Simplex Algorithm

Operations Research

- Operation Research is also called OR for short and it is a scientific approach to decision making which seeks to determine how best to design and operate a system under conditions requiring allocation of scarce resources.
- Operations research as a field, primarily has a set or collection of algorithms which act as tools for problems solving in chosen application areas.
- OR has extensive applications in engineering business and public systems and is also used by manufacturing and service industries to solve their day to day problems.

Contd…

- The history of the OR as a field as goes up to the Second World War. In fact this field operations research started during the Second World War when the British military asked scientists to analyze military problems.
- In fact Second World War was perhaps the first time when people realized that resources were scarce and had to be used effectively and allocated efficiently. The application of mathematics and scientific methods to military applications was called operations research to begin with.
- But today it has a different definition it is also called management science.

Contd…

- Linear programming was first conceived by Dantzig, around 1947 at the end of the Second World War.
- Very historically, the work of a Russian mathematician first had taken place in 1939 but since it was published in 1959,
- Dantzig was still credited with starting linear programming. In fact Dantzig did not use the term linear programming. His first paper was titled ‘Programming in Linear Structure’.
- Much later, the term ‘Linear Programming’ was coined by Koopmans.
- The Simplex method which is the most popular and powerful tool to solve linear programming problems, was published by Dantzig in 1949.

- Two researchers, L. V. Kantorovich of the former Soviet Union and the Dutch-American T. C. Koopmans, were even awarded the Nobel Prize in 1975 for their contributions to linear programming theory and its applications to economics.
- G. B. Dantzig is the father of LP.

In an *optimization problem* one seeks to maximize or minimize a specific quantity, called the *objective*, which depends on a finite number of input variables. These variables may be independent of one another, or they may be related through one or more *constraints*.

$$
\begin{array}{ll}
\text{optimize:} & z = f(x_1, x_2, \ldots, x_n) \\
\text{subject to:} & \left.\begin{array}{l} g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \\ g_m(x_1, x_2, \ldots, x_n) \end{array}\right\} \begin{array}{c} \le \\ = \\ \ge \end{array} \left\{\begin{array}{l} b_1 \\ b_2 \\ \cdots \\ \\ b_m \end{array}\right.
\end{array}
$$

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$g_i(x_1, x_2, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

Optimisation Problem

- Shortest path
- Minimum spanning tree

Iterative Improvement

Algorithm design technique for solving optimization problems

- Start with a feasible solution
- Repeat the following step until no improvement can be found:
  - change the current feasible solution to a feasible solution with a better value of the objective function
- Return the last feasible solution as optimal

Note: Typically, a change in a current solution is “small” (*local search*)

Major difficulty: Local optimum vs. global optimum

Linear programming (LP) problem is to optimize a linear function of several variables subject to linear constraints:

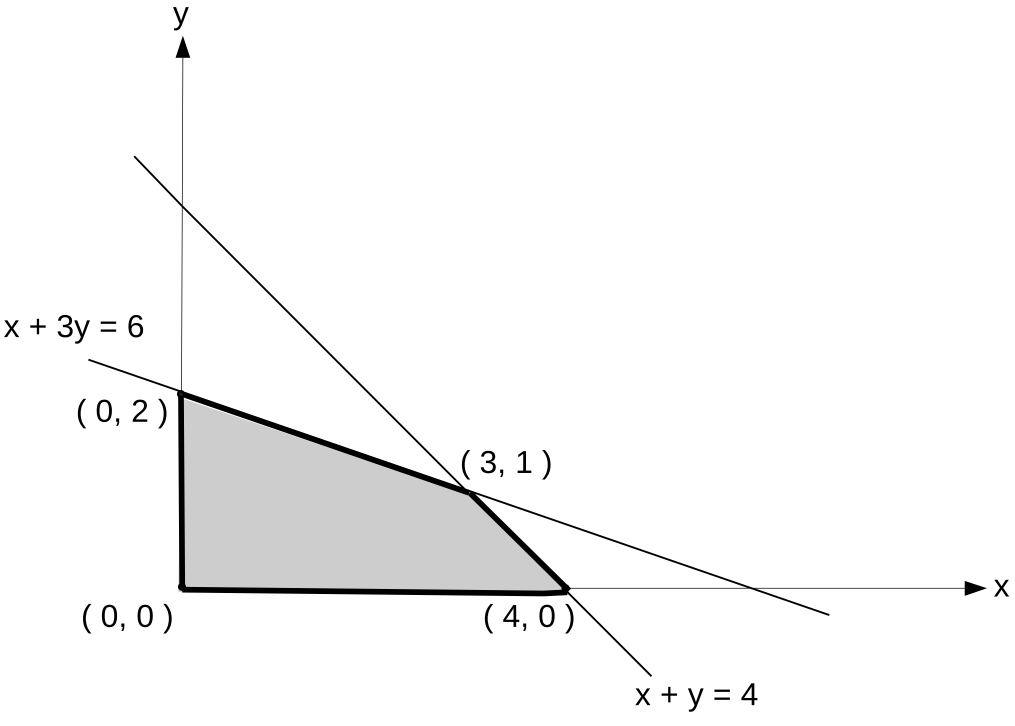
maximize (or minimize) $c_1 x_1 + ...+ c_n x_n$
subject to $a_{i\,1}x_1+ ...+ a_{in} x_n \leq$ (or $\geq$ or $=$) b_i, $i = 1,...,m$
$x_1 \geq 0, ... , x_n \geq 0$

The function $z = c_1 x_1 + ...+ c_n x_n$ is called the *objective function*;

constraints $x_1 \geq 0, ... , x_n \geq 0$ are called *nonnegativity constraints*

$$\begin{aligned} \text{maximize} \quad & 3x + 5y \\ \text{subject to} \quad & x + y \leq 4 \\ & x + 3y \leq 6 \\ & x \geq 0, \; y \geq 0 \end{aligned}$$

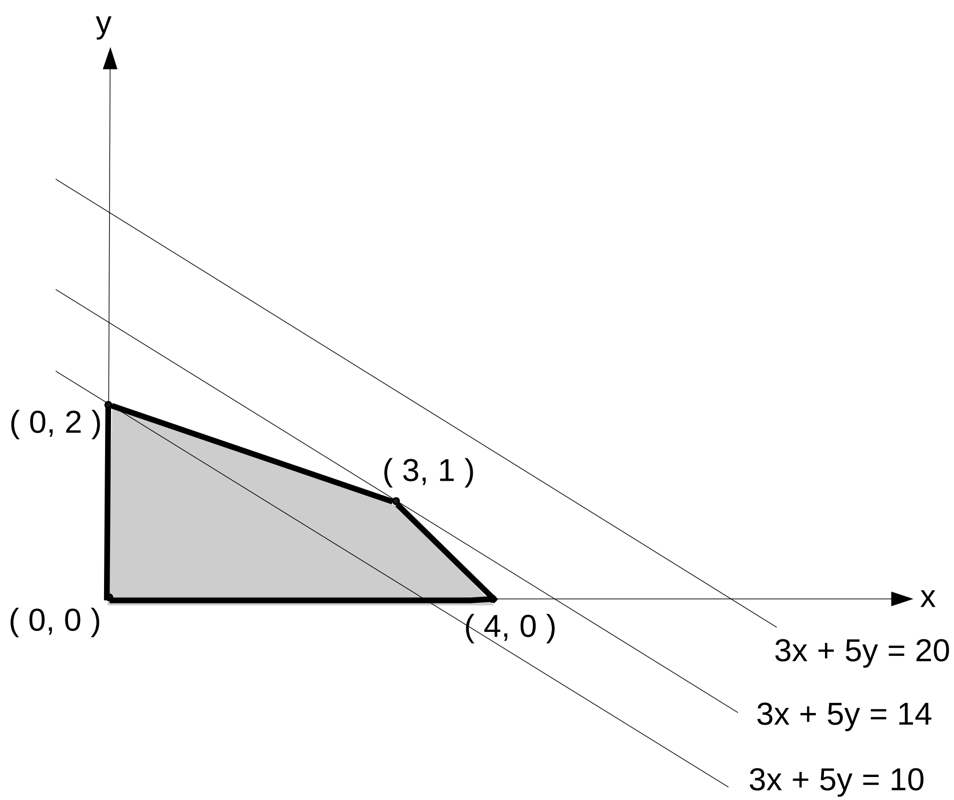
Feasible region



$$\begin{array}{ll} \text{maximize} & 3x + 5y \\ \text{subject to} & x + y \le 4 \\ & x + 3y \le 6 \\ & x \ge 0,\ y \ge 0 \end{array}$$

