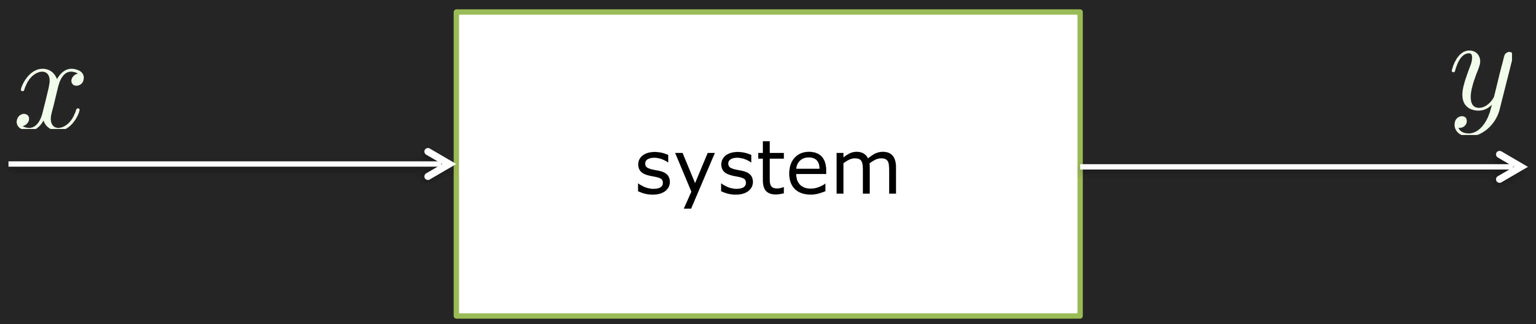
Introduction to Compressive Sensing

Aswin Sankaranarayanan

$$y = \Phi x$$

Is this system linear ?

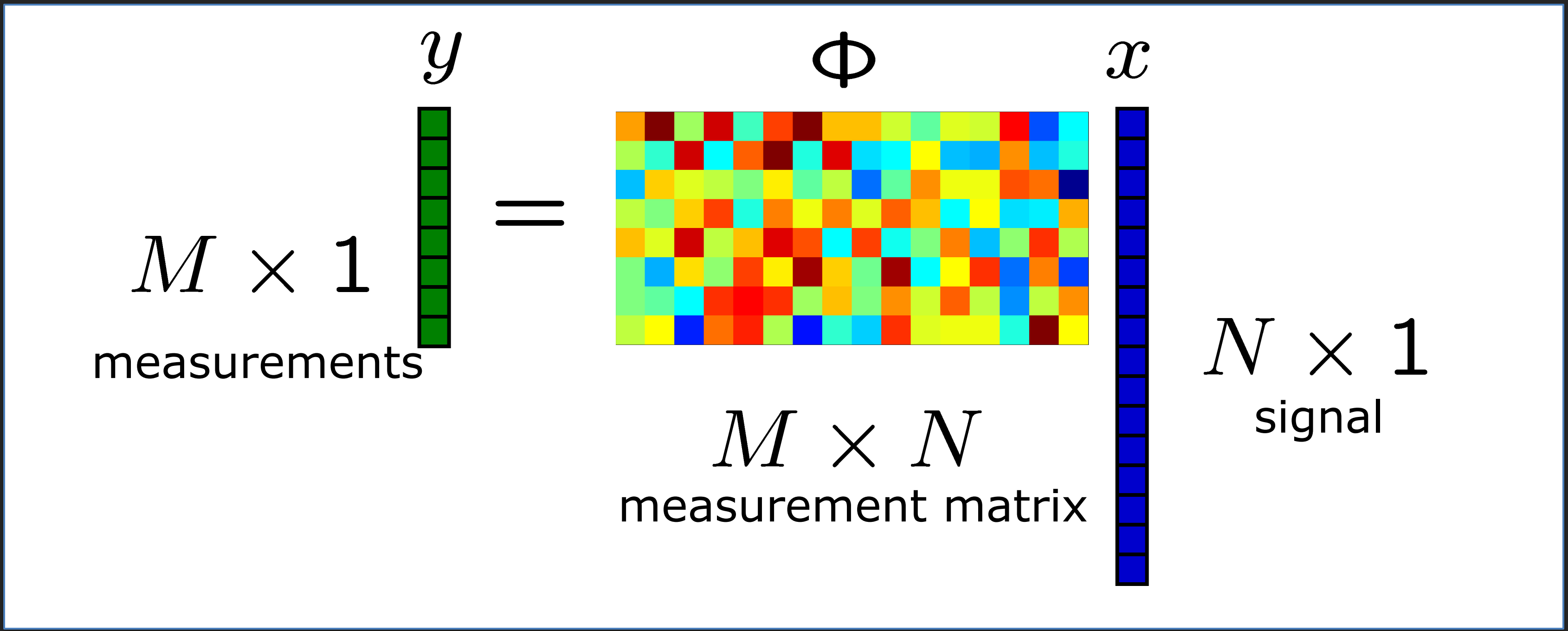


$$y = \Phi x$$

Is this system linear ?

Given *y*, can we recovery *x* ?

Under-determined problems



If M < N, then the system is information lossy

Image credit
Graeme Pope

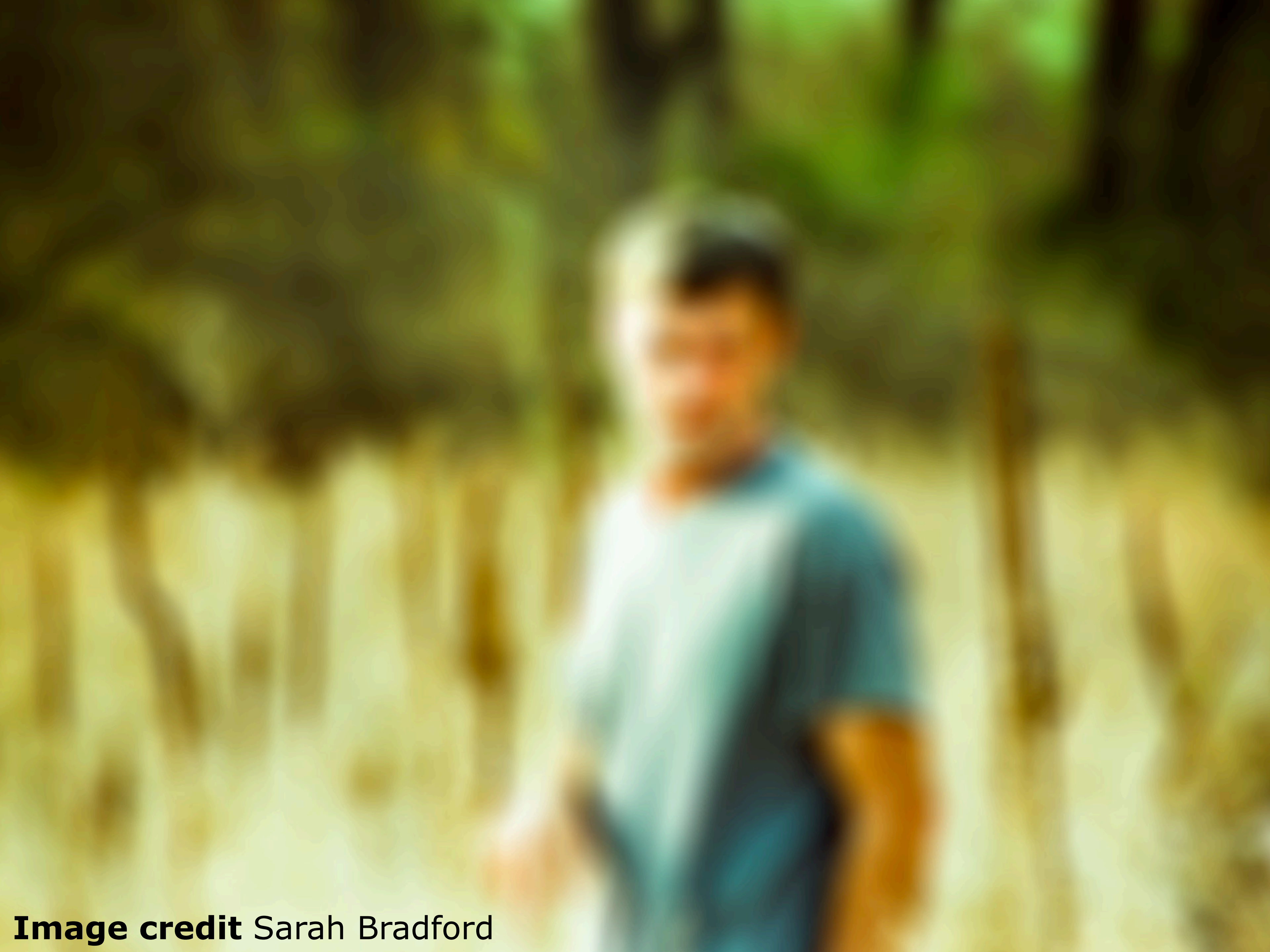
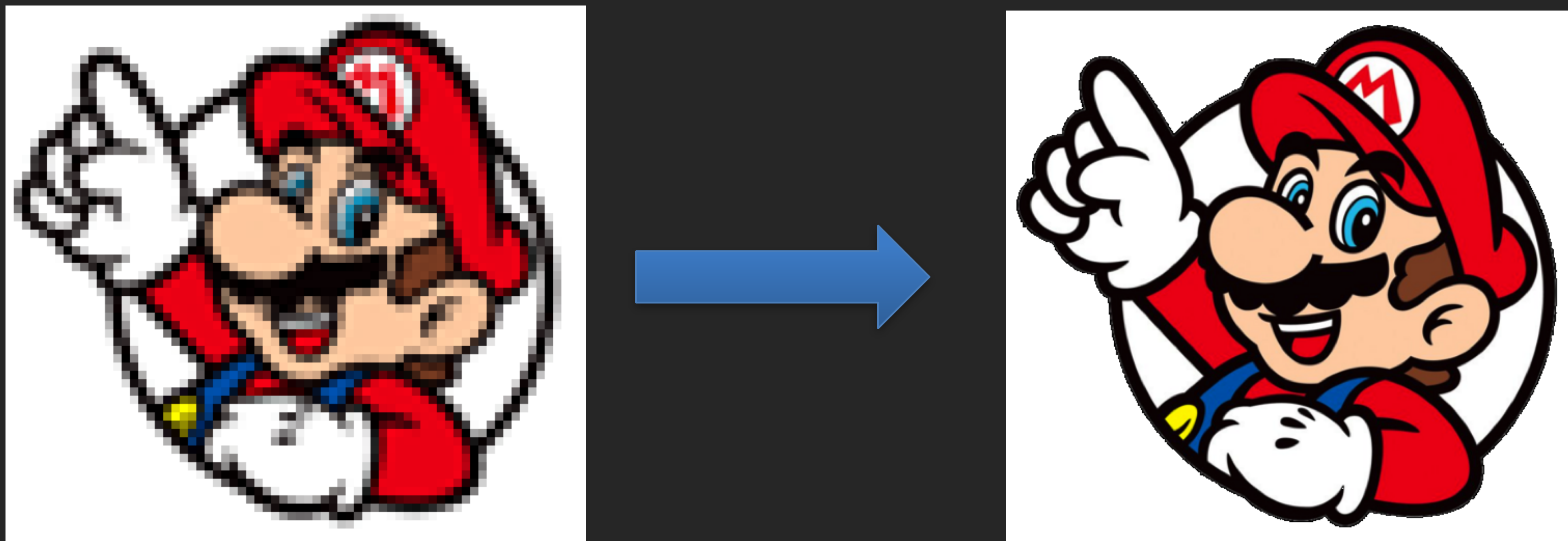


