Similarity of Triangles

Quick Notes (Grade 10)

Clean, colorful, exam-ready

Definition
Use for: precise meanings and notation.

Example
Use for: worked examples with steps.

Theorem/Rule
Use for: key rules and criteria.

Shortcut
Use for: quick tricks and checks.

Property
Use for: consequences and facts.

Warning
Use for: common mistakes and alerts.

Sides in RoyalBlue, angle marks in ForestGreen, parallels in Fuchsia.

Overview: What is Similarity?

Definition

Two figures are **similar** if they have the same shape. For triangles, this means:

- Corresponding **angles are equal**.
- Corresponding **side lengths are in proportion**.

We write: $\triangle ABC \sim \triangle A'B'C'$.

Property

Scale factor k: If $\triangle ABC \sim \triangle A'B'C'$, and AB matches $A'B'$, then $\frac{A'B'}{AB} = k$. Every corresponding side scales by k.

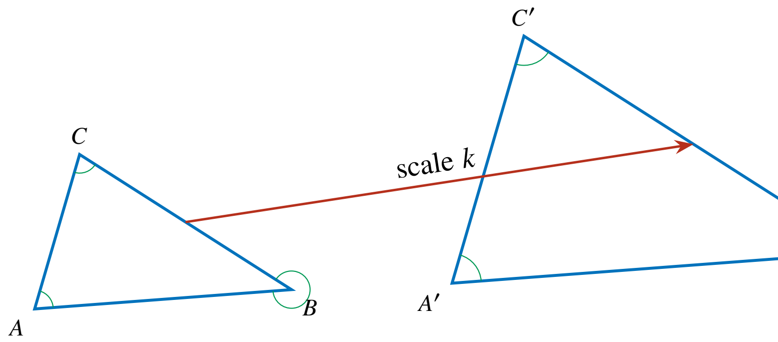
- Perimeters scale by k.
- Areas scale by k^2.

Example

Matching corresponding parts: Keep vertex order consistent. If $\triangle ABC \sim \triangle DEF$, then $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$. So $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

Shortcut

Quick check: If two angles of one triangle match two of another, triangles are similar (AAA). You do not need the third angle.



Warning

Keep order: $\triangle ABC \sim \triangle DEF$ means $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$. Mixing order gives wrong ratios.

Core Definitions and Similarity Criteria

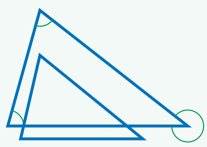
Definition

Similar triangles: $\triangle ABC \sim \triangle A'B'C'$ if $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$ (AAA), or if certain side-angle conditions hold (SAS, SSS).

Theorem/Rule

AAA Similarity

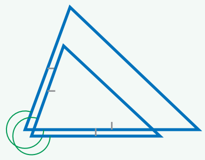
- If three angles of a triangle match the three angles of another, the triangles are similar.
- Ratios of corresponding sides are equal.



Theorem/Rule

SAS Similarity

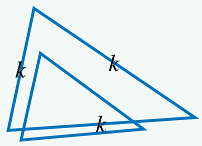
- Two sides in proportion and the included angle equal imply similarity.
- $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ and $\angle A = \angle A'$.



Theorem/Rule

SSS Similarity

- All three side pairs are in the same ratio k.
- $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'} = k$.



Warning

Common mistakes

- Using a non-included angle for SAS.
- Mismatching corresponding sides when writing ratios.
- Forgetting that areas scale by k^2, not by k.

Consequences of Similarity

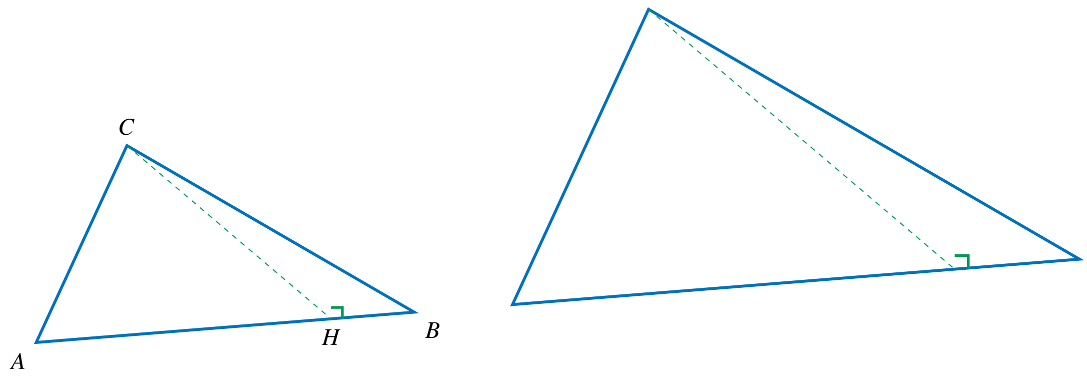
Property

If $\triangle ABC \sim \triangle A'B'C'$ with scale factor k (so $A'B' = k\,AB$), then:

- Corresponding angles are equal.
- $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CA}{C'A'} = \dfrac{1}{k}$.
- Perimeter ratio is $\dfrac{P_{A'B'C'}}{P_{ABC}} = k$.
- Area ratio is $\dfrac{[A'B'C']}{[ABC]} = k^2$.
- Altitudes, medians, and angle bisectors scale by k.

Example

A quick proof (altitudes scale by k): In similar triangles, $\dfrac{AB}{A'B'} = \dfrac{h_{AB}}{h'_{A'B'}}$ because both pairs are opposite equal angles and form similar right triangles when dropping perpendiculars. Hence $h'_{A'B'} = k\,h_{AB}$.



Shortcut

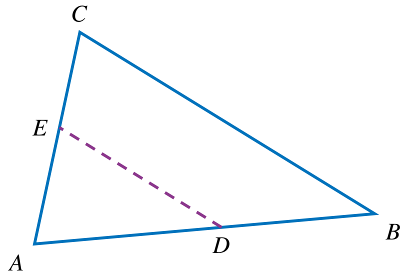
Area shortcut: If sides scale by k, areas scale by k^2. If perimeters scale by 3, areas scale by 9.

Key Theorems from Similarity

Theorem/Rule

Basic Proportionality Theorem (Thales): If a line through a triangle is parallel to one side, it divides the other two sides proportionally.

If $DE \parallel BC$ in $\triangle ABC$, then $\frac{AD}{DB} = \frac{AE}{EC}$ and $\frac{AD}{AB} = \frac{AE}{AC}$.



Example

Proof idea: Since $DE \parallel BC$, $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$ (alternate interior angles). Thus $\triangle ADE \sim \triangle ABC$ (AAA), so $\frac{AD}{AB} = \frac{AE}{AC}$ and $\frac{DE}{BC} = \frac{AD}{AB} = \frac{AE}{AC}$. From the first, $\frac{AD}{DB} = \frac{AE}{EC}$.

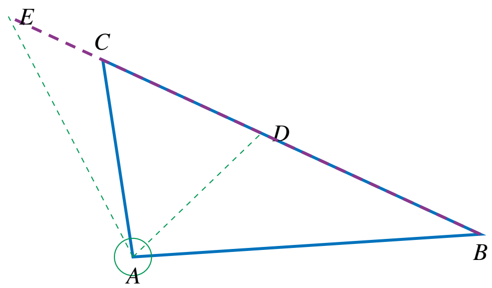
Theorem/Rule

Converse of BPT: If a line through a triangle cuts two sides proportionally, it is parallel to the third side.

Theorem/Rule

Angle Bisector Theorem (internal): In $\triangle ABC$, if AD bisects $\angle A$ with D on BC, then $\frac{BD}{DC} = \frac{AB}{AC}$.

External bisector at A meets the extension of BC at E and gives $\frac{BE}{EC} = \frac{AB}{AC}$.

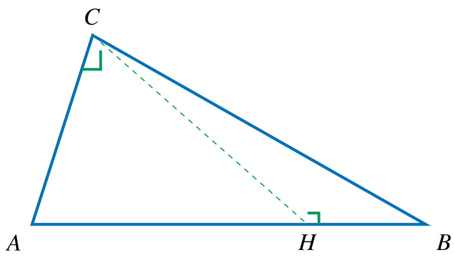


Example

Right-triangle similarity facts:

- In right $\triangle ABC$ with right angle at C, altitude from C to hypotenuse AB meets AB at H. Then $\triangle ACH \sim \triangle ABC \sim \triangle HCB$.
- Mean proportionals: $CH^2 = AH \cdot HB$, $AC^2 = AH \cdot AB$, $BC^2 = HB \cdot AB$.
- **Pythagoras by similarity:** From $AC^2 = AH \cdot AB$ and $BC^2 = HB \cdot AB$, add to get $AC^2 + BC^2 =$

$$(AH + HB) \cdot AB = AB^2.$$

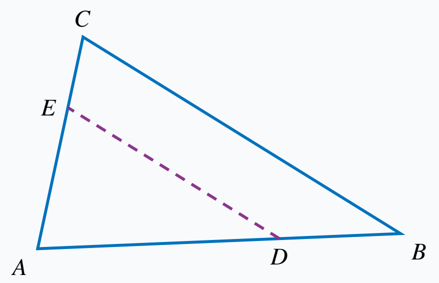


Worked Examples

Example

Type A: Find an unknown side using similarity

In $\triangle ABC$, $DE \parallel BC$ with D on AB and E on AC. If $AD = 6$, $AB = 9$, and $AE = 8$, find AC.



Steps

1. $DE \parallel BC \Rightarrow \triangle ADE \sim \triangle ABC$.
2. $\dfrac{AD}{AB} = \dfrac{AE}{AC} \Rightarrow \dfrac{6}{9} = \dfrac{8}{AC}$.
3. Cross-multiply: $6\,AC = 9 \cdot 8 = 72$.
4. $AC = 12$.

Shortcut

Use BPT: a single proportion solves it.

Example

Type B: Perimeter and area comparisons

Two similar triangles have side ratio $k = \dfrac{5}{3}$. If the smaller has perimeter 24 and area 54, find the larger's perimeter and area.

Steps

1. Perimeter scales by k: $P_{\text{large}} = k \cdot 24 = \dfrac{5}{3} \cdot 24 = 40$.
2. Area scales by k^2: $A_{\text{large}} = k^2 \cdot 54 = \left(\dfrac{5}{3}\right)^2 54 = \dfrac{25}{9} \cdot 54 = 150$.

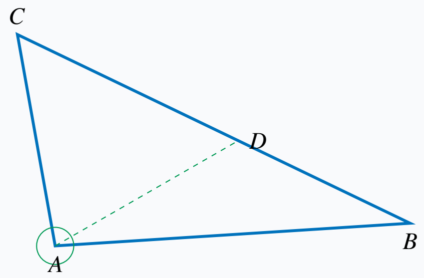
Shortcut

Perimeter $\times k$, area $\times k^2$. Keep which triangle is larger consistent.

Example

Type C: Angle-bisector splits a side

In $\triangle ABC$, AD bisects $\angle A$. If $AB = 7$, $AC = 9$, and $BC = 12$, find BD and DC.



Steps

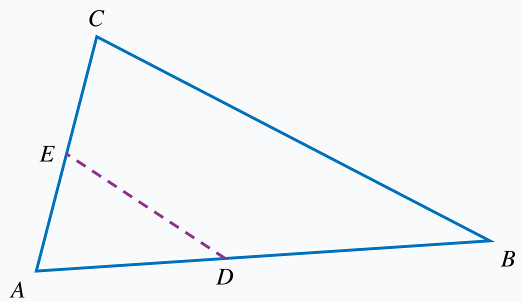
1. Angle Bisector Theorem: $\dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{7}{9}$.
2. Let $BD = 7x$, $DC = 9x$, then $7x + 9x = 12 \Rightarrow x = \dfrac{12}{16} = \dfrac{3}{4}$.
3. $BD = 7x = 5.25$, $DC = 9x = 6.75$.

Shortcut

Split in the ratio of the adjacent sides.

Example

Type D: Parallel line cuts triangle
In $\triangle ABC$, $DE \parallel BC$ with $AD = 5$, $DB = 7$, $AE = 6$. Find EC.



Steps

1. BPT: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$, so $\dfrac{5}{7} = \dfrac{6}{EC}$.
2. Cross-multiply: $5\,EC = 42 \Rightarrow EC = 8.4$.

Shortcut

Set up the proportion once; check order carefully.

Example

Type E: Shadow / scale-model problem
At the same time, a tree casts a 9 m shadow and a 1.5 m stick casts a 1.2 m shadow. How tall is the tree?
Steps

1. Similar right triangles from equal sun elevation: $\dfrac{\text{Tree height}}{9} = \dfrac{1.5}{1.2}$.
2. Tree height $= 9 \cdot \dfrac{1.5}{1.2} = 9 \cdot 1.25 = 11.25\,\text{m}$.

Shortcut

Equal sun angle gives AAA. Compare heights to shadows directly.