Optimal solution: $x = 3$, $y = 1$

Geometric solution



Example

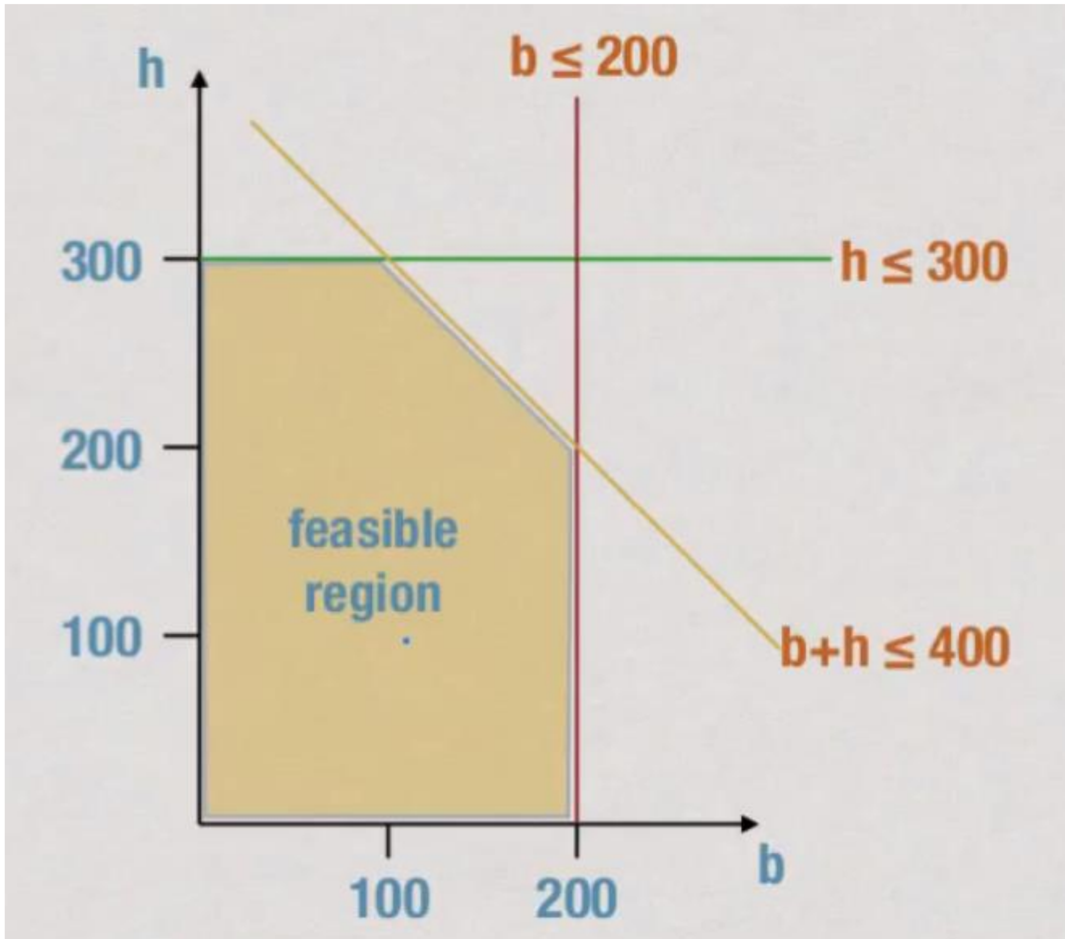
- Sweet shop sells Burfis and Halwas.
- Each box of burfi earns a profit of Rs.100/- and Halwa 600.
- Daily demand for burfi is atmost 200 and halwa is 300.
- Staff can produce 400 boxes a day altogether.
- What is the profitable mix of burfis and halwas to produce?

LP model

- Let b = no. of burfi boxes produced in a day
- Let h = no. of halwa boxes produced in a day
- Profit $z = 100b + 600h$
- Demand constraint: $b \leq 200$, $h \leq 300$
- Production constraint: $b + h \leq 400$
- Implicit constraints: $b, h \geq 0$

Contd…

- Objective function:
- $Z = 100b + 600h$
- Constraints:
- $b \leq 200$,
- $h \leq 300$
- $b + h \leq 400$
- $b, h \geq 0$



h
b ≤ 200
300
h ≤ 300
200
feasible
region
100
b+h ≤ 400
100
200
b

- Optimal value always occurs at the end point vertex.
- Solution:
  - Feasible solution is convex
  - Empty, constraints not satisfiable
  - May be unbounded, no upperlimit

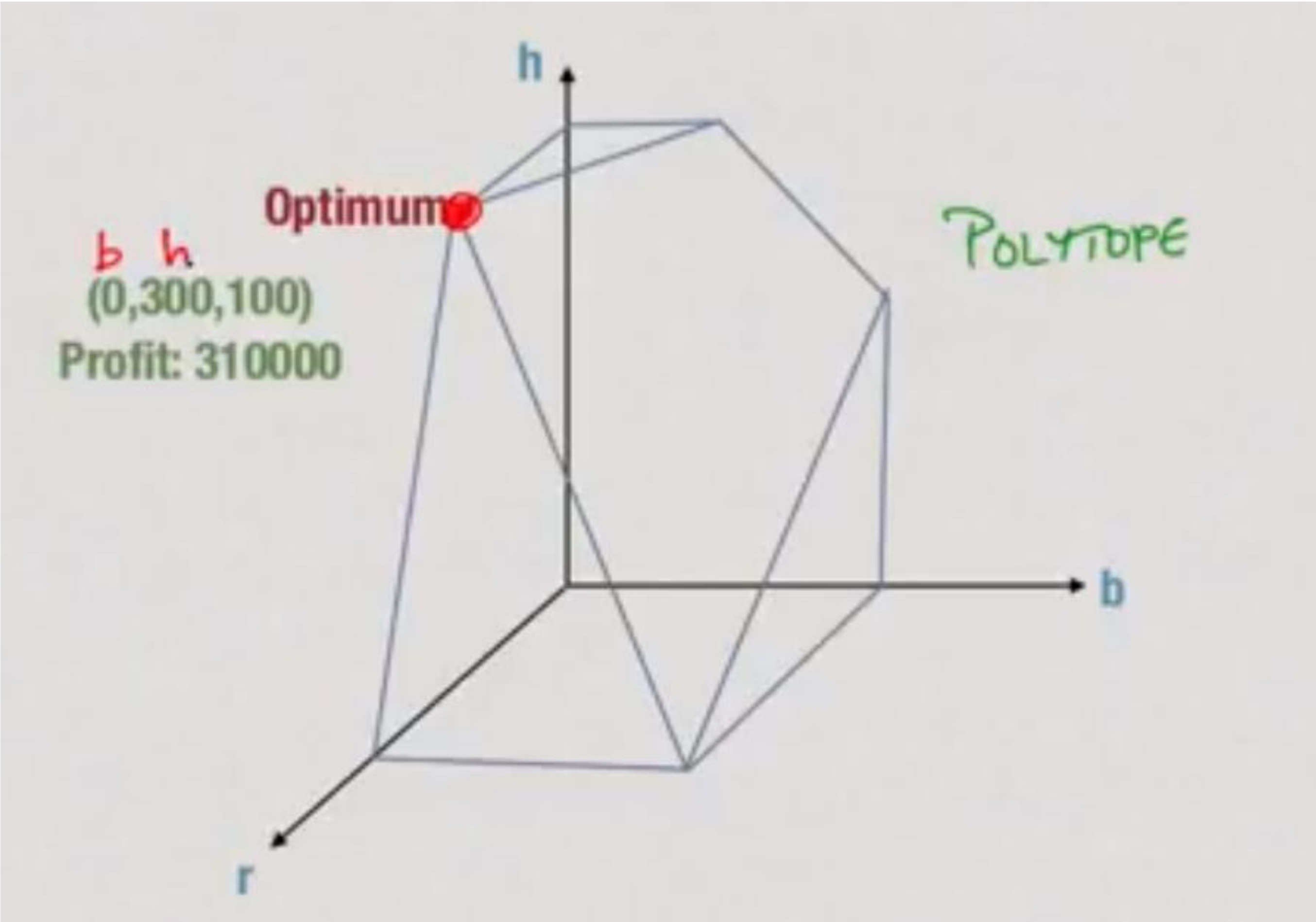
Simplex algorithm

- Start with any vertex, evaluate objective function
- If the adjacent vertex has better value, move.
- If the current vertex has better value than neighbours, stop.
- Can be exponential and efficient in practice.

- Narendra Karmarkar invented a polynomial algorithm for linear programming also known as the interior point method

- Sweet shop sells Burfis and Halwas. It adds rasamalai
- Each box of burfi earns a profit of Rs.100/- and Halwa 600, rasamalai 1300.
- Daily demand for burfi is atmost 200 and halwa is 300, rasamalai unlimited.
- Staff can produce 400 boxes a day altogether.
- Milk supply:600 box of halwa or 200 box of rasamalai or any combination.
- What is the profitable mix of burfis and halwas to produce?

- Objective function:
- $Z = 100b + 600h + 1300r$
- Constraints:
- $b \leq 200$,
- $h \leq 300$
- $b + h + r \leq 400$
- $h + 3r \leq 600$
- $b, h, r \geq 0$


h
Optimum
b h
(0,300,100)
Profit: 310000
POLYTOPE
b
r

Contd…

- Dual LP:
- $h \leq 300$ * 100
- $b+h+r \leq 400$ * 100
- $h+3r \leq 600$ * 400

Example

- For example we have these two products A and B. Two resources are need R one and R2. A requires one unit of R1 and three units of R two. B requires one unit of R one and two units of R two. Manufacturer has 5 units of R one available and 12 units of R two available. The manufacturer also makes a profit of rupees 6 per unit of A sold and rupees 5 per unit of B of sold. So this is the problem setting that we are looking at.

- let X = number of units of A produced
- Y = number of units of B produced.
- $Z = 6X + 5Y$.
- Constraints: $X + Y \leq 5$
- $3X + 2Y \leq 12$
- $X, Y \geq 0$

Production planning

- Let us consider a company making a single product demand of 1000, 800, 1200 and 900 respectively for 4 months. Now the company wants to meet the demand for the product in the next 4 months. The company can use two modes of production. There is something called as Regular and overtime production. Now the regular time capacity is 800/month and overtime capacity is 200/ month. In order to produce one item in regular time, it costs Rs 20 and to produce overtime it costs Rs 25. The company can also produce more in a particular month and carry the excess to the next month. Such a carrying cost is rupees 3.

- Let Xj. as quantity produced using regular time production in month j
- Yj as quantity produce using overtime in month j.
- Now these are our decision variables, Xj and Yj.
- I1 which is the quantity that is carried to the next month.

- Minimize:
- $Z = 20\,(X1+X2+X3+X4) + 25(Y1+Y2+Y3+Y4) + 3(I1+I2+I3)$.
- $X1 + Y1 = 1000 + I1$
- $I1 + X2 + Y2 = 800 + I2$
- $I2 + X3 + Y3 = 1200 + I3$
- $I3 + X4 + y4 = 900$

Agenda

- Linear Programming problem
- Simplex – Algebraic form
- Worth of resource and dual
- Tabular form
- Matrix form
- Minimization/ $\geq$ constraint (dual simplex)
- Limitations – Cycling
- Computational issues - complexity

Simplex algorithm is used to solve Linear Programming problems (LPP)

What is a Linear Programming problem?

Example

A company can make 3 products. Sale price = 250, 180, 300. They are made on two machines – 10, 12, 15 and 24, 14, 20 on M1 and M2. Availability = 2400 and 4800. How much do they produce?

Let X_1, X_2, X_3 be the units of P, Q, R made

Decision variable

Maximize $250X_1 + 180X_2 + 300X_3$

Objective function

$10X_1 + 12X_2 + 15X_3 \leq 2400$

$24X_1 + 14X_2 + 20X_3 \leq 4800$

constraints

Linear

$X_1, X_2 \geq 0$

Non negativity

$$\text{Maximize } 250X_1 + 180X_2 + 300X_3$$
$$10X_1 + 12X_2 + 15X_3 \le 2400$$
$$24X_1 + 14X_2 + 20X_3 \le 4800$$
$$X_1, X_2 \ge 0$$

$$\text{Maximize } 250X_1 + 180X_2 + 300X_3 + 0X_4 + 0X_5$$
$$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$$
$$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$$
$$X_1, X_2 \ge 0$$

$$X_4 = 2400 - 10X_1 - 12X_2 - 15X_3$$
$$X_5 = 4800 - 24X_1 - 14X_2 - 20X_3$$
$$Z = 250X_1 + 180X_2 + 300X_3 + 0X_4 + 0X_5$$

$X_4 = 2400$, $X_5 = 4800$ $Z = 0$

X_3 comes into the solution.