Image credit Sarah Bradford

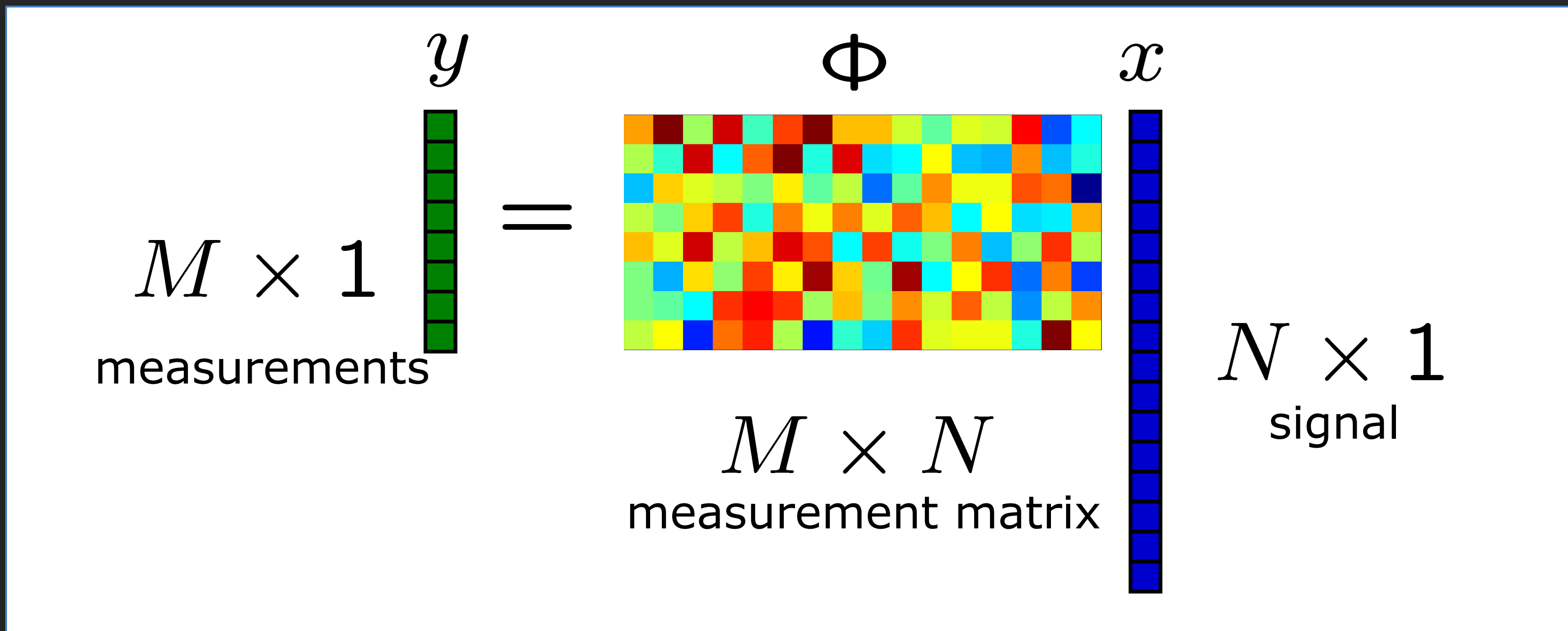
Super-resolution



Can we increase the resolution of this image ?

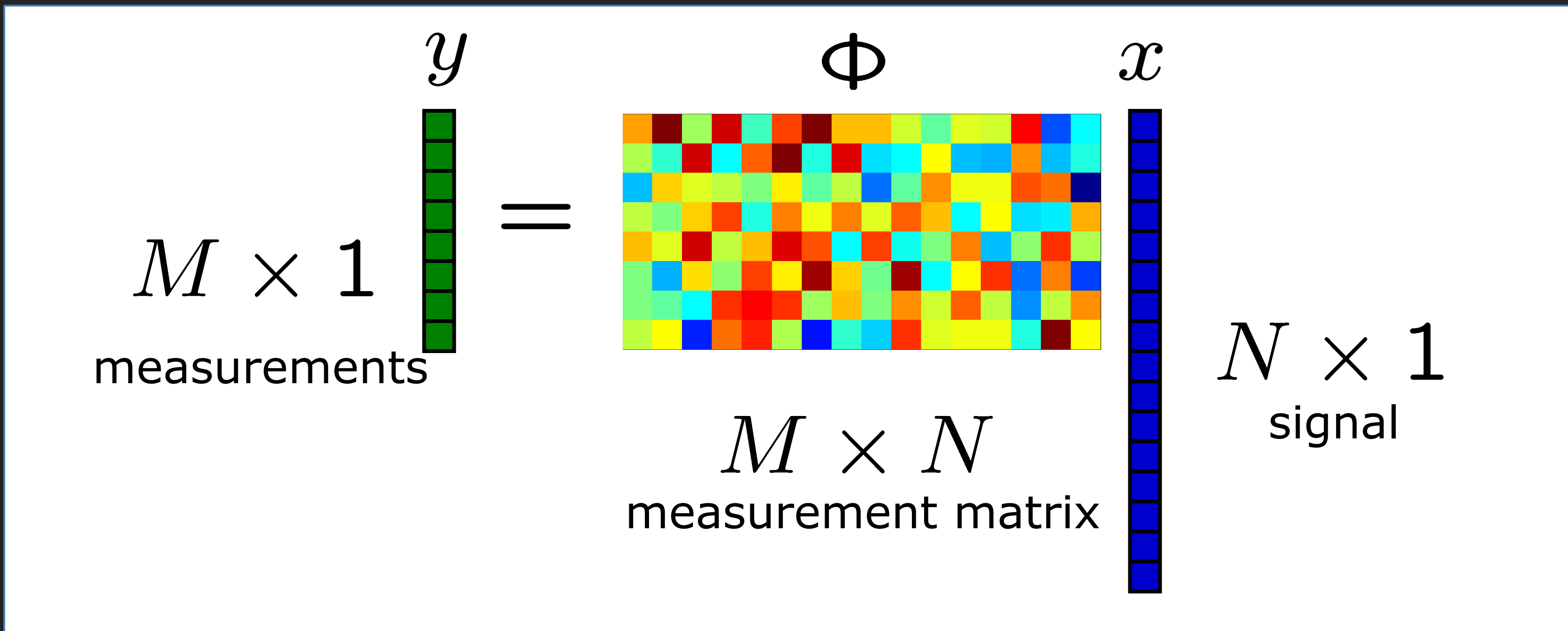
[(Link: Depixelizing pixel art)]