Example

Type F: Coordinate check for similarity
Points: $A(0,0)$, $B(4,2)$, $C(1,3)$ and $A'(0,0)$, $B'(6,3)$, $C'(1.5,4.5)$. Show $\triangle ABC \sim \triangle A'B'C'$.
Steps

1. Slopes show equal angles: $m_{AB} = \dfrac{2}{4} = \frac{1}{2}$ and $m_{A'B'} = \dfrac{3}{6} = \frac{1}{2}$; similarly for other sides.
2. Side ratios: $\overline{AB} = \sqrt{(4)^2 + (2)^2} = \sqrt{20}$, $\overline{A'B'} = \sqrt{(6)^2 + (3)^2} = \sqrt{45}$, ratio $= \sqrt{45/20} = \frac{3}{2}$.
3. Likewise $\dfrac{AC'}{AC} = \dfrac{\sqrt{(1.5)^2 + (4.5)^2}}{\sqrt{(1)^2 + (3)^2}} = \frac{3}{2}$ and for BC.
4. Hence $k = \frac{3}{2}$ and triangles are similar (SSS).

Shortcut

Check any two: either AAA via slopes/angles or SSS via distances.

Formula and Fact Sheet

Property

Topic	Key Facts / Formulas
Similarity notation	$\triangle ABC \sim \triangle A'B'C'$; keep vertex order.
Scale factor	$k = \dfrac{A'B'}{AB} = \dfrac{B'C'}{BC} = \dfrac{C'A'}{CA}$.
Angle equality	$\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.
AAA	Three equal angles $\Rightarrow$ triangles similar.
SAS	Two side ratios equal and included angle equal $\Rightarrow$ similar.
SSS	Three side ratios equal (same k) $\Rightarrow$ similar.
Perimeter	$P_2 = k\, P_1$.
Area	$[\triangle_2] = k^2\, [\triangle_1]$.
Medians/altitudes/bisectors	Scale by k in similar triangles.
BPT	If $DE \parallel BC$ in $\triangle ABC$, then $\dfrac{AD}{DB} = \dfrac{AE}{EC}$ and $\dfrac{AD}{AB} = \dfrac{AE}{AC}$.
Converse BPT	If a line cuts two sides proportionally, it is parallel to the third.
Angle Bisector Thm	Internal: $\dfrac{BD}{DC} = \dfrac{AB}{AC}$. External: $\dfrac{BE}{EC} = \dfrac{AB}{AC}$.
Right-triangle similarity	With altitude to hypotenuse: $CH^2 = AH \cdot HB$, $AC^2 = AH \cdot AB$, $BC^2 = HB \cdot AB$.
Pythagoras (via similarity)	$AC^2 + BC^2 = AB^2$.
Word problems	Heights/shadows or scale models: use height/shadow ratios.

Pitfalls and Checks

Warning

- **Wrong order:** Always match letters correctly when writing ratios.
- **SAS uses included angle:** The equal angle must be between the two compared sides.
- **k vs k^2:** Perimeter uses k, area uses k^2.
- **Rounding:** Keep sufficient precision; round only at the end.
- **Not to scale:** Diagrams may mislead; rely on given measures.