Best value = min{2400/15, 4800/20} = min {160, 240} = 160

X_3 replaces X_4.

$X_4 = 2400 - 10X_1 - 12X_2 - 15X_3$
$X_5 = 4800 - 24X_1 - 14X_2 - 20X_3$
$Z = 250X_1 + 180X_2 + 300X_3 + 0X_4 + 0X_5$

$15X_3 = 2400 - 10X_1 - 12X_2 - X_4$

$$\boldsymbol{X_3 = 160 - \frac{2X_1}{3} - \frac{4X_2}{5} - \frac{X_4}{15}}$$

$$X_5 = 4800 - 24X_1 - 14X_2 - 20\left(160 - \frac{2X_1}{3} - \frac{4X_2}{5} - \frac{X_4}{15}\right) = 1600 - \frac{32X_1}{3} + 2X_2 - \frac{4X_4}{3}$$

$$Z = 250X_1 + 180X_2 + 300\left(160 - \frac{2X_1}{3} - \frac{4X_2}{5} - \frac{X_4}{15}\right) = 48000 + 50X_1 - 60X_2 - 20X_4$$

$X_3 = 160$, $X_5 = 4800$ $Z = 48000$
X_1 comes into the solution.
Best value = min{160/(2/3), 1600/(32/3)} = min {240, 150} = 150
X_1 replaces X_5.

$$\frac{32X_1}{3} = 1600 + 2X_2 + \frac{4X_4}{3} - X_5$$

$$X_1 = 150 + \frac{3X_2}{16} + \frac{X_4}{8} - \frac{3X_5}{32}$$

Optimum solution

$X_1 = 150$, $X_3 = 60$

Revenue = 55500

$$X_3 = 160 - \frac{2\left(150 + \frac{3X_2}{16} + \frac{X_4}{8} - \frac{3X_5}{32}\right)}{3} - \frac{4X_2}{5} - \frac{X_4}{15}$$

$$X_3 = 60 - \frac{37X_2}{40} - \frac{3X_4}{20} + \frac{X_5}{16}$$

$$Z = 48000 + 50\left(150 + \frac{3X_2}{16} + \frac{X_4}{8} - \frac{3X_5}{32}\right) - 60X_2 - 20X_4$$

$$Z = 55500 - \frac{405X_2}{8} - \frac{55}{4}X_4 - \frac{75X_5}{16}$$

Why did we not produce X_2 ?

We associate a worth for each resource. Let Y_1 and Y_2 be the worth of the two resources. Since the resources have been converted to products, the total revenue should be equal to the total worth of the resources.

When we produce, the unit revenue should be equal to unit worth
$10Y_1 + 24Y_2 = 250$, $15Y_1 + 20Y_2 = 300$; Solving, we get $Y_1 = 55/4$ $Y_2 = 75/16$
Total worth = 2400x55/4 + 4800x75/16 = 55500
For X_2, price = 180, value of resources = 12x55/4 + 14x75/16 = 1875/8 < 180 by 405/8

In general, we wish to find out Y_1 and Y_2 such that we minimize the total value of the resources and yet try to use them to meet the unit price. We therefore
Minimize $2400Y_1 + 4800Y_2$
such that $10Y_1 + 24Y_2 \geq 250$; $12Y_1 + 14Y_2 \geq 180$; $15Y_1 + 20Y_2 \geq 300$; Y_1 $Y_2 \geq 0$

The above problem is called the **dual** and has the solution $Y_1 = 55/4$ $Y_2 = 75/16$ with W = 55500

SIMPLEX SOLVES THE GIVEN PROBLEM (PRIMAL) AND ITS DUAL

		$-X_1$	$-X_2$	$-X_3$
		-250	-180	-300
X_4	2400	10	12	15
X_5	4800	24	14	20

		$-X_1$	$-X_2$	$-X_4$
	48000	-50	60	20
X_3	160	2/3	4/5	1/15
X_5	1600	32/3	-2	-4/3

		$-X_5$	$-X_2$	$-X_4$
	55500	75/16	405/8	55/4
X_3	60	-1/16	37/40	3/20
X_1	150	3/32	-3/16	-1/8

Tabular method is used for quick hand computation and class room teaching

Matrix method – How do I write a computer program for Simplex?

Maximize $250X_1 + 180X_2 + 300X_3$
$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$
$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$

$$X_B = \begin{bmatrix} X_4 \\ X_5 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I; BX_B = b;\ X_B = B^{-1}b = b = \begin{bmatrix} 2400 \\ 4800 \end{bmatrix}$$

$$yB = C_{B;}y = C_B = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Find whether X_1 X_2 X_3 can enter the solution

$$C_1 - yP_1 = 250 - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 24 \end{bmatrix} = 250$$

$C_2 - yP_2 = 180$; $C_3 - yP_3 = 300$;

Variable X_3 enters. Takes value = min {2400/15, 4800/24} = 160.
Replaces X_4

Maximize $250X_1 + 180X_2 + 300X_3$

$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$

$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$

$$X_B = \begin{bmatrix} X_3 \\ X_5 \end{bmatrix} = B^{-1}b \qquad B = \begin{bmatrix} 15 & 0 \\ 20 & 1 \end{bmatrix}; = B^{-1} = \frac{1}{15}\begin{bmatrix} 1 & 0 \\ -20 & 15 \end{bmatrix}$$

$$X_B = \begin{bmatrix} X_3 \\ X_5 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 1 & 0 \\ -20 & 15 \end{bmatrix}\begin{bmatrix} 2400 \\ 4800 \end{bmatrix} = \begin{bmatrix} 160 \\ 1600 \end{bmatrix}$$

$$yB = C_{B;} y = C_B B^{-1} \qquad y = C_B B^{-1} = [300 \quad 0]\frac{1}{15}\begin{bmatrix} 1 & 0 \\ -20 & 15 \end{bmatrix} = [20 \quad 0]$$

Find whether X_1 X_2 X_4 can enter the solution

$$C_1 - yP_1 = 250 - [20 \quad 0]\begin{bmatrix} 10 \\ 24 \end{bmatrix} = 50$$

$C_2 - yP_2 = -60$; $C_4 - yP_3 = -20$;

Variable X_1 enters. New X_1 column is given by$B^{-1}P_1$.

$$B^{-1}P_1 = \frac{1}{15}\begin{bmatrix} 1 & 0 \\ -20 & 15 \end{bmatrix}\begin{bmatrix} 10 \\ 24 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 32/3 \end{bmatrix}$$

X_1 takes value = min {160/(2/3), 1600/(32/3) = 150 and replaces X_5

Maximize $250X_1 + 180X_2 + 300X_3$

$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$

$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$

$$X_B = \begin{bmatrix} X_3 \\ X_1 \end{bmatrix} = B^{-1}b \qquad B = \begin{bmatrix} 15 & 10 \\ 20 & 24 \end{bmatrix}; = B^{-1} = \frac{1}{160}\begin{bmatrix} 24 & -10 \\ -20 & 15 \end{bmatrix}$$

$$X_B = \begin{bmatrix} X_3 \\ X_1 \end{bmatrix} = \frac{1}{160}\begin{bmatrix} 24 & -10 \\ -20 & 15 \end{bmatrix}\begin{bmatrix} 2400 \\ 4800 \end{bmatrix} = \begin{bmatrix} 60 \\ 150 \end{bmatrix}$$

$$yB = C_{B;}y = C_B B^{-1} \qquad y = C_B B^{-1} = [300 \quad 250]\frac{1}{160}\begin{bmatrix} 24 & -10 \\ -20 & 15 \end{bmatrix} = \begin{bmatrix} \frac{55}{4} & \frac{75}{16} \end{bmatrix}$$

Find whether X_2 X_4 X_5 can enter the solution

$$C_2 - yP_2 = 180 - \begin{bmatrix} \frac{55}{4} & \frac{75}{16} \end{bmatrix}\begin{bmatrix} 12 \\ 14 \end{bmatrix} = \frac{-405}{8}$$

$C_4 - yP_4 = \frac{-55}{4}$; $C_5 - yP_5 = \frac{-75}{16}$

No entering variable. Algorithm terminates

Variations in simplex algorithm – handling ≥ constraints

Minimize $4X_1 + 5X_2$
$X_1 + X_2 \geq 8$
$3X_1 + 5X_2 \geq 34$
$X_1, X_2 \geq 0$

$3X_1 + 5X_2 \geq 34$ becomes $3X_1 + 5X_2 – X_4 = 34$ with $X_4 \geq 0$. We cannot start simplex with X_4 as a basic variable because $X_4 = - 34 + 3X_1 + 5X_2$ violates $X_4 \geq 0$

Two phase method – uses artificial variables and eliminates them in first phase

Big M method – uses artificial variables till the end

Dual simplex method – Starts with infeasible solution and reaches feasibility

Dual Simplex algorithm

Minimize $4X_1 + 5X_2$
$X_1 + X_2 \geq 8$
$3X_1 + 5X_2 \geq 34$
$X_1, X_2 \geq 0$

Minimize $4X_1 + 5X_2 + 0X_3 + 0X_4$
$X_1 + X_2 - X_3 \geq 8$
$3X_1 + 5X_2 - X_4 = 34$
$X_1, X_2, X_3, X_4 \geq 0$

$X_3 = -8 + X_1 + X_2$
$X_4 = -34 + 3X_1 + 5X_2$
$Z = 4X_1 + 5X_2$

$X_3 = -8$, $X_4 = -34$ $Z = 0$ is infeasible. To make it feasible one of the infeasible variables has to leave. We push out X_4 (most negative). X_2 comes in because the rate of increase of objective function is smaller – minimum {4/3, 5/5}

