UCS1524 – Logic Programming

Propositional Logic : Resolution and Semantic Entailment

Session Meta Data

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Session Objectives

- Understanding the concept of resolution and semantic entailment in propositional logic (PL)
- Learning correctness and completeness in resolution with respect to PL

Session Outcomes

- At the end of this session, participants will be able to
  - Apply resolution and semantic entailment in PL

Agenda

- Resolution
- Resolvent
- Semantic entailment
- Correctness and completeness

Propositional Resolution

Pivot

$$\frac{C \vee p \qquad\qquad D \vee \neg p}{C \vee D}$$

Resolvent

Res({C, p}, {D, ¬p}) = {C, D}

Given two clauses (C, p) and (D, ¬p) that contain a literal p of different polarity, create a new clause by taking the union of literals in C and D

Resolution Lemma

Let F be a CNF formula.

Let R be a resolvent of two clauses C1 and C2 in F.

Then, $F \cup \{ R \}$ is equivalent to F.

- i.e., R is implied by F. Adding it to F does not change the meaning of F

Resolution Theorm

Let F be a set of clauses

$$Res(F) = F \cup \{R \mid R \text{ is a resolvent of two clauses in } F\}$$

Define Res^n recursively as follows:

$$Res^0(F) = F$$

$$Res^{n+1}(F) = Res(Res^n(F)), \text{ for } n \geq 0$$

$$Res^*(F) = \bigcup_{n \geq 0} Res^n(F)$$

Theorem: A CNF F is UNAT iff Res*(F) contains an empty clause

Proof System

$$P_1, \ldots, P_n \vdash C$$

An inference rule is a tuple (P_1, …, P_n, C)

- where, P_1, …, P_n, C are formulas
- P_i are called premises and C is called a conclusion
- intuitively, the rules says that the conclusion is true if the premises are

A proof system P is a collection of inference rules

A proof in a proof system P is a tree (or a DAG) such that

- nodes are labeled by formulas
- for each node *n*, (parents(n), n) is an inference rule in P

Propositional Resolution

Definition:

Consider the two clauses C_1 and C_2 containing the literals L_1 and L_2 respectively, where L_1 and L_2 are complementary. The procedure of resolution *proceeds as follows:*

(1) Delete L_1 from C_1 and L_2 from C_2, yielding the clauses C_1' and C_2';

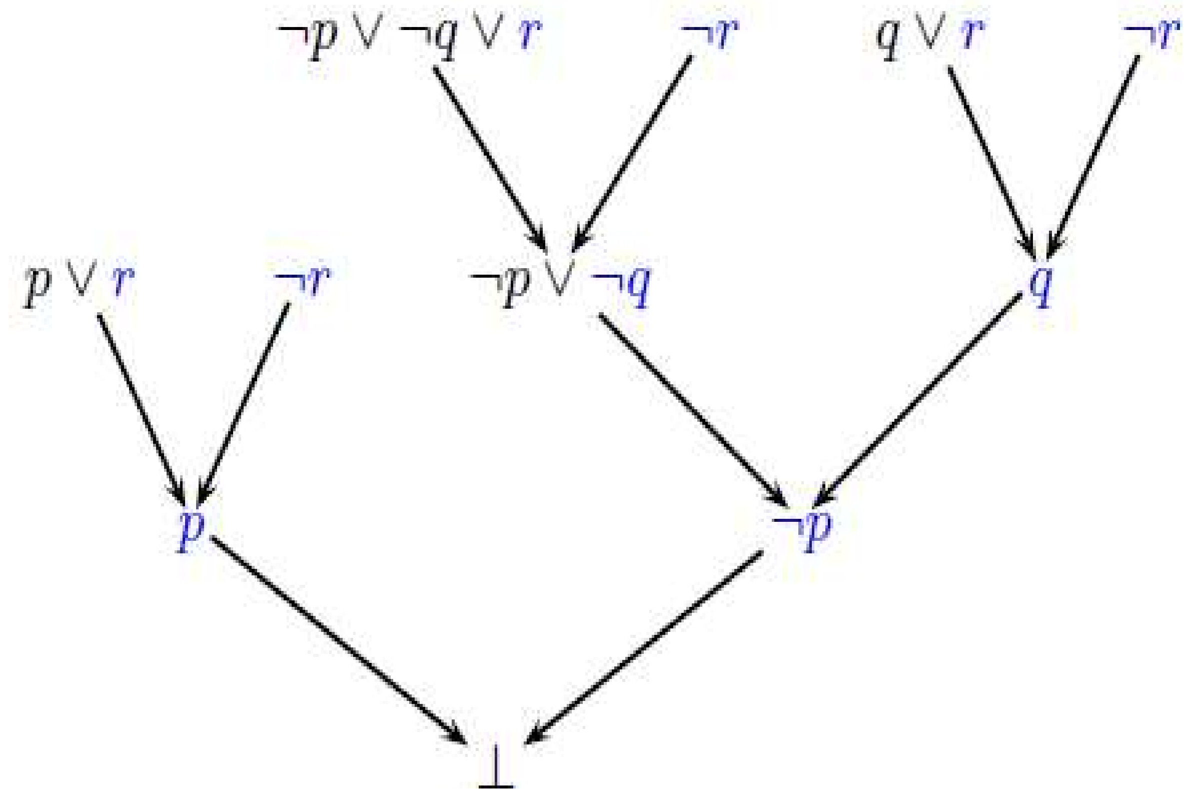
(2) Form the disjunction C' of C_1' and C_2';