Under-determined problems



Fewer knowns than unknowns!

Under-determined problems



Fewer knowns than unknowns!

An infinite number of solutions to such problems

Credit: Rob Fergus and Antonio Torralba

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Is there anything we can do about this ?

Complete the sentences

I cnt blv I m bl t rd ths sntnc.

Gv m frdm, r gv m dth

Hy, I m slvng n ndr-dtrmnd lnr systm.

how: ?

Complete the matrix

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

how: ?

Complete the image

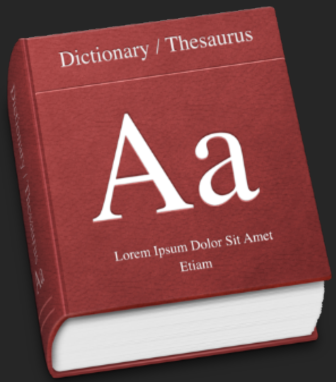
Model ?

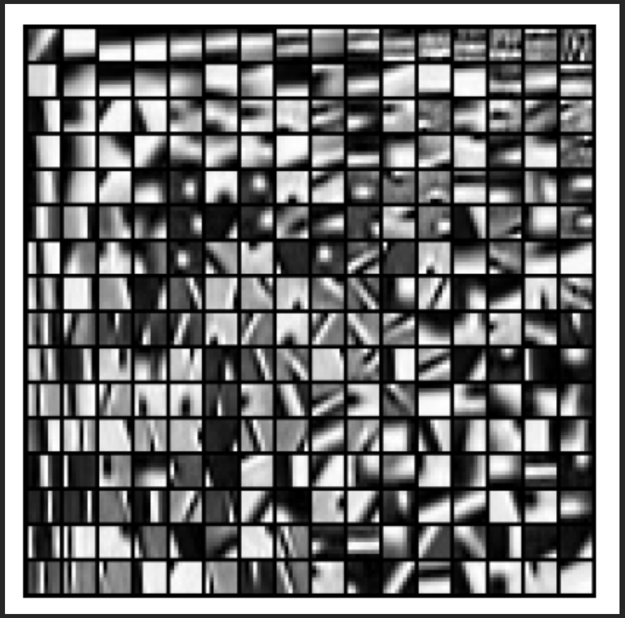
Dictionary of visual words

I cnt blv I m bl t rd ths sntnc.

Shrlck s th vc f th drgn

Hy, I m slvng n ndr-dtrmnd lnr systm.





Dictionary of visual words

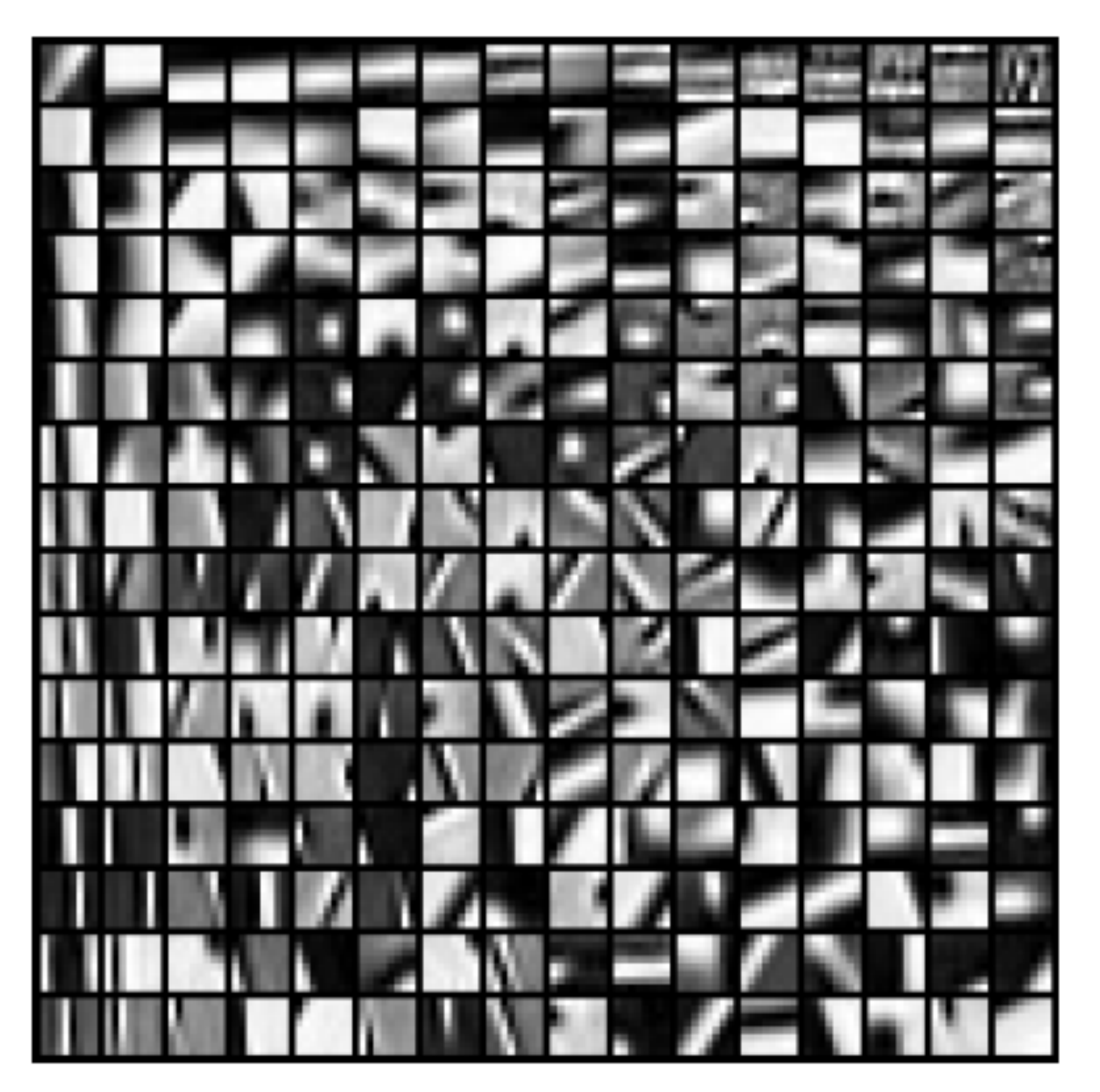
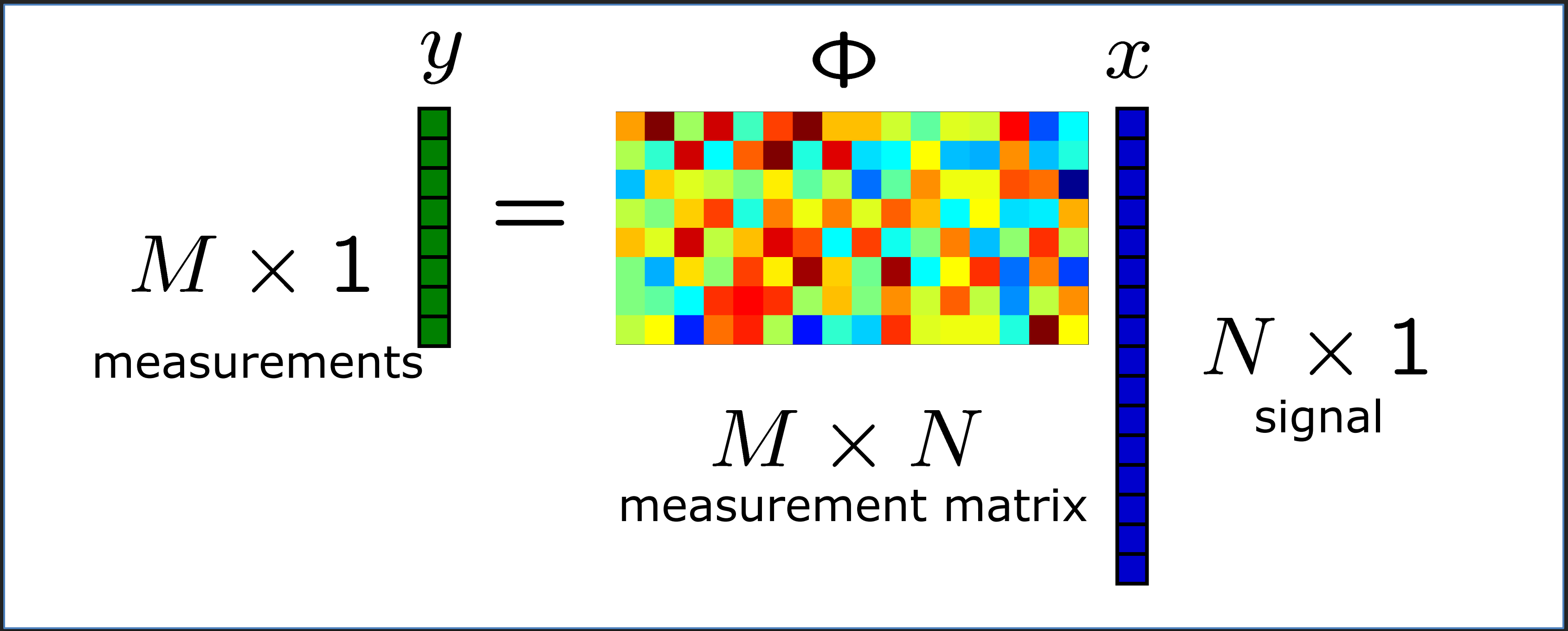



