SSN COLLEGE OF ENGINEERING
Department of Mathematics

UMA1477 & UMA1478

UNIT – 2 TWO DIMENSIONAL RANDOM VARIABLES

- **Prepared by Dr.N.Padmapriya**

One dimensional random variable has been considered to a random experiment for which the outcome has only one characteristic. In many situations, we will be interested in recording two or more characteristics of the outcome of a random experiment. i.e., two or more random variables will be defined on the same space.

Two dimensional random variable

Let S be the sample space of a random experiment. Let X and Y be two random variables defined on S. Then the pair (X, Y) is called a *two dimensional random variable* or a bivariate random variable. Let $s \in S$ be a sample point. Since X and Y are random variables associated with S we have real numbers x and y. The bivariate random variable (X, Y) can be considered as a function which assigns to each point $s \in S$ appoint (x, y) in the 2D plane. The range space of a bivariate random variable (X,Y) is denoted by

$$R_{xy} = \{(x, y) / s \in S and X(s) = x, Y(s) = y\}$$

Discrete bivariate random variable

If both the random variables X and Y are discrete then (X,Y) is called a *discrete bivariate random variable.*

Continuous bivariate random variable

If both the random variables X and Y are continuous then (X,Y) is called a *continuous bivariate random variable*.

Mixed bivariate random variable

If one of X and Y is discrete while the other one is continuous, then (X,Y) is called a *mixed bivariate random variable*.

Example

Consider the random experiment in which two fair dice are thrown simultaneously. Define a random variable X as the number on the first die and Y as the number of second die. Then (X,Y) is a *bivariate random variable*.

Joint probability mass function

Let X and Y be random variables defined on sample space S with respective image sets $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_m\}$. The probability of the ordered pair (x_i, y_j) is $P(X = x_i, Y = y_j) = P(X = x_i \cap Y = y_j)$ which is written as $P_{XY}(x_i, y_j)$ or simple P_{ij} is called the joint probability mass function of X and Y and is represented in the following form.

X \ Y	y_1	y_2	$\cdots$	y_j	$\cdots$	y_m	Total
x_1	p_{11}	p_{12}	$\cdots$	p_{1j}	$\cdots$	p_{1m}	$p_{1.}$
x_2	p_{21}	p_{22}	$\cdots$	p_{2j}	$\cdots$	p_{2m}	$p_{2.}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_i	p_{i1}	p_{i2}	$\cdots$	p_{ij}	$\cdots$	p_{im}	$p_{i.}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	$\cdots$	p_{nj}	$\cdots$	p_{nm}	$p_{n.}$
Total	$p_{.1}$	$p_{.2}$	$\cdots$	$p_{.j}$	$\cdots$	$p_{.m}$	1

Properties of joint probability mass function

1. $0 \le P_{XY}(x_i, y_j) \le 1, i = 1,\dots,n, j = 1,\dots,m$
2. $\sum_{x_i} \sum_{y_j} P_{XY}(x_i, y_j) = 1$

Marginal probability mass function

Suppose $P_{XY}(x_i, y_j)$ is the joint probability mass function of (X, Y). Suppose for a fixed value $X = x_i$ the random variable Y can take the possible values $y_j, j = 1,2,\dots,m$. Then, the probability distribution of X is

$$P_X(x_i) = P(X = x_i) = \sum_{y_j} P_{XY}(x_i, y_j)$$

Where the summation is taken over all possible pairs (x_i, y_j) with x_i is fixed. The function $P_X(x_i), i = 1,2,\dots,n$ is called the marginal probability mass function of X.

Similarly,

$$P_Y(y_j) = P(Y = y_j) = \sum_{x_i} P_{XY}(x_i, y_j)$$

Where the summation is taken over all possible pairs (x_i, y_j) with y_j fixed $(j = 1,2,\dots,m)$ is called the marginal probability mass function of Y.