$X_3 = -8 + X_1 + X_2$
$X_4 = -34 + 3X_1 + 5X_2$
$Z = 4X_1 + 5X_2$

$5X_2 = 34 - 3X_1 - X_4$

$$\boldsymbol{X_2 = \frac{34}{5} - \frac{3X_1}{5} + \frac{X_4}{5}}$$

$$X_3 = -8 + X_1 + \left(\frac{34}{5} - \frac{3X_1}{5} + \frac{X_4}{5}\right) = \frac{-6}{5} + \frac{2X_1}{5} + \frac{X_4}{5}$$

$$Z = 40X_1 + 5\left(\frac{34}{5} - \frac{3X_1}{5} + \frac{X_4}{5}\right) = 34 + X_1 + X_4$$

$X_2 = 34/5$, $X_3 = -6/5$ $Z = 34$ is infeasible. To make it feasible, the infeasible variable has to leave. We push out X_3. X_1 comes in because the rate of increase of objective function is smaller – minimum {1/(2/5), 1/(1/5}.

$$X_3 = \frac{-6}{5} + \frac{2X_1}{5} + \frac{X_4}{5}$$

$$\frac{2}{5}X_1 = \frac{6}{5} + X_3 - \frac{X_4}{5}$$

$$X_1 = 3 + \frac{5}{2}X_3 - \frac{X_4}{2}$$

$$X_2 = \frac{34}{5} - \frac{3}{5}(3 + \frac{5}{2}X_3 - \frac{X_4}{2}) + \frac{X_4}{5}$$

$$X_2 = 5 - \frac{3}{2}X_3 + \frac{X_4}{2}$$

$$Z = 34 + X_1 + X_4 = 34 + 3 + \frac{5}{2}X_3 - \frac{X_4}{2} + X_4 = 37 + \frac{5}{2}X_3 + \frac{X_4}{2}$$

Limitations in simplex algorithm – cycling

Maximize $\frac{3}{4}X_1 - 20X_2 + \frac{1}{2}X_3 - 6X_4$

$$\frac{1}{4}X_1 - 8X_2 - X_3 + 9X_4 \le 0$$

$$\frac{1}{2}X_1 - 12X_2 - \frac{1}{2}X_3 + 3X_4 \le 0$$

$$X_3 \le 1$$

$$X_j \ge 0$$

Add X_5 X_6 X_7 as slack variables and start simplex.
Use Largest coefficient rule and first variable rule for entering and leaving variables

Successive basis are
$\{X_5, X_6, X_7\}$
$\{X_1, X_6, X_7\}$
$\{X_1, X_2, X_7\}$
$\{X_3, X_2, X_7\}$
$\{X_3, X_4, X_7\}$
$\{X_5, X_4, X_7\}$
$\{X_5, X_6, X_7\}$
Comes back to starting basis

Lexicographic rule
Smallest subscript rule (Bland)
$\{X_5, X_6, X_7\}$
$\{X_1, X_6, X_7\}$
$\{X_1, X_2, X_7\}$
$\{X_3, X_2, X_7\}$
$\{X_3, X_4, X_7\}$
$\{X_3, X_4, X_1\}$
$\{X_3, X_5, X_1\}$
$X_1 = X_3 = 1$ with $Z = 5/4$ is optimum

Making simplex algorithm faster and better

Memory and Speed

Memory – matrix method, column generation, bounded variables, decomposition

Speed – Time taken per iteration, number of iterations

Matrix inversion – Solving equations

- Substitution
- Gauss- Jordan
- Gaussian Elimination
- Eta factorization of the basis
- Sparse and dense matrices
- LU factorization

Eta factorization of the basis - example

Maximize $250X_1 + 180X_2 + 300X_3$
$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$
$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$

$$X_B = \begin{bmatrix} X_3 \\ X_1 \end{bmatrix} = B^{-1}b \qquad B_2 = \begin{bmatrix} 15 & 10 \\ 20 & 24 \end{bmatrix}; = B^{-1} = \frac{1}{160}\begin{bmatrix} 24 & -10 \\ -20 & 15 \end{bmatrix}$$

$$yB_2 = C_B; [y_1 \quad y_2]B_0E_1E_2 = [300 \quad 250] \qquad E_1 = \begin{bmatrix} 15 & 0 \\ 20 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 2/3 \\ 0 & 32/3 \end{bmatrix}$$

$$uE_2 = C_B; [u_1 \quad u_2]\begin{bmatrix} 1 & 2/3 \\ 0 & 32/3 \end{bmatrix} = [300 \quad 250] from\ which\ u = \begin{bmatrix} 300 & \frac{75}{16} \end{bmatrix}$$

$$yE_1 = u; [y_1 \quad y_2]\begin{bmatrix} 15 & 0 \\ 20 & 1 \end{bmatrix} = \begin{bmatrix} 300 & \frac{75}{16} \end{bmatrix} from\ which\ y = \begin{bmatrix} \frac{55}{4} & \frac{75}{16} \end{bmatrix}$$

Eta factorization of the basis - example

We do not explicitly invert the matrix but use the E matrices to substitute and get the values. We use this method to get y and entering column where we used $y = C_B B^{-1}$ and entering column = $B^{-1}P_j$

In the kth iteration $B_k = B_0\ E_1\ E_2\ E_3 \ldots E_k$

Time per iteration of revised simplex = 32m + 10n
Time per iteration of standard simplex = mn/4
Revised simplex is better when m > 100

Average number of iterations of simplex (sample) – Avis and Chvatal (1978)

m, n	10	20	30	40	50
10	9.4	14.2	17.4	19.4	20.2
20		25.2	30.7	38	41.5
30			44.4	52.7	62.9
40				67.6	78.7
50					95.2

Largest coefficient rule

m, n	10	20	30	40	50
10	7.02	9.17	10.8	12.1	12.6
20		16.2	20.2	24.2	27.3
30			28.7	34.5	39.4
40				43.3	39.9
50					58.9

Largest increase rule

With $m \leq 50$ and $m + n \leq 200$, Dantzig (1963) reported that the number of iterations is usually $\leq 3m/2$, rarely going up to $3m$.

Complexity issues

1. Algebraic method – ${}^nC_m$ iterations.
2. Klee and Minty problems - 2^{n-1} iterations

$$Maximize \sum_{i=1}^{n} 10^{n-j} X_j$$

$$2\sum_{j=1}^{i-1} 10^{i-j} X_j + X_i \le 100^{i-1}$$

$X_j \ge 0.$

Maximize $100 X_1 + 10 X_2 + X_3$
Subject to $X_1 \le 1$
$20 X_1 + X_2 \le 100$
$200 X_1 + 20X_2 + X_3 \le 10000$
$X_1, X_2, X_3 \ge 0.$

Polynomially bounded algorithms

1. Ellipsoid Algorithm (1979)
2. Karmarkar's algorithm (1984)

Simplex Method

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Iterative Improvement

Algorithm design technique for solving optimization problems

- Start with a feasible solution
- Repeat the following step until no improvement can be found:
  - change the current feasible solution to a feasible solution with a better value of the objective function
- Return the last feasible solution as optimal

Note: Typically, a change in a current solution is “small” (*local search*)

Major difficulty: Local optimum vs. global optimum

Examples

- simplex method
- Ford-Fulkerson algorithm for maximum flow problem
- maximum matching of graph vertices
- Gale-Shapley algorithm for the stable marriage problem
- local search heuristics

Linear Programming

Linear programming (LP) problem is to optimize a linear function of several variables subject to linear constraints:

maximize (or minimize) $c_1 x_1 + \ldots + c_n x_n$

subject to $a_{i\,1} x_1 + \ldots + a_{in} x_n \leq (\text{or} \geq \text{or} =)\ b_i$,

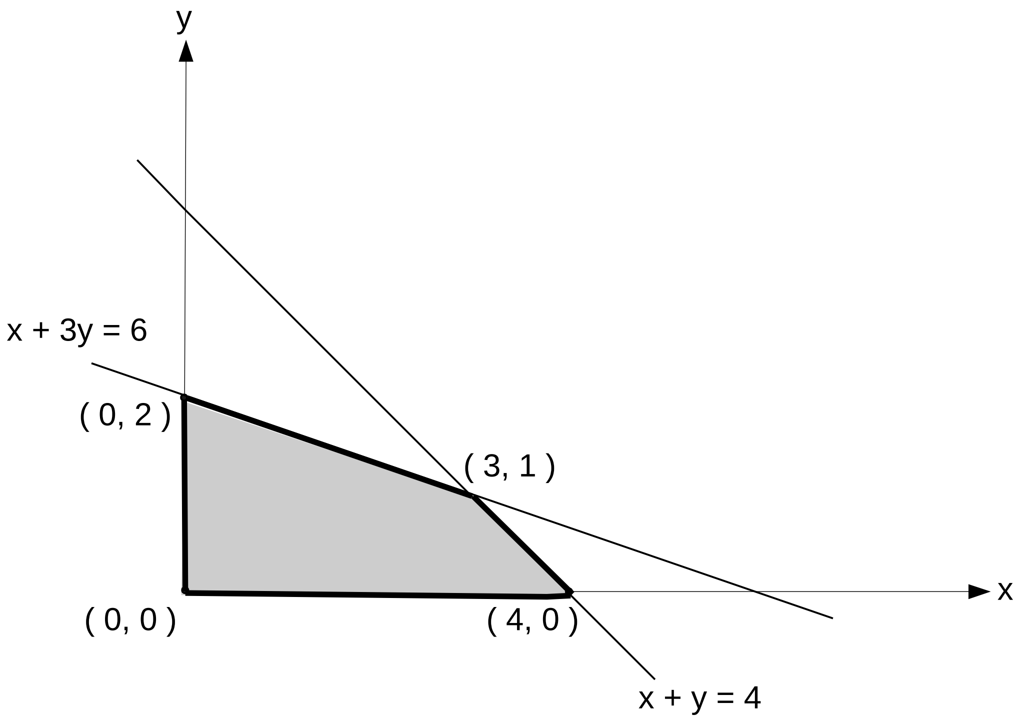
$i = 1,\ldots,m$ $x_1 \geq 0, \ldots, x_n \geq 0$

The function $z = c_1 x_1 + \ldots + c_n x_n$ is called the *objective function*;

constraints $x_1 \geq 0, \ldots, x_n \geq 0$ are called *nonnegativity constraints*

$$\begin{array}{ll} \text{maximize} & 3x + 5y \\ \text{subject to} & x + y \le 4 \\ & x + 3y \le 6 \\ & x \ge 0, \quad y \ge 0 \end{array}$$

Feasible Region



3 possible outcomes in solving an LP problem

- has a finite optimal solution, which may not be unique

- *unbounded*: the objective function of maximization (minimization) LP problem is unbounded from above (below) on its feasible region

- *infeasible*: there are no points satisfying all the constraints, i.e. the constraints are contradictory maximize $3x + 5y$

subject to $x + y >= 4$

$x + 3y >= 6$

$x \geq 0, \; y \geq 0$

THEOREM 9.1 Optimal Solution of a Linear Programming Problem

If a linear programming problem has an optimal solution, then it must occur at a vertex of the set of feasible solutions. If the problem has more than one optimal solution, then at least one of them must occur at a vertex of the set of feasible solutions. In either case, the value of the objective function is unique.