Image credit
Graeme Pope


Image credit
Graeme Pope

Result
Studer, Baraniuk, ACHA 2012

Compressive Sensing



A toolset to solve under-determined systems by exploiting additional structure/models on the signal we are trying to recover.

Key Theoretical Ideas in Compressive Sensing

Linear Inverse Problems

- Many classic problems in computer can be posed as linear inverse problems
- Notation
  - **Signal** of interest $x \in \mathbb{R}^N$
  - **Observations** $y \in \mathbb{R}^M$
  - Measurement model $y = \Phi x + e$

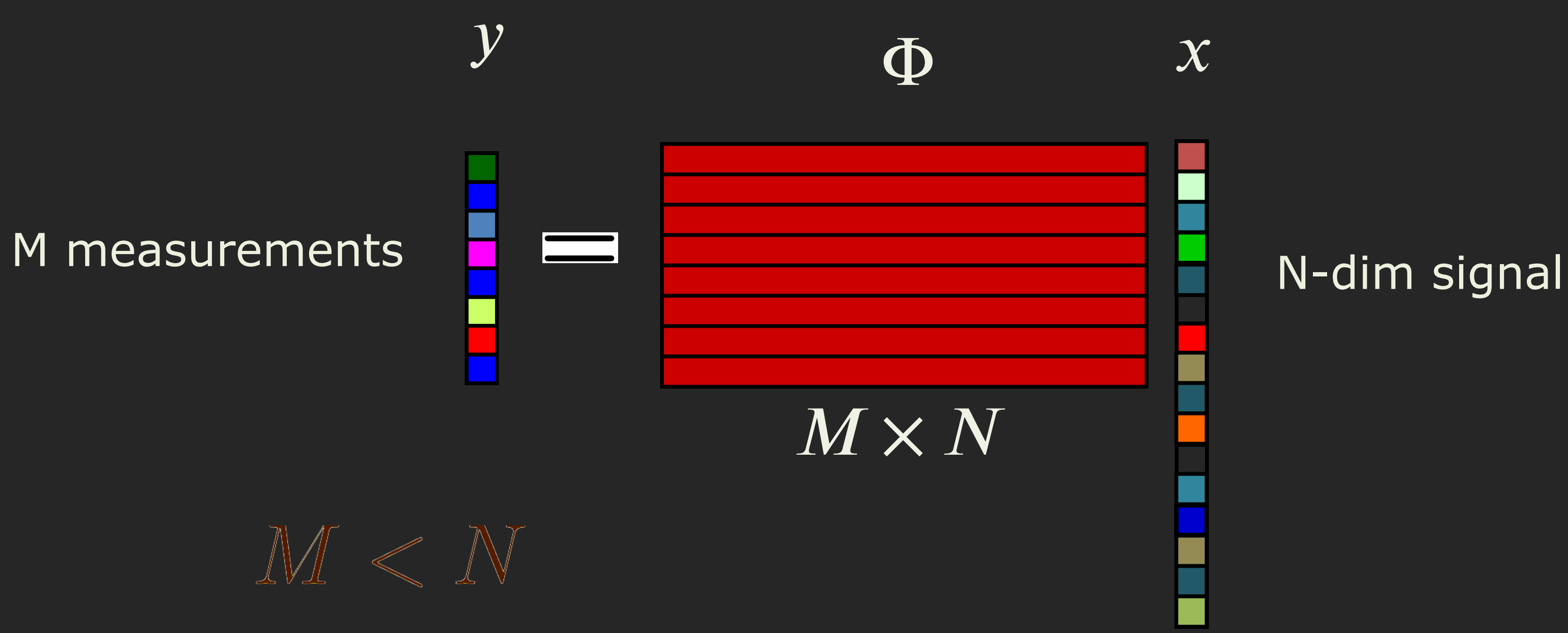
measurement matrix

measurement noise

- Problem definition: given y, recover x

Linear Inverse Problems

$$y = \Phi x$$

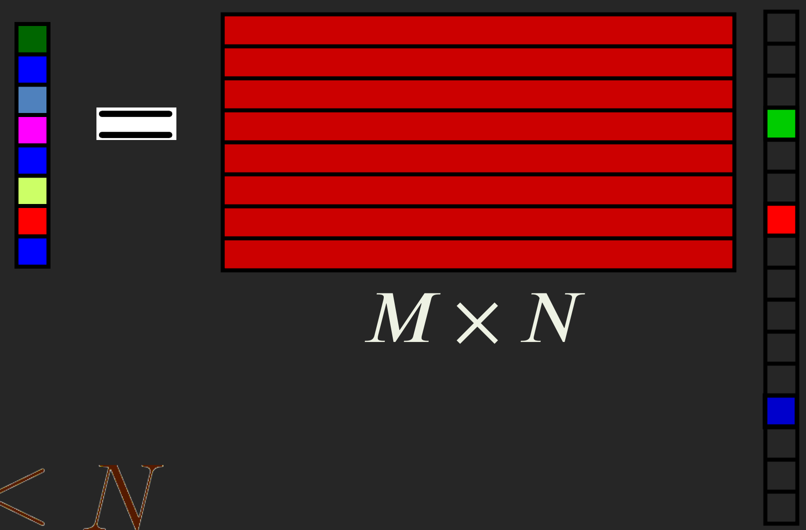


Measurement matrix has a *(N-M)* dimensional **null-space**

Solution is no longer **unique**

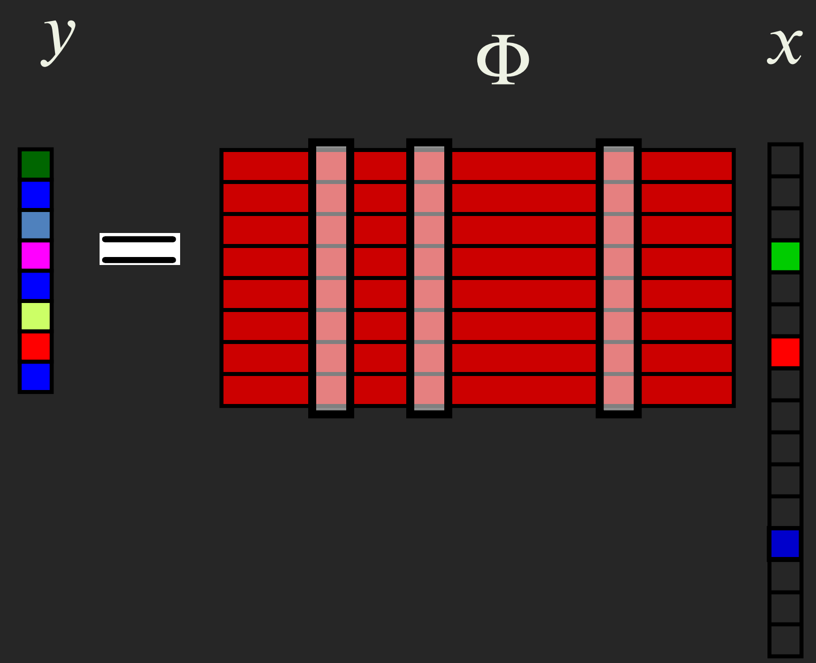
Sparse Signals

$$y = \Phi x$$



$$K < M \le N$$

How Can It Work?



