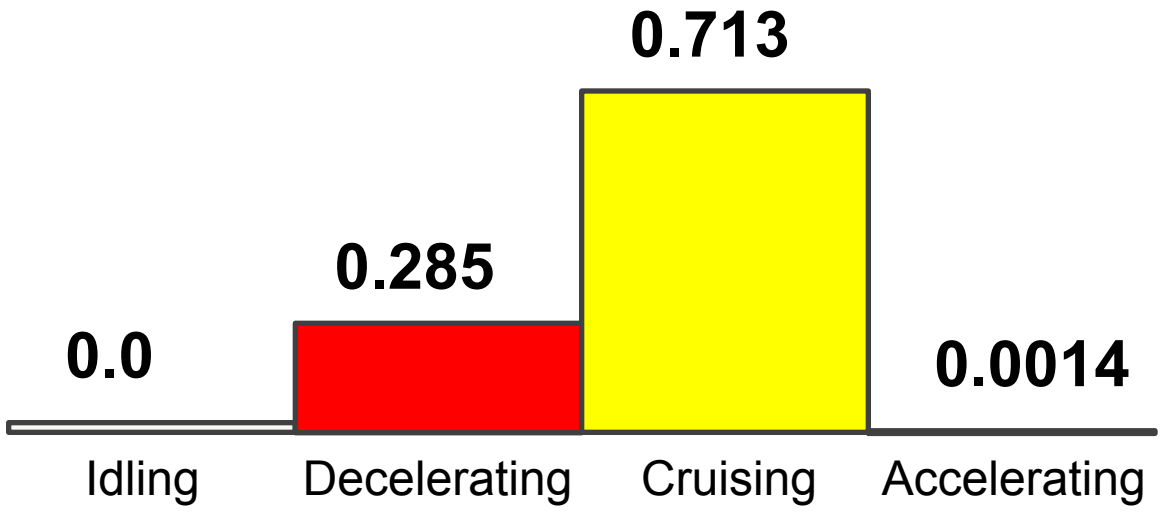
$$P(state \mid \mathbf{x}_0, \mathbf{x}_1) = \frac{P(state \mid \mathbf{x}_0) P(\mathbf{x}_1 \mid state)}{\sum_{state'} P(state' \mid \mathbf{x}_0) P(\mathbf{x}_1 \mid state')}$$

The Posterior at T = 1



- Multiply the two, term by term, and normalize them so that they sum to 1.0

Estimating the state at T = 1+

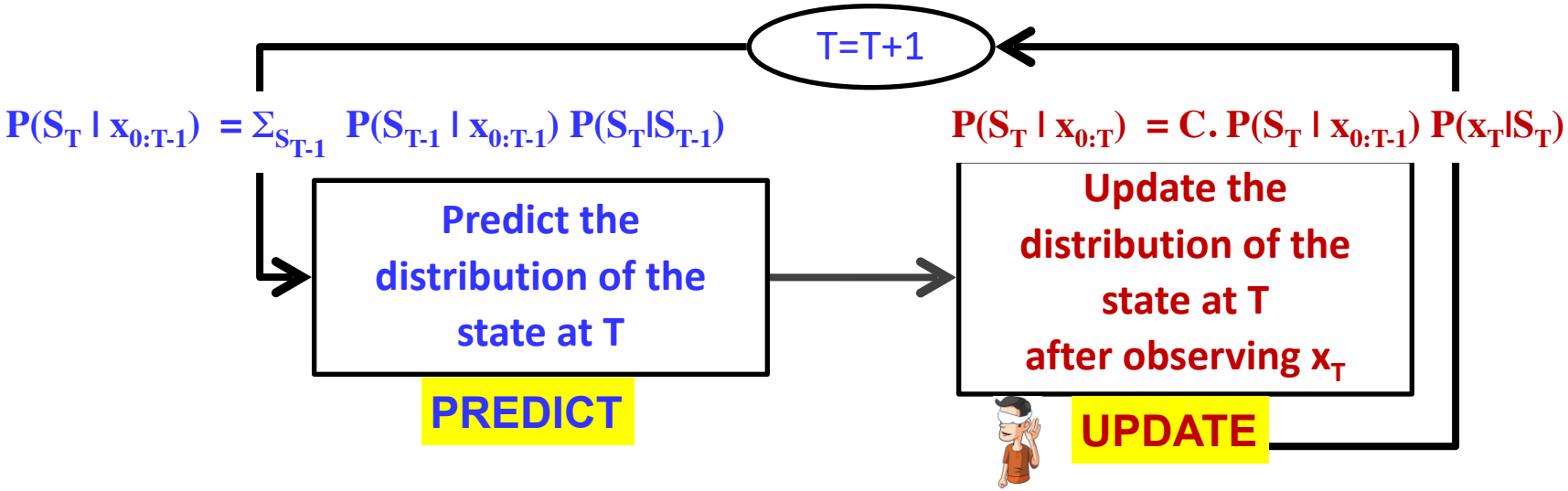


- The updated probability at T=1 incorporates information from both x_0 and x_1
  - It is NOT a local decision based on x_1 alone
  - Because of the Markov nature of the process, the state at T=0 affects the state at T=1
    - x_0 provides evidence for the state at T=1

Overall Process

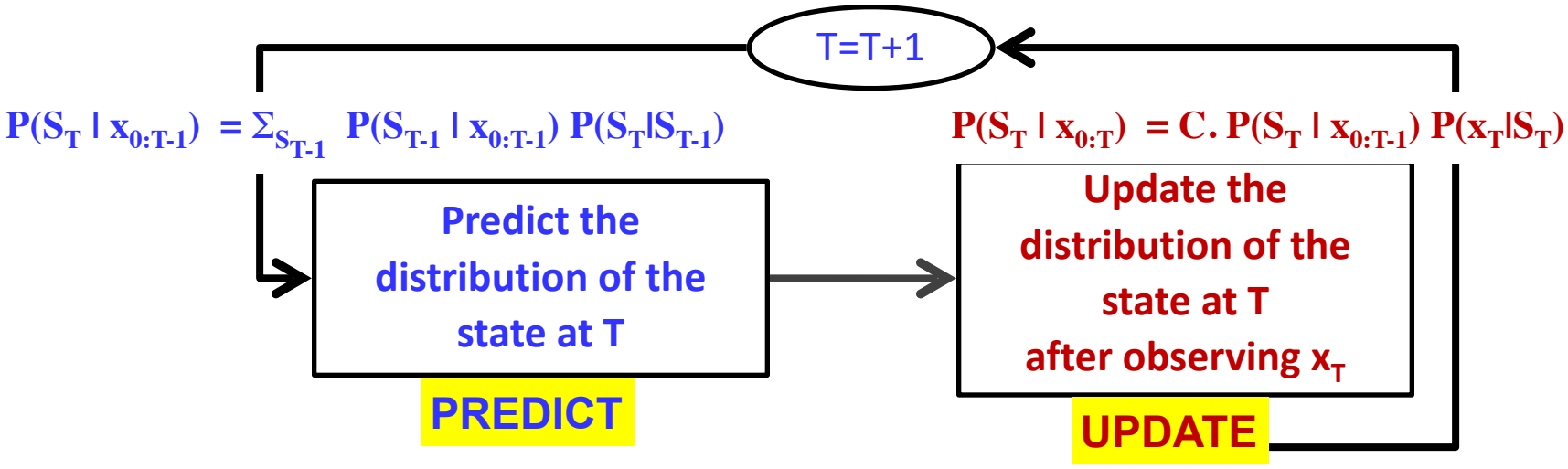
Time	Computation
T=0- : A priori probability	$P(S0) = P(S)$
T = 0+: Update after X_0	$P(S0 \mid X0) = C \,.\, P(S0)P(X0 \mid S0)$
T=1- (Prediction before X_1)	$P(S1 \mid X0) = \sum_{S0} P(S1 \mid S0)P(S0 \mid X0)$
T = 1+: Update after X_1	$P(S1 \mid X_{0:1}) = C \,.\, P(S1 \mid X0)P(X1 \mid S1)$
T=2- (Prediction before X_2)	$P(S2 \mid X_{0:1}) = \sum_{S1} P(S2 \mid S1)P(S1 \mid X_{0:1})$
T = 2+: Update after X_2	$P(S2 \mid X_{0:2}) = C \,.\, P(S2 \mid X_{0:1})P(X2 \mid S2)$
…	…
T= t- (Prediction before X_t)	$P(S_t \mid X_{0:t-1}) = \sum_{S_{t-1}} P(St \mid S_{t-1})P(S_{t-1} \mid X_{0:t-1})$
T = t+: Update after X_t	$P(St \mid X_{0:t}) = C \,.\, P(St \mid X_{0:t-1})P(Xt \mid St)$

Overall procedure

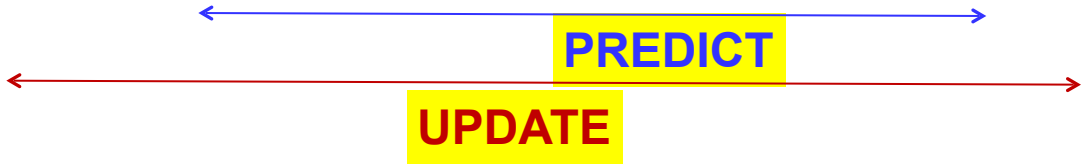


- At T=0 the predicted state distribution is the initial state probability
- At each time T, the current estimate of the distribution over states considers *all* observations x_0 ... x_T
  - A natural outcome of the Markov nature of the model
- The prediction+update is identical to the forward computation for HMMs to within a normalizing constant

Comparison to Forward Algorithm



- Forward Algorithm:
  - $P(x_{0:T}, S_T) = P(x_T \mid S_T) \Sigma_{S_{T-1}} P(x_{0:T-1}, S_{T-1}) P(S_T \mid S_{T-1})$



- Normalized:
  - $P(S_T \mid x_{0:T}) = (\Sigma_{S'_T} P(x_{0:T}, S'_T))^{-1} P(x_{0:T}, S_T) = C\, P(x_{0:T}, S_T)$

Decomposing the Algorithm

$$P(S_t, X_{0:t}) = P(Xt \mid S_t) \sum_{S_{t-1}} P(St \mid S_{t-1}) P(S_{t-1}, X_{0:t-1})$$

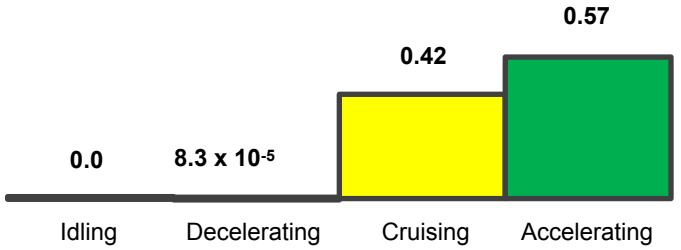
$$0{:}t-1) = \sum_{S_{t-1}} P(St \mid S_{t-1}) P(S_{t-1} \mid X_{0:t-1})$$

Update: $$P(St \mid X_{0:t}) = \frac{P(St \mid X_{0:t-1}) P(Xt \mid St)}{\sum_S P(S \mid X_{0:t-1}) P(Xt \mid S)}$$

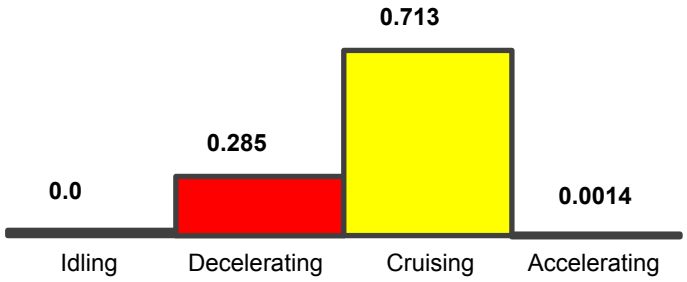
Estimating a Unique state

- What we have estimated is a *distribution* over the states
- If we had to guess $\boldsymbol{\alpha}$ state, we would pick the most likely state from the distributions