(3) Delete (possibly) redundant literals from C', thus obtaining the clause C.

The resulting clause C is called the resolvent *of C_1 and C_2. The clauses C_1 and C_2 are said to be the* parent clauses *of the resolvent.*

Resolution Proof Example

A refutation of $\neg p \vee \neg q \vee r$, $p \vee r$, $q \vee r$, $\neg r$:



Resolution Proof Example

Show by resolution that the following CNF is UNSAT

$$\neg b \land (\neg a \lor b \lor \neg c) \land a \land (\neg a \lor c)$$

$$\dfrac{\dfrac{\dfrac{\neg a \lor b \lor \neg c \qquad a}{b \lor \neg c} \qquad \neg b}{\neg c} \qquad \dfrac{a \qquad \neg a \lor c}{c}}{\bot}$$

Propositional Resolution

- Resolution rule:

$$\frac{\begin{array}{l}\alpha \vee \beta \\ \neg\beta \vee \gamma\end{array}}{\alpha \vee \gamma}$$

- Resolution refutation:
  - Convert all sentences to CNF
  - Negate the desired conclusion (converted to CNF)
  - Apply resolution rule until either
    - Derive false (a contradiction)
    - Can't apply any more
- Resolution refutation is sound and complete
  - If we derive a contradiction, then the conclusion follows from the axioms
  - If we can't apply any more, then the conclusion cannot be proved from the axioms.

Propositional Resolution - Example

Prove R

1	P v Q
2	$P \to R$
3	$Q \to R$

Step	Formula	Derivation

Propositional Resolution - Example

Prove R

1	P v Q
2	$P \rightarrow R$
3	$Q \rightarrow R$

false v R

¬ R v false

~~false v false~~

Step	Formula	Derivation
1	P v Q	Given
2	¬ P v R	Given
3	¬ Q v R	Given
4	¬ R	Negated conclusion
5	Q v R	1,2
6	¬ P	2,4
7	¬ Q	3,4
8	R	5,7
9	•	4,8

Exercise

Using resolution show that

$$A \wedge B \wedge C$$

is a consequence of

$$\neg A \vee B$$

$$\neg B \vee C$$

$$A \vee \neg C$$

$$A \vee B \vee C$$

Properties of resolution

- Correctness (or consistency): If the application of the syntactic rules say that the semantic property holds, then this is indeed the case.
- If the empty clause can be derived from F then F is unsatisfiable.