Independent random variables

Two random variables X and Y are said to be independent if

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j), i = 1,2,\dots,n, \quad j = 1,2,\dots,m$$

Otherwise they are said to be dependent. In other words, if $P_{XY}(x_i, y_j) = P_X(x_i)P_Y(y_j)$ then X and Y are independent.

Joint probability distribution

Let (X,Y) be a bivariate random variable. Then their joint distribution function or joint cumulative distribution function is denoted by $F_{XY}(x, y)$ or $F(x, y)$ and it represents the probability that simultaneously the observation (X, Y) will have

$$F_{xy}(x, y) = P(X \le x, Y \le y) = \sum_{x_i \le x} \sum_{y_j \le y} p(x_i, y_j) - discrete$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y) dxdy - continous$$

Properties - jcdf

$1. F(-\infty, y) = 0$

$2. F(x, \infty) = 0$

$3. F(-\infty, \infty) = 1$

$4. \frac{\partial^2 F}{\partial x \partial y} = f(x, y)$

Conditional probability function

Let (X, Y) be a discrete bivariate random variable with joint probability mass function

$$P_{X/Y}\left(x_i / y_j\right) = P\left(X = x_i / Y = y_j\right) = \frac{P(X = x \cap Y = y_j)}{P(Y = y_j)}$$

then,
$$= \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)}, P_Y(y_j) > 0$$

is called the conditional probability mass function of X given Y=y_j. Similarly the conditional probability mass function of Y given X= x_i is defined as

$$P_{Y/X}\left(y_j / x_i\right) = \frac{P_{XY}(x_i, y_j)}{P_X(x_j)}, P_X(x_j) > 0 .$$

Continuous Random Variable

If (X, Y) can assume all values in a specified region R in the xy-plane (X,Y) is called a two dimensional continuous random variable.

Joint probability density function

If (X, Y) is a 2D continuous random variable, then

$P\left\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2} \;\&\; y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right\} = f(x, y)dxdy$ then f(x,y) is called the joint pdf of (X,Y) provided f(x,y) satisfies the following conditions.

Properties-jpdf

1. $f(x, y) \ge 0$
2. $\iint_R f(x, y)dxdy = 1$

.

Marginal probability density function

Let (X, Y) be a continuous bivarate random variable with joint pdf $f_{XY}(x,y)$. The marginal probability density functions of X and Y are defined as

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

respectively.

Conditional probability function

If (X, Y) is a continuous bivariate random variable with joint pdf $f_{XY}(x,y)$ then the conditional probability density function *of Y* given that X = x is defined by

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_x(x)},\ f_x(x) > 0$$

Similarly, the conditional probability density function *of X* given Y=y is

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)},\ f_Y(y) > 0$$

Properties

1. $f_{Y/X}(y/x) \geq 0$

2. $\int_{-\infty}^{\infty} f_{Y/X}(y/x)dy = 1$

Independent random variables

The continuous random variables X and Y with joint pdf $f_{XY}(x,y)$ are said be independent if $f_{XY}(x,y) = f_X(x)f_Y(y)$.

Formulas

Cov(X,Y) = E(XY)-E(X)E(Y)

V(aX+b) = a^2Var(X) +b^2Var(Y) +2abCov(X.Y)

$$\sigma_x^2 = E(X^2) - [E(X)]^2$$

$$\sigma_y^2 = E(Y^2) - [E(Y)]^2$$

Correlation

Karl Pearson's coefficient of correlation = $\rho(X,Y) = r(X,Y) = \dfrac{Cov(X,Y)}{\sigma_x \sigma_y}$, $-1 \le r \le 1$

Lines of regression

Line of regression of X on Y

$$X - \bar{x} = r\frac{\sigma_y}{\sigma_x}(Y - \bar{y})$$

Line of regression of Y on X

$$Y - \bar{y} = r\frac{\sigma_x}{\sigma_y}(X - \bar{x})$$