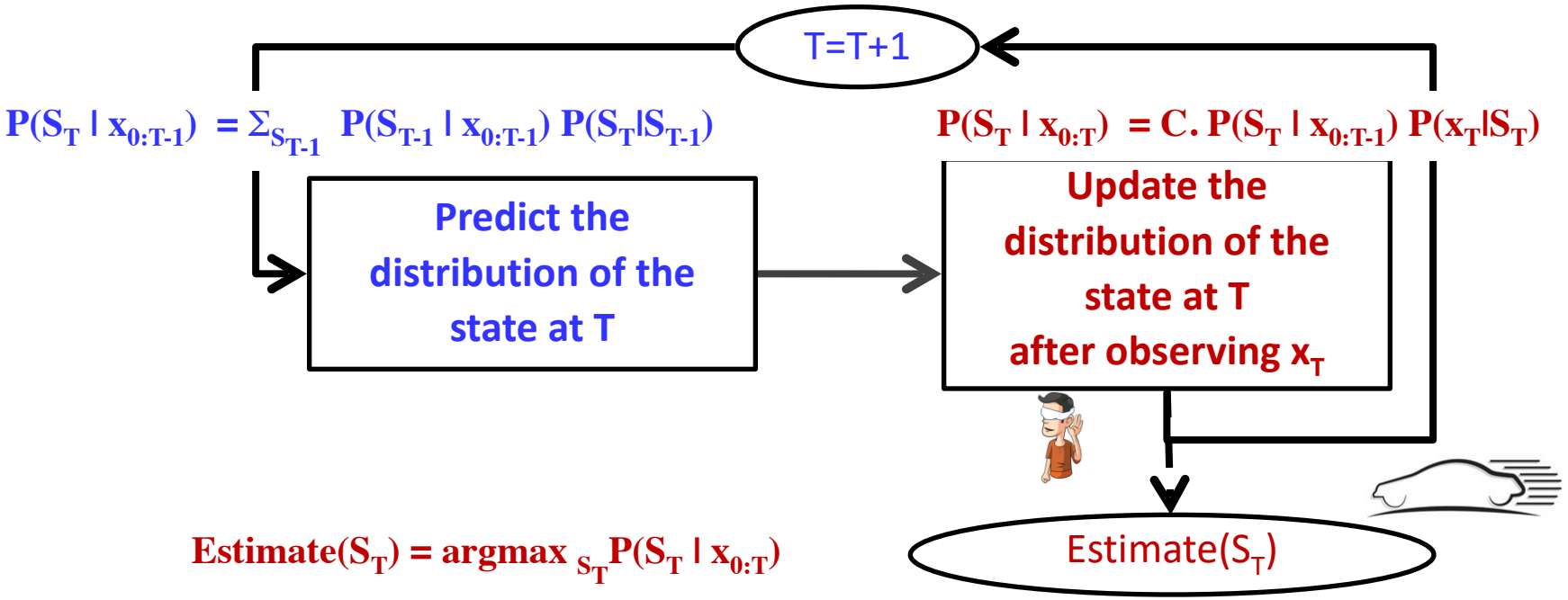
- State(T=0) = Accelerating



- State(T=1) = Cruising

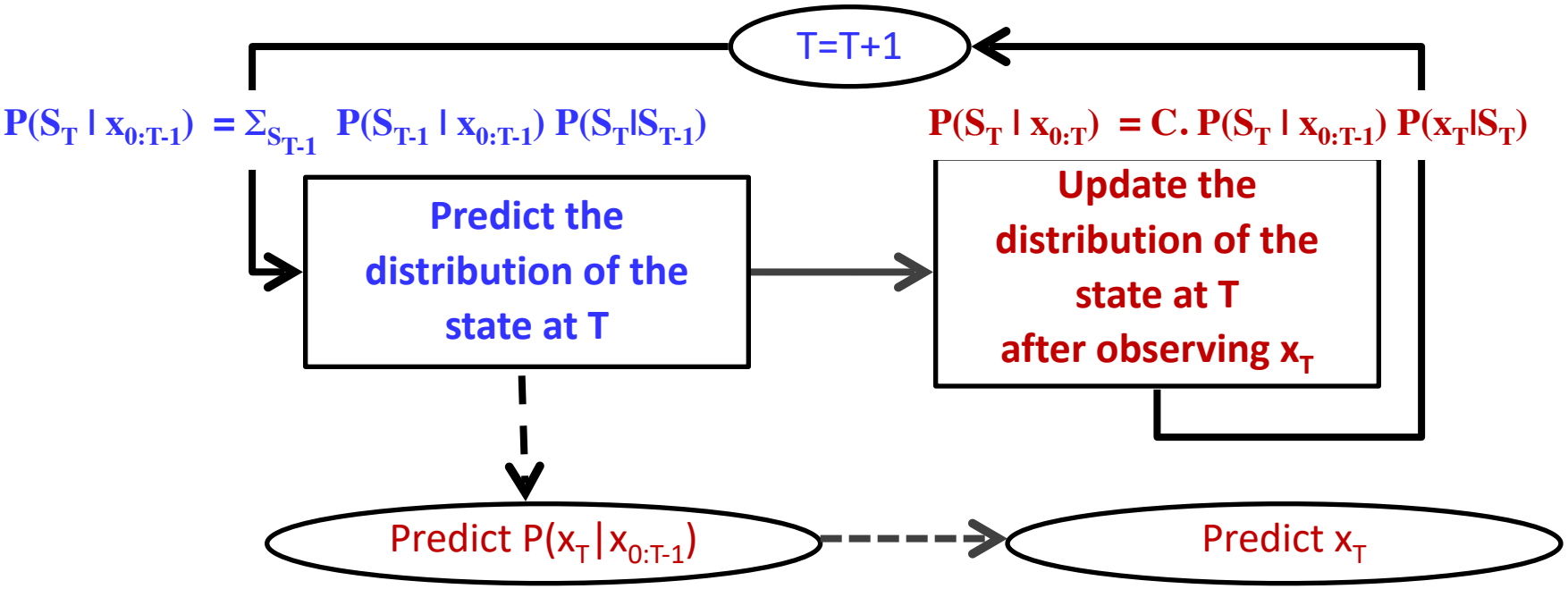


Estimating the *state*



- The state is estimated from the updated distribution
  - The updated distribution is propagated into time, not the state

Predicting the *next observation*



- The probability distribution for the observations at the next time is a mixture:
- $$P(X_t \mid X_{0:t-1}) = \sum_{S_t} P(Xt \mid S_t) P(S_t \mid X_{0:t-1})$$
- The actual observation can be predicted from $P(x_T \mid x_{0:T-1})$

Predicting the next observation

- Can use any of the various estimators of x_T from $P(x_T|x_{0:T-1})$
- MAP estimate:
  - $\text{argmax}_{x_T} P(x_T|x_{0:T-1})$
- MMSE estimate:
  - $\text{Expectation}(x_T|x_{0:T-1})$

Difference from Viterbi decoding

- Estimating only the *current* state at any time
  - Not the state sequence
  - Although we are considering all past observations
- The most likely state at T and T+1 may be such that there is no valid transition between S_T and S_{T+1}

A *continuous* state model

- HMM assumes a very coarsely quantized state space
  - Idling / accelerating / cruising / decelerating
- Actual state can be finer
  - Idling, accelerating at various rates, decelerating at various rates, cruising at various speeds
- Solution: Many more states (one for each acceleration /deceleration rate, crusing speed)?
- Solution: A *continuous* valued state

Tracking and Prediction: The wind and the target

- Aim: measure wind velocity
- Using a noisy wind speed sensor
  - E.g. arrows shot at a target

- **State:** Wind speed at time t depends on speed at time t-1

$$S_t = S_{t-1} + \epsilon_t$$

- **Observation:** Arrow position at time t depends on wind speed at time t

$$Y_t = AS_t + \gamma_t$$

The real-valued state model

- A state equation describing the dynamics of the system

$$s_t = f(s_{t-1}, \varepsilon_t)$$

  - s_t is the state of the system at time t
  - ε_t is a driving function, which is assumed to be random
- The state of the system at any time depends only on the state at the previous time instant and the driving term at the current time

- An observation equation relating state to observation

$$o_t = g(s_t, \gamma_t)$$

  - o_t is the observation at time t
  - γ_t is the noise affecting the observation (also random)
- The observation at any time depends only on the current state of the system and the noise

States are still “hidden”

$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

- The state is a continuous valued parameter that is not directly seen
  - The state is the position of the automobile or the star
- The observations are dependent on the state and are the only way of knowing about the state
  - Sensor readings (for the automobile) or recorded image (for the telescope)

Statistical Prediction and Estimation

- Given an *a priori* probability distribution for the state
  - $P_0(s)$: Our belief in the state of the system before we observe any data
    - Probability of state of navlab
    - Probability of state of stars
- Given a sequence of observations $o_0..o_t$
- Estimate state at time t

Prediction and update at t = 0

- Prediction
  - Initial probability distribution for state
  - $P(s_0) = P_0(s_0)$
- Update:
  - Then we observe o_0
  - We must update our belief in the state

$$P(s_0 \mid o_0) = \frac{P(s_0)P(o_0 \mid s)}{P(o_0)} = \frac{P_0(s_0)P(o_0 \mid s_0)}{P(o_0)}$$

- $P(s_0 \mid o_0) = C.P_0(s_0)P(o_0 \mid s_0)$

Prediction and update at t = 0

- Prediction
  - Initial probability distribution for state
  - $P(s_0) = P_0(s_0)$
- Update:
  - Then we observe o_0
  - We must update our belief in the state

$$P(s_0 \mid o_0) = \frac{P(s_0)P(o_0 \mid s)}{P(o_0)} = \frac{P_0(s_0)P(o_0 \mid s_0)}{P(o_0)}$$

- $P(s_0 \mid o_0) = C.P_0(s_0)P(o_0 \mid s_0)$

The observation probability: P(o|s)

- $o_t = g(s_t, \gamma_t)$
  - This is a (possibly many-to-one) stochastic function of state s_t and noise γ_t
  - Noise γ_t is random. Assume it is the same dimensionality as o_t

- Let $P_\gamma(\gamma_t)$ be the probability distribution of γ_t
- Let $\{\gamma: g(s_t, \gamma) = o_t\}$

$$P(o_t \mid s_t) = \sum_{\gamma: g(s_t,\gamma)=o_t} \frac{P_\gamma(\gamma)}{|J_\gamma(g(s_t,\gamma))|}$$

The observation probability

- $P(o|s) = ?$ $\quad o_t = g(s_t, \gamma_t)$

$$P(o_t \mid s_t) = \sum_{\gamma : g(s_t,\gamma)=o_t} \frac{P_\gamma(\gamma)}{|J_\gamma(g(s_t,\gamma))|}$$

- The J is a Jacobian

$$|J_\gamma(g(s_t,\gamma))| = \begin{vmatrix} \frac{\partial o_t(1)}{\partial \gamma(1)} & \cdots & \frac{\partial o_t(1)}{\partial \gamma(n)} \\ ? & ? & ? \\ \frac{\partial o_t(n)}{\partial \gamma(1)} & ? & \frac{\partial o_t(n)}{\partial \gamma(n)} \end{vmatrix}$$

- For scalar functions of scalar variables, it is simply a derivative:

$$|J_\gamma(g(s_t,\gamma))| = \left|\frac{\partial o_t}{\partial \gamma}\right|$$

Predicting the next state at t=1

- Given $P(s_0|o_0)$, what is the probability of the state at t=1

$$P(s_1 \mid o_0) = \int_{\{s_0\}} P(s_1, s_0 \mid o_0) ds_0 = \int_{\{s_0\}} P(s_1 \mid s_0) P(s_0 \mid o_0) ds_0$$

- State progression function:

$$s_t = f(s_{t-1}, \varepsilon_t)$$

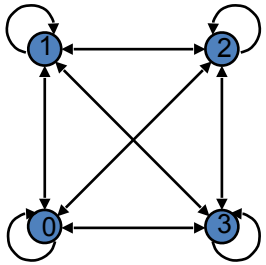
  - ε_t is a driving term with probability distribution $P_\varepsilon(\varepsilon_t)$

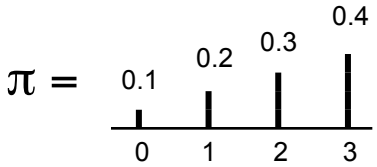
- $P(s_t|s_{t-1})$ can be computed similarly to $P(o|s)$
  - $P(s_1|s_0)$ is an instance of this

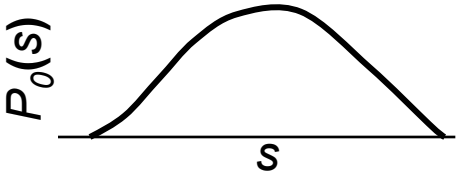
And moving on

- $P(s_1|o_0)$ is the predicted state distribution for t=1
- Then we observe o_1
  - We must update the probability distribution for s_1
  - $P(s_1|o_{0:1}) = CP(s_1|o_0)P(o_1|s_1)$
- We can continue on

