Ɛ-NFA / NFA WITH Ɛ MOVES

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AP/CSE

LEARNING OBJECTIVE

- To construct finite automata for any given pattern and find its equivalent regular expressions
  - To learn the basic concept of Ɛ-NFA

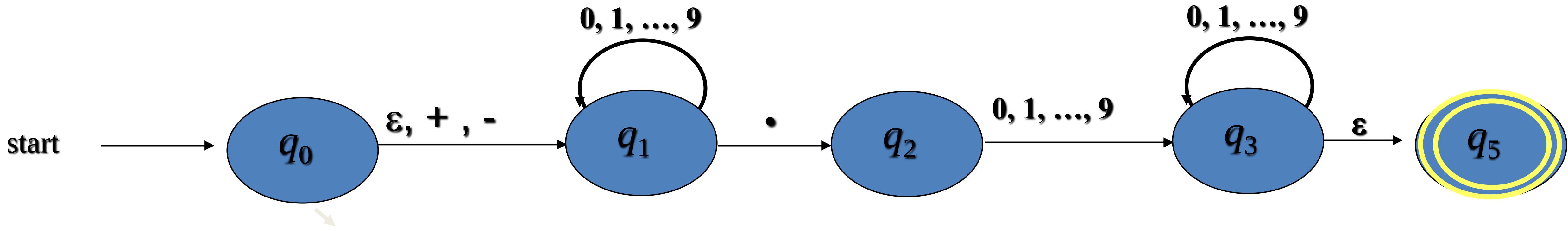
INTRODUCTION

- The NFA can be extended to include transitions on empty input ε
- The NFA with ε moves is defined by 5 tuple (Q, Σ, δ, q_0, F), with all components as in NFA except δ

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow 2^{Q}$$

- The intention is that δ (q, a) will consists of all states p such that there is a transition labeled 'a' from q to p, where a is either ε or any symbol in Σ.

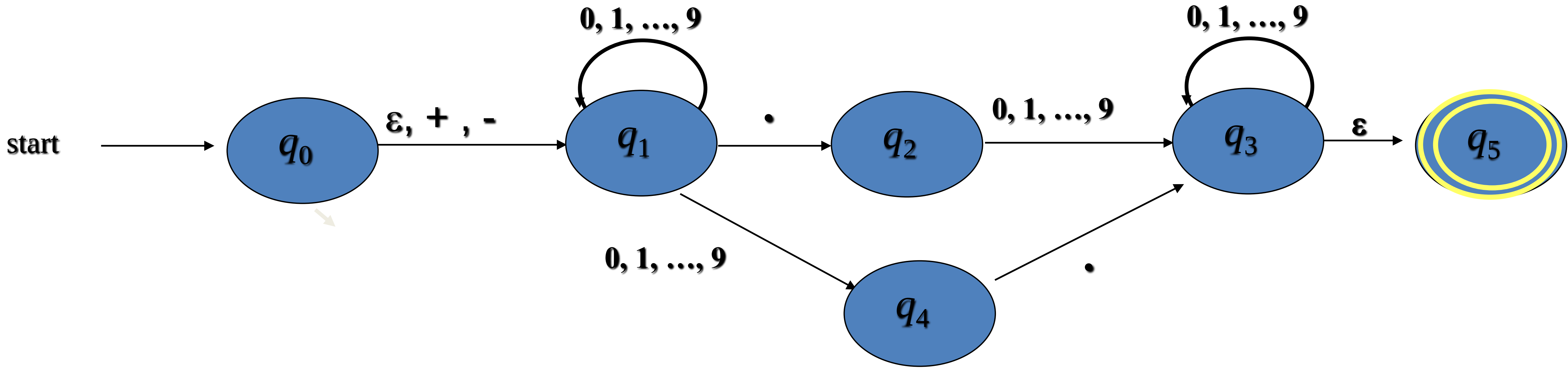
EXAMPLE



- An ε-NFA accepting decimal numbers like 2.15, .125, +1.4, -0.501…

- To accept a number like "+5." (nothing after the decimal point), add new state q_4.