Graphical Method for Solving a Linear Programming Problem

To solve a linear programming problem involving two variables by the graphical method, use the steps listed below.

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are *feasible solutions*.)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and maximum value will exist. (For an unbounded region, *if* an optimal solution exists, then it will occur at a vertex.)

Example

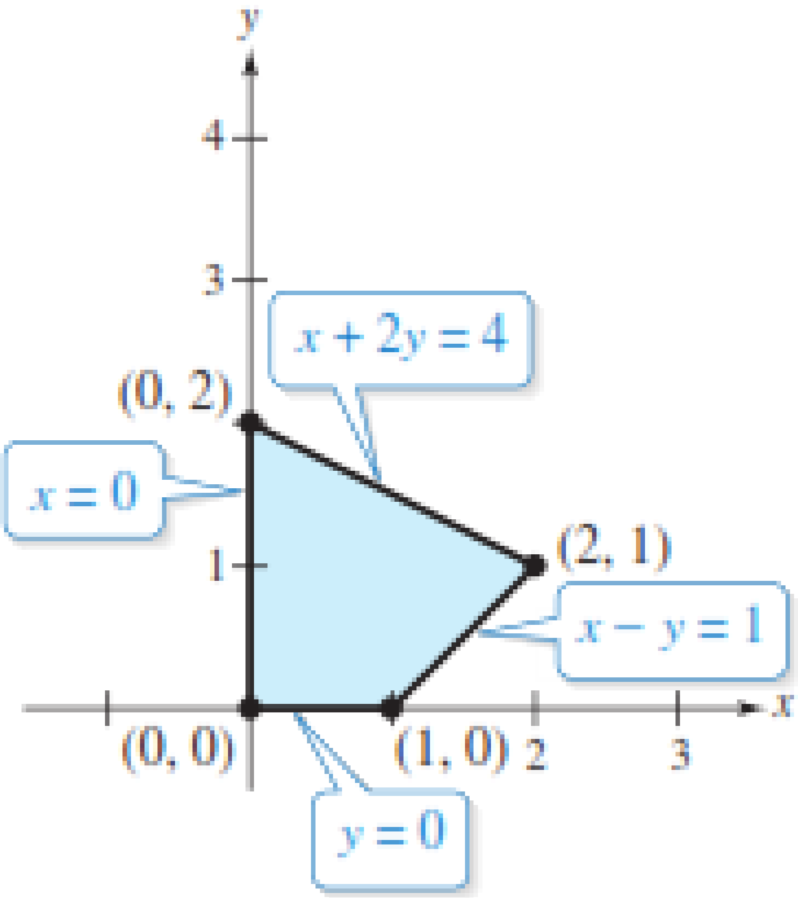
Find the maximum value of

$$z = 3x + 2y \qquad \text{Objective function}$$

subject to the constraints listed below.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}\right\} \quad \text{Constraints}$$

Example (2,1)



Example

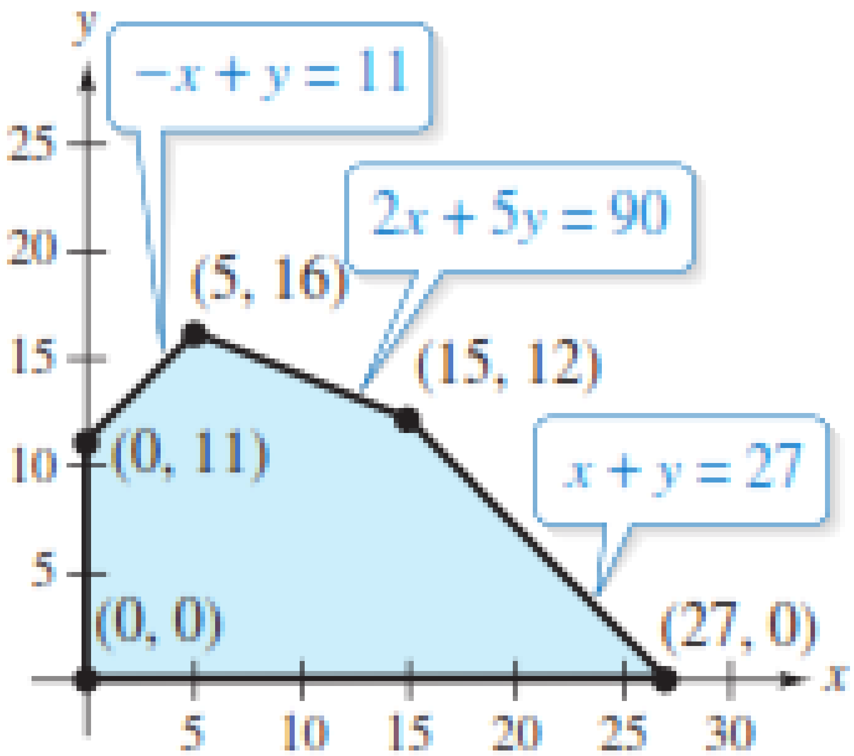
Find the maximum value of the objective function

$z = 4x + 6y$ Objective function

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left.\begin{aligned} -x + y &\leq 11 \\ x + y &\leq 27 \\ 2x + 5y &\leq 90. \end{aligned}\right\} \quad \text{Constraints}$$

(15,12)



y
$-x + y = 11$
25
$2x + 5y = 90$
20
(5, 16)
15
(15, 12)
10
(0, 11)
$x + y = 27$
5
(0, 0)
(27, 0)
x
5
10
15
20
25
30

Example

2. A farmer has a 320 acre farm on which she plants two crops: corn and soybeans. For each acre of corn planted, her expenses are $50 and for each acre of soybeans planted, her expenses are $100. Each acre of corn requires 100 bushels of storage and yields a profit of $60. Each acre of soybeans requires 40 bushels of storage and yields a profit of $90. If the total amount of storage space available is 19,200 bushels and the farmer has only $20,000 on hand, how many acres of each crop should she plant in order to maximize her profit? What will her profit be if she follows this strategy?

Solution

x: acres of corn
y: acres of soybeans

$P = 60x + 90y$

$$\begin{cases} x + y \le 320 & \text{(acres)} \\ 50x + 100y \le 20{,}000 & \text{(expenses)} \\ 100x + 40y \le 19{,}200 & \text{(storage)} \\ x \ge 0 \\ y \ge 0 \end{cases}$$

1. A potter is making cups and plates. It takes her 6 minutes to make a cup and 3 minutes to make a plate. Each cup uses ¾ lb of clay and each plate use one lb of clay. She has 20 hours available for making the cups and plates and has 250 lbs of clay on hand. She makes a profit of $2 on each cup and $1.50 on each plate. How many cups and how many plates should she make in order to maximize her profit?

x: # of cups
y: # of plates

$P = 2x + 1.50y$

$$\begin{cases} 6x + 3y \leq 1200 & \text{(time)} \\ \frac{3}{4}x + y \leq 250 & \text{(clay)} \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Application

LINEAR ALGEBRA APPLIED

When financial institutions replenish automatic teller machines (ATMs), they need to take into account a large number of variables and constraints to keep the machines stocked appropriately. Demand for cash machines fluctuates with such factors as weather, economic conditions, day of the week, and even road construction. Further complicating the matter in the United States is a penalty for depositing and withdrawing money from the Federal Reserve in the same week. To address this complex problem, a company that specializes in providing financial services technology can create high-end optimization software to set up and solve a linear programming problem with many variables and constraints. The company determines an equation for the objective function to minimize total cash in ATMs, while establishing constraints on travel routes, service vehicles, penalty fees, and so on. The optimal solution generated by the software allows the company to build detailed ATM restocking schedules.

Simplex Method

Standard Form of a Linear Programming Problem

A linear programming problem is in **standard form** when it seeks to *maximize* the objective function $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ subject to the constraints

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m
\end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. After adding slack variables, the corresponding system of **constraint equations** is

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 \qquad\qquad &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad + s_2 \qquad &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + s_m &= b_m
\end{aligned}$$

where $s_i \geq 0$.

Example

maximum value of $z = 4x_1 + 6x_2$,

$$\begin{aligned} -x_1 + x_2 &\le 11 \\ x_1 + x_2 &\le 27 \\ 2x_1 + 5x_2 &\le 90. \end{aligned}$$

Slack Variables

$$
\begin{aligned}
-x_1 + x_2 + s_1 &= 11 \\
x_1 + x_2 + s_2 &= 27 \\
2x_1 + 5x_2 + s_3 &= 90.
\end{aligned}
$$

$$-c_1x_1 - c_2x_2 - \cdot \cdot \cdot - c_nx_n + (0)s_1 + (0)s_2 + \cdot \cdot \cdot + (0)s_m + z = 0.$$

slack variable" called an **artificial variable**

X1,x2 = Non basic variable

$$z = 4x_1 + 6x_2$$ Objective function

$$\left.\begin{aligned} -x_1 + x_2 + s_1 \quad\quad &= 11 \\ x_1 + x_2 \quad + s_2 \quad &= 27 \\ 2x_1 + 5x_2 \quad\quad + s_3 &= 90 \end{aligned}\right\}$$ Constraints

is shown below.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
−1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
−4	−6	0	0	0	0	

↑ Current z-value

Initial Solution

$$x_1 = 0$$
$$x_2 = 0$$
$$s_1 = 11$$
$$s_2 = 27$$
$$s_3 = 90.$$

This solution is a basic feasible solution and is often written as

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90).$$

Step 2

- To perform an **optimality check for a solution represented by a simplex tableau,**
- look at the entries in the bottom row of the tableau. If any of these entries are negative (as above), then the current solution is *not optimal.*

Pivoting

1. The **entering variable** corresponds to the least (the most negative) entry in the bottom row of the tableau, excluding the "b-column."
2. The **departing variable** corresponds to the least nonnegative ratio b_i/a_{ij} in the column determined by the entering variable, when $a_{ij} > 0$.
3. The entry in the simplex tableau in the entering variable's column and the departing variable's row is the **pivot.**

Contd…

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
−1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
−4	−6	0	0	0	0	

SOLUTION

The objective function for this problem is

$$z = 4x_1 + 6x_2.$$

Note that the current solution

$$(x_1 = 0, x_2 = 0, s_1 = 11, s_2 = 27, s_3 = 90)$$

corresponds to a z-value of 0. To improve this solution, choose x_2 as the entering variable, because −6 is the least entry in the bottom row.