Discrete vs. Continuous state systems







$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

Prediction at time 0:

$$P(S_0) = \pi(S_0) \qquad P(S_0) = P_0(S_0)$$

Update after O_0:

$$P(S_0 \mid O_0) = C \,.\, \pi(S_0) P(O_0 \mid S_0) \qquad P(S_0 \mid O_0) = C \,.\, P(S_0) P(O_0 \mid S_0)$$

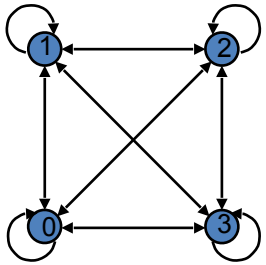
Prediction at time 1:

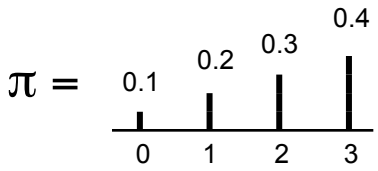
$$P(S_1 \mid O_0) = \sum_{S_0} P(S_0 \mid O_0) P(S_1 \mid S_0) \qquad P(S_1 \mid O_0) = \int_{-\infty}^{\infty} P(S_0 \mid O_0) P(S_1 \mid S_0) dS_0$$

Update after O_1:

$$P(S_1 \mid O_{0:1}) = C \,.\, P(S_1 \mid O_0) P(O_1 \mid S_1) \qquad P(S_1 \mid O_{0:1}) = C \,.\, P(S_1 \mid O_0) P(O_1 \mid S_1)$$

Discrete vs. Continuous State Systems





$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

Prediction at time t:

$$P(S_t \mid O_{0:t-1}) = \sum_{S_{t-1}} P(S_{t-1} \mid O_{0:t-1}) P(S_t \mid S_{t-1})$$

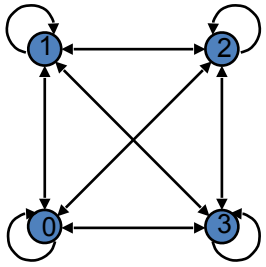
$$P(S_t \mid O_{0:t-1}) = \int_{-\infty}^{\infty} P(S_{t-1} \mid O_{0:t-1}) P(S_t \mid S_{t-1}) dS_{t-1}$$

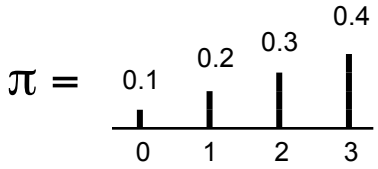
Update after observing O_t:

$$P(S_t \mid O_{0:t}) = C \,.\, P(S_t \mid O_{0:t-1}) P(O_t \mid S_t)$$

$$P(S_t \mid O_{0:t}) = C \,.\, P(S_t \mid O_{0:t-1}) P(O_t \mid S_t)$$

Discrete vs. Continuous State Systems





$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

Parameters		
Initial state prob.	π	$P(s)$
Transition prob	$P(s_t = j \mid s_{t-1} = i)$	$P(s_t \mid s_{t-1})$
Observation prob	$P(O \mid s)$	$P(O \mid s)$

Special case: Linear Gaussian model

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$P(\varepsilon) = \frac{1}{\sqrt{(2\pi)^d \,|\Theta_\varepsilon|}} \exp\left(-0.5(\varepsilon - \mu_\varepsilon)^T \Theta_\varepsilon^{-1} (\varepsilon - \mu_\varepsilon)\right)$$

$$o_t = B_t s_t + \gamma_t$$

$$P(\gamma) = \frac{1}{\sqrt{(2\pi)^d \,|\Theta_\gamma|}} \exp\left(-0.5(\gamma - \mu_\gamma)^T \Theta_\gamma^{-1} (\gamma - \mu_\gamma)\right)$$

- A *linear* state dynamics equation
  - Probability of state driving term ε is Gaussian
  - Sometimes viewed as a driving term μ_ε and additive zero-mean noise
- A *linear* observation equation
  - Probability of observation noise γ is Gaussian
- A_t, B_t and Gaussian parameters assumed known
  - May vary with time

Linear model example
The wind and the target

- **State:** Wind speed at time t depends on speed at time t-1

$$S_t = S_{t-1} + \epsilon_t$$

- **Observation:** Arrow position at time t depends on wind speed at time t

$$O_t = BS_t + \gamma_t$$

Model Parameters: The initial state probability

$$P_0(s) = \frac{1}{\sqrt{(2\pi)^d \mid R \mid}} \exp\left(-0.5(s-\bar{s})R^{-1}(s-\bar{s})^T\right)$$

$$P_0(s) = Gaussian(s; \bar{s}, R)$$

- We also assume the *initial* state distribution to be Gaussian
  - Often assumed zero mean

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Model Parameters: The observation probability

$$o_t = B_t s_t + \gamma_t \qquad P(\gamma) = Gaussian(\gamma; \mu_\gamma, \Theta_\gamma)$$

$$P(o_t \mid s_t) = Gaussian(o_t; \mu_\gamma + B_t s_t, \Theta_\gamma)$$

- The probability of the observation, given the state, is simply the probability of the noise, with the mean shifted
  - Since the only uncertainty is from the noise
- The new mean is the mean of the distribution of the noise + the value of the observation in the absence of noise

Model Parameters:
State transition probability

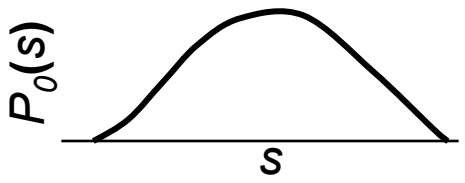
$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$P(\varepsilon) = Gaussian(\varepsilon; \mu_\varepsilon, \Theta_\varepsilon)$$

$$P(s_{t+1} \mid s_t) = Gaussian(s_t; \mu_\varepsilon + A_t s_t, \Theta_\varepsilon)$$

- The probability of the state at time t, given the state at t-1, is simply the probability of the driving term, with the mean shifted

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time 0:

$$P(S_0) = P_0(S_0)$$

Update after O_0:

$$P(S_0 \mid O_0) = C \,.\, P(S_0) P(O_0 \mid S_0)$$

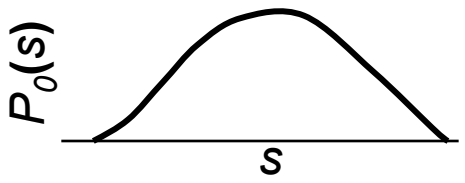
Prediction at time 1:

$$P(S_1 \mid O_0) = \int_{-\infty}^{\infty} P(S_0 \mid O_0) P(S_1 \mid S_0) dS_0$$

Update after O_1:

$$P(S_1 \mid O_{0:1}) = C \,.\, P(S_1 \mid O_0) P(O_1 \mid S_1)$$

Continuous state systems



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Update after O_1:

$$P(S_1 \mid O_{0:1}) = C \,.\, P(S_1 \mid O_0) P(O_1 \mid S_1)$$

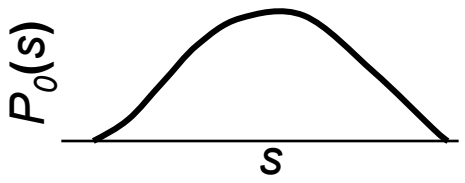
Model Parameters: The initial state probability

$$P_0(s) = \frac{1}{\sqrt{(2\pi)^d \mid R_0 \mid}} \exp\left(-0.5(s-\bar{s}_0)R_0^{-1}(s-\bar{s}_0)^T\right)$$

$$P_0(s) = Gaussian(s; \bar{s}_0, R_0)$$

- We assume the *initial* state distribution to be Gaussian
  - Often assumed zero mean

Continuous state systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

a priori probability distribution of state s

$$= N(\bar{s}0, R0)$$

Prediction at time 0:

$$P(S_0) = P_0(S_0)$$

Update after O_0:

$$P(S_0 \mid O_0) = C \,.\, P(S_0) P(O_0 \mid S_0)$$

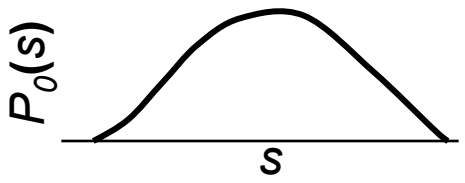
Prediction at time 1:

$$P(S_1 \mid O_0) = \int_{-\infty}^{\infty} P(S_0 \mid O_0) P(S_1 \mid S_0) dS_0$$

Update after O_1:

$$P(S_1 \mid O_{0:1}) = C \,.\, P(S_1 \mid O_0) P(O_1 \mid S_1)$$

Continuous state systems



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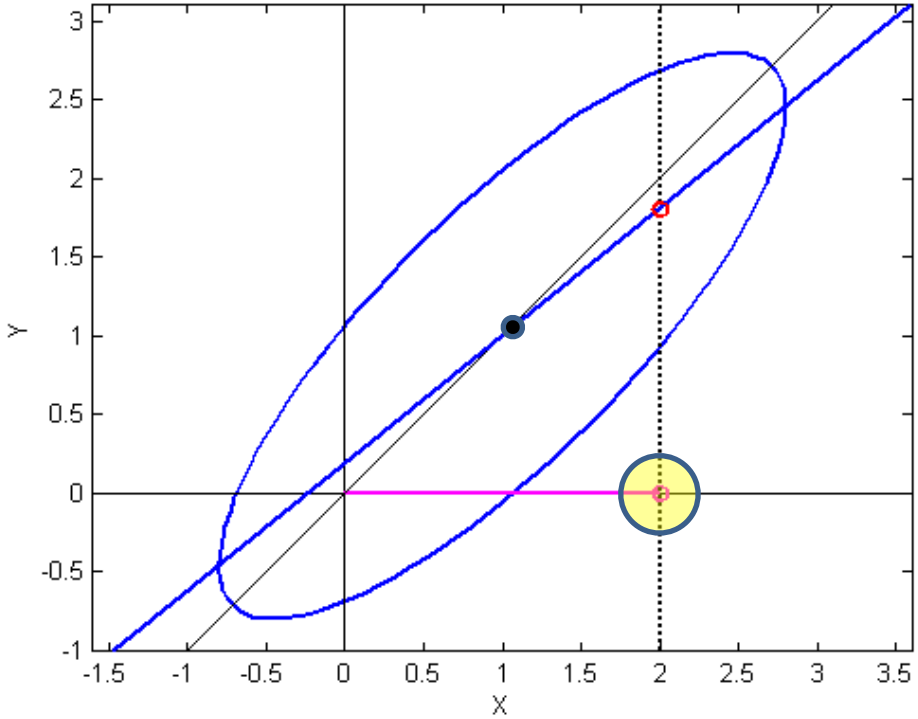
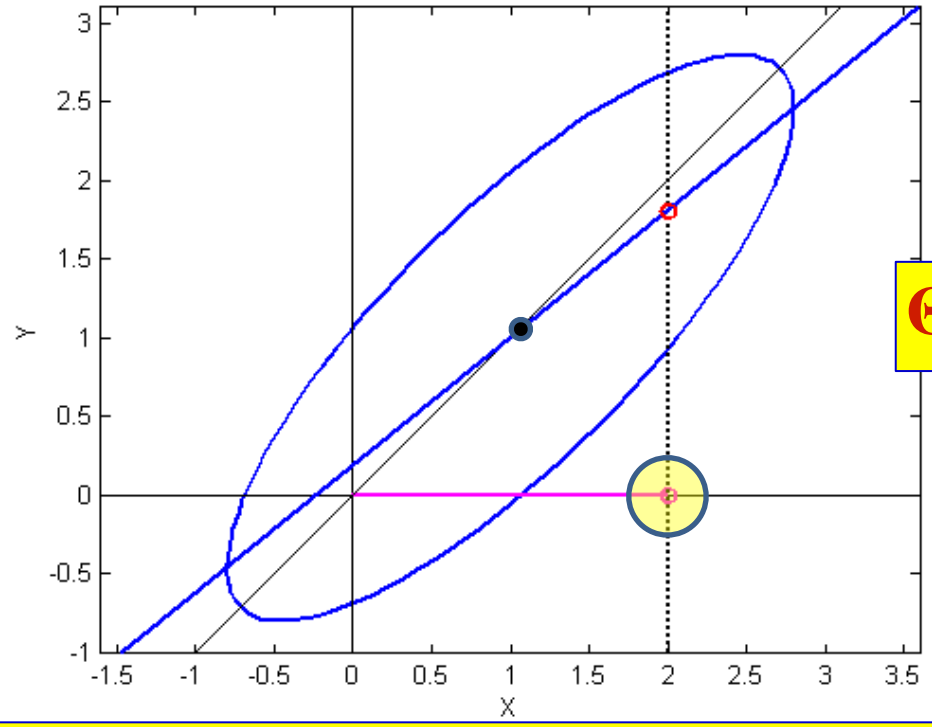
Recap: Conditional of S given O: P(S|O) for Gaussian RVs



$$O = BS + \gamma$$

$$P(S|O) = N(\mu_S + \Theta_{SO}\Theta_O^{-1}(O - \mu_O), \quad \Theta_S - \Theta_{SO}\Theta_O^{-1}\Theta_{OS})$$