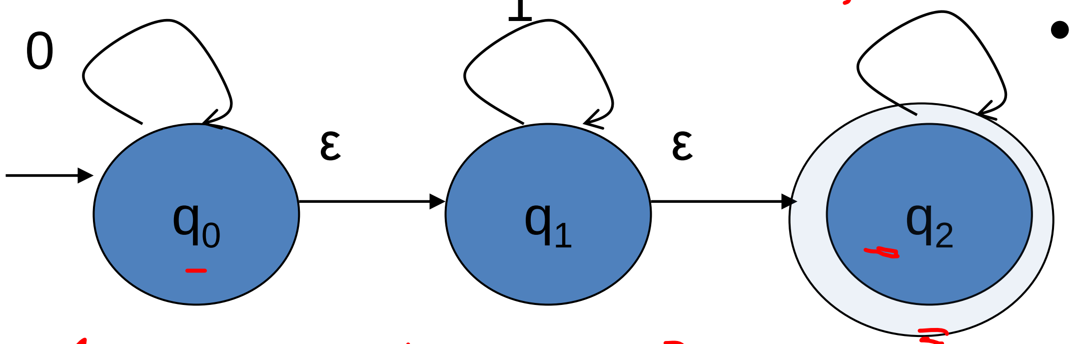
EXAMPLE



- An ε-NFA accepting decimal numbers like 2.15, .125, +1.4, -0.501…
- To accept a number like "+5." (nothing after the decimal point), we have to add q_4.

EXAMPLE

ε-closure(q_0) = {q_0, q_1, q_2}



L = {000, 00111, 0001122..., 0022, 112, 22, 11, ...}

States δ	Inputs			
	0	1	2	ε
→ q_0	{q_0}	ϕ	ϕ	{q_1}
q_1	ϕ	{q_1}	ϕ	{q_2}
* q_2	ϕ	ϕ	q_2	ϕ

M = Q = {q_0, q_1, q_2} Σ = ~~{0, 1, 2}~~ {0, 1, 2} q_0 {q_2}

- The transition diagram of the NFA accepts the language consisting of any number of 0's followed by any number of 1's followed by any number of 2's.
- For example, the string w = 002 is accepted by the NFA along the path – q_0, q_0, q_0, q_1, q_2, q_2, with arcs labeled 0, 0, ε, ε, 2.

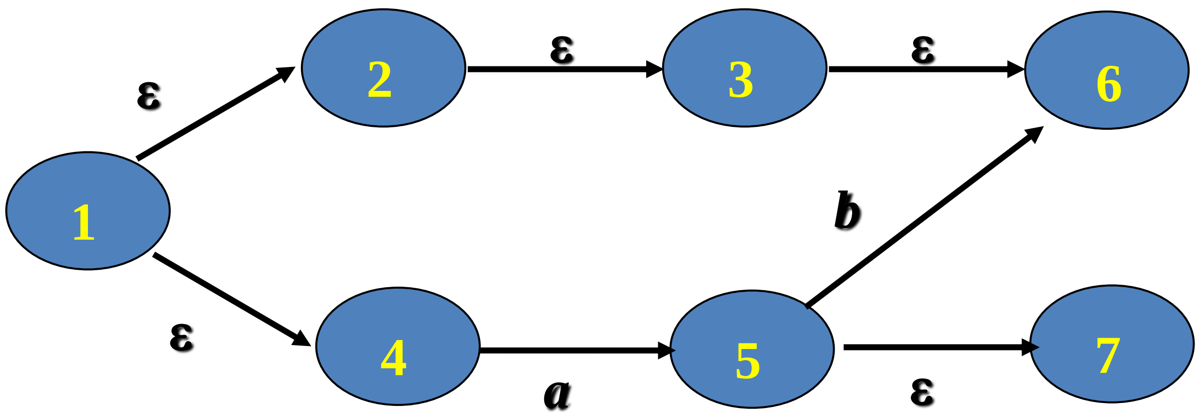
ε - CLOSURE

- We have to define the ε-closure to define the extended transition function for the ε-NFA.
- Formal recursive definition of the set ε–closure(q) for q:
  - State q is in ε-closure(q) (including the state itself);
  - If p is in ε-closure(q), then all states accessible from p through paths of ε*'s* are also in ε-closure(q).