- Matrix Φ
  not full rank…

  $M < N$

  … and so
  loses information in general

- **But we are only interested in recovering sparse signals**

Two Key Ideas

- Idea 1 --- An invertible mapping on the space of sparse signals!!!

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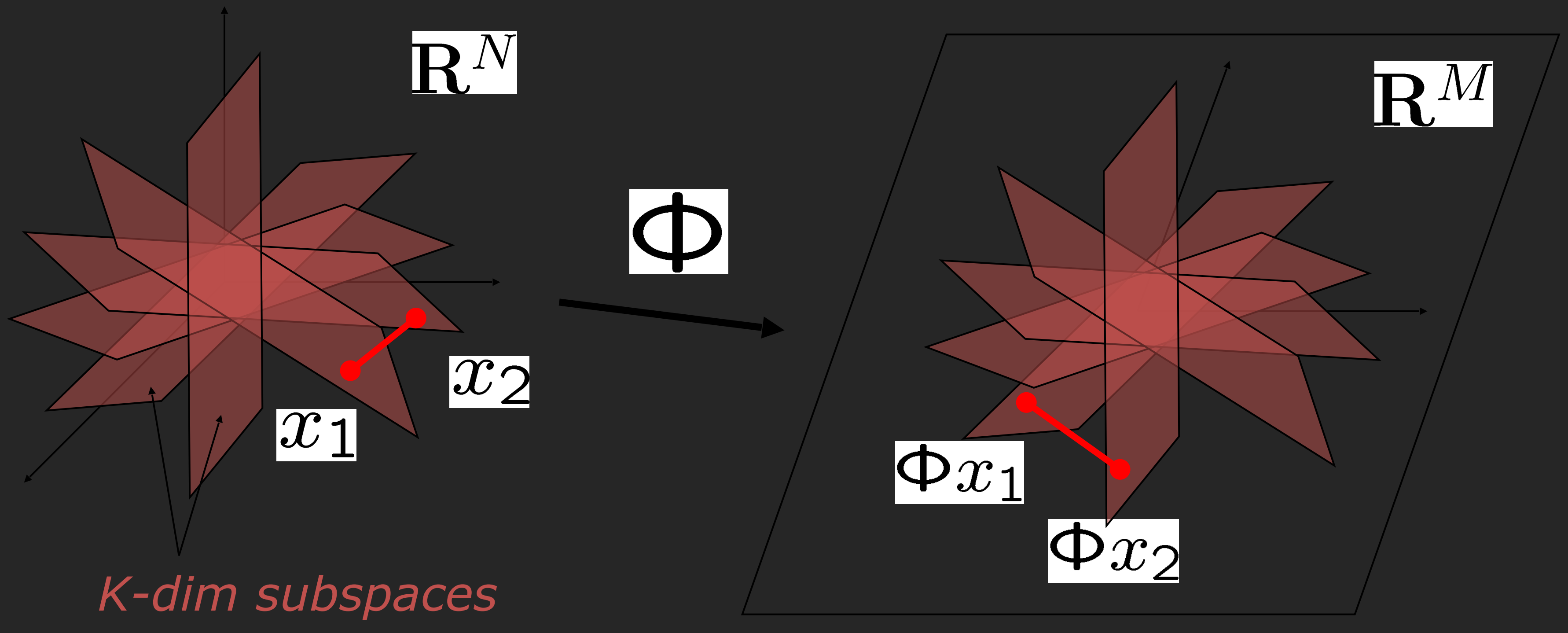
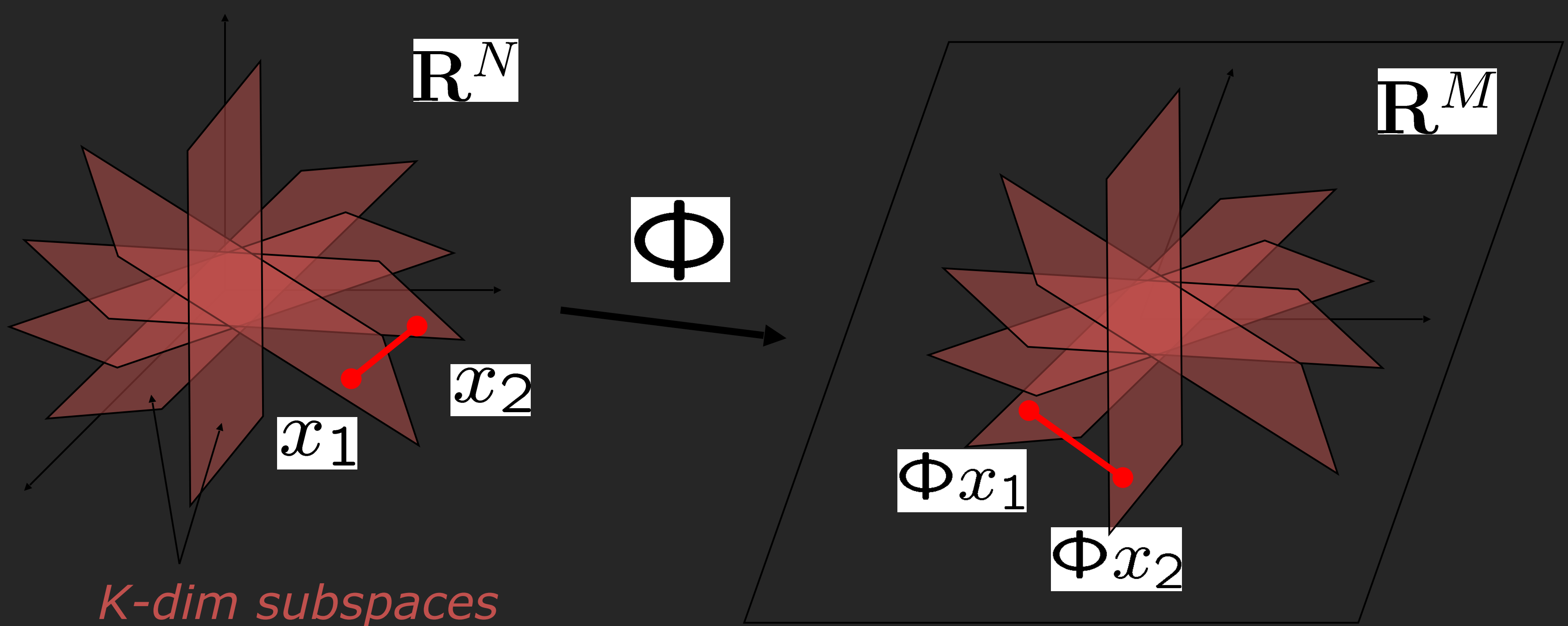
- Design Φ such that no two sparse signals x_1 and x_2 such that $\Phi x_1 = \Phi x_2$