- Completeness: If the semantic property holds, then this can be shown with the help of the syntactic rules.
- If F is unsatisfiable then the empty clause can be derived from F.

Propositional Resolution

Theorem: Propositional resolution is sound and complete for propositional logic

Proof: Follows from Resolution Theorem

A set of clauses F is unsatisfiable iff $\Box \in Res^*(F)$

Completeness Proof

Completeness: F is unsatisfiable $\Rightarrow \square \in Res^*(F)$
By induction on the number of atomic formulas in F.

Here: Induction step with $n+1=4$

$$F = \{\{A_1\}, \{\neg A_2, \cancel{A_4}\}, \{\neg A_1, A_2, \cancel{A_4}\}, \cancel{\{A_3, \neg A_4\}}, \cancel{\{\neg A_1, \neg A_3, \neg A_4\}}\}$$

$$F_0 = \{\{A_1\}, \{\neg A_2\}, \{\neg A_1, A_2\}\}$$

F(false/A4) :- Remove A4 from a clause if positive and Remove clause if negative

Completeness Proof

Completeness: F is unsatisfiable $\Rightarrow \square \in Res^*(F)$

By induction on the number of atomic formulas in F.

Here: Induction step with $n + 1 = 4$

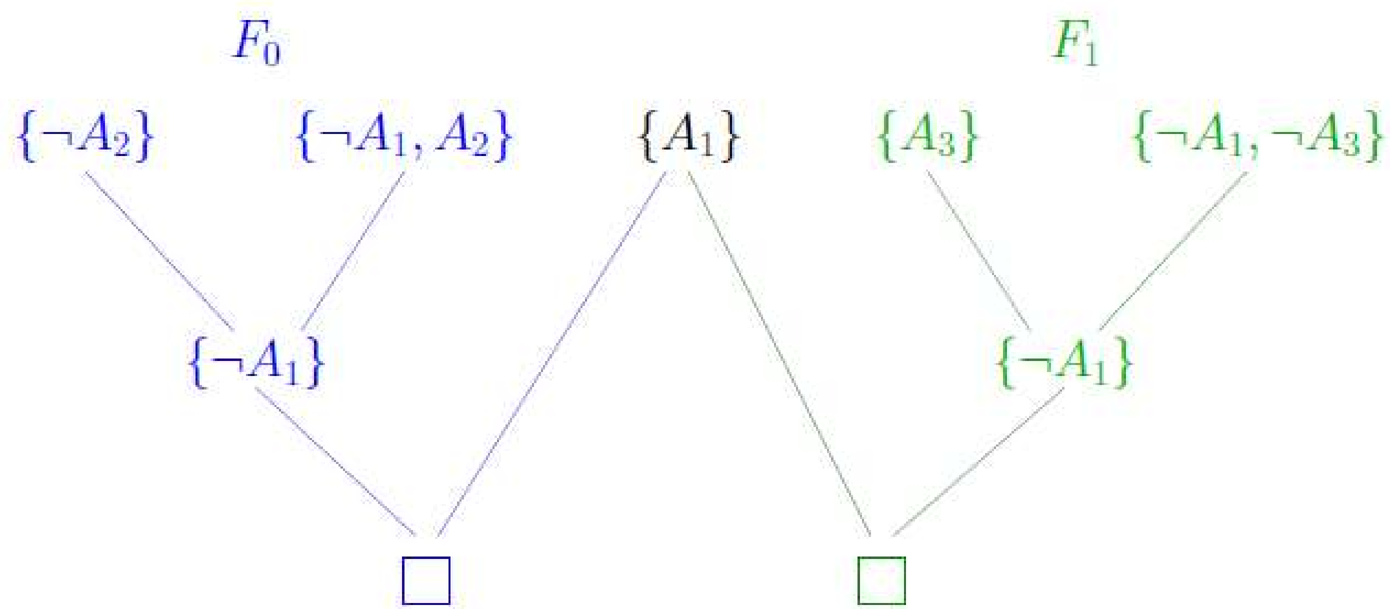
$$F = \{\{A_1\}, \cancel{\{\neg A_2, A_4\}}, \cancel{\{\neg A_1, A_2, A_4\}}, \{A_3, \cancel{\neg A_4}\}, \{\neg A_1, \neg A_3, \cancel{\neg A_4}\}\}$$