Contd…

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	
	↑ Entering					

To see *why* you choose x_2 as the entering variable, remember that $z = 4x_1 + 6x_2$. So, it appears that a unit change in x_2 produces a change of 6 in z, whereas a unit change in x_1 produces a change of only 4 in z.

Contd…

- In the event of a tie when choosing entering and/or departing variables, any of the tied variables may be chosen.

To find the departing variable, locate the b_i's that have corresponding positive elements in the entering variable's column and form the ratios

$$\frac{11}{1} = 11, \quad \frac{27}{1} = 27, \quad \text{and} \quad \frac{90}{5} = 18.$$

Here the least nonnegative ratio is 11, so choose s_1 as the departing variable.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables	
−1	1	1	0	0	11	s_1	← Departing
1	1	0	1	0	27	s_2	
2	5	0	0	1	90	s_3	
−4	−6	0	0	0	0		
	↑ Entering						

Note that the pivot is the entry in the first row and second column. Now, use Gauss-Jordan elimination to obtain the improved solution shown below.

Before Pivoting **After Pivoting**

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ -4 & -6 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \begin{array}{l} \\ -R_1 + R_2 \\ -5R_1 + R_3 \\ 6R_1 + R_4 \end{array}$$

The new tableau is shown below.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	66	

Note that x_2 has replaced s_1 in the basic variables column and the improved solution

$$(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$$

has a z-value of

$$z = 4x_1 + 6x_2 = 4(0) + 6(11) = 66.$$

II Iteration

In Example 1, the improved solution is not optimal because the bottom row has a negative entry. So, apply another iteration of the simplex method to improve the solution further. Choose x_1 as the entering variable. Moreover, the lesser of the ratios $16/2 = 8$ and $35/7 = 5$ is 5, so s_3 is the departing variable. Gauss-Jordan elimination produces the matrices shown below.

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \begin{array}{l} \\ \\ \frac{1}{7}R_3 \\ \\ \end{array}$$

$$\longrightarrow \begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & \frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix} \begin{array}{l} R_1 + R_3 \\ R_2 - 2R_3 \\ \\ R_4 + 10R_3 \end{array}$$

So, the new simplex tableau is as shown below.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

In this tableau, there is still a negative entry in the bottom row. So, choose s_1 as the entering variable and s_2 as the departing variable, as shown in the next tableau.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2 ← Departing
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	
		↑ Entering				

One more iteration of the simplex method gives the tableau below. (Check this.)

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	x_2
0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	14	s_1
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	x_1
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132	← Maximum z-value

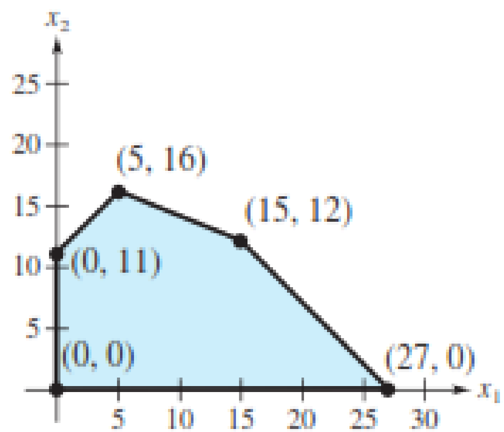
In this tableau, there are no negative elements in the bottom row. So, the optimal solution is

$$(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$$

with

$$z = 4x_1 + 6x_2 = 4(15) + 6(12) = 132.$$

$$\begin{array}{ccccccc} (0, 0) & & (0, 11) & & (5, 16) & & (15, 12) \\ & \rightarrow & & \rightarrow & & \rightarrow & \\ z = 0 & & z = 66 & & z = 116 & & z = 132 \end{array}$$



The Simplex Method (Standard Form)

To solve a linear programming problem in standard form, use the steps below.

1. Convert each inequality in the set of constraints to an equation by adding slack variables.
2. Create the initial simplex tableau.
3. Locate the most negative entry in the bottom row, excluding the "b-column." This entry is called the entering variable, and its column is the **entering column.** (If ties occur, then any of the tied entries can be used to determine the entering column.)
4. Form the ratios of the entries in the "b-column" with their corresponding positive entries in the entering column. (If all entries in the entering column are 0 or negative, then there is no maximum solution.) The **departing row** corresponds to the least nonnegative ratio b_i/a_{ij}. (For ties, choose any corresponding row.) The entry in the departing row and the entering column is called the **pivot.**
5. Use elementary row operations to change the pivot to 1 and all other entries in the entering column to 0. This process is called **pivoting.**
6. When all entries in the bottom row are zero or positive, this is the final tableau. Otherwise, go back to Step 3.
7. If you obtain a final tableau, then the linear programming problem has a maximum solution. The maximum value of the objective function is the entry in the lower right corner of the tableau.

$$
\begin{aligned}
\text{maximize} \quad & 3x + 5y \\
\text{subject to} \quad & x + y \le 4 \\
& x + 3y \le 6 \\
& x \ge 0, \quad y \ge 0.
\end{aligned}
$$

$$\begin{array}{ll} \text{maximize} & 3x + 5y + 0u + 0v \\ \text{subject to} & x + y + u \phantom{{}+v} = 4 \\ & x + 3y + + v = 6 \\ & x,\, y,\, u,\, v \geq 0. \end{array}$$

(3,1) z=14

	x	y	u	v	
u	1	1	1	0	4
v	1	3	0	1	6
	−3	−5	0	0	0

(2,0) Z = 6

$$\begin{aligned} \text{maximize} \quad & 3x + y \\ \text{subject to} \quad & -x + y \leq 1 \\ & 2x + y \leq 4 \\ & x \geq 0,\ y \geq 0 \end{aligned}$$

	x	y	u	v	
u	-1	1	1	0	1
v	2	1	0	1	4
	-3	-1	0	0	0

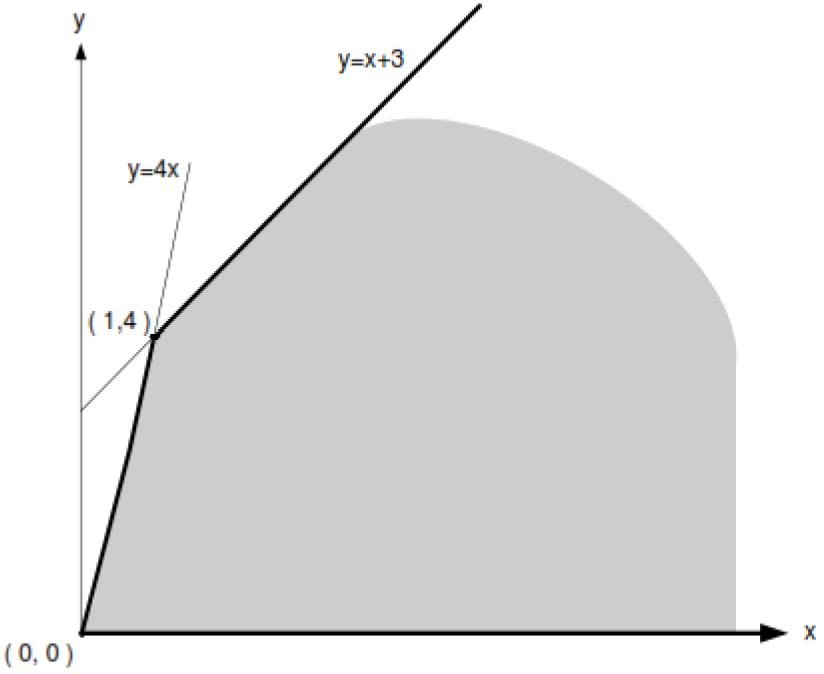
$\theta_v = \frac{4}{2}$

	x	y	u	v	
u	0	$\frac{3}{2}$	1	$\frac{1}{2}$	3
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	2
	0	$\frac{1}{2}$	0	$\frac{3}{2}$	6

The optimal solution found is $x = 2$, $y = 0$, with the maximal value of the objective function equal to 6.

$$
\begin{aligned}
\text{maximize} \quad & x + 2y \\
\text{subject to} \quad & 4x \geq y \\
& y \leq 3 + x \\
& x \geq 0,\ y \geq 0
\end{aligned}
$$

Unbounded



$$
\begin{array}{rl}
\text{maximize} & x + 2y \\
\text{subject to} & -4x + y + u \qquad\quad = 0 \\
& -x + y + \qquad +v = 3 \\
& x, y, u, v \geq 0.
\end{array}
$$

	x	y	u	v		
← u	−4	1	1	0	0	$\theta_u = \frac{0}{1}$
v	−1	1	0	1	3	$\theta_v = \frac{3}{1}$
	−1	−2	0	0	0	
		↑				

	x	y	u	v		
y	−4	1	1	0	0	
← v	3	0	−1	1	3	$\theta_v = \frac{3}{3}$
	−9	0	2	0	0	
	↑					

	x	y	u	v	
y	0	1	$-\frac{1}{3}$	$\frac{4}{3}$	4
x	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	1
	0	0	−1	3	9
			↑		

Find the dual of the linear programming problem

$$\begin{aligned} \text{maximize} \quad & x_1 + 4x_2 - x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \le 6 \\ & x_1 - x_2 - 2x_3 \le 2 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Solution

$$\begin{array}{rl} \text{minimize} & 6y_1 + 2y_2 \\ \text{subject to} & y_1 + y_2 \geq 1 \\ & y_1 - y_2 \geq 4 \\ & y_1 - 2y_2 \geq -1 \\ & y_1, y_2 \geq 0. \end{array}$$

Example

Use the simplex method to find the maximum value of

$$z = 2x_1 - x_2 + 2x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 2x_1 + x_2 \phantom{{}- 2x_3} &\le 10 \\ x_1 + 2x_2 - 2x_3 &\le 20 \\ x_2 + 2x_3 &\le 5 \end{aligned}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 10, 20, 5)$$

the initial and subsequent simplex tableaus for this problem are shown below. (Check the computations, and note the "tie" that occurs when choosing the first entering variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	0	1	0	0	10	s_1	
1	2	−2	0	1	0	20	s_2	
0	1	2	0	0	1	5	s_3	← Departing
−2	1	−2	0	0	0	0		
		↑ Entering						

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	0	1	0	0	10	s_1	← Departing
1	3	0	0	1	1	25	s_2	
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3	
−2	2	0	0	0	1	5		
↑ Entering								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	5	x_1
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	1	20	s_2
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3
0	3	0	1	0	1	15	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = \left(5, 0, \tfrac{5}{2}, 0, 20, 0\right)$$

and the maximum value of z is 15.

