# 2D Transformations - Reflection and Shearing
# 2D Composite Transformations

# Overview

Homogeneous Coordinates

Reflection

Shearing

# HOMOGENEOUS CO-ORDINATES

Graphics applications involves sequences of geometric transformations.

Efficient approach is needed to combine the transformations so that the final coordinates are obtained directly.

Combine the multiplicative and the translational terms for 2d geometric transformations into single matrix multiplication by homogenous coordinates.

***Homogeneous coordinates seem unintuitive, but they make graphics operations much easier***

Represent each 2D coordinate position $(x, y)$ with the homogenous coordinate triple $(x_h, y_h, h)$.

# HOMOGENEOUS CO-ORDINATES

Represent each 2D coordinate position ($x, y$) with the homogenous coordinate triple ($x_h$, $y_h$, $h$). Where

$$x = \frac{x_h}{h} \quad y = \frac{y_h}{h} \quad P = \begin{bmatrix} x_h \\ y_h \\ h \end{bmatrix} = \begin{bmatrix} h \cdot x \\ h \cdot y \\ h \end{bmatrix}$$

General homogeneous representation can also written as (h.x,h.y,h) set h=1.

Transformations of translation, scaling and rotation can be represented using Homogeneous coordinates.

# Homogeneous Transformation Coordinates

Translation

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = T(t_x, t_y) \cdot P$$

Rotation

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = R(\theta) \cdot P$$

Scaling

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = S(s_x, s_y) \cdot P$$

# Composite Transformations

Application of a sequence of transformations to a point:

$$\mathbf{P}' = \mathbf{M_2} \cdot \mathbf{M_1} \cdot \mathbf{P}$$
$$= \mathbf{M} \cdot \mathbf{P}$$

**Composite transformations** is formed by calculating the matrix product of the individual transformations and forming products of transformation matrix.

# Composite Transformations-Translation

First: composition of similar type transformations

If we apply to successive translations to a point:

$$\mathbf{P}' = \mathbf{T}(t_{2x}, t_{2y}) \cdot \{\mathbf{T}(t_{1x}, t_{1y}) \cdot \mathbf{P}\}$$
$$= \{\mathbf{T}(t_{2x}, t_{2y}) \cdot \mathbf{T}(t_{1x}, t_{1y})\} \cdot \mathbf{P}$$

P AND P' are represented as homogenous coordinate values.

$$T(t_{2x}, t_{2y}) \cdot T(t_{1x}, t_{1y}) = \begin{bmatrix} 1 & 0 & t_{2x} \\ 0 & 1 & t_{2y} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{1x} \\ 0 & 1 & t_{1y} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{1x} + t_{2x} \\ 0 & 1 & t_{1y} + t_{2y} \\ 0 & 0 & 1 \end{bmatrix} = T(t_{1x} + t_{2x}, t_{1y} + t_{2y})$$

Successive translations are additive

# Composite Transformations-Rotation

Two successive rotations applied  to the point p produce the transformed position

P'=R(θ) {R (Φ).P}={R(θ) .R (Φ)}.P

$$\mathbf{R}(\theta)\cdot\mathbf{R}(\varphi)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}=$$

$$\begin{bmatrix}\cos\theta\cos\varphi-\sin\theta\sin\varphi & -\cos\theta\sin\varphi-\sin\theta\cos\varphi & 0\\ \sin\theta\cos\varphi+\cos\theta\sin\varphi & -\sin\theta\sin\varphi+\cos\theta\cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos(\theta+\varphi) & -\sin(\theta+\varphi) & 0\\ \sin(\theta+\varphi) & \cos(\theta+\varphi) & 0\\ 0 & 0 & 1\end{bmatrix}=\mathbf{R}(\theta+\varphi)$$

Two successive rotations are additive.

# Composite Transformations-Scaling

Two successive scaling operations produces the following composite scaling matrix

$$\mathbf{S}(s_{2x}, s_{2y}) \cdot \mathbf{S}(s_{1x}, s_{1y}) = \begin{bmatrix} s_{2x} & 0 & 0 \\ 0 & s_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{1x} & 0 & 0 \\ 0 & s_{1y} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{1x} \cdot s_{2x} & 0 & 0 \\ 0 & s_{1y} \cdot s_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{S}(s_{1x} \cdot s_{2x}, s_{1y} \cdot s_{2y})$$

•The resulting matrix indicates the successive operations are multiplicative.

# Composite Transformations

Combining transformations reduces to matrix multiplication, e.g.

- R(r,) = T(r)* R(θ)* T(-r)

In general: multiplication of a 3x3 with another 3x3 matrix requires 3*3*3 = 27 multiplications and 2*3*3 additions.

In 2D transformations, the third row of the matrices is always [0 0 1] and should never be calculated.

In addition, in homogeneous coordinates the third component of the vectors is always one: (x,y,1).