Recap: Conditional of S given O: P(S|O) for Gaussian RVs



$$O = BS + \gamma$$

$$\Theta_{SO} = \Theta_S B^{\mathrm{T}}$$

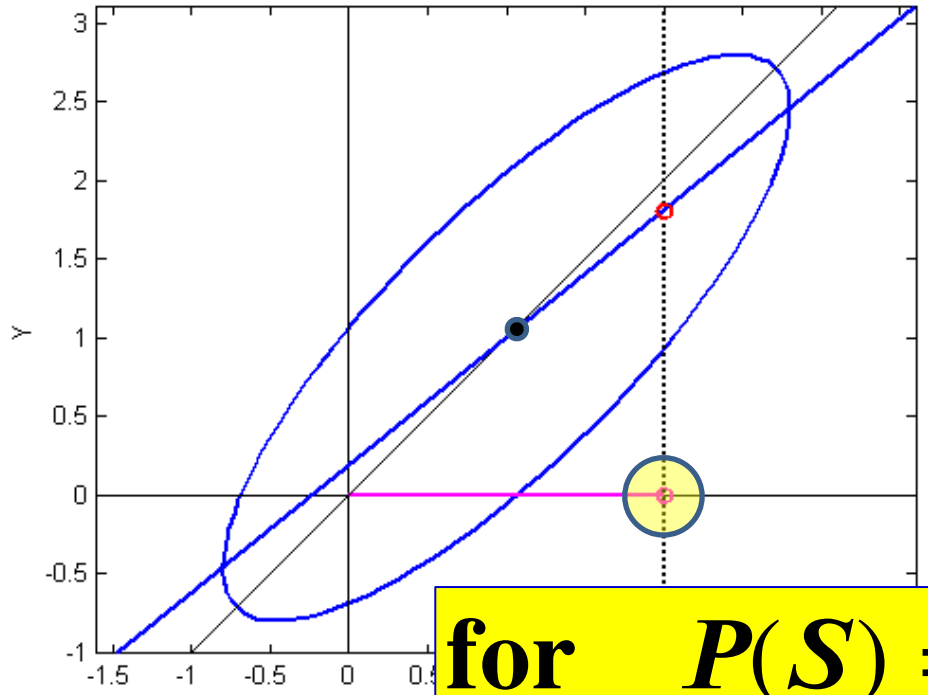
$$\Theta_O = B\Theta_S B^{\mathrm{T}} + \Theta_\gamma$$

$$P(S|O) = N(\mu_S + \Theta_{SO}\Theta_O^{-1}(O - \mu_O),\quad \Theta_S - \Theta_{SO}\Theta_O^{-1}\Theta_{OS})$$

$$P(S|O) = N(\mu_S + \Theta_S B^{\mathrm{T}}\left(B\Theta_S B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}(O - B\mu_s - \mu_\gamma),$$

$$\Theta_S - \Theta_S B^{\mathrm{T}}\left(B\Theta_S B^{\mathrm{T}} + \Theta_\gamma\right)^{-1} B\Theta_S)$$

Recap: Conditional of S given O: P(S|O) for Gaussian RVs



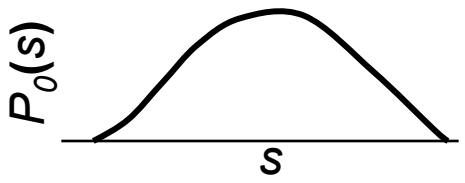
$$O = BS + \varepsilon$$

for $P(S) = N(\bar{s}0, R0)$

$$P(S0|O0) = N(\bar{s0} + R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_{\gamma}\right)^{-1}(O0 - B\bar{s}0 - \mu_{\gamma}),$$

$$R0 - R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_{\gamma}\right)^{-1}BR0)$$

Continuous state systems



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$$o_t = B_t s_t + \gamma_t$$

Prediction at time 0:

$$P(S_0) = N(\bar{s}0, R0)$$

Update after O_0:

$$P(S_0 | O_0) = C \,.\, P(S_0) P(O_0 | S_0)$$

$$P(S_0 | O_0) = N(\hat{s}_0, \hat{R}_0)$$

Prediction at time 1:

$$P(S_1 | O_0) = \int_{-\infty}^{\infty} P(S_0 | O_0) P(S_1 | S_0) dS_0$$

Update after O_1:

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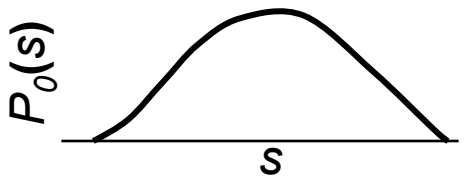
Continuous state systems



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$$K_0 = R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}$$

$$\hat{s}_0 = \bar{s}0 + K_0(O0 - B\bar{s}0 - \mu_\gamma)$$

$$\hat{R}_0 = (I - K_0)\, R0$$

Prediction at time 1:

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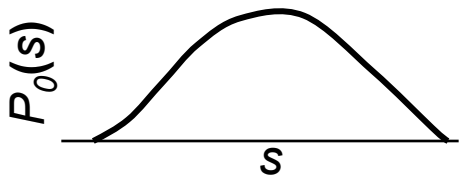
Continuous state systems



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$$P(S_0 | O_0) = C \,.\, P(S_0) P(O_0 | S_0)$$

$$= N(\bar{s}0 + R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}(O0 - B\bar{s}0 - \mu_$$

$$R0 - R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1} BR0)$$

Prediction at time 1:

$$P(S_1 | O_0) = \int_{-\infty}^{\infty} P(S_0 | O_0) P(S_1 | S_0) dS_0$$

Update after O_1:

$$P(S_1 | O_{0:1}) = C \,.\, P(S_1 | O_0) P(O_1 | S_1)$$

Introducting shorthand notation

$$P(S0 \mid O0) = N(\bar{s}0 + R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_{\gamma}\right)^{-1}(O0 - B\bar{s}0 - \mu_{\gamma}),$$
$$R0 - R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_{\gamma}\right)^{-1}BR0)$$

$$\hat{s}_0 = \bar{s}0 + R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_{\gamma}\right)^{-1}(O - B\bar{s}0 - \mu_{\gamma})$$

$$\hat{R}_0 = R0 - R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_{\gamma}\right)^{-1}BR0$$

$$P\left(S0 \mid O0\right) = N(\hat{s}_0, \hat{R}_0)$$

Introducting shorthand notation

$$P(S0 \mid O0) = N(\bar{s}0 + R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_{\gamma}\right)^{-1}(O0 - B\bar{s}0 - \mu_{\gamma}),$$
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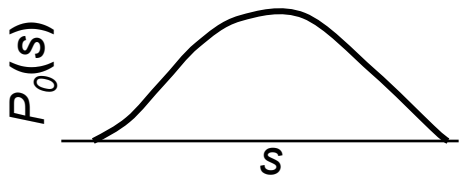
$$K_0 = R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_{\gamma}\right)^{-1}$$

$$\hat{s}_0 = \bar{s}0 + K_0(O - B\bar{s}0 - \mu_{\gamma})$$

$$\hat{R}_0 = (I - K0B)R0$$

$$P(S0 \mid O0) = N(\hat{s}_0, \hat{R}_0)$$

Continuous state systems



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Update after O_0:

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Prediction at time 1:

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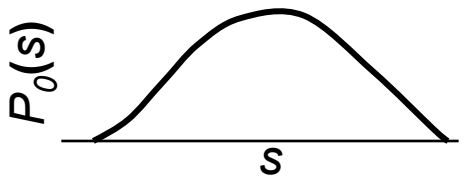
Update after O_1:

$$P(S_1 | O_{0:1}) = C \,.\, P(S_1 | O_0) P(O_1 | S_1)$$

Continuous state systems



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Prediction at time 1:

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Upda

$$P(S_1 \mid O_{0:1}) = C \,.\, P(S_1 \mid O_0) P(O_1 \mid S_1)$$

The prediction equation

$$P(S_1 \mid O_0) = \int_{-\infty}^{\infty} P(S_0 \mid O_0) P(S_1 \mid S_0) dS_0$$

$$P(S_0 \mid O_0) = N(\hat{s}_0, \hat{R}_0)$$

$$P(S_1 \mid S_0) = N(AS_0 + \mu_\varepsilon, \Theta_\varepsilon)$$

$$P(\varepsilon) = N(\mu_\varepsilon, \Theta_\varepsilon)$$

$$s_{t+1} = A_t s_t + \varepsilon_t$$

- The integral of the product of two Gaussians

$$P(S_1 \mid O_0) = \int_{-\infty}^{\infty} Gaussian(S_0; \hat{s}_0, \hat{R}_0) Gaussian(S_1; AS_0, \Theta_\varepsilon) dS_0$$

The Prediction Equation

- The integral of the product of two Gaussians is Gaussian!

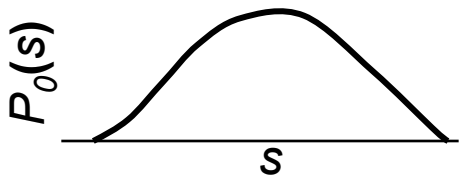
$$P(S_1 \mid O_0) = \int_{-\infty}^{\infty} Gaussian\left(S_0; \hat{s}_0, \hat{R}_0\right) Gaussian(S_1; AS_0 + \mu_\varepsilon, \Theta_\varepsilon) dS_0$$

$$= \int_{-\infty}^{\infty} C_1 exp\left(-0.5\left(S_0 - \hat{s}_0\right)\hat{R}_0^{-1}\left(S_0 - \hat{s}_0\right)^T\right) . C_2 exp(-0.5(S_1 - AS_0 - \mu_\varepsilon)\Theta_\varepsilon^{-1}(S_1 - AS_0 - \mu_\varepsilon)^T) d$$

$$= Gaussian(S_1; A\hat{s}_0 + \mu_\varepsilon, \Theta_\varepsilon + A\hat{R}_0 A^T)$$

$$P(S_1 \mid O_0) = N(A\hat{s}_0 + \mu_\varepsilon, \Theta_\varepsilon + A\hat{R}_0 A^T)$$

Continuous state systems



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Update after O_0:

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$$K_0 = R0B^{\mathrm{T}}\left(BR0B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}$$

$$\hat{s}_0 = \bar{s}0 + K_0(O0 - B\bar{s}0 - \mu_\gamma) \qquad \hat{R}_0 = (I - K_0)\,R0$$

Prediction at time 1:

$$P(S_1 \mid O_0) = \int_{-\infty}^{\infty} P(S_0 \mid O_0) P(S_1 \mid S_0) dS_0$$

$$= N(A\hat{s}_0 + \mu_\varepsilon, \Theta_\varepsilon + A\hat{R}_0 A^T)$$

Update after O_1:

$$P(S_1 \mid O_{0:1}) = C \,.\, P(S_1 \mid O_0) P(O_1 \mid S_1)$$

More shorthand notation

$$P(S1 \mid O0) = N(A\hat{s}_0 + \mu_\varepsilon, \Theta_\varepsilon + A\hat{R}_0 A^T)$$

$$\bar{s}_1 = A\hat{s}_0 + \mu_\varepsilon$$

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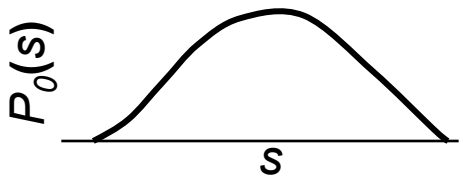
$$P(S1 \mid O0) = N(\bar{s}_1, R_1)$$