Example

Use the simplex method to find the maximum value of

$$z = 3x_1 + 2x_2 + x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 30 \\ 2x_1 + 3x_2 + x_3 &\leq 60 \\ x_1 + 2x_2 + 3x_3 &\leq 40 \end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

Using the basic feasible solution $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 30, 60, 40)$, the initial and subsequent simplex tableaus for this problem are shown below. (Note that s_1 is an artificial variable, rather than a slack variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
4	1	1	1	0	0	30	s_1	← Departing
2	3	1	0	1	0	60	s_2	
1	2	3	0	0	1	40	s_3	
−3	−2	−1	0	0	0	0		
↑ Entering								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{15}{2}$	x_1	
0	$\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	45	s_2	← Departing
0	$\frac{7}{4}$	$\frac{11}{4}$	$-\frac{1}{4}$	0	1	$\frac{65}{2}$	s_3	
0	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	0	0	$\frac{45}{2}$		
	↑ Entering							

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	0	$\frac{1}{5}$	$\frac{3}{10}$	$-\frac{1}{10}$	0	3	x_1
0	1	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	0	18	x_2
0	0	$\frac{12}{5}$	$\frac{1}{10}$	$-\frac{7}{10}$	1	1	s_3
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	45	

So the optimal solution is $(x_1, x_2, x_3, s_1, s_2, s_3) = (3, 18, 0, 0, 0, 1)$ and the maximum value of z is 45. This solution satisfies the equation provided in the constraints, because $4(3) + 1(18) + 1(0) = 30$.

A manufacturer produces three types of plastic fixtures. The table below shows the times required for molding, trimming, and packaging. (Times are in hours per dozen fixtures, and profits are in dollars per dozen fixtures.)

Process	*Type A*	*Type B*	*Type C*
Molding	1	2	$\frac{3}{2}$
Trimming	$\frac{2}{3}$	$\frac{2}{3}$	1
Packaging	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
Profit	\$11	\$16	\$15

The maximum amounts of production time that the manufacturer can allocate to each component are listed below.

Molding: 12,000 hours

Trimming: 4600 hours

Packaging: 2400 hours

How many dozen units of each type of fixture should the manufacturer produce to obtain a maximum profit?

Solution

Let x_1, x_2, and x_3 represent the numbers of dozens of types A, B, and C fixtures, respectively. The objective function to be maximized is

$$\text{Profit} = 11x_1 + 16x_2 + 15x_3$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. Moreover, using the information in the table, you can write the constraints below.

$$\begin{aligned} x_1 + 2x_2 + \tfrac{3}{2}x_3 &\leq 12{,}000 \\ \tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 + x_3 &\leq 4600 \\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 &\leq 2400 \end{aligned}$$

So, the initial simplex tableau with the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 12{,}000, 4600, 2400)$$

is as shown below.

A minimization problem is in **standard form** when the objective function

$$w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

is to be minimized, subject to the constraints

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m
\end{aligned}$$

Minimisation problem

Minimization Problem: Find the minimum value of

$w = 0.12x_1 + 0.15x_2$ Objective function

subject to the constraints

$$\left.\begin{aligned} 60x_1 + 60x_2 &\geq 300 \\ 12x_1 + 6x_2 &\geq 36 \\ 10x_1 + 30x_2 &\geq 90 \end{aligned}\right\} \quad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

$$\begin{bmatrix} 60 & 60 & 300 \\ 12 & 6 & 36 \\ 10 & 30 & 90 \\ 0.12 & 0.15 & 0 \end{bmatrix}$$

Next, form the transpose of this matrix.

$$\begin{bmatrix} 60 & 12 & 10 & 0.12 \\ 60 & 6 & 30 & 0.15 \\ 300 & 36 & 90 & 0 \end{bmatrix}$$

To interpret the transposed matrix as a maximization problem, introduce new variables, y_1, y_2, and y_3. This corresponding maximization problem is called the **dual** of the original minimization problem.

Dual Maximization Problem: Find the maximum value of

$$z = 300y_1 + 36y_2 + 90y_3$$ Dual objective function

subject to the constraints

$$\left.\begin{aligned} 60y_1 + 12y_2 + 10y_3 &\le 0.12 \\ 60y_1 + 6y_2 + 30y_3 &\le 0.15 \end{aligned}\right\}$$ Constaints

where $y_1 \geq 0$, $y_2 \geq 0$, and $y_3 \geq 0$.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
60	12	10	1	0	0.12	s_1	← Departing
60	6	30	0	1	0.15	s_2	
−300	−36	−90	0	0	0		
↑ Entering							

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
1	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{60}$	0	$\frac{1}{500}$	y_1	
0	−6	20	−1	1	$\frac{3}{100}$	s_2	← Departing
0	24	−40	5	0	$\frac{3}{5}$		
		↑ Entering					

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{4}$	0	$\frac{1}{40}$	$-\frac{1}{120}$	$\frac{7}{4000}$	y_1
0	$-\frac{3}{10}$	1	$-\frac{1}{20}$	$\frac{1}{20}$	$\frac{3}{2000}$	y_3
0	12	0	3	2	$\frac{33}{50}$	
			↑ x_1	↑ x_2		

THEOREM 9.2 The von Neumann Duality Principle

The objective value w of a minimization problem in standard form has a minimum value if and only if the objective value z of the dual maximization problem has a maximum value. Moreover, the minimum value of w is equal to the maximum value of z.

Summary

Solving a Minimization Problem

A minimization problem is in standard form when the objective function

$$w = c_1x_1 + c_2x_2 + \cdot \cdot \cdot + c_nx_n$$

is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdot \cdot \cdot + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdot \cdot \cdot + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdot \cdot \cdot + a_{mn}x_n &\geq b_m \end{aligned}$$

where

$$x_i \geq 0 \quad \text{and} \quad b_i \geq 0.$$

To solve this problem, use the steps below.

1. Form the **augmented matrix** for the system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\ c_1 & c_2 & \dots & c_n & 0 \end{bmatrix}$$

2. Form the transpose of this matrix.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} & c_1 \\ a_{12} & a_{22} & \cdots & a_{m2} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} & c_n \\ b_1 & b_2 & \cdots & b_m & 0 \end{bmatrix}$$

3. Form the **dual maximization problem** corresponding to this transposed matrix. That is, find the maximum of the objective function

$$z = b_1y_1 + b_2y_2 + \cdots + b_my_m$$

subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\le c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &\le c_2 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &\le c_n \end{aligned}$$

where

$$y_1 \ge 0, \quad y_2 \ge 0, \quad \ldots, \quad \text{and} \quad y_m \ge 0.$$

Apply the **simplex method** to the dual maximization problem. The maximum value of z will be the minimum value of w. Moreover, the values of $x_1, x_2, \ldots, x_n$ will occur in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

Example

Find the minimum value of

$$w = 3x_1 + 2x_2 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2x_1 + x_2 &\geq 6 \\ x_1 + x_2 &\geq 4 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

The augmented matrices corresponding to this problem are shown below.

$$\begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 4 \\ 3 & 2 & 0 \end{bmatrix}$$

Minimization Problem

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 6 & 4 & 0 \end{bmatrix}$$

Dual Maximization Problem

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 4y_2 \qquad \text{Dual objective function}$$

subject to the constraints

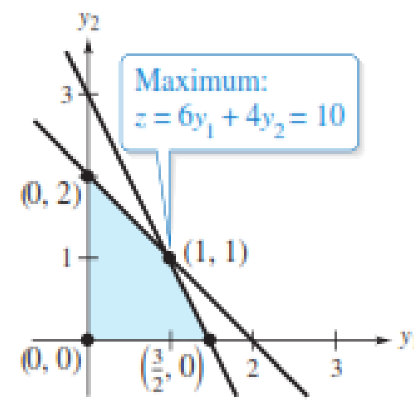
$$\left.\begin{aligned} 2y_1 + y_2 &\le 3 \\ y_1 + y_2 &\le 2 \end{aligned}\right\} \qquad \text{Dual constraints}$$

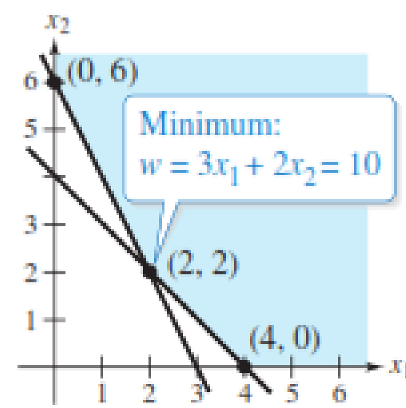
where $y_1 \ge 0$ and $y_2 \ge 0$. Now apply the simplex method to the dual problem, as shown below.

y_1	y_2	s_1	s_2	b	Basic Variables	
2	1	1	0	3	s_1	← Departing
1	1	0	1	2	s_2	
−6	−4	0	0	0		
↑ Entering						

y_1	y_2	s_1	s_2	b	Basic Variables	
1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	y_1	
0	$\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	s_2	← Departing
0	−1	3	0	9		
	↑ Entering					

y_1	y_2	s_1	s_2	b	Basic Variables
1	0	1	−1	1	y_1
0	1	−1	2	1	y_2
0	0	2	2	10	
		↑ x_1	↑ x_2		





Example

Find the minimum value of $w = 2x_1 + 10x_2 + 8x_3$ subject to the constraints

$$\left.\begin{aligned} x_1 + x_2 + x_3 &\ge 6 \\ x_2 + 2x_3 &\ge 8 \\ -x_1 + 2x_2 + 2x_3 &\ge 4 \end{aligned}\right\} \quad \text{Constraints}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

The augmented matrices corresponding to this problem are shown below.

