UCS1524 – Logic Programming

Propositional Logic : Normal Forms
and Horn Logic

Session Meta Data

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Session Objectives

- Understanding the concepts of normal forms in propositional logic (PL)
- Learning conjunctive normal forms (CNF) and disjunctive normal forms (DNF)
- Learning Horn clauses and satisfiability

Session Outcomes

- At the end of this session, participants will be able to
  - apply the representation of statements in PL using CNF and DNF
  - Apply CNF to convert propositional formula to Horn clauses

Agenda

- Normal forms
- Conjunctive normal forms (CNF)
- Disjunctive normal forms (DNF)
- Horn Clauses
- Satisfiability

Normal Form

A canonical or standard fundamental form of a statement to which others can be reduced

Normal Form: DNF

A *literal* is either an atomic proposition v or its negation ¬v

A *cube* is a conjunction of literals

- e.g., (v1 ∧ ¬ v2 ∧ v3)

A formula F is in *Disjunctive Normal Form* (DNF) if F is a disjunction of conjunctions of literals

$$\bigvee_{i=1}^{n} \left(\bigwedge_{j=1}^{m_i} L_{i,j} \right)$$

(Fun) Fact: determining whether a DNF formula F is satisfiable is easy

- easy == linear in the size of the formula

Normal Form: CNF

A *literal* is either an atomic proposition v or its negation ¬v

A *clause* is a disjunction of literals

- e.g., (v1 v ¬v2 v v3)

A formula F is in *Conjunctive Normal Form* (CNF) if F is a conjunction of disjunctions of literals

$$\bigwedge_{i=1}^{n} \left(\bigvee_{j=1}^{m_i} L_{i,j} \right)$$

(Fun) Fact: determining whether a CNF formula F is satisfiable is hard

- hard == NP-complete

Normal Form Theorem

Theorem: For every formula F, there is an equivalent formula F_1 in CNF. For every formula F, there is an equivalent formula F_2 in DNF.

That is, CNF and DNF are normal forms:

- Every propositional formula can be converted to CNF and to DNF without affecting its meaning (i.e., semantics)!

Proof: (by induction on the structure of the formula F)

Converting a formula to CNF

Given a formula F

1. Substitute in F every occurrence of a sub-formula of the form

   $\neg\neg G$ by G

   $\neg(G \wedge H)$ by $(\neg G \vee \neg H)$

   $\neg(G \vee H)$ by $(\neg G \wedge \neg H)$

   This is called Negation Normal Form (NNF)

2. Substitute in F each occurrence of a sub-formula of the form

   $(F \vee (G \wedge H))$ by $((F \vee G) \wedge (F \vee H))$

   $((F \wedge G) \vee H)$ by $((F \vee H) \wedge (G \vee H))$

The resulting formula F is in CNF

- the result in CNF might be exponentially bigger than original formula F

Example

- Convert the given formula to normal forms

$$
\begin{aligned}
\neg((\neg p \to \neg q) \land \neg r) &\equiv \neg((\neg\neg p \lor \neg q) \land \neg r) && \text{[definition]} \\
&\equiv \neg((p \lor \neg q) \land \neg r) && \text{[double negation]} \\
&\equiv \neg(p \lor \neg q) \lor \neg\neg r && \text{[DeMorgan's]} \\
&\equiv \neg(p \lor \neg q) \lor r && \text{[double negation]} \\
&\equiv (\neg p \land \neg\neg q) \lor r && \text{[DeMorgan's]} \\
\text{DNF} \quad &\equiv (\neg p \land q) \lor r && \text{[double negation]} \\
\text{CNF} \quad &\equiv (\neg p \lor r) \land (q \lor r) && \text{[distributive]}
\end{aligned}
$$

Example: From Truth Table to CNF and DNF

DNF: Each row of the truth table with value 1 yields a conjunction, a 0 in column *A* yields ¬*A*, and a 1 yields *A*

CNF: Each row of the truth table with value 0 yields a disjunction, a 0 in column *A* yields *A*, and a 1 yields ¬*A*

DNF

$$(\neg A \wedge \neg B \wedge \neg C) \vee$$
$$(A \wedge \neg B \wedge \neg C) \vee$$
$$(A \wedge \neg B \wedge C)$$