Composite converts all to matrix multiplications thus improving computational efficiency

# Rotation around a pivot point

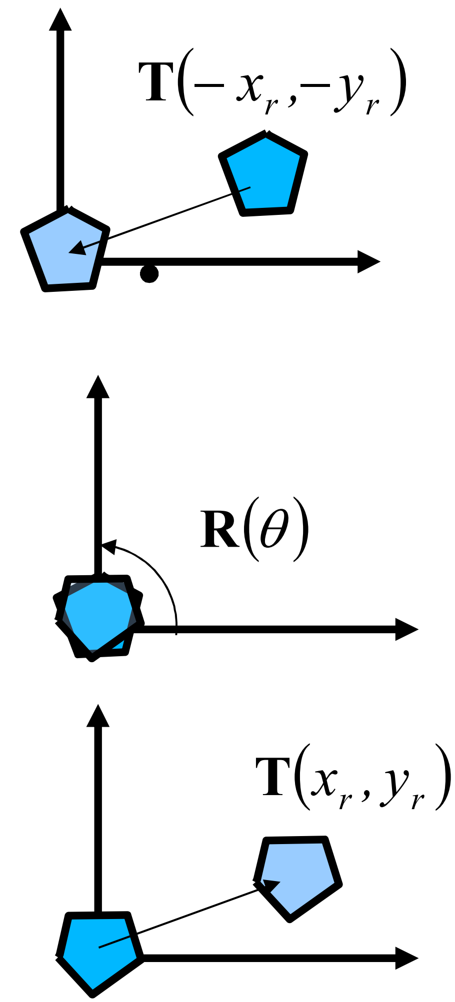
Rotations about any selected pivot point $(x_r, y_r)$ by performing the following sequence:

- Translate the object so that the pivot point moves to the origin
- Rotate around origin
- Translate the object so that the pivot point is back to its original position

$$\mathbf{T}(x_r, y_r)\cdot\mathbf{R}(\theta)\cdot\mathbf{T}(-x_r,-y_r)=$$

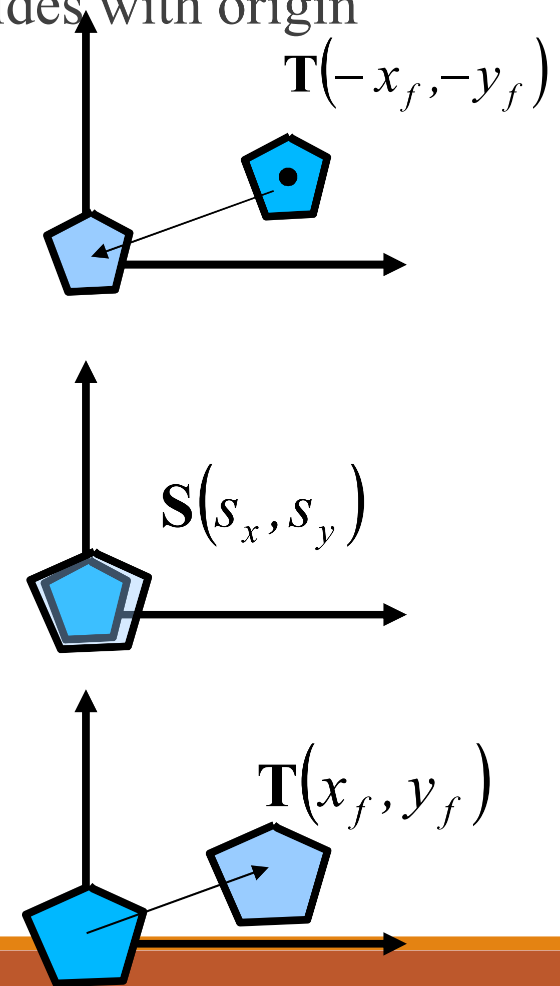
$$\begin{bmatrix}1 & 0 & x_r\\ 0 & 1 & y_r\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & -x_r\\ 0 & 1 & -y_r\\ 0 & 0 & 1\end{bmatrix}=$$

$$\begin{bmatrix}\cos\theta & -\sin\theta & x_r(1-\cos\theta)+y_r\sin\theta\\ \sin\theta & \cos\theta & y_r(1-\cos\theta)-x_r\sin\theta\\ 0 & 0 & 1\end{bmatrix}$$



# Scaling with respect to a Fixed Point

- Translate object to origin so fixed point coincides with origin
- Scale the object with respect to origin
- Translate back by inverse translation.




$$\mathbf{T}(x_f, y_f) \cdot \mathbf{S}(s_x, s_y) \cdot \mathbf{T}(-x_f, -y_f) =$$