Practice Problems

1. In $\triangle ABC$, $DE \parallel BC$, $AD = 3$, $AB = 9$, $AE = 5$. Find AC.
2. Two similar triangles have $k = \frac{4}{3}$. If the smaller area is 27, find the larger area.
3. In $\triangle ABC$, AD bisects $\angle A$. If $AB = 8$, $AC = 10$, $BC = 18$, find BD and DC.
4. In right $\triangle ABC$ with right angle at C, altitude to hypotenuse meets at H, $AH = 5$, $HB = 9$. Find CH, AC, BC.
5. $\triangle ABC \sim \triangle DEF$ with $AB = 7$, $BC = 9$, $DE = 14$. Find EF.
6. A model uses scale $1 : 50$. A real building is 45 m tall. How tall is the model?
7. If $\triangle ABC \sim \triangle PQR$ and $\angle A = 40°$, $\angle C = 65°$, find $\angle Q$.
8. In $\triangle ABC$, $DE \parallel BC$ with $AD = 4$, $DB = 8$, $AE = 6$. Find EC.
9. $\triangle ABC \sim \triangle A'B'C'$ and $k = 2.5$. If $P_{ABC} = 28$, find $P_{A'B'C'}$.
10. If $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{3}{5}$, what is the area ratio $\frac{[DEF]}{[ABC]}$?
11. Sides $AB = 6$, $AC = 10$. If AD bisects $\angle A$ meeting BC at D and $BD = 7.2$, find BC.
12. Points $A(0, 0)$, $B(3, 1)$, $C(0, 2)$. Points $D(0, 0)$, $E(6, 2)$, $F(0, 4)$. Are $\triangle ABC$ and $\triangle DEF$ similar? State k.
13. In $\triangle ABC$, a line through A meets BC at D with $\frac{BD}{DC} = \frac{2}{3}$. Prove the line is parallel to AB or AC? Which?
14. $\triangle ABC \sim \triangle PQR$ with $AB = 12$, $BC = 15$, $CA = 9$ and $PQ = 16$. Find k and QR, RP.
15. In right $\triangle ABC$ with $\angle C = 90°$, $AC = 9$, $BC = 12$. Find AB using similarity (not directly by Pythagoras).
16. A 2 m stick casts 1.6 m shadow. A building casts 24 m shadow at the same time. Find the building height.
17. If $\triangle ABC \sim \triangle DEF$ and $\frac{AB}{DE} = \frac{BC}{EF} = k$, express $\frac{[DEF]}{[ABC]}$ in terms of k.
18. In $\triangle ABC$, $DE \parallel BC$ with $AD = 7$, $AE = 10$, $AC = 15$. Find AB.
19. In $\triangle ABC$, AD bisects $\angle A$ and meets BC at D. If $AB : AC = 5 : 7$ and $BC = 18$, find BD and DC.
20. Coordinates: $A(1, 1)$, $B(5, 3)$, $C(2, 6)$ and $A'(2, 2)$, $B'(10, 6)$, $C'(4, 12)$. Check similarity and find k.

Answers are listed at the end.

Answer Key (Condensed)

1. $AC = 15$.
2. $k^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$, so area $= 27 \cdot \frac{16}{9} = 48$.
3. $BD : DC = 8 : 10 = 4 : 5$, so $BD = 8$, $DC = 10$ (since $BC = 18$).
4. $CH = \sqrt{5 \cdot 9} = \sqrt{45} = 3\sqrt{5}$, $AC = \sqrt{5 \cdot 14} = \sqrt{70}$, $BC = \sqrt{9 \cdot 14} = \sqrt{126} = 3\sqrt{14}$.
5. $k = \frac{DE}{AB} = \frac{14}{7} = 2$, so $EF = 2 \cdot 9 = 18$.
6. Model height $= \frac{45}{50} = 0.9$ m.
7. $\angle Q = \angle B = 180° - 40° - 65° = 75°$.
8. $\dfrac{AD}{DB} = \dfrac{AE}{EC} \Rightarrow \frac{4}{8} = \frac{6}{EC} \Rightarrow EC = 12$.
9. $P = 2.5 \cdot 28 = 70$.
10. $\left(\frac{5}{3}\right)^2 = \frac{25}{9}$.
11. $BD/DC = 6/10 = 3/5$ with $BD = 7.2 \Rightarrow 7.2/DC = 3/5 \Rightarrow DC = 12$, so $BC = 19.2$.
12. Yes, $k = 2$ (every coordinate doubled).
13. Line is parallel to AB if it passes through A and cuts BC in ratio of $BA : AC$ only if ratio matches; given $BD : DC = 2 : 3$, it is *not* generally parallel to AB or AC unless sides match that ratio. Insufficient data; typical answer: parallel to *a side* only when the side ratios match.
14. $k = \frac{PQ}{AB} = \frac{16}{12} = \frac{4}{3}$; $QR = \frac{4}{3} \cdot 15 = 20$, $RP = \frac{4}{3} \cdot 9 = 12$.
15. $k = \dfrac{AB}{\sqrt{AC^2 + BC^2}}$, or compute: similar to a scaled 3-4-5, so $AB = 15$.
16. Height $= 24 \cdot \frac{2}{1.6} = 30$ m.
17. $\dfrac{[DEF]}{[ABC]} = k^2$.
18. $\dfrac{AD}{AB} = \dfrac{AE}{AC} \Rightarrow \frac{7}{AB} = \frac{10}{15} \Rightarrow AB = 10.5$.
19. $BD : DC = 5 : 7$; with $BC = 18$, get $BD = \frac{5}{12} \cdot 18 = 7.5$, $DC = 10.5$.
20. Yes, each coordinate multiplied by 2; $k = 2$.