PUMPING LEMMA FOR CF LANGUAGES

Dr. A. Beulah

AP/CSE

LEARNING OBJECTIVE

- To Design pushdown automata for any CFL (K3)
  - To Understand the concept of pumping lemma for CFL

STATEMENT OF THE CFL PUMPING LEMMA

For every context-free language L

There is an integer n, such that

For every string w in L of length $\geq$ n

There exists w = uvxyz such that:

1. $|vxy| \leq n$.
2. $|vy| > 0$.
3. For all $i \geq 0$, uv^ixy^iz is in L.

CFL

PDA → CFG

'n'

$|w| \geq n$

$w = uvxyz$

$|vxy| \leq n$

$vy \neq \varepsilon$, $|vy| > 0$

$\forall i$ $uv^ixy^iz \in L$

n

$|w| \geq n$

$w = xyz$

$|xy| < n$

$\cancel{|y| \neq \varepsilon}$

$y \neq \varepsilon$ $|y| > 0$

$xy^iz \in L$ → PL

$\left.\begin{array}{l} \text{DFA} \\ \text{NFA} \\ \varepsilon\text{-NFA} \\ \text{RE} \end{array}\right\}$ RL

EXAMPLE

- $L=\{a^n b^n c^n \mid n>=0\}$

'n'

$|w| \geq n$

$w = uvxyz$

$|vxy| \leq n$

$|vy| > 0$ i.e. $vy \neq \varepsilon$

$uv^i x y^i z \in L$

$w = a^p b^p c^p$

$|w| \geq n$

$w = a^p$

$vxy = b^p$ $\quad |vxy| < n$

$vy = b^{p-r}$ $\quad |vy| > 0$

$z = c^p$

$uv^i x y^i z$

$= u\, vxy\, (vy)^{i-1} z$

$= a^p b^p (b^{p-r})^{i-1} c^p$

$i=0$

$a^p b^p (b^{p-r})^{-1} c^p$

$= a^p \cancel{b^p} \cancel{b^{-p}} b^r c^p$

$= a^p b^r c^p \notin L$

$i=1$

$a^p b^p (b^{p-r})^0 c^p$

$a^p b^p c^p \in L$

$i=2$

$a^p b^p (b^{p-r})^1 c^p$

$a^p b^p . b^p . b^{-r} c^p$

$a^p b^{2p-r} c^p \notin L$

$a^n b^n c^n$

PDA ✗ CFL ✗

u

vxy

vy z

$\therefore$ L is not a CFL

EXAMPLE

- $L=\{a^k b^j c^k d^j \mid k,j>=1\}$

$w = a^P$

$vxy = b^q$

$vy = b^{q-r}$

$z = c^P d^q$

$|vxy| < n$

$uv^i xy^i z$

$= u\, vxy\, (vy)^{i-1} z$

$= a^P b^q (b^{q-r})^{i-1} c^P d^q$

i=0

$uv^0 x y^0 z = a^P b^q (b^{q-r})^{-1} c^P d^q$

$= a^P b^q b^{-q} b^r c^P d^q$

$= a^P b^r c^P d^q \notin L$

i=1

$uv^1 x y^1 z = a^P b^q (b^{q-r})^0 c^P d^q$

$= a^P b^q c^P d^q \in L$

i=2

$uv^2 x y^2 z = a^P b^q (b^{q-r})^{2-1} c^P d^q$

$= a^P b^q b^q . b^{-r} . c^P . d^q$

$= a^P b^{2q-r} c^P d^q \notin L$

$\therefore$ The language is not a CFL.

EXAMPLE

- $L=\{a^n b^{n+1} c^{n+2} \mid n>=0\}$

INTUITION

- Recall the pumping lemma for regular languages.
- It told us that if there was a string long enough to cause a cycle in the DFA for the language, then we could “pump” the cycle and discover an infinite sequence of strings that had to be in the language.