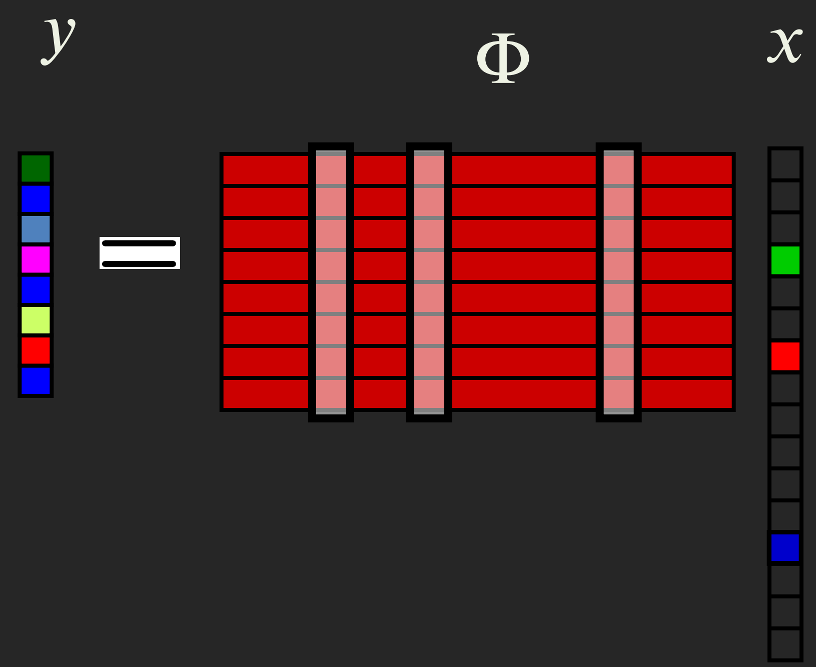
Restricted Isometry Property (RIP)

- RIP of order *2K* implies: for all *K*-sparse x_1 and x_2

$$(1-\delta_{2K}) \leq \frac{\|\Phi x_1 - \Phi x_2\|_2^2}{\|x_1 - x_2\|_2^2} \leq (1+\delta_{2K})$$



How Can It Work?



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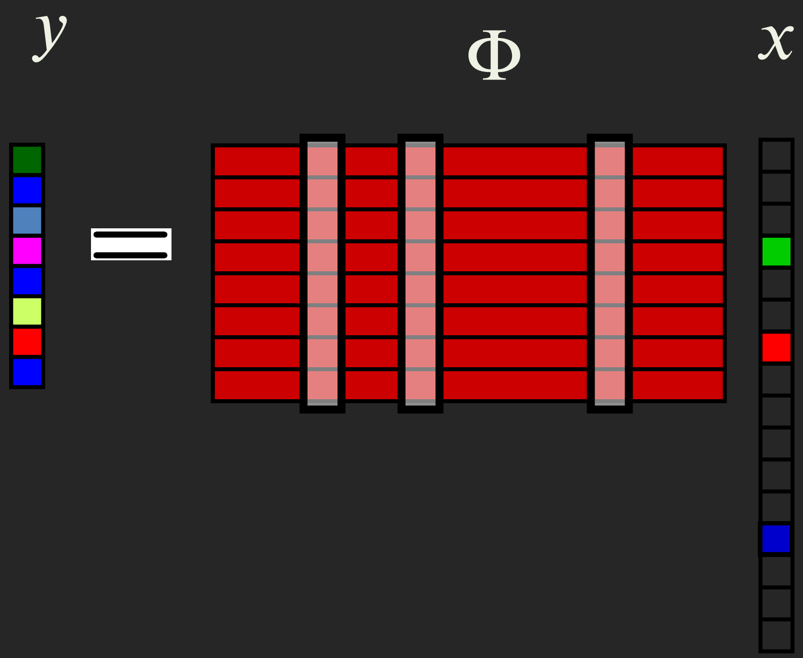
- **Design** Φ so that each of its Mx$2K$ submatrices are full rank (RIP)

How Can It Work?



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- **Design** Φ so that each of its Mx$2K$ submatrices are full rank (RIP)

- Random measurements provide RIP with

$$M \sim K \log(N/K)$$

Two Key Ideas

- Idea 1 --- An invertible mapping on the space of sparse signals!!!

- Idea 2 --- Recovery of signals: use sparse priors!

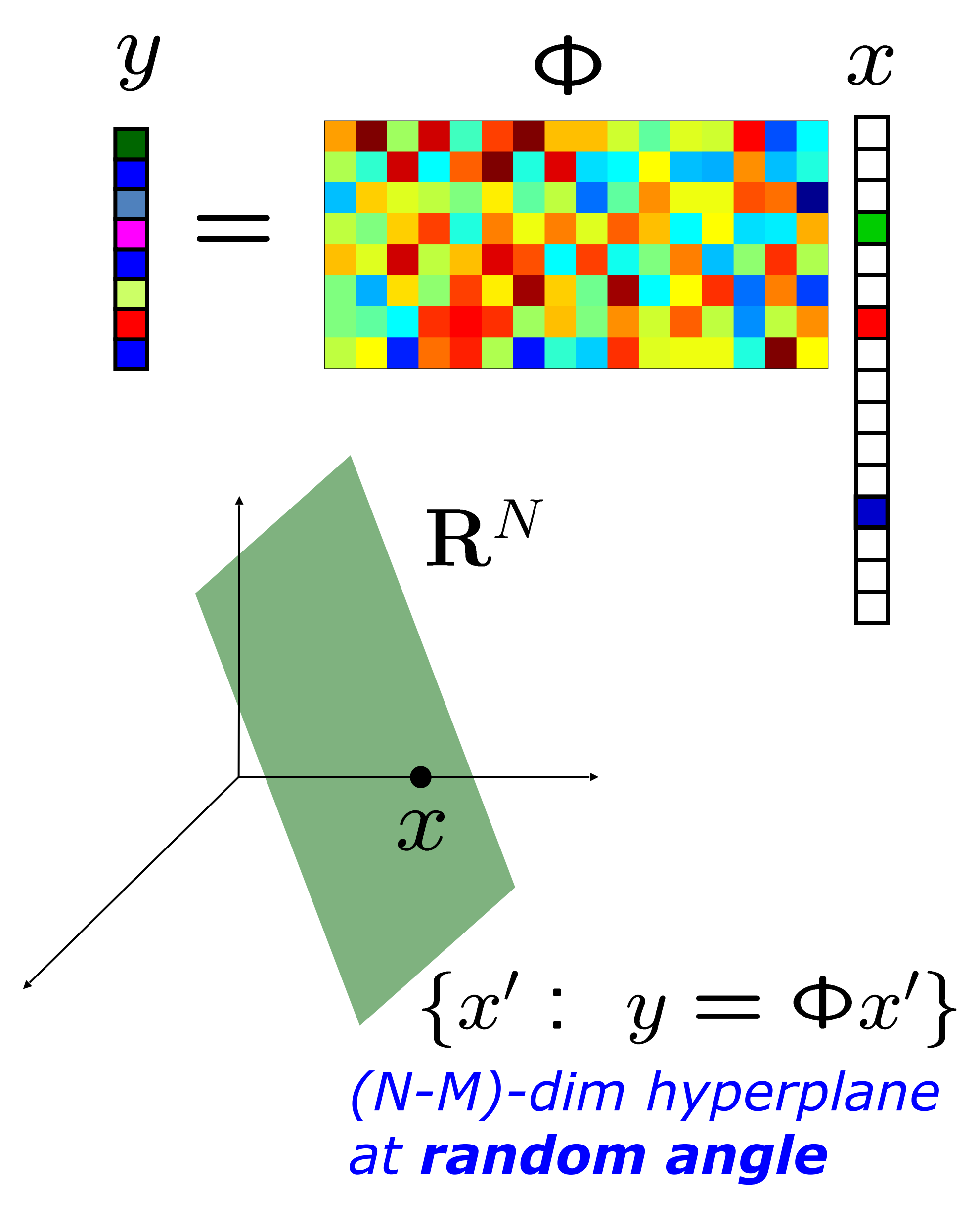
CS Signal Recovery

- Random projection Φ not full rank

- Recovery problem: given $y = \Phi x$ find x

- **Null space**

- Search in null space for the "sparsest" x

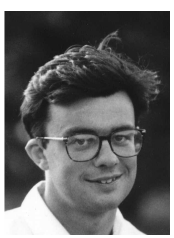


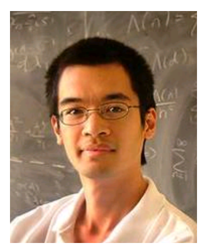
ℓ_1 Signal Recovery

- Recovery: given $y = \Phi x$
  (ill-posed inverse problem) find x (sparse)