Continuous state systems



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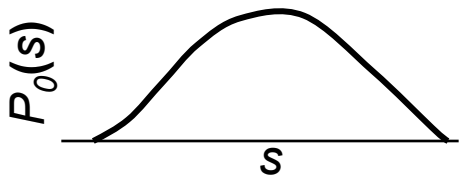
Continuous state systems



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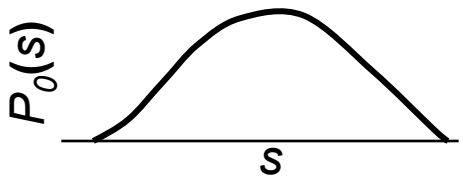
Continuous state systems



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Update after O_1:

$$P(S_1 \mid O_{0:1}) = C \,.\, P(S_1 \mid O_0) P(O_1 \mid S_1) = N(\hat{s}_1, \hat{R}_1)$$

$$K_1 = R1B^{\mathrm{T}}\left(BR1B^{\mathrm{T}} + \Theta_\gamma\right)^{-1}$$

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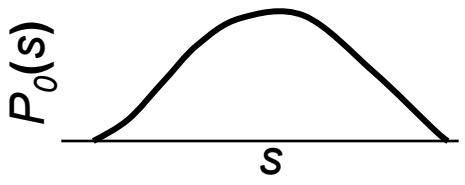
$$\hat{R}_1 = (I - K_1 B)\, R1$$

Continuous state systems



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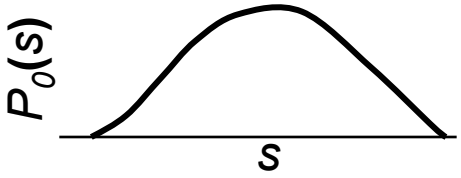
Gaussian Continuous State
Linear Systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

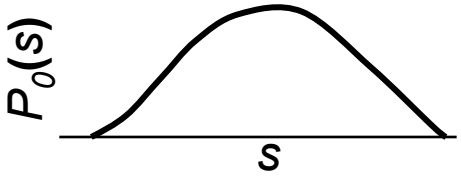
Prediction at time t:

$$P(S_t \mid O_{0:t-1}) = \int_{-\infty}^{\infty} P(S_{t-1} \mid O_{0:t-1}) P(S_t \mid S_{t-1}) dS_{t-1}$$

Update after observing O_t:

$$P(S_t \mid O_{0:t}) = C \,.\, P(S_t \mid O_{0:t-1}) P(O_t \mid S_t)$$

Gaussian Continuous State
Linear Systems



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Prediction at time t:

$$P(S_t \mid O_{0:t-1}) = N(\bar{s}_t, R_t)$$

$$\bar{s}_t = A\hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A\hat{R}_{t-1}A^T$$

Update after observing O_t:

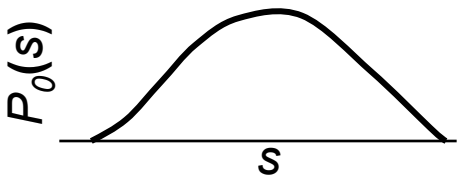
$$P(S_t \mid O_{0:t}) = N(\hat{s}_t, \hat{R}_t)$$

$$K_t = R1B^T\left(BR1B^T + \Theta_\gamma\right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + Kt\,(Ot - B\bar{s}_t - \mu_\gamma)$$

$$\hat{R}_t = (I - KtB)\,Rt$$

Gaussian Continuous State
Linear Systems



$$s_{t+1} = A_t s_t + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

Prediction at time t:

$$P(S_t \mid O_{0:t-1}) = N(\bar{s}_t, R_t)$$

Update after observing O_t:

$$P(S_t \mid O_{0:t}) = N(\hat{s}_t, \hat{R}_t)$$

KALMAN FILTER

$$\bar{s}_t = A\hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A\hat{R}_{t-1}A^T$$

$$K_t = R1B^T\left(BR1B^T + \Theta_\gamma\right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + Kt\,(Ot - B\bar{s}_t - \mu_\gamma)$$

$$\hat{R}_t = (I - KtB)\,Rt$$

The Kalman filter

- Prediction (based on state equation)

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- Update (using observation and observation equation)

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

$$o_t = B_t s_t + \gamma_t$$

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - B_t \bar{s}_t - \mu_\gamma \right)$$

$$\hat{R}_t = \left(I - K_t B_t \right) R_t$$

Explaining the Kalman Filter

- Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- The Kalman filter can be explained intuitively without working through the math

$$\hat{R}_t = (I - K_t B_t) R_t$$

The Kalman filter

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

The predicted state at time t is obtained simply by propagating the estimated state at t-1 through the state dynamics equation

$$\hat{s}_t = \bar{s}_t + K_t\left(o_t - B_t \bar{s}_t - \mu_\gamma\right)$$

$$\hat{R}_t = \left(I - K_t B_t\right) R_t$$

The Kalman filter

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

This is the uncertainty in the prediction. The variance of the predictor = variance of ε_t + variance of As_{t-1}

The two simply add because ε_t is not correlated with s_t

The Kalman filter

- Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$o_t = B_t s_t + \gamma_t$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

We can also predict the observation from the predicted state using the observation equation

$$\hat{R}_t = (I - K_t B_t) R_t$$

The Kalman filter

- Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$o_t = B_t s_t + \gamma_t$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

- Update

Actual observation

o_t

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

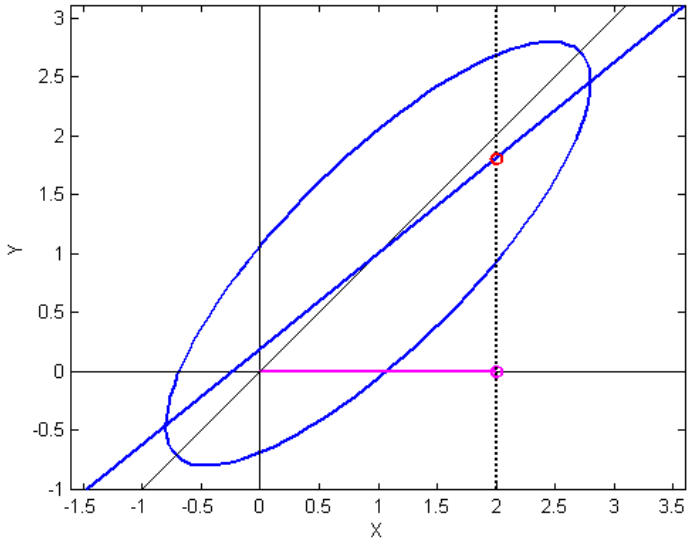
$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - B_t \bar{s}_t \right)$$

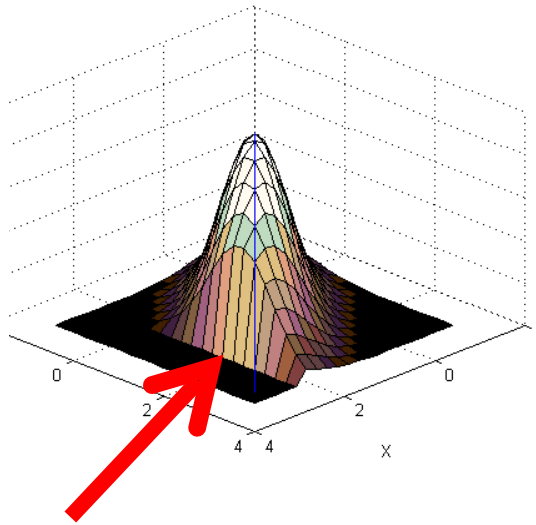
$$\hat{R}_t = \left(I - K_t B_t \right) R_t$$

MAP Recap (for Gaussians)

- If P(x,y) is Gaussian:

$$P(\mathbf{x},\mathbf{y}) = N\left(\begin{bmatrix}\mu_x\\ \mu_y\end{bmatrix}, \begin{bmatrix}C_{xx} & C_{xy}\\ C_{yx} & C_{yy}\end{bmatrix}\right)$$





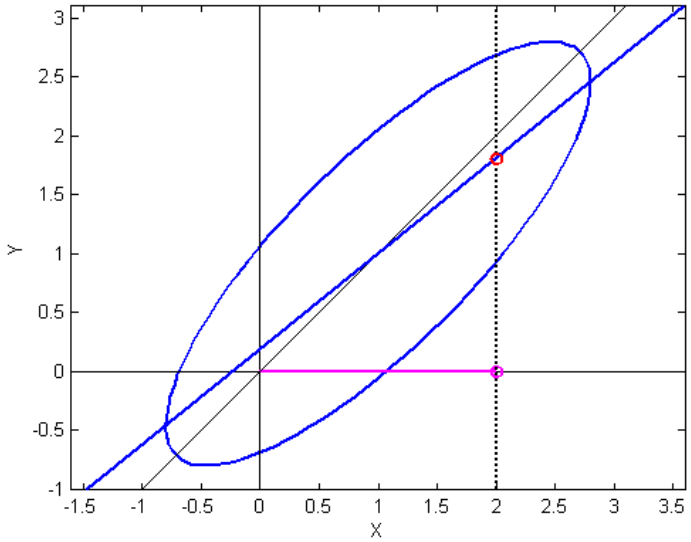
$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x-\mu_x), C_{yy} - C_{yx}^{T}C_{xx}^{-1}C_{xy})$$

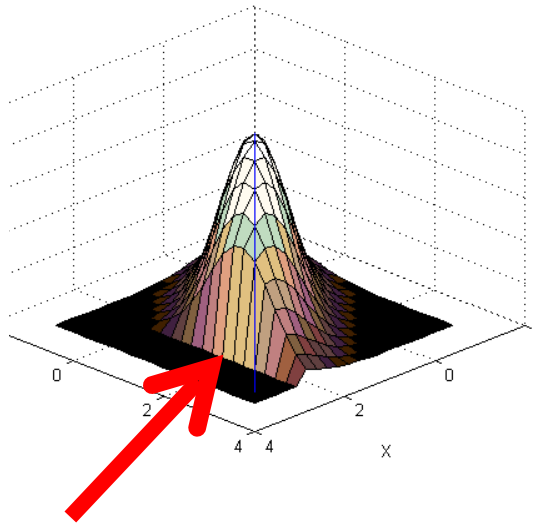
$$\hat{y} = \mu_y + C_{yx}C_{xx}^{-1}(x-\mu_x)$$

MAP Recap: For Gaussians

- If P(x,y) is Gaussian:

$$P(\mathbf{y},\mathbf{x}) = N\left(\begin{bmatrix}\mu_x\\ \mu_y\end{bmatrix}, \begin{bmatrix}C_{xx} & C_{xy}\\ C_{yx} & C_{yy}\end{bmatrix}\right)$$





$$P(y \mid x) = N(\mu_y + C_{yx}C_{xx}^{-1}(x-\mu_x), C_{yy} - C_{yx}^{T}C_{xx}^{-1}C_{xy})$$

$$\hat{y} = \mu_y + C_{yx}C_{xx}^{-1}(x-\mu_x)$$

"Slope" of the line

The Kalman filter

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

- Update

o_t

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

This is the slope of the MAP estimator that predicts s from o

RB^T = C (BRB^T+Θ) = C

This is also called the **Kalman Gain**

The Kalman filter

- Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

We must correct the predicted value of the state after making an observation

$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

o_t

$$K_t = R_t B_t \left(B_t R_t B_t + \Theta_\gamma \right)$$

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - \hat{o}_t \right)$$

The correction is the difference between the actual observation and the predicted observation, scaled by the Kalman Gain

The Kalman filter

- Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$\hat{o}_t = B_t \bar{s}_t + \mu_\gamma$$

o_t

We must correct the predicted value of the state after making an observation

$$K_t = K_t B_t \left(B_t K_t B_t + \Theta_\gamma \right)$$

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - B_t \bar{s}_t - \mu_\gamma \right)$$

The correction is the difference between the actual observation and the predicted observation, scaled by the Kalman Gain

The Kalman filter

- Prediction

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

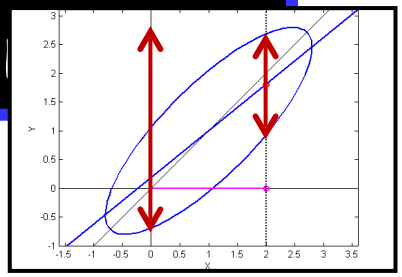
$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- Update:

The uncertainty in state decreases if we observe the data and make a correction

The reduction is a multiplicative "shrinkage" based on Kalman gain a



The Kalman filter

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- Prediction

$$\bar{s}_t = A_t \hat{s}_{t-1} + \mu_\varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- Update:

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

- Update

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - B_t \bar{s}_t - \mu_\gamma \right)$$

$$\hat{R}_t = \left(I - K_t B_t \right) R_t$$

The Kalman Filter

- Very popular for tracking the state of processes
  - Control systems
  - Robotic tracking
    - Simultaneous localization and mapping
  - Radars
  - Even the stock market..
- What are the parameters of the process?