CNF

$$(A \vee B \vee \neg C) \wedge$$
$$(A \vee \neg B \vee C) \wedge$$
$$(A \vee \neg B \vee \neg C) \wedge$$
$$(\neg A \vee \neg B \vee C) \wedge$$
$$(\neg A \vee \neg B \vee \neg C)$$

Truth table

A	B	C	F
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

2-CNF Fragment

A formula F is in 2-CNF iff

- F is in CNF
- every clause of F has at most 2 literals

Theorem: There is a polynomial algorithm for deciding wither a a 2-CNF formula F is satisfiable

Precedence

Operator precedence:

$\leftrightarrow$	binds weaker than
$\rightarrow$	which binds weaker than
$\vee$	which binds weaker than
$\wedge$	which binds weaker than
$\neg$	.

So we have

$$A \leftrightarrow B \vee \neg C \rightarrow D \wedge \neg E \equiv (A \leftrightarrow ((B \vee \neg C) \rightarrow (D \wedge \neg E)))$$

But: well chosen parenthesis help to visually parse formulas.

3-CNF Fragment

A formula F is in 3-CNF iff

- F is in CNF
- every clause of F has at most 3 literals

Theorem: Deciding whether a 3-CNF formula F is satisfiable is at least as hard as deciding satisfiability of an arbitrary CNF formula G

Proof: by effective *reduction* from CNF to 3-CNF

Let G be an arbitrary CNF formula. Replaced every clause of the form

$$(\ell_0 \vee \cdots \vee \ell_n)$$

with 3-literal clauses

$$(\ell_0 \vee b_0) \wedge (\neg b_0 \vee \ell_1 \vee b_1) \wedge \cdots \wedge (\neg b_{n-1} \vee \ell_n)$$

where $\{b_i\}$ are fresh atomic propositions not appearing in F

Homework on Normal Forms

1. Convert the following formulas into DNF and CNF form

$$(p \to q) \to (\neg r \land q)$$

2. Find the DNF and CNF for the formula F given in the truth table

p	q	r	F
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

Homework on Normal Forms

1. Convert the following formulas into DNF and CNF form

$$(p \to q) \to (\neg r \wedge q)$$

DNF $(p \wedge \neg q) \vee (\neg r \wedge q)$

CNF $(p \vee \neg r) \wedge (p \vee q) \wedge (\neg q \vee \neg r)$

2. Find the DNF and CNF for the formula F given in the truth table

p	q	r	F
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

DNF of F: $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$

CNF : $(\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee q \vee \neg r)$

Horn Fragment

A formula F in CNF is a Horn formula if
every disjunction in F
contains at most one positive literal.

A formula F is in Horn fragment iff

- F is in CNF
- in every clause, at most one literal is positive

$$(\neg A \vee \neg B \vee C) \text{ becomes } (A \wedge B \to C)$$
$$(\neg A \vee \neg B) \text{ becomes } (A \wedge B \to 0)$$
$$A \text{ becomes } (1 \to A)$$

$$(A \vee \neg B) \wedge (\neg C \vee \neg A \vee D) \wedge (\neg A \vee \neg B) \wedge D \wedge \neg E$$

- Note that each clause can be written as an implication
  - e.g. $C \wedge A \Rightarrow D$, $A \wedge B \Rightarrow$ False, True $\Rightarrow D$

$$(B \to A) \wedge (A \wedge C \to D) \wedge (A \wedge B \to 0) \wedge (1 \to D) \wedge (E \to 0)$$

Theorem: There is a polynomial time algorithm for deciding satisfiability of a Horn formula F

Horn Satisfiability

Function HORN (ϕ)
//precondition: ϕ is a Horn formula
//postcondition: HORN (ϕ) decides the satisfiability for ϕ
{
1
mark all occurrences of T in ϕ
while there is a conjunct $p_1 \wedge p_2 \wedge \dots p_n \rightarrow P$ of ϕ
such that all p_j are marked but P is not
mark P
end while
if $\perp$ is marked
0 **return** 'unsatisfiable'
else
return 'satisfiable'
end if
}