- Optimization: $\widehat{x} = \arg\min_{y=\Phi x} \|x\|_1$

- **Convexify** the ℓ_0 optimization






Candes Romberg Tao Donoho

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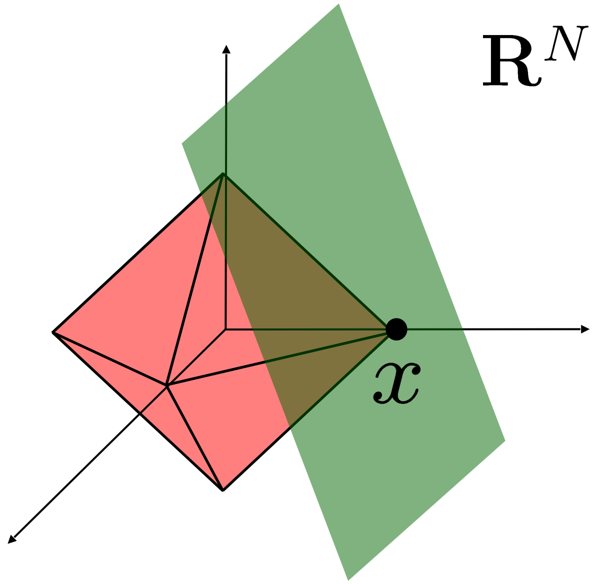
- **Polynomial time** alg
  (linear programming)



Compressive Sensing

Let. $y = \Phi x_0 + e$

$$\widehat{x} = \arg\min_{x} \|x\|_1 \quad s.t. \quad \|y - \Phi x\|_2 \leq \|e\|$$

If Φ satisfies RIP with $\delta_{2K} \leq \sqrt{2} - 1$,

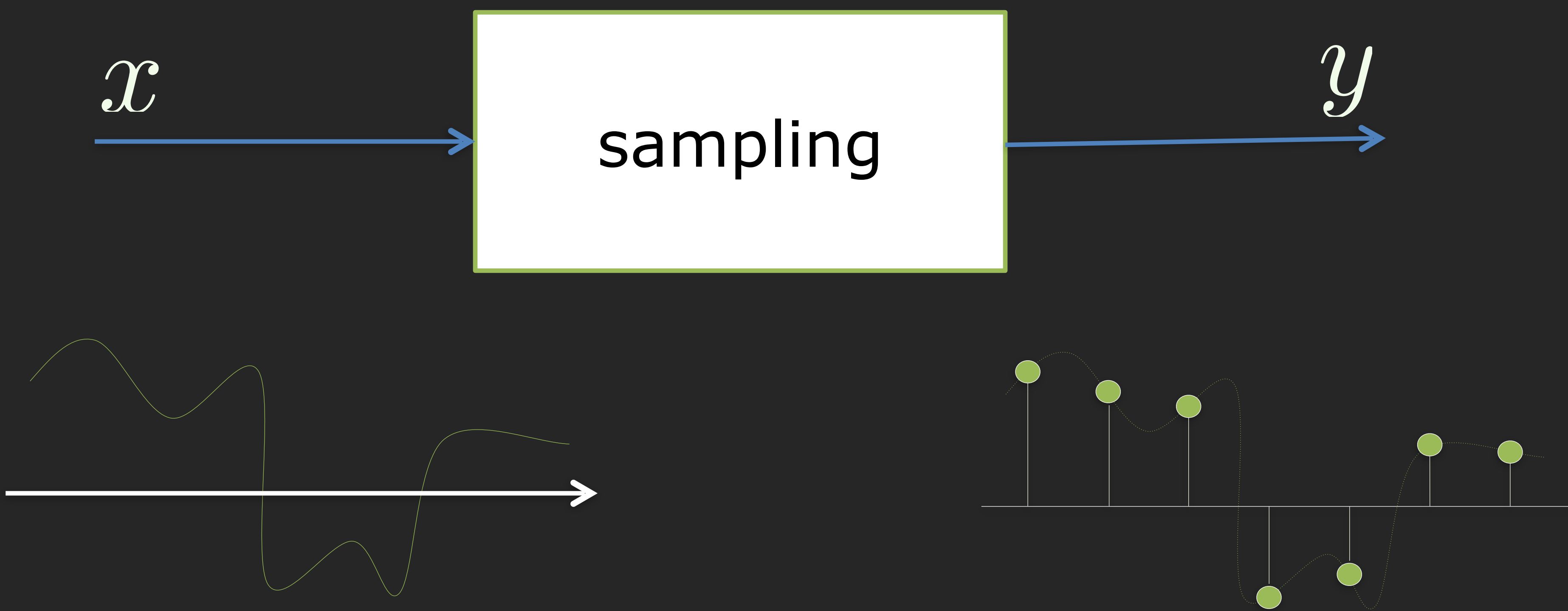
Then

$$\|\hat{x} - x_0\|_1 \leq C_1 \|e\|_2 + C_2 \|x_0 - x_{0,K}\|_2 / \sqrt{K}$$

Best K-sparse approximation

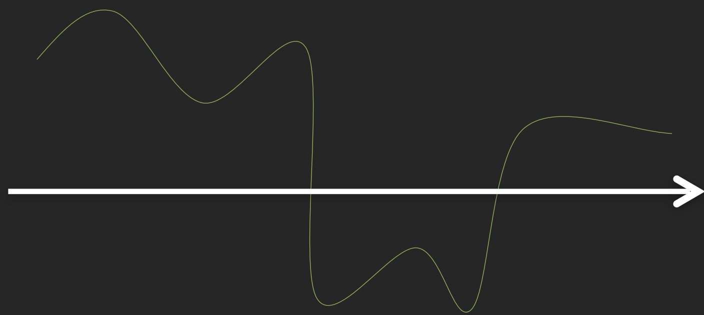
modern sensors are linear systems!!!

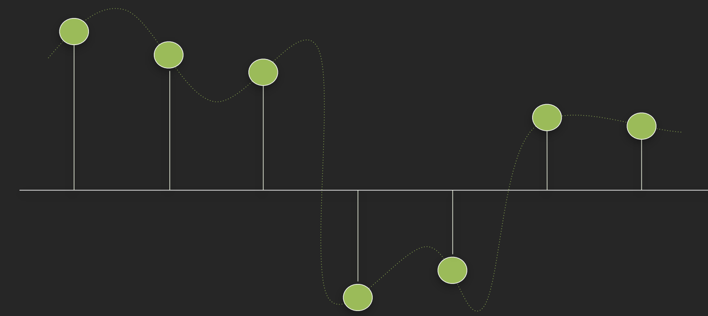
Sampling

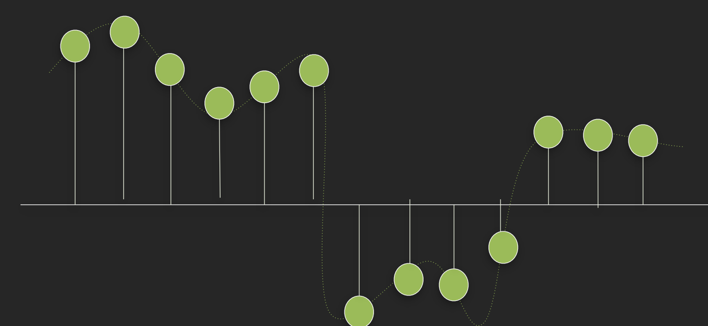


Can we recover the analog signal from its discrete time samples ?

Nyquist Theorem







An analog signal can be reconstructed perfectly from discrete samples *provided you sample it densely*.

The Nyquist Recipe

sample faster

sample denser

the more you sample,
the more detail is preserved

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But what happens if you do not follow the Nyquist recipe ?

Credit: Rob Fergus and Antonio Torralba

Image credit: Boston.com

The Nyquist Recipe

sample faster

sample denser

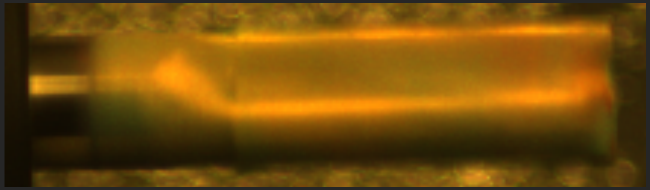
the more you sample,
the more detail is preserved

But what happens if you do not follow the Nyquist recipe ?

