Propositional Logic

Propositional logic

- **Logical constants**: true, false
- **Propositional symbols**: P, Q, S, ... (**atomic sentences**)
- Wrapping **parentheses**: (…)
- Sentences are combined by **connectives**:

$\wedge$ ...and	[conjunction]
$\vee$ ...or	[disjunction]
$\Rightarrow$...implies	[implication / conditional]
$\Leftrightarrow$..is equivalent	[biconditional]
$\neg$ ...not	[negation]

- **Literal**: atomic sentence or negated atomic sentence

Examples of PL sentences

- P means "It is hot."
- Q means "It is humid."
- R means "It is raining."
- $(P \wedge Q) \rightarrow R$

  "If it is hot and humid, then it is raining"
- $Q \rightarrow P$

  "If it is humid, then it is hot"
- A better way:

  Hot = "It is hot"

  Humid = "It is humid"

  Raining = "It is raining"

Propositional logic (PL)

- A simple language useful for showing key ideas and definitions
- User defines a set of propositional symbols, like P and Q.
- User defines the **semantics** of each propositional symbol:
  - P means "It is hot"
  - Q means "It is humid"
  - R means "It is raining"
- A sentence (well formed formula) is defined as follows:
  - A symbol is a sentence
  - If S is a sentence, then $\neg$S is a sentence
  - If S is a sentence, then (S) is a sentence
  - If S and T are sentences, then (S $\vee$ T), (S $\wedge$ T), (S $\rightarrow$ T), and (S $\leftrightarrow$ T) are sentences
  - A sentence results from a finite number of applications of the above rules

A BNF grammar of sentences in propositional logic

```
S := <Sentence> ;
<Sentence> := <AtomicSentence> | <ComplexSentence> ;
<AtomicSentence> := "TRUE" | "FALSE" |
                    "P" | "Q" | "S" ;
<ComplexSentence> := "(" <Sentence> ")" |
          <Sentence> <Connective> <Sentence> |
          "NOT" <Sentence> ;
<Connective> := "NOT" | "AND" | "OR" | "IMPLIES" |
  "EQUIVALENT" ;
```

Some terms

- The meaning or **semantics** of a sentence determines its **interpretation**.
- Given the truth values of all symbols in a sentence, it can be "evaluated" to determine its **truth value** (True or False).
- A **model** for a KB is a "possible world" (assignment of truth values to propositional symbols) in which each sentence in the KB is True.

More terms

- A **valid sentence** or **tautology** is a sentence that is True under all interpretations, no matter what the world is actually like or how the semantics are defined. Example: “It’s raining or it’s not raining.”
- An **inconsistent sentence** or **contradiction** is a sentence that is False under all interpretations. The world is never like what it describes, as in “It’s raining and it’s not raining.”
- **P entails Q**, written P |= Q, means that whenever P is True, so is Q. In other words, all models of P are also models of Q.