EXAMPLE

- ε-closure for a set of states *S*:

$$\varepsilon\text{-closure}(S) = \bigcup_{q \in S} \varepsilon\text{-closure}(q)$$



ε-closure(2)
{2, 3, 6}

- ε-closure(1) = {1, 2, 3, 4, 6}
- ε-closure({3, 5}) = ε-closure(3)∪ ε-closure(5) = {3, 6} ∪ {5, 7} = {3, 5, 6, 7}

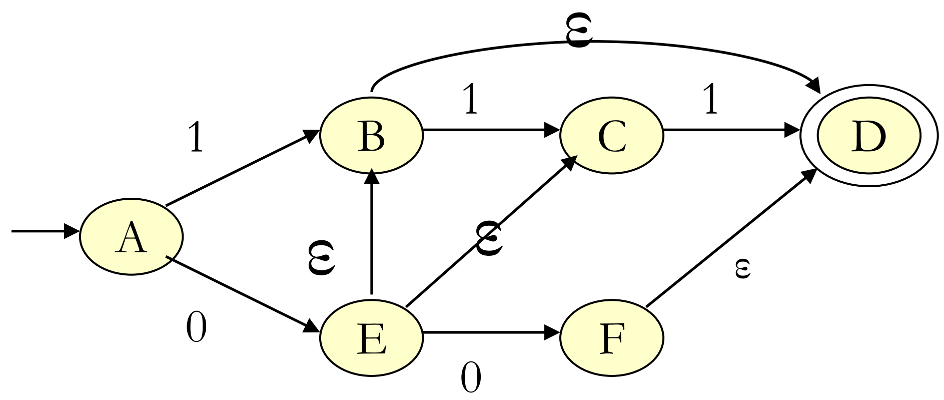
EXAMPLE

- ε-closure(A) ={A}
- ε-closure(E)={E,B,C,D}
- ε-closure({C, D}) = {C, D}

ε-clos (C) ∪ ε-clo (D)

{C} ∪ {D}

ε-closure (B) = {B, D}



EXAMPLE

- Find $\delta(q_0, 01)$

ε01

1. $\hat{\delta}(q_0, \varepsilon) = \varepsilon\text{-closure}(q_0)$

$= \{q_0, q_1, q_2\}$

2 $\hat{\delta}(q_0, 0) = \varepsilon\text{-closure}(\delta(\hat{\delta}(q_0, \varepsilon), 0))$

ε0

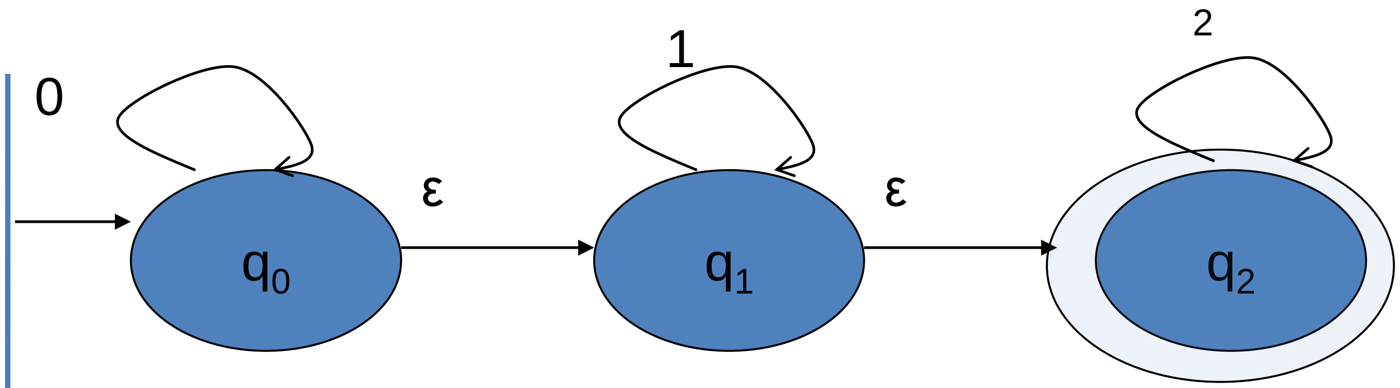
$= \varepsilon\text{-closure}(\delta(\{q_0, q_1, q_2\}, 0))$

$= \varepsilon\text{-closure}(\delta(q_0, 0) \cup \delta(q_1, 0) \cup \delta(q_2, 0))$