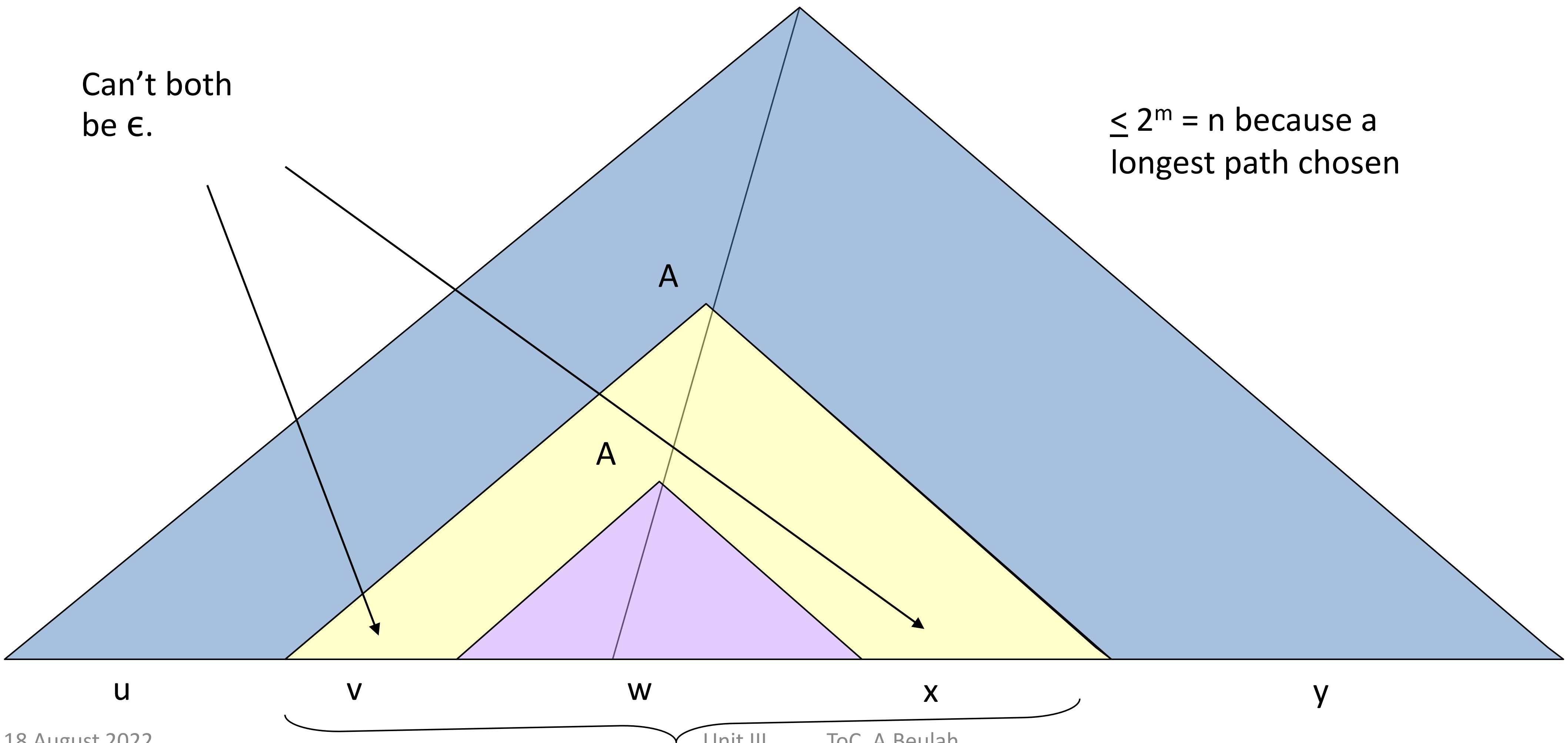
INTUITION – (2)

- For CFL's the situation is a little more complicated.
- We can always find two pieces of any sufficiently long string to "pump" in tandem.
  - That is: if we repeat each of the two pieces the same number of times, we get another string in the language.