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ -1 & 2 & 2 & 4 \\ 2 & 10 & 8 & 0 \end{bmatrix}$$

Minimization Problem

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 2 & 10 \\ 1 & 2 & 2 & 8 \\ 6 & 8 & 4 & 0 \end{bmatrix}$$

Dual Maximization Problem

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 8y_2 + 4y_3 \qquad \text{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} y_1 \qquad\quad - \;\; y_3 &\le 2 \\ y_1 + \;\; y_2 + 2y_3 &\le 10 \\ y_1 + 2y_2 + 2y_3 &\le 8 \end{aligned}\right\} \qquad \text{Dual constraints}$$

where $y_1 \ge 0$, $y_2 \ge 0$, and $y_3 \ge 0$. Now apply the simplex method to the dual problem as shown below.

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables	
1	0	−1	1	0	0	2	s_1	
1	1	2	0	1	0	10	s_2	
1	2	2	0	0	1	8	s_3	← Departing
−6	−8	−4	0	0	0	0		
	↑ Entering							

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables	
1	0	−1	1	0	0	2	s_1	← Departing
$\frac{1}{2}$	0	1	0	1	$-\frac{1}{2}$	6	s_2	
$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	4	y_2	
−2	0	4	0	0	4	32		
↑ Entering								

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables
1	0	-1	1	0	0	2	y_1
0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	5	s_2
0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	3	y_2
0	0	2	2	0	4	36	
			$\uparrow$	$\uparrow$	$\uparrow$		
			x_1	x_2	x_3		

From this final simplex tableau, the maximum value of z is 36. So, the minmum value of w is 36, and this occurs when $x_1 = 2$, $x_2 = 0$, and $x_3 = 4$.

Example

A petroleum company owns two refineries. Refinery 1 costs \$20,000 per day to operate, and refinery 2 costs \$25,000 per day to operate. The table shows the numbers of barrels of each grade of oil the refineries can produce each day.

Grade	*Refinery 1*	*Refinery 2*
High-grade	400	300
Medium-grade	300	400
Low-grade	200	500

The company has orders totaling 25,000 barrels of high-grade oil, 27,000 barrels of medium-grade oil, and 30,000 barrels of low-grade oil. How many days should it run each refinery to minimize its costs and still refine enough oil to meet its orders?

Let x_1 and x_2 represent the numbers of days the two refineries operate. Then the total cost is

$$C = 20{,}000x_1 + 25{,}000x_2. \qquad \text{Objective function}$$

The constraints are

$$\left.\begin{array}{rl} \text{(High-grade)} & 400x_1 + 300x_2 \geq 25{,}000 \\ \text{(Medium-grade)} & 300x_1 + 400x_2 \geq 27{,}000 \\ \text{(Low-grade)} & 200x_1 + 500x_2 \geq 30{,}000 \end{array}\right\} \quad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$. The augmented matrices corresponding to this problem are shown below.

$$\begin{bmatrix} 400 & 300 & 25{,}000 \\ 300 & 400 & 27{,}000 \\ 200 & 500 & 30{,}000 \\ 20{,}000 & 25{,}000 & 0 \end{bmatrix}$$

Minimization Problem

$$\begin{bmatrix} 400 & 300 & 200 & 20{,}000 \\ 300 & 400 & 500 & 25{,}000 \\ 25{,}000 & 27{,}000 & 30{,}000 & 0 \end{bmatrix}$$

Dual Maximization Problem

Now apply the simplex method to the dual problem.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
400	300	200	1	0	20,000	s_1	
300	400	500	0	1	25,000	s_2	← Departing
−25,000	−27,000	−30,000	0	0	0		
		↑ Entering					

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
280	140	0	1	$-\frac{2}{5}$	10,000	s_1	← Departing
$\frac{3}{5}$	$\frac{4}{5}$	1	0	$\frac{1}{500}$	50	y_3	
−7000	−3000	0	0	60	1,500,000		
↑ Entering							

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{280}$	$-\frac{1}{700}$	$\frac{250}{7}$	y_1
0	$\frac{1}{2}$	1	$-\frac{3}{1400}$	$\frac{1}{350}$	$\frac{200}{7}$	y_3
0	500	0	25	50	1,750,000	
			↑ x_1	↑ x_2		

From this final simplex tableau, the minimum cost is

$C = \$1,750,000$ Minimum cost

and this occurs when

$x_1 = 25$ and $x_2 = 50.$

So, the two refineries should be operated for the numbers of days shown below.

Refinery 1: 25 days

Refinery 2: 50 days

Note that by operating the two refineries for these numbers of days, the company produces the amounts of oil listed below.

High-grade oil: $400(25) + 300(50) = 25,000$ barrels
Medium-grade oil: $300(25) + 400(50) = 27,500$ barrels
Low-grade oil: $200(25) + 500(50) = 30,000$ barrels

So, the company refines enough of each grade of oil to meet its orders (with a surplus of 500 barrels of medium-grade oil).

Mixed-Constraint Problem: Find the maximum value of

$$z = x_1 + x_2 + 2x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 2x_1 + x_2 + x_3 &\le 50 \\ 2x_1 + x_2 \phantom{{}+x_3} &\ge 36 \\ x_1 \phantom{{}+x_2} + x_3 &\ge 10 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$. This is a maximization problem,

$$2x_1 + x_2 + x_3 + s_1 = 50.$$

For the other two inequalities, a new type of variable, a **surplus variable,** is introduced, as shown below.

$$\begin{aligned} 2x_1 + x_2 \quad\quad - s_2 \quad\quad &= 36 \\ x_1 \quad\quad + x_3 \quad\quad - s_3 &= 10 \end{aligned}$$

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
2	1	1	1	0	0	50	s_1
2	1	0	0	−1	0	36	s_2
1	0	1	0	0	−1	10	s_3
−1	−1	−2	0	0	0	0	

To eliminate the surplus variables from the current solution, use "trial and error." That is, in an effort to find a feasible solution, arbitrarily choose new entering variables. For example, it seems reasonable to select x_3 as the entering variable, because its column has the most negative entry in the bottom row.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	1	1	0	0	50	s_1	
2	1	0	0	−1	0	36	s_2	
1	0	1	0	0	−1	10	s_3	← Departing
−1	−1	−2	0	0	0	0		
		↑ Entering						

After pivoting, the new simplex tableau is as shown below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
1	1	0	1	0	1	40	s_1	
2	1	0	0	−1	0	36	s_2	← Departing
1	0	1	0	0	−1	10	x_3	
1	−1	0	0	0	−2	20		
	↑ Entering							

The current solution $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 10, 40, -36, 0)$ is still not feasible, so choose x_2 as the entering variable and pivot to obtain the simplex tableau below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
-1	0	0	1	1	1	4	s_1	← Departing
2	1	0	0	-1	0	36	x_2	
1	0	1	0	0	-1	10	x_3	
3	0	0	0	-1	-2	56		
					↑ Entering			

At this point, you obtain a feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 36, 10, 4, 0, 0).$$

From here, continue by applying the simplex method as usual. Note that the next entering variable is s_3. After pivoting one more time, you obtain the final simplex tableau shown below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
-1	0	0	1	1	1	4	s_3
2	1	0	0	-1	0	36	x_2
0	0	1	1	1	0	14	x_3
1	0	0	2	1	0	64	

Note that this tableau is final because it represents a feasible solution and there are no negative entries in the bottom row. So, the maximum value of the objective function is $z = 64$ and this occurs when $x_1 = 0$, $x_2 = 36$, and $x_3 = 14$.

Find the maximum value of

$$z = 3x_1 + 2x_2 + 4x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 3x_1 + 2x_2 + 5x_3 &\le 18 \\ 4x_1 + 2x_2 + 3x_3 &\le 16 \\ 2x_1 + x_2 + x_3 &\ge 4 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

$$
\begin{aligned}
3x_1 + 2x_2 + 5x_3 + s_1 \qquad\qquad &= 18 \\
4x_1 + 2x_2 + 3x_3 \qquad + s_2 \qquad &= 16 \\
2x_1 + x_2 + x_3 \qquad\qquad - s_3 &= 4
\end{aligned}
$$

Next form the initial simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
3	2	5	1	0	0	18	s_1	
4	2	3	0	1	0	16	s_2	
2	1	1	0	0	−1	4	s_3	← Departing
−3	−2	−4	0	0	0	0		

This tableau does not represent a feasible solution because the value of s_3 is negative. So, s_3 should be the departing variable. There are no real guidelines as to which variable should enter the solution, and in fact, any choice will work. However, some entering variables will require more tedious computations than others. For example, choosing x_1 as the entering variable produces the tableau below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
0	$\frac{1}{2}$	$\frac{7}{2}$	1	0	$\frac{3}{2}$	12	s_1
0	0	1	0	1	2	8	s_2
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	2	x_1
0	$-\frac{1}{2}$	$-\frac{5}{2}$	0	0	$-\frac{3}{2}$	6	

Choosing x_2 as the entering variable on the initial tableau instead produces the tableau shown below, which contains "nicer" numbers.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
-1	0	3	1	0	2	10	s_1
0	0	1	0	1	2	8	s_2
2	1	1	0	0	-1	4	x_2
1	0	-2	0	0	-2	8	

Choosing x_3 as the entering variable on the initial tableau will also produce a tableau that does not contain fractions. (Verify this.)

X2 entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
-1	0	3	1	0	2	10	s_1	← Departing
0	0	1	0	1	2	8	s_2	
2	1	1	0	0	-1	4	x_2	
1	0	-2	0	0	-2	8		
		↑ Entering						

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
$-\frac{1}{3}$	0	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{10}{3}$	x_3	
$\frac{1}{3}$	0	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{14}{3}$	s_2	← Departing
$\frac{7}{3}$	1	0	$-\frac{1}{3}$	0	$-\frac{5}{3}$	$\frac{2}{3}$	x_2	
$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{44}{3}$		
					↑ Entering			

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$-\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	x_3
$\frac{1}{4}$	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	1	$\frac{7}{2}$	s_3
$\frac{11}{4}$	1	0	$-\frac{3}{4}$	$\frac{5}{4}$	0	$\frac{13}{2}$	x_2
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	17	

So, the maximum value of the objective function is $z = 17$, and this occurs when

$x_1 = 0$, $x_2 = \frac{13}{2}$, and $x_3 = 1$.