Truth tables

And

p	q	$p \cdot q$
T	T	T
T	F	F
F	T	F
F	F	F

Or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

If . . . then

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Not

p	$\sim p$
T	F
F	T

Truth tables II

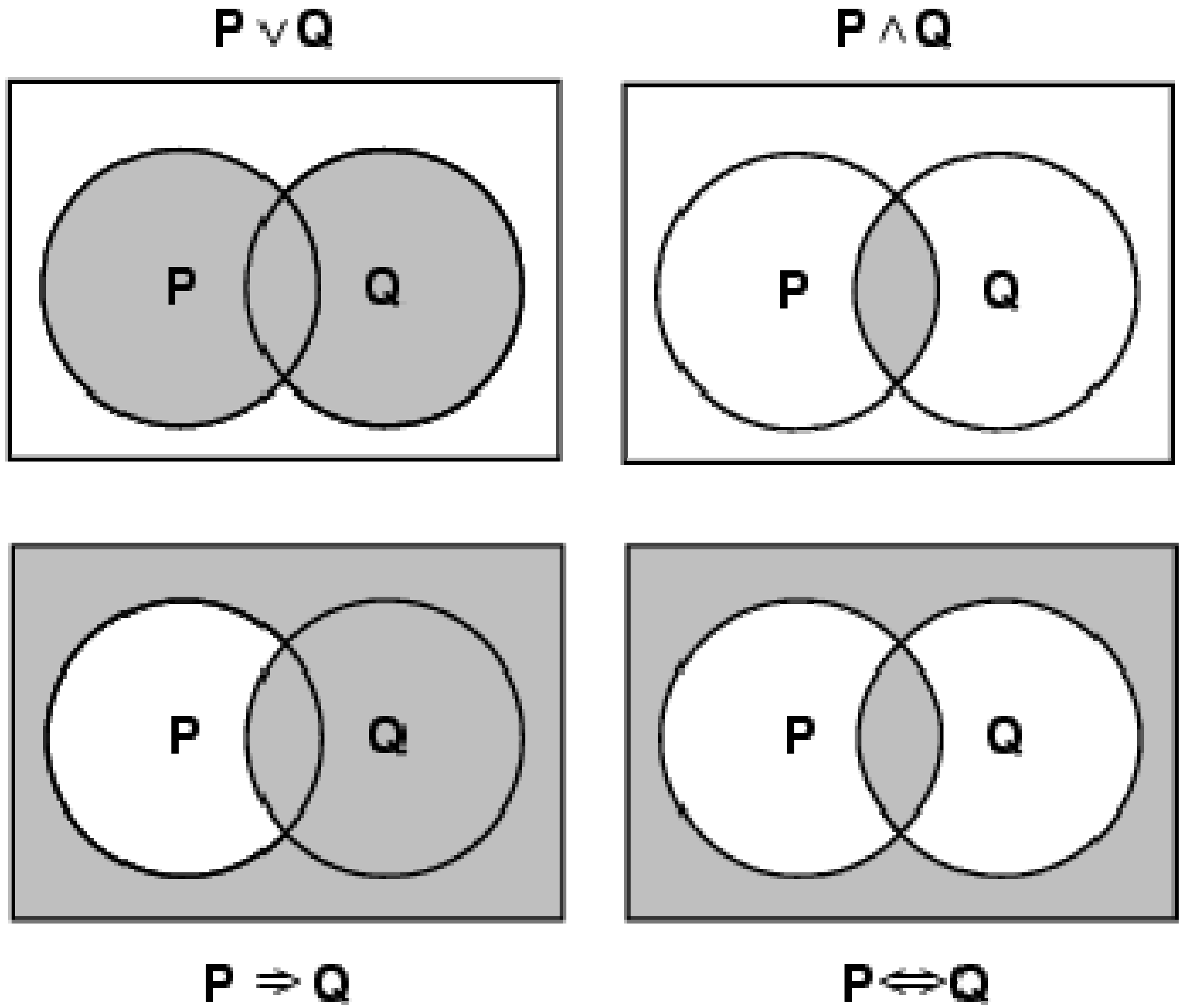
The five logical connectives:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	*False*	*True*	*False*	*False*	*True*	*True*
False	*True*	*True*	*False*	*True*	*True*	*False*
True	*False*	*False*	*False*	*True*	*False*	*False*
True	*True*	*False*	*True*	*True*	*True*	*True*

A complex sentence:

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$((P \vee H) \wedge \neg H) \Rightarrow P$
False	*False*	*False*	*False*	*True*
False	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
True	*True*	*True*	*False*	*True*

Models of complex sentences



Inference rules

- **Logical inference** is used to create new sentences that logically follow from a given set of predicate calculus sentences (KB).
- An inference rule is **sound** if every sentence X produced by an inference rule operating on a KB logically follows from the KB. (That is, the inference rule does not create any contradictions)
- An inference rule is **complete** if it is able to produce every expression that logically follows from (is entailed by) the KB. (Note the analogy to complete search algorithms.)

Sound rules of inference

- Here are some examples of sound rules of inference
  - *A rule is sound if its conclusion is true whenever the premise is true*
- Each can be shown to be sound using a truth table

RULE	**PREMISE**	**CONCLUSION**
Modus Ponens	$A, A \rightarrow B$	B
And Introduction	A, B	$A \wedge B$
And Elimination	$A \wedge B$	A
Double Negation	$\neg\neg A$	A
Unit Resolution	$A \vee B, \neg B$	A
Resolution	$A \vee B, \neg B \vee C$	$A \vee C$

Soundness of modus ponens

A	B	$A \rightarrow B$	OK?
True	True	True	√
True	False	False	√
False	True	True	√
False	False	True	√

Soundness of the resolution inference rule

α	β	γ	$\alpha \vee \beta$	$\neg\beta \vee \gamma$	$\alpha \vee \gamma$
False	*False*	*False*	*False*	*True*	*False*
False	*False*	*True*	*False*	*True*	*True*
False	*True*	*False*	*True*	*False*	*False*
False	*True*	*True*	*True*	*True*	*True*
True	*False*	*False*	*True*	*True*	*True*
True	*False*	*True*	*True*	*True*	*True*
True	*True*	*False*	*True*	*False*	*True*
True	*True*	*True*	*True*	*True*	*True*

Proving things

- A **proof** is a sequence of sentences, where each sentence is either a premise or a sentence derived from earlier sentences in the proof by one of the rules of inference.
- The last sentence is the **theorem** (also called goal or query) that we want to prove.
- Example for the "weather problem" given above.