Kalman filter contd.

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$

- Model parameters A and B must be known
  - Often the state equation includes an *additional* driving term: $s_t = A_t s_{t-1} + G_t u_t + \varepsilon_t$
  - The parameters of the driving term must be known
- The initial state distribution must be known

Defining the parameters

- State state must be carefully defined
  - E.g. for a robotic vehicle, the state is an extended vector that includes the current velocity and acceleration
    - $S = [X, dX, d^2X]$
- State equation: Must incorporate appropriate constraints
  - If state includes acceleration and velocity, velocity at next time = current velocity + acc. * time step
  - $St = AS_{t-1} + e$
    - $A = [1\ t\ 0.5t^2;\ 0\ 1\ t;\ 0\ 0\ 1]$

Parameters

- Observation equation:
  - Critical to have accurate observation equation
  - Must provide a valid relationship between state and observations
- Observations typically high-dimensional
  - May have higher or lower dimensionality than state

Problems

$$s_t = f(s_{t-1}, \varepsilon_t)$$

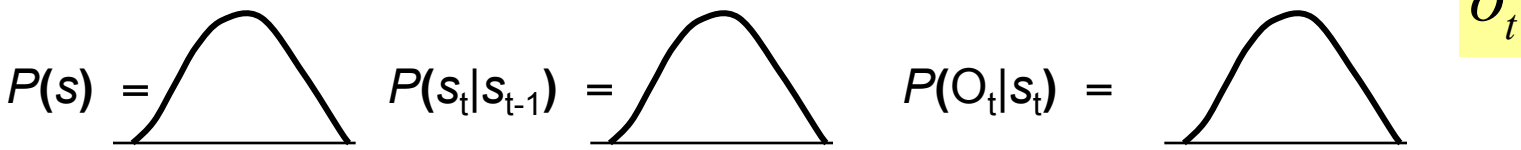
$$o_t = g(s_t, \gamma_t)$$

- f() and/or g() may not be nice linear functions
  - Conventional Kalman update rules are no longer valid
- ε and/or γ may not be Gaussian
  - Gaussian based update rules no longer valid

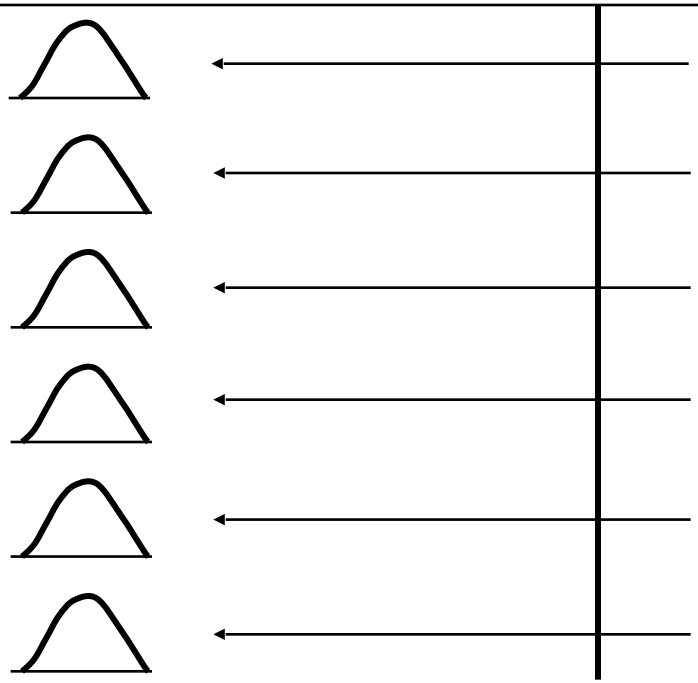
Linear Gaussian Model

$$s_t = A_t s_{t-1} + \varepsilon_t$$

$$o_t = B_t s_t + \gamma_t$$



a priori Transition prob. State output prob



$P(s_0) = P(s)$

$P(s_0 | O_0) = C\, P(s_0)\, P(O_0 | s_0)$

$$P(s_1 | O_0) = \int_{-\infty}^{\infty} P(s_0 | O_0) P(s_1 | s_0) ds_0$$

$P(s_1 | O_{0:1}) = C\, P(s_1 | O_0)\, P(O_1 | s_0)$

$$P(s_2 | O_{0:1}) = \int_{-\infty}^{\infty} P(s_1 | O_{0:1}) P(s_2 | s_1) ds_1$$

$P(s_2 | O_{0:2}) = C\, P(s_2 | O_{0:1})\, P(O_2 | s_2)$

All distributions remain Gaussian

Problems

$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

- Nonlinear f() and/or g() : The Gaussian assumption breaks down
  - Conventional Kalman update rules are no longer valid

The problem with non-linear functions

$$s_t = f(s_{t-1}, \varepsilon_t)$$

$$o_t = g(s_t, \gamma_t)$$

$$P(s_t \mid o_{0:t-1}) = \int_{-\infty}^{\infty} P(s_{t-1} \mid o_{0:t-1}) P(s_t \mid s_{t-1}) ds_{t-1}$$

$$P(s_t \mid o_{0:t}) = CP(s_t \mid o_{0:t-1}) P(o_t \mid s_t)$$

- Estimation requires knowledge of $P(o|s)$
  - Difficult to estimate for nonlinear $g()$
  - Even if it can be estimated, may not be tractable with update loop
- Estimation also requires knowledge of $P(s_t|s_{t-1})$
  - Difficult for nonlinear $f()$
  - May not be amenable to closed form integration

The problem with nonlinearity

$$o_t = g(s_t, \gamma_t)$$

- The PDF may not have a closed form

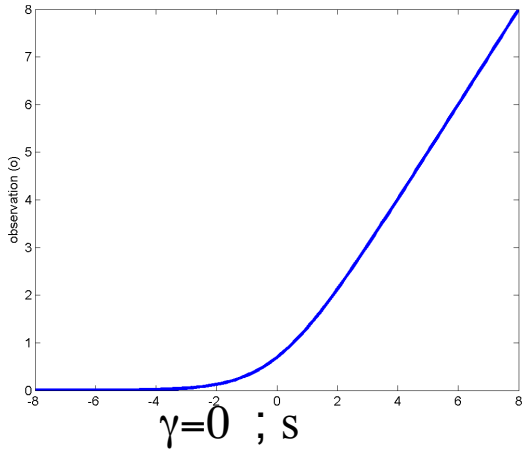
$$P(o_t \mid s_t) = \sum_{\gamma : g(s_t,\gamma)=o_t} \frac{P_\gamma(\gamma)}{|J_{g(s_t,\gamma)}(o_t)|}$$

$$|J_{g(s_t,\gamma)}(o_t)| = \begin{vmatrix} \frac{\partial o_t(1)}{\partial \gamma(1)} & \cdots & \frac{\partial o_t(1)}{\partial \gamma(n)} \\ ? & ? & ? \\ \frac{\partial o_t(n)}{\partial \gamma(1)} & ? & \frac{\partial o_t(n)}{\partial \gamma(n)} \end{vmatrix}$$

- Even if a closed form exists initially, it will typically become intractable very quickly

Example: a simple nonlinearity

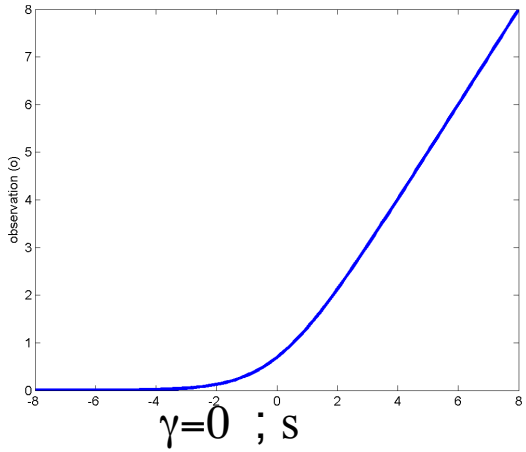
$$o = \gamma + \log(1 + \exp(s))$$



- P(o|s) = ?
  - Assume γ is Gaussian
  - $P(\gamma) = Gaussian(\gamma; \mu_\gamma, \Theta_\gamma)$

Example: a simple nonlinearity

$$o = \gamma + \log(1 + \exp(s))$$



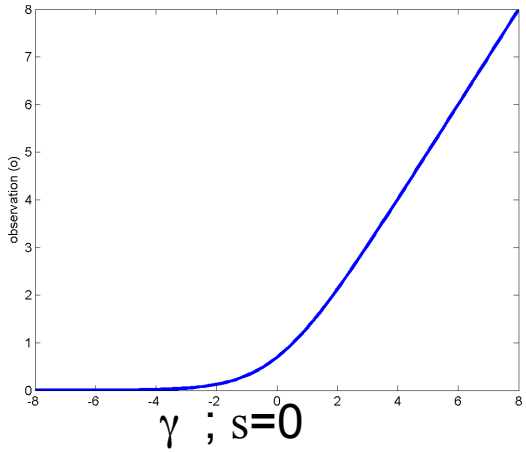
- P(o|s) = ?

$$P(\gamma) = Gaussian(\gamma; \mu_\gamma, \Theta_\gamma)$$

$$P(o \mid s) = Gaussian(o; \mu_\gamma + \log(1 + \exp(s)), \Theta_\gamma)$$

Example: At T=0.

$$o = \gamma + \log(1 + \exp(s))$$



- Assume initial probability P(s) is Gaussian

$$P(s_0) = P_0(s) = Gaussian(s; \bar{s}, R)$$

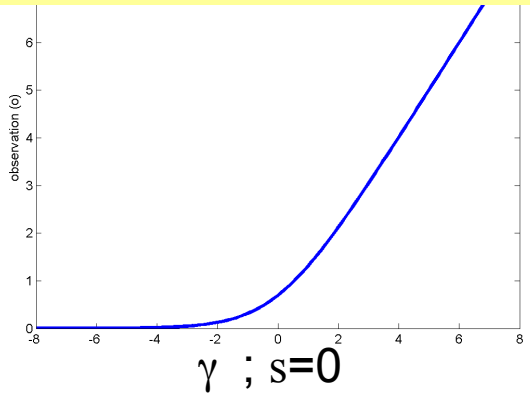
- Update

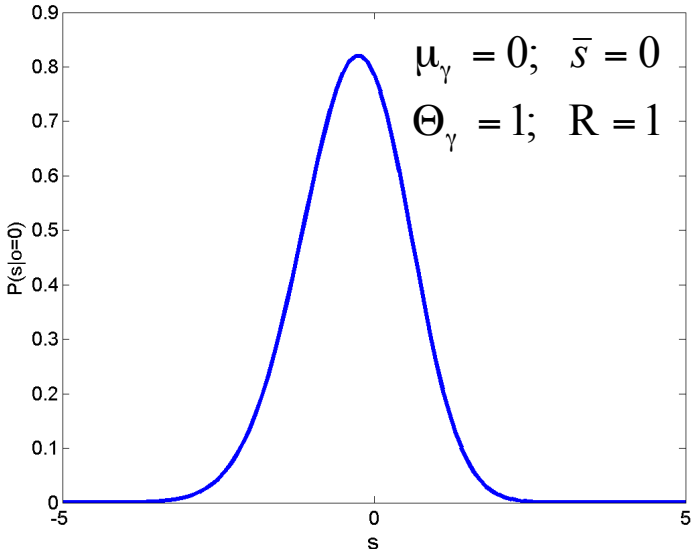
$$P(s_0 \mid o_0) = CP(o_0 \mid s_0)P(s_0)$$

$$P(s_0 \mid o_0) = CGaussian(o; \mu_\gamma + \log(1 + \exp(s_0)), \Theta_\gamma)Gaussian(s_0; \bar{s}, R)$$

UPDATE: At T=0.