Breaking resolution barriers

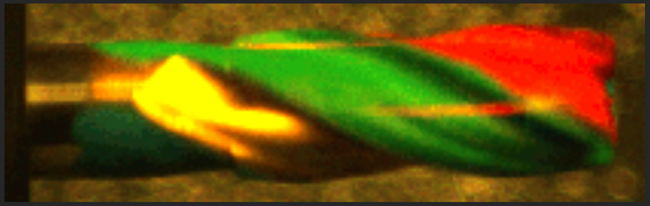
- Observing a 2000 fps spinning tool with a 25 fps camera

Normal Video:
25fps



Compressively obtained video:
25fps

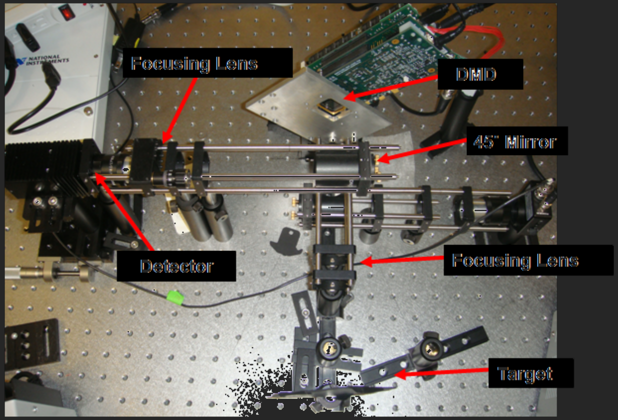
Recovered Video:
2000fps



Slide/Image credit: Reddy et al. 2011

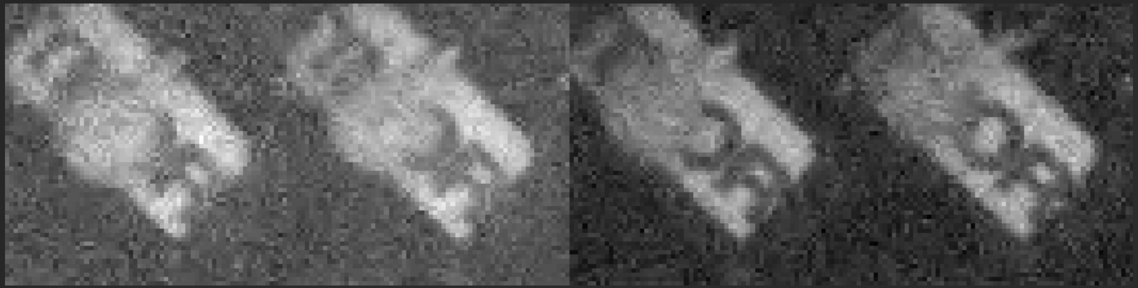
Compressive Sensing

Use of **motion flow-models** in the context of compressive video recovery



single pixel camera

128x128 images sensed at 61x comp.

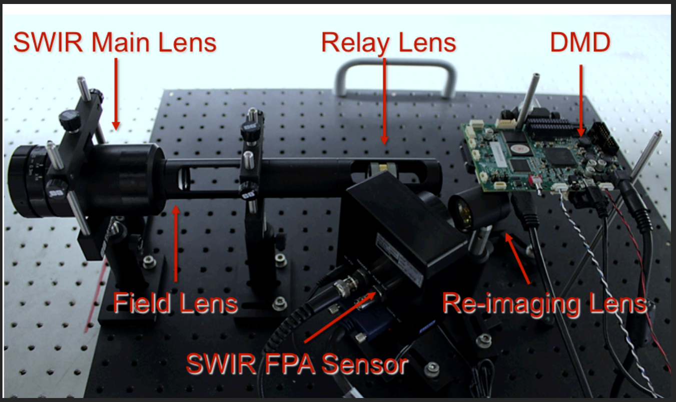


Naïve frame-to-frame recovery

CS-MUVI at 61x compression

Sankaranarayanan et al. ICCP 2012, SIAM J. Imaging Sciences, 2015*

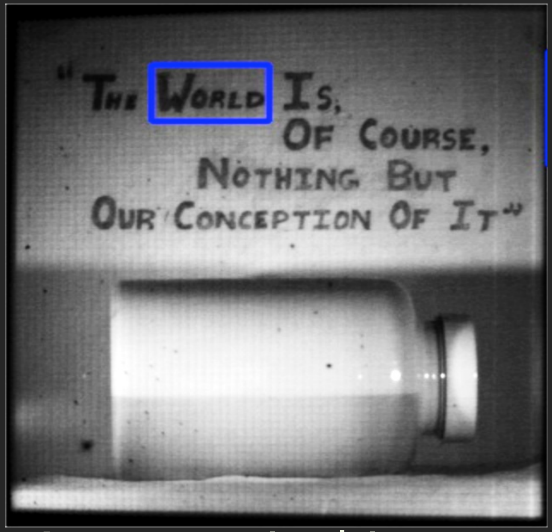
Compressive Imaging Architectures



Scalable imaging architectures that deliver videos at **mega-pixel resolutions** in infrared

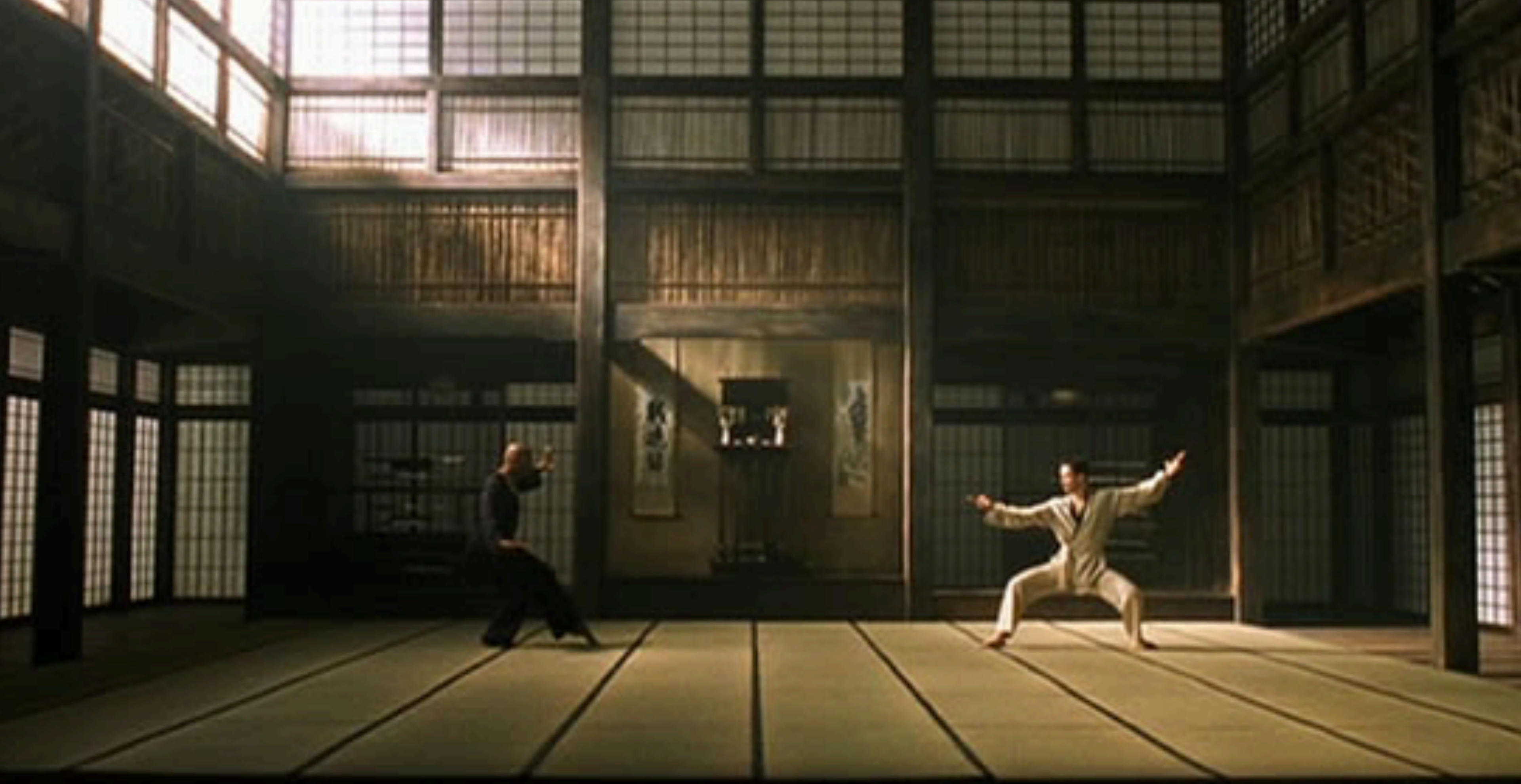
visible image

SWIR image



A mega-pixel image obtained from a 64x64 pixel array sensor

Chen et al. CVPR 2015, Wang et al. ICCP 2015



What you must learn is that these rules are no different than the rules of a computer system. Some of them can be bent. Others can be broken.