$= \varepsilon\text{-closure}(\{q_0\} \cup \phi \cup \phi)$

$= \varepsilon\text{-closure}(\{q_0\})$

$= \{q_0, q_1, q_2\}$



EXAMPLE CONT…

3. $\hat{\delta}(q_0, 01) = \varepsilon\text{-closure}($

Sol: $\varepsilon\text{-closure}(\delta(\hat{\delta}(q_0, 0), 1))$

$= \varepsilon\text{-closure}(\delta(\{q_0, q_1, q_2\}, 1))$

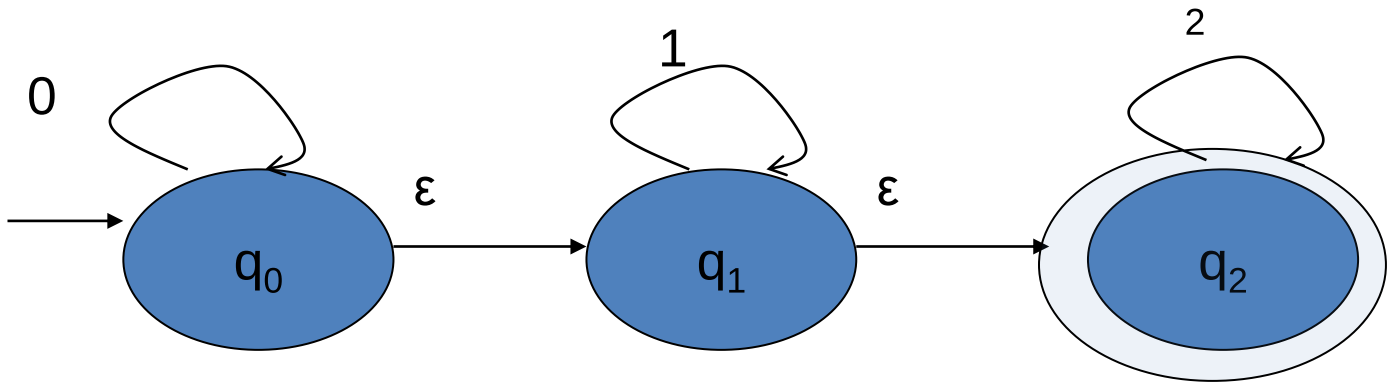
$= \varepsilon\text{-closure}(\delta(q_0, 1) \cup \delta(q_1, 1) \cup \delta(q_2, 1))$

$= \varepsilon\text{-closure}(\phi \cup \{q_1\} \cup \phi)$

$= \varepsilon\text{-closure}(q_1)$

$= \{q_1, q_2\}$ $\quad q_2 \in F$ ∴ The string is accepted

01 =



EXTENDED TRANSITIONS OF ε-NFA

- Basis: $\hat{\delta}(q, \varepsilon) = \varepsilon\text{-closure}(q)$.
- Induction:

  $\hat{\delta}(q, xa)$ is computed as:

  If $\hat{\delta}(q, x) = \{p_1, p_2, \ldots, p_k\}$ and

  $$\bigcup_{i=1}^{k} \delta(p_i, a) = \{r_1, r_2, \ldots, r_m\},$$

  $$\text{then } \hat{\delta}(q, xa) = \varepsilon\text{-closure}(\{r_1, r_2, \ldots, r_m\})$$

  $$= \varepsilon\text{-closure}\left(\bigcup_{i=1}^{k} \delta(p_i, a)\right)$$

LANGUAGE OF ε-NFA

- The language accepted by NFA with ε - move is defined as:
- L(M) = $\{w \mid \hat{\delta}(q_0, w) \cap F \neq \varphi\}$

$\{\ \} \cap F \neq$

$\{\ \} = \Phi$

EQUIVALENCE OF NFA & ε-NFA

- **Theorem**

  If L is accepted by NFA with ε -transitions, than L is accepted by an NFA without ε -transitions.

- **Proof**

- Let M = (Q, ∑ , δ , q_0, F) be an NFA with ε - transitions. Construct M^1 which is NFA without ε - transition.

  $M^{|} = (Q, \sum, \delta^{|}, q_0, F^{|})$ where

$$F^{|} = \begin{cases} F \cup \{q_0\} & \text{if } \varepsilon\text{-CLOSURE } (q_0) \text{ contains a state of F.} \\ F & \text{otherwise} \end{cases}$$

PROOF

By induction :

$\delta^{|}$ and $\hat{\delta}$ are same

δ and $\hat{\delta}$ are different

Let x be any string

$\delta^{|}(q_0, x) = \hat{\delta}(q_0, x)$

This statement is not true if

$x = \varepsilon$ because $\delta^{|}(q, \varepsilon) = \{q\}$ and

$\hat{\delta}(q_0, \varepsilon) = \varepsilon$ - CLOSURE (q_0)