$$F_0 = \{\{A_1\}, \{\neg A_2\}, \{\neg A_1, A_2\}\}$$

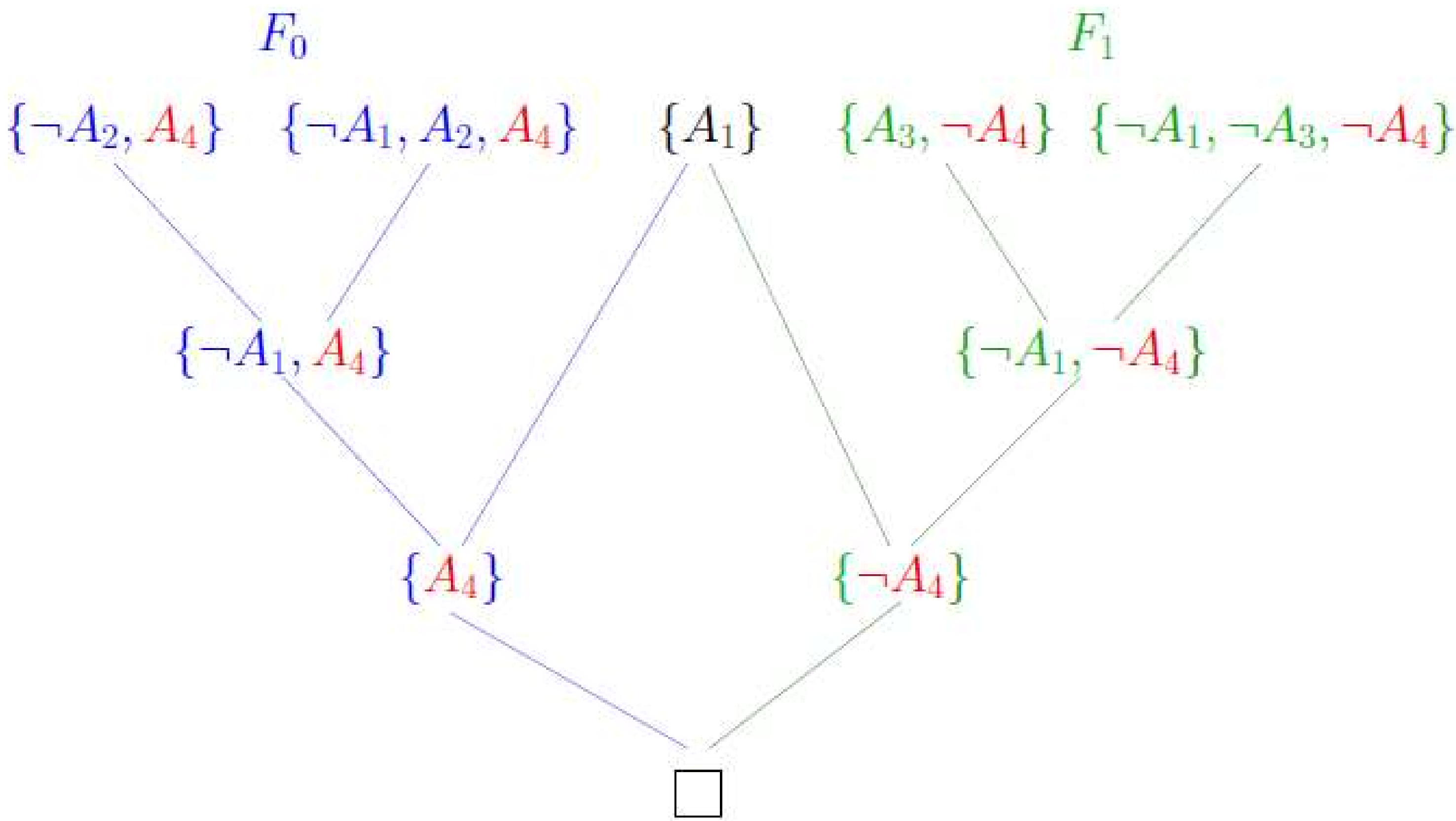
$$F_1 = \{\{A_1\}, \{A_3\}, \{\neg A_1, \neg A_3\}\}$$