Example

- $(p \wedge q \wedge w \rightarrow \bot) \wedge (t \rightarrow \bot) \wedge (r \rightarrow p) \wedge (T \rightarrow r) \wedge (T \rightarrow q) \wedge (u \rightarrow s) \wedge (T \rightarrow u)$
- *Alg: mark all occurrences of T in* ϕ
- Mark: r, q, u through $(T \rightarrow r)$, $(T \rightarrow q)$, $(T \rightarrow u)$
- $(p \wedge q \wedge w \rightarrow \bot) \wedge (t \rightarrow \bot) \wedge (r \rightarrow p) \wedge (T \rightarrow r) \wedge (T \rightarrow q) \wedge (u \rightarrow s) \wedge (T \rightarrow u)$
- *Alg: while loop: mark P where all* p_j *are marked*
- Mark: p through $(r \rightarrow p)$, s through $(u \rightarrow s)$
- $(p \wedge q \wedge w \rightarrow \bot) \wedge (t \rightarrow \bot) \wedge (r \rightarrow p) \wedge (T \rightarrow r) \wedge (T \rightarrow q) \wedge (u \rightarrow s) \wedge (T \rightarrow u)$
- Return?
  - Satisfiable

Example

- $(p \wedge q \wedge w \rightarrow \bot) \wedge (t \rightarrow \bot) \wedge (r \rightarrow p) \wedge (T \rightarrow r) \wedge (T \rightarrow q) \wedge (r \wedge u \rightarrow w) \wedge (u \rightarrow s) \wedge (T \rightarrow u)$
- Mark: r, q, u through $(T \rightarrow r)$, $(T \rightarrow q)$,$(T \rightarrow u)$
- $(p \wedge q \wedge w \rightarrow \bot) \wedge (t \rightarrow \bot) \wedge (r \rightarrow p) \wedge (T \rightarrow r) \wedge (T \rightarrow q) \wedge (r \wedge u \rightarrow w) \wedge (u \rightarrow s) \wedge (T \rightarrow u)$
- Mark: p through $(r \rightarrow p)$, s through $(u \rightarrow s)$, w through $(r \wedge u \rightarrow w)$
- $(p \wedge q \wedge w \rightarrow \bot) \wedge (t \rightarrow \bot) \wedge (r \rightarrow p) \wedge (T \rightarrow r) \wedge (T \rightarrow q) \wedge (r \wedge u \rightarrow w) \wedge (u \rightarrow s) \wedge (T \rightarrow u)$
- Mark $\bot$ through $(p \wedge q \wedge w \rightarrow \bot)$
- $(p \wedge q \wedge w \rightarrow \bot) \wedge (t \rightarrow \bot) \wedge (r \rightarrow p) \wedge (T \rightarrow r) \wedge (T \rightarrow q) \wedge (r \wedge u \rightarrow w) \wedge (u \rightarrow s) \wedge (T \rightarrow u)$
- Return?
  - Unsatisfiable

Summary

- Normal forms
- Conjunctive normal forms (CNF)
- Disjunctive normal forms (DNF)
- Horn Clauses
- Satisfiability

Check your understanding

Check the following horn formula are satisfiable or unsatisfiable

1. $(p_5 \rightarrow p_{11}) \wedge (p_2 \wedge p_3 \wedge p_5 \rightarrow p_{13}) \wedge (T \rightarrow p_5) \wedge (p_5 \wedge p_{11} \rightarrow \perp)$
2. $(T \rightarrow q) \wedge (T \rightarrow s) \wedge (w \rightarrow \perp) \wedge (p \wedge q \wedge s \rightarrow \perp) \wedge (v \rightarrow s) \wedge (T \rightarrow r) \wedge (r \rightarrow p)$
3. $(T \rightarrow q) \wedge (T \rightarrow s) \wedge (w \rightarrow \perp) \wedge (p \wedge q \wedge s \rightarrow v) \wedge (v \rightarrow s) \wedge (T \rightarrow r) \wedge (r \rightarrow p)$
4. $\neg b \wedge (\neg a \vee b \vee \neg c) \wedge a \wedge (\neg a \vee c)$ Hint: Convert to Horn formula