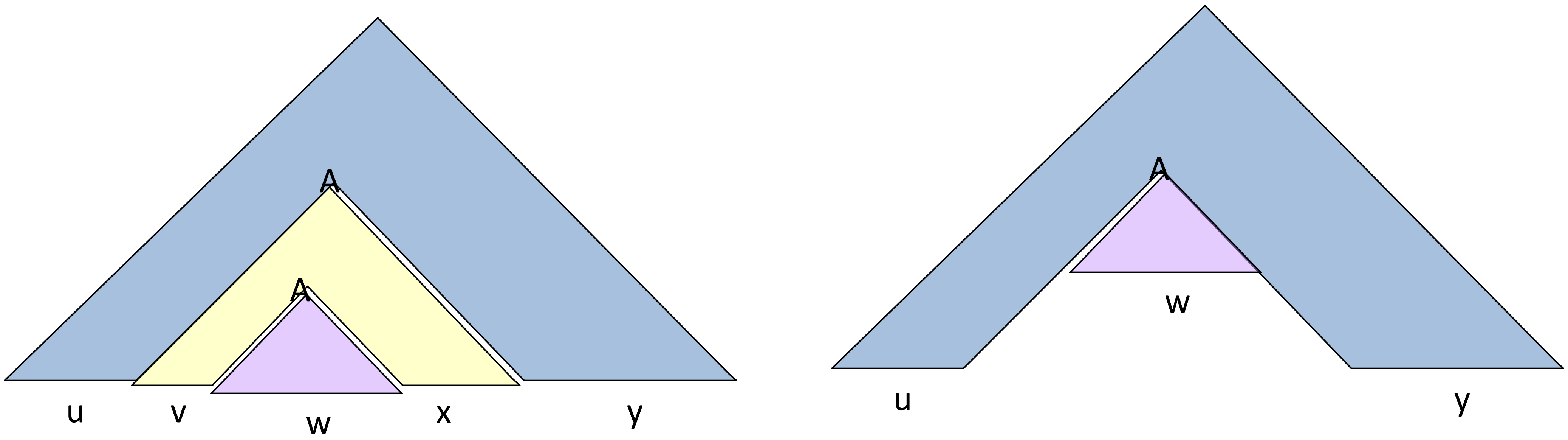
PROOF OF THE PUMPING LEMMA

- Let the grammar have m variables.
- Pick $n = 2^m$.
- Let $|z| \geq n$.