$$o = \gamma + \log(1 + \exp(s))$$





$$P(s_0 \mid o_0) = CGaussian(o; \mu_\gamma + \log(1 + \exp(s_0)), \Theta_\gamma) Gaussian(s_0; \bar{s}, R)$$

$$P(s_0 \mid o_0) = C \exp\left(\begin{array}{c} -0.5(\mu_\gamma + \log(1 + \exp(s_0)) - o)^T \Theta_\gamma^{-1} (\mu_\gamma + \log(1 + \exp(s_0)) - o) \\ -0.5(s_0 - \bar{s})^T R^{-1} (s_0 - \bar{s}) \end{array} \right)$$

- = Not Gaussian

Prediction for T = 1

$$s_t = s_{t-1} + \varepsilon \qquad P(\varepsilon) = Gaussian(\varepsilon; 0, \Theta_\varepsilon)$$

- Trivial, linear state transition equation

$$P(s_t \mid s_{t-1}) = Gaussian(s_t; s_{t-1}, \Theta_\varepsilon)$$

- Prediction $$P(s_1 \mid o_0) = \int_{-\infty}^{\infty} P(s_0 \mid o_0) P(s_1 \mid s_0) ds_0$$

$$P(s_1 \mid o_0) = \int_{-\infty}^{\infty} C \exp\left(\begin{array}{c} -0.5(\mu_\gamma + \log(1+\exp(s_0)) - o)^T \Theta_\gamma^{-1} (\mu_\gamma + \log(1+\exp(s_0)) - o) \\ -0.5(s_0 - \bar{s})^T R^{-1} (s_0 - \bar{s}) \end{array} \right) \exp\left((s_1 - s_0)^T \Theta_\varepsilon^{-1} (s_1 - s_0) \right) ds_0$$

- = intractable

Update at T=1 and later

- Update at T=1

$$P(s_t \mid o_{0:t}) = CP(s_t \mid o_{0:t-1})P(o_t \mid s_t)$$

  - Intractable

- Prediction for T=2

$$P(s_t \mid o_{0:t-1}) = \int_{-\infty}^{\infty} P(s_{t-1} \mid o_{0:t-1})P(s_t \mid s_{t-1})ds_{t-1}$$

  - Intractable

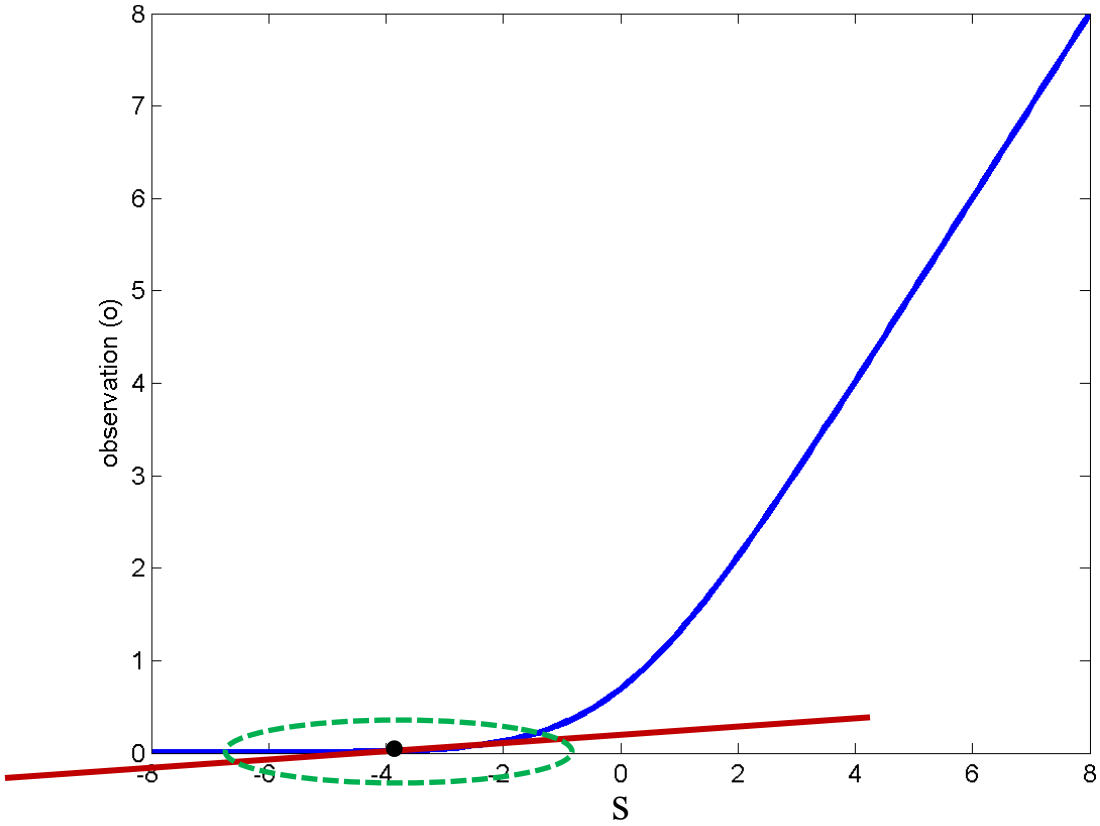
The State prediction Equation

$$s_t = f(s_{t-1}, \varepsilon_t)$$

- Similar problems arise for the state prediction equation

- $P(s_t|s_{t-1})$ may not have a closed form
- Even if it does, it may become intractable within the prediction and update equations
  - Particularly the prediction equation, which includes an integration operation

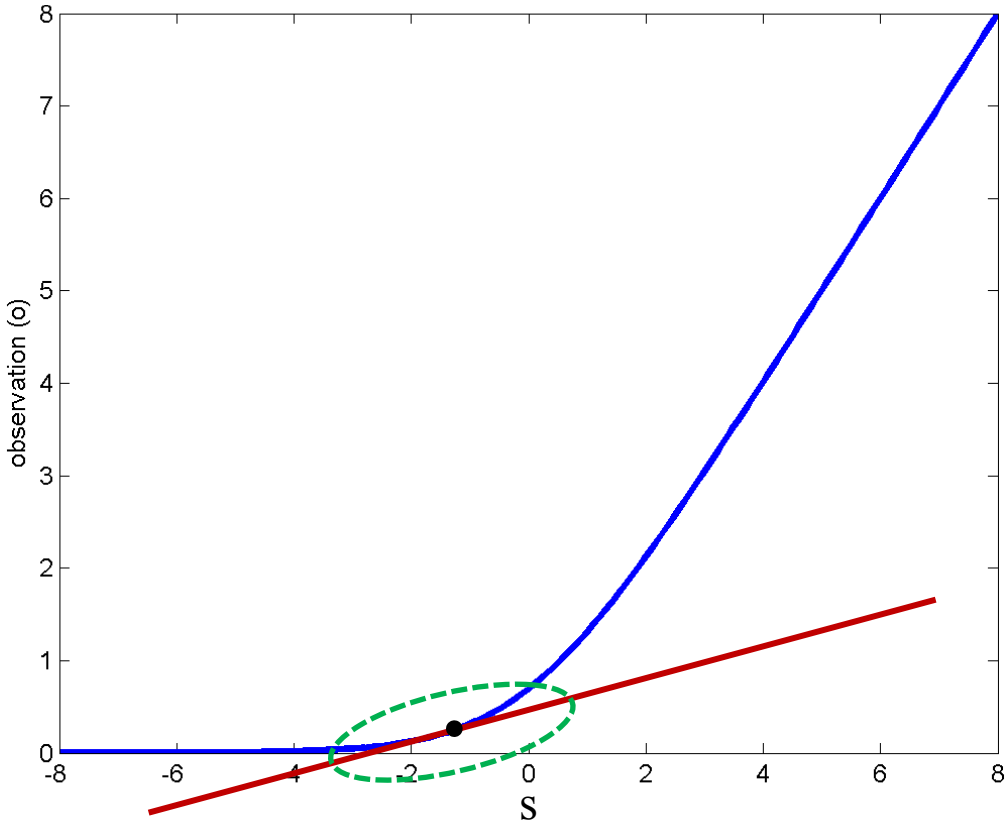
Simplifying the problem: Linearize



$$o = \gamma + \log(1 + \exp(s))$$

- The *tangent* at any point is a good *local* approximation if the function is sufficiently smooth

Simplifying the problem: Linearize



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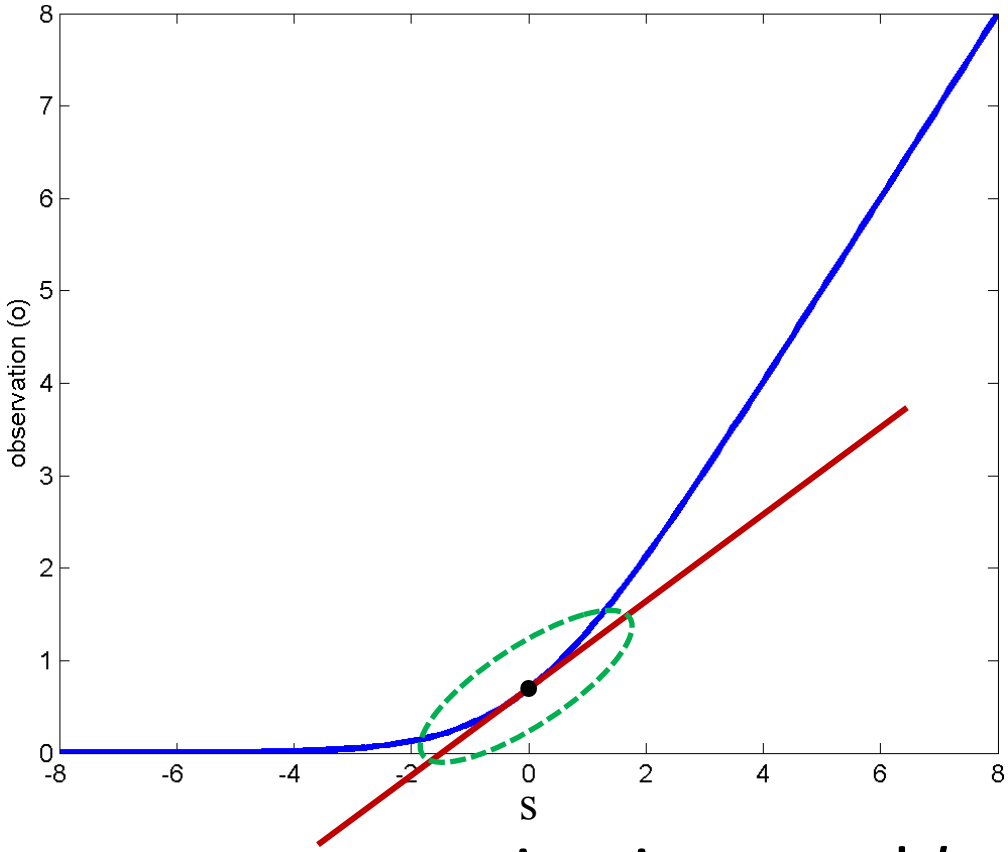