F(true/ A4):- Remove clause if A4 is positive and Remove A4 from a clause if A4 is negative

Completeness Proof



Completeness Proof



Semantic Entailment

Let $\Sigma = \{p_1, p_2, ..., p_n\}$ be a set of premises and let α be the conclusion that we want to derive.

Σ *semantically entails* α, denoted $\Sigma \vDash \alpha$, if and only if

- Whenever all the premises in Σ are true, then the conclusion α is true.
- For any truth valuation t, if every premise in Σ is true under t, then the conclusion α is true under t.
- For any truth valuation t, if t satisfies Σ (denoted $\Sigma^t = \mathtt{T}$), then t satisfies α ($\alpha^t = \mathtt{T}$).
- $(p_1 \wedge p_2 \wedge ... \wedge p_n) \rightarrow \alpha$ is a tautology.

If Σ semantically entails α, then we say that the argument (with the premises in Σ and the conclusion α) is valid.

Semantic Entailment

Let $\Sigma = \{(\neg(p \wedge q)), (p \rightarrow q)\}$, $x = (\neg p)$, and $y = (p \leftrightarrow q)$. Based on the truth table, which of the following statements is true?

a. $\Sigma \models x$ and $\Sigma \models y$.

b. $\boxed{\Sigma \models x \text{ and } \Sigma \not\models y.}$

c. $\Sigma \not\models x$ and $\Sigma \models y$.

d. $\Sigma \not\models x$ and $\Sigma \not\models y$.

p	q	$(\neg(p \wedge q))$	$(p \rightarrow q)$	$x = (\neg p)$	$y = (p \leftrightarrow q)$
0	0	1	1	1	1
0	1	1	1	1	0
1	0	1	0	0	0
1	1	0	1	0	1

Equivalence and Entailment

Equivalence can be expressed using the notion of entailment.

Lemma. $\alpha \equiv \beta$ if and only if both $\{\alpha\} \models \beta$ and $\{\beta\} \models \alpha$.

Entailment and Derivation

A set of formulas F entails a set of formulas G iff every model of F and is a model of G

$$F \models G$$

A formula G is derivable from a formula F by a proof system P if there exists a proof whose leaves are labeled by formulas in F and the root is labeled by G

$$F \vdash_P G$$

Soundness and Completeness

A proof system P is sound iff

$$(F \vdash_P G) \implies (F \models G)$$

A proof system P is complete iff

$$(F \models G) \implies (F \vdash_P G)$$

Summary

- Resolution in propositional logic
- Semantics entailment
- Soundness and completeness

Check your understanding

Check the last formula is the consequence of the 1st two using resolution

$$P \rightarrow Q$$

$$\neg P \rightarrow R$$

$$\neg Q \rightarrow \neg R$$

Check your understanding

- Prove that the following formula is unsatisfiable

$$F = \{\{A, B, \neg C\}, \{\neg A\}, \{A, B, C\}, \{A, \neg B\}\}$$

Check your understanding

Prove R using resolution

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Check your understanding

What is $\{(\neg(p \land q)), (p \land q)\} \vDash (p \leftrightarrow q)$?

a. True

b. False

p	q	$(\neg(p \land q))$	$(p \land q)$	$(p \leftrightarrow q)$
0	0	1	0	1
0	1	1	0	0
1	0	1	0	0
1	1	0	1	1