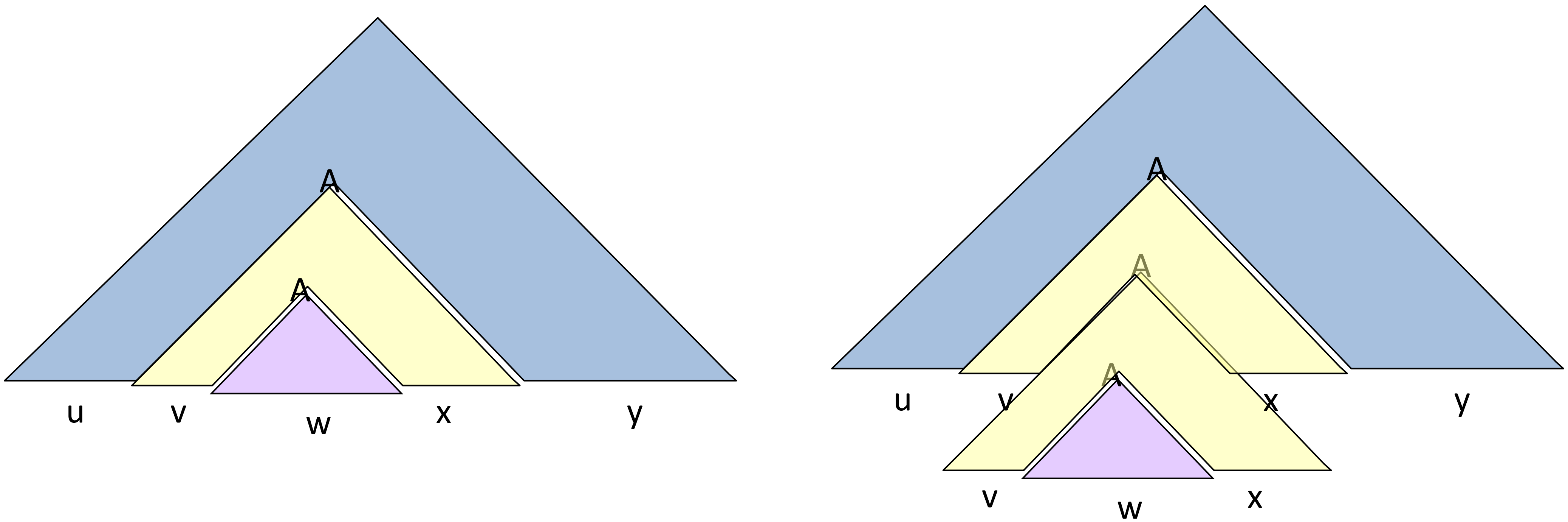
Parse Tree in the Pumping-Lemma Proof



Pump Zero Times

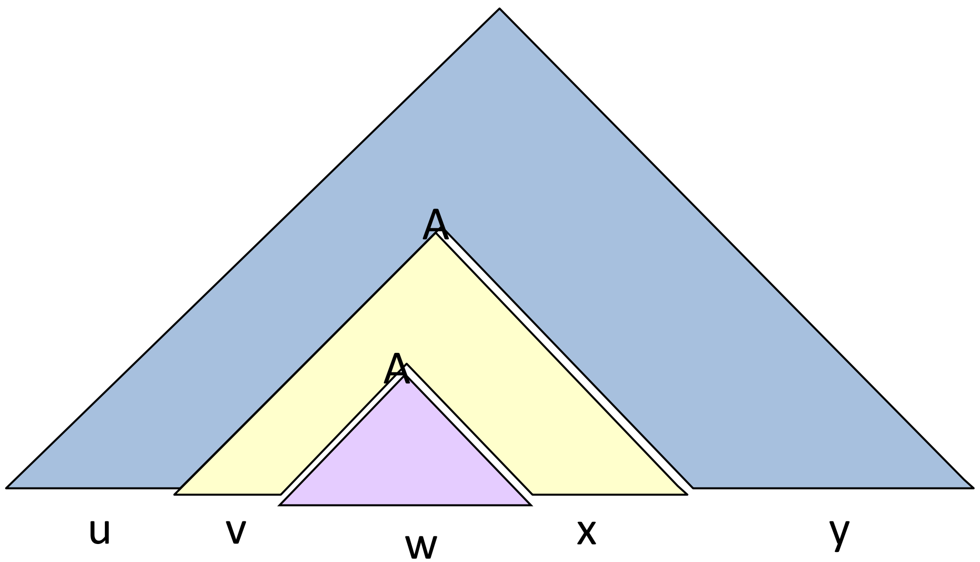


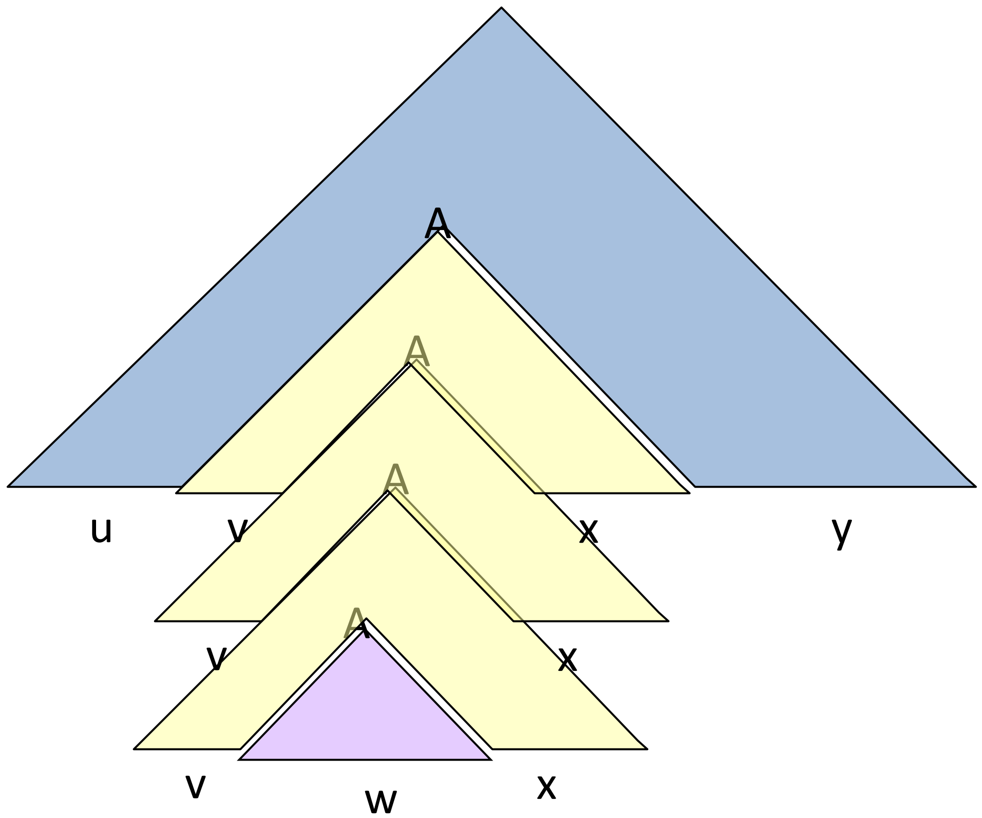
Pump Twice



Pump Thrice

Etc., Etc.





USING THE PUMPING LEMMA

- Non-CFL's typically involve trying to match two pairs of counts or match two strings.
- Example: The text uses the pumping lemma to show that {ww | w in (**0**+**1**)*} is not a CFL.

- $\{0^i10^i \mid i \geq 1\}$ is a CFL.
  - We can match one pair of counts.
- But $L = \{0^i10^i10^i \mid i \geq 1\}$ is not.
  - We can't match two pairs, or three counts as a group.
- Proof using the pumping lemma.
- Suppose L were a CFL.
- Let n be L's pumping-lemma constant.

- Consider $z = 0^n10^n10^n$.
- We can write $z = uvwxy$, where $|vwx| \leq n$, and $|vx| \geq 1$.
- Case 1: vx has no 0's.
  - Then at least one of them is a 1, and uwy has at most one 1, which no string in L does.

- Still considering $z = 0^n10^n10^n$.
- Case 2: vx has at least one 0.
  - vwx is too short (length $\leq$ n) to extend to all three blocks of 0's in $0^n10^n10^n$.
  - Thus, uwy has at least one block of n 0's, and at least one block with fewer than n 0's.
  - Thus, uwy is not in L.

SUMMARY

- Pumping lemma for Context free languages
  - Lemma – to prove the given language is not CFL
  - Problems
  - Proof

TEST YOUR KNOWLEDGE

- Pumping lemma for context free grammar is used for

a) Proving certain languages are not context free

b) Proving language is infinite

c) Both (a) and (b)

d) None of these

REFERENCE

- Hopcroft J.E., Motwani R. and Ullman J.D, “Introduction to Automata Theory, Languages and Computations”, Second Edition, Pearson Education, 2008