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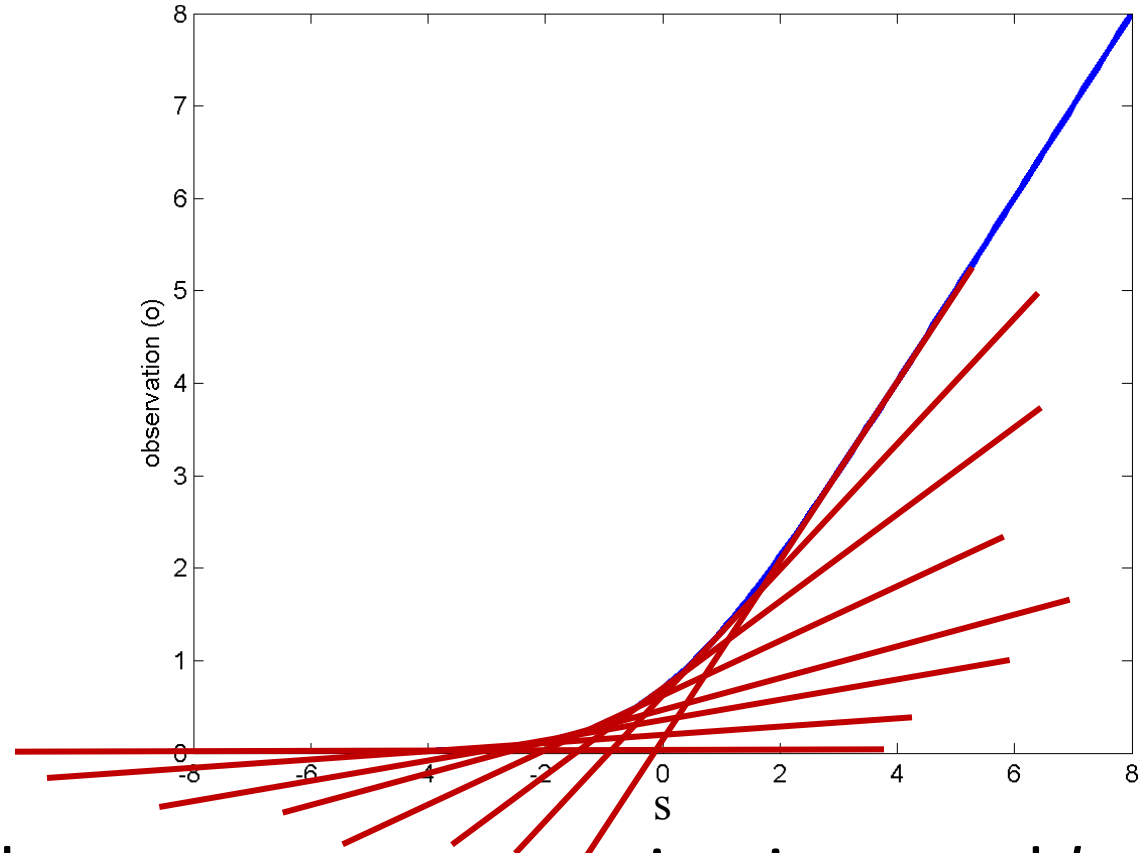
Simplifying the problem: Linearize



- The *tangent* at any point is a good *local* approximation if the function is sufficiently smooth

Linearizing the observation function

$$P(s_t \mid o_{0:t-1}) = Gaussian(\bar{s}_t, R_t)$$

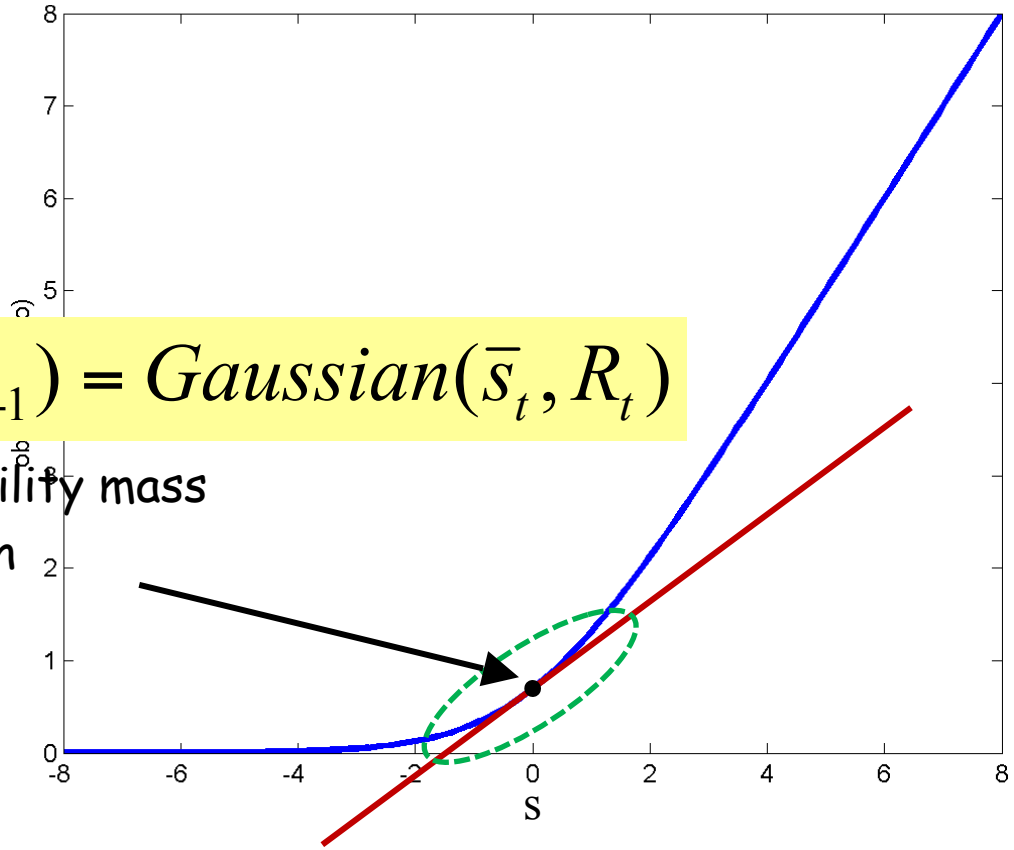
$$o = \gamma + g(s) \quad \Rightarrow \quad o \approx \gamma + g(\bar{s}_t) + J_g(\bar{s}_t)(s - \bar{s}_t)$$

- Simple first-order Taylor series expansion
  - J() is the Jacobian matrix
    - Simply a determinant for scalar state
- Expansion around *current* predicted *a priori* (or predicted) mean of the state
  - Linear approximation changes with time

Most probability is in the low-error region

$$P(s_t \mid o_{0:t-1}) = Gaussian(\bar{s}_t, R_t)$$

Most probability mass close to mean



- $P(s_t)$ is small where approximation error is large
  - Most of the probability mass of *s* is in low-error regions

Linearizing the observation function

$$P(s_t \mid o_{0:t-1}) = Gaussian(\bar{s}_t, R_t)$$

$$o = \gamma + g(s) \quad \Rightarrow \quad o \approx \gamma + g(\bar{s}_t) + J_g(\bar{s}_t)(s - \bar{s}_t)$$

- With the linearized approximation the system becomes "linear"
- The observation PDF becomes Gaussian

$$P(\gamma) = Gaussian(\gamma; 0, \Theta_\gamma)$$

$$P(o \mid s) = Gaussian(o; g(\bar{s}) + J_g(\bar{s})(s - \bar{s}), \Theta_\gamma)$$

The state equation?

$$s_t = f(s_{t-1}) + \varepsilon \qquad P(\varepsilon) = Gaussian(\varepsilon; 0, \Theta_\varepsilon)$$

- Again, direct use of f() can be disastrous
- Solution: Linearize

$$P(s_{t-1} \mid o_{0:t-1}) = Gaussian(s_{t-1}; \hat{s}_{t-1}, \hat{R}_{t-1})$$

$$s_t = f(s_{t-1}) + \varepsilon \quad \Rightarrow \quad s_t \approx \varepsilon + f(\hat{s}_{t-1}) + J_f(\hat{s}_{t-1})(s_{t-1} - \hat{s}_{t-1})$$

- Linearize around the mean of the updated distribution of s at t-1
  - Converts the system to a linear one

Linearized System

$$o = \gamma + g(s)$$

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o \approx \gamma + g(\bar{s}_t) + J_g(\bar{s}_t)(s - \bar{s}_t)$$

$$s_t \approx \varepsilon + f(\hat{s}_{t-1}) + J_f(\hat{s}_{t-1})(s_{t-1} - \hat{s}_{t-1})$$

- Now we have a simple time-varying linear system
- Kalman filter equations directly apply

The Extended Kalman filter

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \gamma$$

- Prediction

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$A_t = J_f(\hat{s}_{t-1})$$

$$B_t = J_g(\bar{s}_t)$$

Jacobians used in Linearization

- Update

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - g(\bar{s}_t) \right)$$

$$\hat{R}_t = \left(I - K_t B_t \right) R_t$$

Assuming ε and γ are 0 mean for simplicity

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$o_t = g(s_t) + \gamma$$

The predicted state at time t is obtained simply by propagating the estimated state at t-1 through the state dynamics equation

$$\hat{s}_t = \bar{s}_t + K_t\left(o_t - g(\bar{s}_t)\right)$$

$$\hat{R}_t = \left(I - K_t B_t\right)R_t$$

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$A_t = J_f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$B_t = J_g(\bar{s}_t)$$

Uncertainty of prediction.
The variance of the predictor = variance of ε_t + variance of As_{t-1}

A is obtained by linearizing f()

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

- Update

$$B_t = J_g(\bar{s}_t)$$

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

The Kalman gain is the slope of the MAP estimator that predicts s from o

$RBT = C_{so}$, $(BRB^T+\Theta) = C_{oo}$

B is obtained by linearizing g()

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1}) \qquad o_t = g(s_t) + \varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

We can also predict the observation from the predicted state using the observation equation

$$s_t = \bar{s}_t + K_t(o_t - g(\bar{s}_t))$$

$$\hat{R}_t = (I - K_t B_t) R_t \qquad \bar{o}_t = g(\bar{s}_t)$$

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

We must correct the predicted value of the state after making an observation

$$\hat{s}_t = \bar{s}_t + K_t\left(o_t - g(\bar{s}_t)\right)$$

$$\bar{o}_t = g(\bar{s}_t)$$

The correction is the difference between the actual observation and the predicted observation, scaled by the Kalman Gain

The Extended Kalman filter

- Prediction

$$s_t = f(s_{t-1}) + \varepsilon$$

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$o_t = g(s_t) + \varepsilon$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$B_t = J_g(\bar{s}_t)$$

The uncertainty in state decreases if we observe the data and make a correction

The reduction is a multiplicative "shrinkage" based on Kalman gain and B

The Extended Kalman filter

$$s_t = f(s_{t-1}) + \varepsilon$$

$$o_t = g(s_t) + \varepsilon$$

- Prediction

$$\bar{s}_t = f(\hat{s}_{t-1})$$

$$R_t = \Theta_\varepsilon + A_t \hat{R}_{t-1} A_t^T$$

$$A_t = J_f(\hat{s}_{t-1})$$

$$B_t = J_g(\bar{s}_t)$$

- Update

$$K_t = R_t B_t^T \left(B_t R_t B_t^T + \Theta_\gamma \right)^{-1}$$

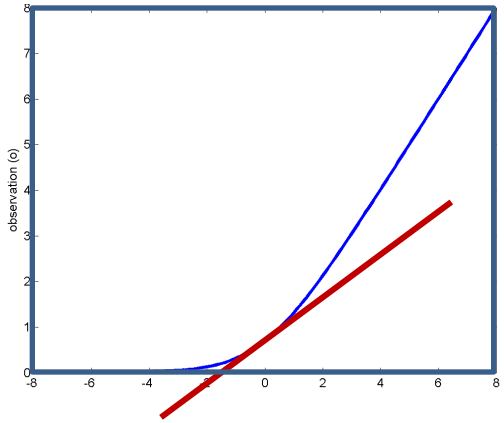
$$\hat{s}_t = \bar{s}_t + K_t \left(o_t - g(\bar{s}_t) \right)$$

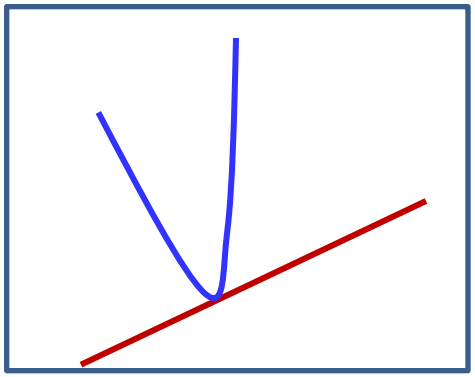
$$\hat{R}_t = \left(I - K_t B_t \right) R_t$$

EKFs

- EKFs are probably the most commonly used algorithm for tracking and prediction
  - Most systems are non-linear
  - Specifically, the relationship between state and observation is usually nonlinear
  - The approach can be extended to include non-linear functions of noise as well
- The term “Kalman filter” often simply refers to an *extended* Kalman filter in most contexts.
- But..

EKFs have limitations





- If the non-linearity changes too quickly with s, the linear approximation is invalid
  - Unstable
- The estimate is often biased
  - The true function lies entirely on one side of the approximation
- Various extensions have been proposed:
  - Invariant extended Kalman filters (IEKF)
  - Unscented Kalman filters (UKF)

Conclusions

- HMMs are predictive models
- Continuous-state models are simple extensions of HMMs
  - Same math applies
- Prediction of linear, Gaussian systems can be performed by Kalman filtering
- Prediction of non-linear, Gaussian systems can be performed by Extended Kalman filtering