PROOF

Basis step

$| x | = 1$

x is a symbol whose value is a

$\delta^{|}(q_0, a) = \hat{\delta}(q_0, a)$ (because by definition of $_{\delta}{}^{\wedge}$)

PROOF

Induction step

let $x = wa$ where a is in Σ.

$$\delta^{|}(q_0, wa) = \delta^{|}(\delta^{|}(q_0, w), a)$$

$$=\delta^{|}(\hat{\delta}\,(q_0, w), a)$$

$$=\delta^{|}(p, a)$$ [because by inductive hypothesis

$$\delta\,(q_0, w)=\hat{\delta}(q_0,w) = p(\text{say})]$$

Now we must show that

$$\delta^{|}(p, a) =\hat{\delta}\,(q_0, wa)$$

PROOF

But

$$\delta^{|}(p, a) = \bigcup_{q in P} \delta^{|}(q, a)$$

$$= \bigcup_{q in P} \hat{\delta}(q, a)$$

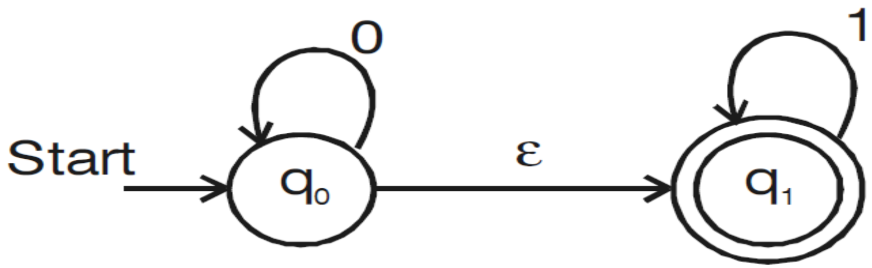
$$= \hat{\delta}(\hat{\delta}(q_0, w), a)$$

$$= \hat{\delta}(q_0, wa)$$

$$= \hat{\delta}(q_0, x)$$

Hence $\delta^{|}(q_0, x) = \hat{\delta}(q_0, x)$

EXAMPLE



Convert ε-NFA to a NFA

$M = (Q, \Sigma, q_0, \delta_E, F)$ ε-NFA

$Q = \{q_0, q_1\}$ | $q_0 \rightarrow IS$

$\Sigma = \{0, 1\}$ | $F = \{q_1\}$

NFA

$M' = (Q, \Sigma, q_0, \delta_N, F')$

$\varepsilon\text{ closure}(q_0) = \{q_0, q_1\}$

$F' = \{q_0, q_1\}$

$Q, \Sigma, q_0, F', \delta_N$

$q_0, q_1 \quad 0, 1$

$\delta_N(q_0, 0) = \delta_E(q_0, 0)$

$= \varepsilon\text{-closure}(\delta(\hat{\delta}(q_0, \varepsilon), 0))$

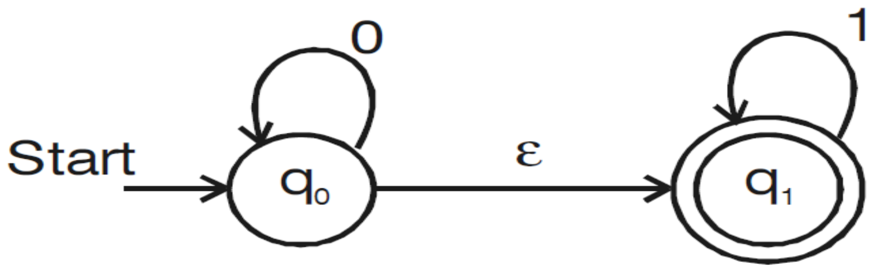
$= \varepsilon\text{-closure}(\delta(\{q_0, q_1\}, 0))$