1 Humid	Premise	"It is humid"
2 $Humid \rightarrow Hot$	Premise	"If it is humid, it is hot"
3 Hot	Modus Ponens(1,2)	"It is hot"
4 $(Hot \wedge Humid) \rightarrow Rain$	Premise	"If it's hot & humid, it's raining"
5 $Hot \wedge Humid$	And Introduction(1,2)	"It is hot and humid"
6 Rain	Modus Ponens(4,5)	"It is raining"

Horn sentences

- A **Horn sentence** or **Horn clause** has the form:

  $P1 \land P2 \land P3 \ldots \land Pn \rightarrow Q$

or alternatively

$\boxed{(P \rightarrow Q) = (\neg P \lor Q)}$

  $\neg P1 \lor \neg P2 \lor \neg P3 \ldots \lor \neg Pn \lor Q$

where Ps and Q are non-negated atoms

- To get a proof for Horn sentences, apply Modus Ponens repeatedly until nothing can be done
- We will use the Horn clause form later

Entailment and derivation

- **Entailment: KB |= Q**
  - Q is entailed by KB (a set of premises or assumptions) if and only if there is no logically possible world in which Q is false while all the premises in KB are true.
  - Or, stated positively, Q is entailed by KB if and only if the conclusion is true in every logically possible world in which all the premises in KB are true.
- **Derivation: KB |- Q**
  - We can derive Q from KB if there is a proof consisting of a sequence of valid inference steps starting from the premises in KB and resulting in Q

Two important properties for inference

Soundness: If KB |- Q then KB |= Q

- If Q is derived from a set of sentences KB using a given set of rules of inference, then Q is entailed by KB.
- Hence, inference produces only real entailments, or any sentence that follows deductively from the premises is valid.

Completeness: If KB |= Q then KB |- Q

- If Q is entailed by a set of sentences KB, then Q can be derived from KB using the rules of inference.
- Hence, inference produces all entailments, or all valid sentences can be proved from the premises.

Propositional logic is a weak language

- Hard to identify "individuals" (e.g., Mary, 3)
- Can't directly talk about properties of individuals or relations between individuals (e.g., "Bill is tall")
- Generalizations, patterns, regularities can't easily be represented (e.g., "all triangles have 3 sides")
- First-Order Logic (abbreviated FOL or FOPC) is expressive enough to concisely represent this kind of information

  FOL adds relations, variables, and quantifiers, e.g.,

  - "*Every elephant is gray*": $\forall$ x (elephant(x) $\rightarrow$ gray(x))
  - "*There is a white alligator*": $\exists$ x (alligator(X) ^ white(X))

Example

- Consider the problem of representing the following information:
  - Every person is mortal.
  - Confucius is a person.
  - Confucius is mortal.
- How can these sentences be represented so that we can infer the third sentence from the first two?

Example II

- In PL we have to create propositional symbols to stand for all or part of each sentence. For example, we might have:

  P = “person”; Q = “mortal”; R = “Confucius”

- so the above 3 sentences are represented as:

  $P \rightarrow Q$; $R \rightarrow P$; $R \rightarrow Q$

- Although the third sentence is entailed by the first two, we needed an explicit symbol, R, to represent an individual, Confucius, who is a member of the classes “person” and “mortal”
- To represent other individuals we must introduce separate symbols for each one, with some way to represent the fact that all individuals who are “people” are also “mortal”

Summary

- The process of deriving new sentences from old one is called **inference**.
  - **Sound** inference processes derives true conclusions given true premises
  - **Complete** inference processes derive all true conclusions from a set of premises
- A **valid sentence** is true in all worlds under all interpretations
- If an implication sentence can be shown to be valid, then—given its premise—its consequent can be derived
- Different logics make different **commitments** about what the world is made of and what kind of beliefs we can have regarding the facts
  - Logics are useful for the commitments they do not make because lack of commitment gives the knowledge base engineer more freedom
- **Propositional logic** commits only to the existence of facts that may or may not be the case in the world being represented
  - It has a simple syntax and simple semantics. It suffices to illustrate the process of inference
  - Propositional logic quickly becomes impractical, even for very small worlds