UCS1524 – Logic Programming

Normal Forms in First Order Logic

Session Meta Data

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Session Objectives

- Understanding the concept of normal forms in first order logic (FOL)
- Learning the conversion of a formula in FOL to clausal form

Session Outcomes

- At the end of this session, participants will be able to
  - Apply rules to convert the given formula in FOL into clausal form.

Agenda

- Rectified formula
- Prenex normal form
- Skolemization
- Clausal form

Rectified Formula

- A formula is rectified if no variable occurs both bound and free and if all quantifiers in the formula refer to different variables.
- Example

$$F = (\neg \exists x(P(x, z) \vee \forall y Q(x, f(y))) \vee \forall y P(g(x, y), z))$$

- Rectified Formula (by renaming y to w)

$$(\neg \exists x(P(x, z) \vee \forall y Q(x, f(y))) \vee \forall w P(g(x, w), z))$$

Prenex Normal Form

- Because of the quantifiers, we cannot produce the CNF form of a formula directly.
- First step: Produce the prenex normal form
- Equivalences can be used to produce prenex normal form

quantifier prefix + (quantifier-free) matrix

$$Qx_1 Qx_2 Qx_3 \ldots Qx_n \ \varphi$$

- Example

$$\exists z((\neg \exists x(P(x,z) \vee \forall y Q(x, f(y))) \vee \forall w P(g(x,w), z))).$$

- Prenex Normal Form

$$\exists z \forall x \exists y \forall w((\neg(P(x,z) \wedge \neg Q(x, f(y))) \vee P(g(x,w), z))$$

Skolemization

- Existentials are replaced by constant
- Examples:

$\exists x \exists y[Philo(x) \wedge StudentOf(y, x)]$
replaced to
$Philo(a) \wedge StudentOf(b, a)$

Every philosopher writes at least one book.
$\forall x[Philo(x) \rightarrow \exists y[Book(y) \wedge Write(x, y)]]$:
$\forall x[\neg Philo(x) \vee \exists y[Book(y) \wedge Write(x, y)]]$: eliminate implication
$g(x)$ $\forall x[\neg Philo(x) \vee [Book(g(x)) \wedge Write(x, g(x))]]$: Skolem form

$\forall x(\neg P(x) \vee Q(x)) \wedge \forall x P(x) \wedge \exists x \neg Q(x)$
$\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \exists z \neg Q(z)$ – rectified formula
$\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \neg Q(a)$ – Skolemization: existential is replaced by constant

SNF to Clausal Form

- Skolem Normal Form: Prenex normal form without existential quantiers.
- Theorem: It is possible to calculate the Skolem normal form of every closed formula

Skolem Normal Form

quantifier prefix + (quantifier-free) matrix

$$\forall x_1 \forall x_2 \forall x_3 \cdots \forall x_n \; \varphi$$

- Put Matrix into CNF using distribution rule
- Eliminate universal quantifiers
- Eliminate conjunction symbol
- Rename variables so that no variable appears in more than one clause.

Steps to convert to CNF

- Eliminate biconditionals and implications
- Move : ⌐inwards
- Standardize variables: each quantifier should use a different one
- Skolemize: a more general form of existential instantiation. Each existential variable is replaced by a Skolem function of the enclosing universally quantified variables
- Drop universal quantifiers
- Distribute ∧ over V

Steps to convert to CNF - Example

Everyone who loves all animals is loved by someone

$$\forall x[\forall y Animal(y) \Rightarrow Loves(x,y)] \Rightarrow [\exists y Loves(y,x)]$$

- Eliminate biconditionals and implications

$$\forall x \neg[\forall y \neg Animal(y) \vee Loves(x,y)] \vee [\exists y Loves(y,x)]$$

- Move : ⌐inwards $\neg\forall x p \equiv \exists x \neg p,\ \neg\exists x p \equiv \forall x \neg p$

$$\forall x[\exists y \neg(\neg Animal(y) \vee Loves(x,y))] \vee [\exists y Loves(y,x)]$$

$$\forall x[\exists y \neg\neg Animal(y) \wedge \neg Loves(x,y)] \vee [\exists y Loves(y,x)]$$

$$\forall x[\exists y Animal(y) \wedge \neg Loves(x,y)] \vee [\exists y Loves(y,x)]$$

Steps to convert to CNF - Example

- Standardize variables (rectified formula): each quantifier should use a different one

$$\forall x [\exists y Animal(y) \land \neg Loves(x, y)] \lor [\exists z Loves(z, x)]$$

- Skolemize: a more general form of existential instantiation. Each existential variable is replaced by a Skolem function of the enclosing universally quantified variables

$$\forall x [Animal(F(x)) \land \neg Loves(x, F(x))] \lor [Loves(G(x), x)]$$

Steps to convert to CNF - Example

- Drop universal quantifiers

$$[Animal(F(x)) \wedge \neg Loves(x, F(x))] \vee [Loves(G(x), x)]$$

- Distribute ^ over V

$$[Animal(F(x)) \vee Loves(G(x), x)] \wedge [\neg Loves(x, F(x)) \vee Loves(G(x), x)]$$

Summary

- Rectified formula
- Prenex normal form
- Skolemization
- Clausal form

Check your understanding

Which formulas are rectified, in prenex, Skolem, or clause form?

	R	P	S	C
$\forall x(\mathit{Tet}(x) \vee \mathit{Cube}(x) \vee \mathit{Dodec}(x))$				
$\exists x \exists y(\mathit{Cube}(y) \vee \mathit{BackOf}(x, y))$				
$\forall x(\neg \mathit{FrontOf}(x, x) \wedge \neg \mathit{BackOf}(x, x))$				
$\neg \exists x \mathit{Cube}(x) \leftrightarrow \forall x \neg \mathit{Cube}(x)$				
$\forall x(\mathit{Cube}(x) \rightarrow \mathit{Small}(x)) \rightarrow \forall y(\neg \mathit{Cube}(y) \rightarrow \neg \mathit{Small}(y))$				
$(\mathit{Cube}(a) \wedge \forall x \mathit{Small}(x)) \rightarrow \mathit{Small}(a)$				
$\exists x(\mathit{Larger}(a, x) \wedge \mathit{Larger}(x, b)) \rightarrow \mathit{Larger}(a, b)$				

Check your understanding

- Convert the formula to rectified form

$$\forall x\ \exists y\ P(x, f(y)) \land \forall y\ (Q(x, y) \lor R(x))$$

- Convert the given formula to clausal form

$$F = (\neg \exists x (P(x, z) \lor \forall y Q(x, f(y))) \lor \forall y P(g(x, y), z))$$