$= \varepsilon\text{-closure}(\delta(q_0, 0) \cup \delta(q_1, 0))$

$= \varepsilon\text{-closure}(q_0)$

$= \{q_0, q_1\}$

EXAMPLE CONT…



$\delta_N(q_0, 1) = \delta_E(q_0, 1)$

$= \varepsilon\text{-closure}(\delta(\underline{\delta(q_0, \varepsilon)}, 1))$

$= \varepsilon\text{-closure}(\delta(\{q_0, q_1\}, 1))$

$= \varepsilon\text{-closure}(\ldots\ldots$

$= \varepsilon\text{-closure}(q_1)$

$= \cancel{\varepsilon\text{-closure}} \{q_1\}$

$\delta_N(q_1, 0) = \delta_E(q_1, 0)$

$= \varepsilon\text{-closure}(\delta(\delta(q_1, \varepsilon), 0))$

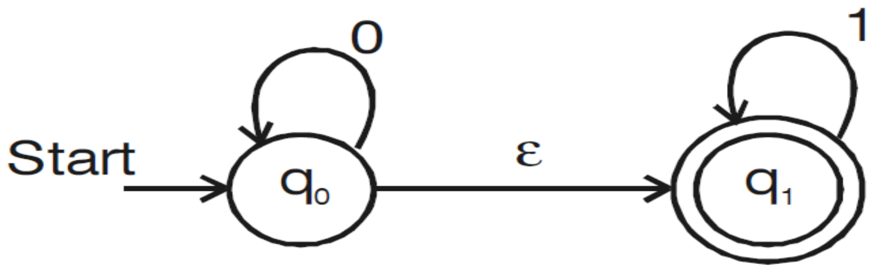
$= \phi$

$\delta_N(q_1, 1) = \delta_E(q_1, 1)$

$= \{q_1\}$

δ_N	0	1
$\rightarrow * q_0$	$\{q_0, q_1\}$	$\{q_1\}$
$* q_1$	ϕ	$\{q_1\}$

EXAMPLE



States δ_N	Inputs 0	Inputs 1
$\rightarrow * q_0$	$\{q_0, q_1\}$	$\{q_1\}$
$* q_1$	ϕ	$\{q_1\}$

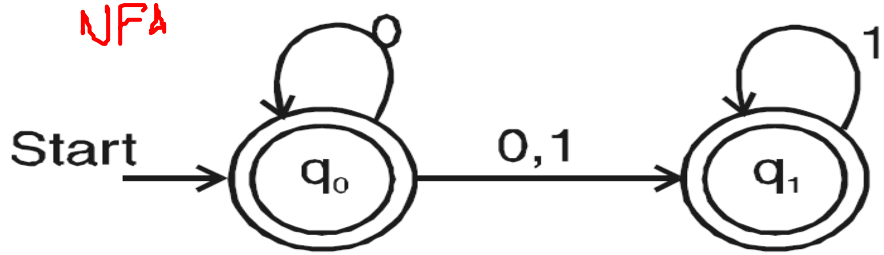
$M' = (Q, \Sigma, q_0, F', \delta_N)$

$Q = \{q_0, q_1\}$

$\Sigma = \{0, 1\}$

$F' = \{q_0, q_1\}$

NFA



EXAMPLE

- Convert the following ε-NFA to NFA

States	Inputs			
	0	1	2	ε
$\rightarrow q_0$	q_0	—	—	q_1
q_1	—	q_1	—	q_2
$* q_2$	—	—	q_2	—

EXAMPLE

- Convert the following ε-NFA to NFA

States	Inputs			
	0	1	2	ε
→ q_0	q_0	—	—	q_1
q_1	—	q_1	—	q_2
* q_2	—	—	q_2	—

States	Inputs		
	0	1	2
q_0	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_2\}$
q_1	—	$\{q_1, q_2\}$	$\{q_2\}$
q_2	—	—	$\{q_2\}$

TEST YOUR KNOWLEDGE

- State true or false?
  An NFA can be modified to allow transition without input alphabets, along with one or more transitions on input symbols.
- According to the given transitions, which among the following are the epsilon closures of q1 for the given NFA?
  $\delta\ (q1, \varepsilon) = \{q2, q3, q4\}$
  $\delta\ (q4, 1) = q1$
  $\delta\ (q1, \varepsilon) = q1$
  a) q4 b) q2 c) q1 d) q1, q2, q3, q4

SUMMARY

- Definition of ε-NFA
- Transition diagram, transition function and properties of transition function for ε-NFA.
- Equivalence of NFA & ε-NFA

LEARNING OUTCOME

On successful completion of this topic, the student will be able to:

- Understand the basic concept of Ɛ-NFA (K3)
- Equivalence of NFA and Ɛ-NFA (K3)

REFERENCE

- Hopcroft J.E., Motwani R. and Ullman J.D, “Introduction to Automata Theory, Languages and Computations”, Second Edition, Pearson Education, 2008