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} =$$

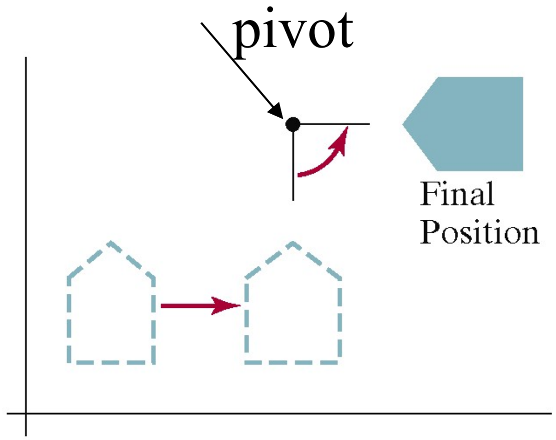
$$\begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

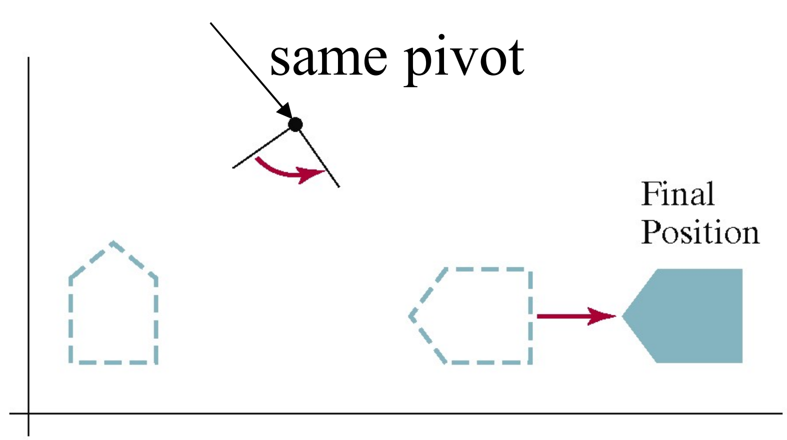
# Concatenation Properties

Matrix multiplication is associative, evaluate matrix products using left-to-right or right-to-left associative grouping.

Matrix composition is not commutative. So careful when applying a sequence of transformations.

Reversing the order in which the sequence of transformations is performed may affect the transformed position of an object.



# REFLECTION

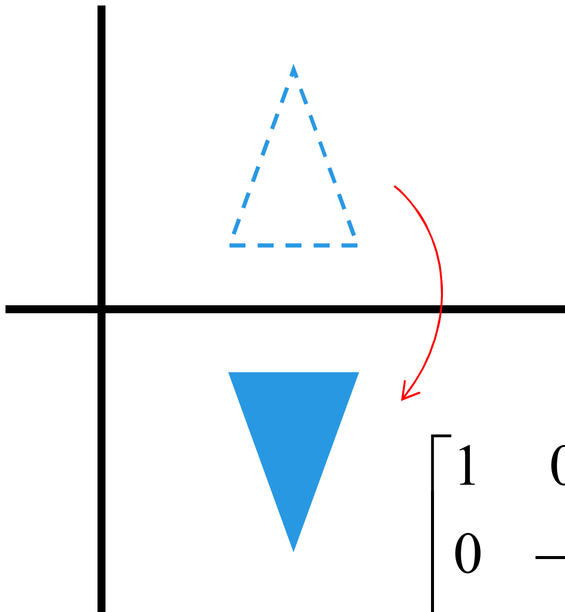
A transformation that produces a mirror image of an object

Image is generated relative to an axis of reflection by rotating the object 180° about the reflection axis

- *Reflection axis is xy* plane – rotation path about the axis is in the plane perpendicular to *xy* plane
- Reflection axis perpendicular to *xy* plane – rotation path is in the *xy* plane

# 2D Reflection

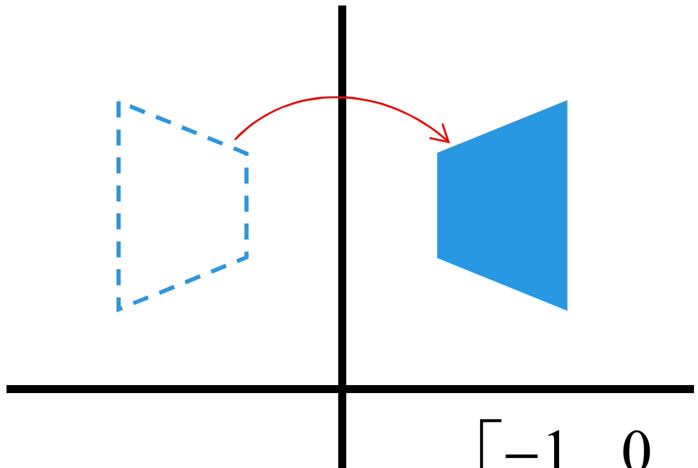
x-axis
Reflection about the line
y=0



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation keeps x values but flips the y values

y-axis
Reflection about the line
x=0

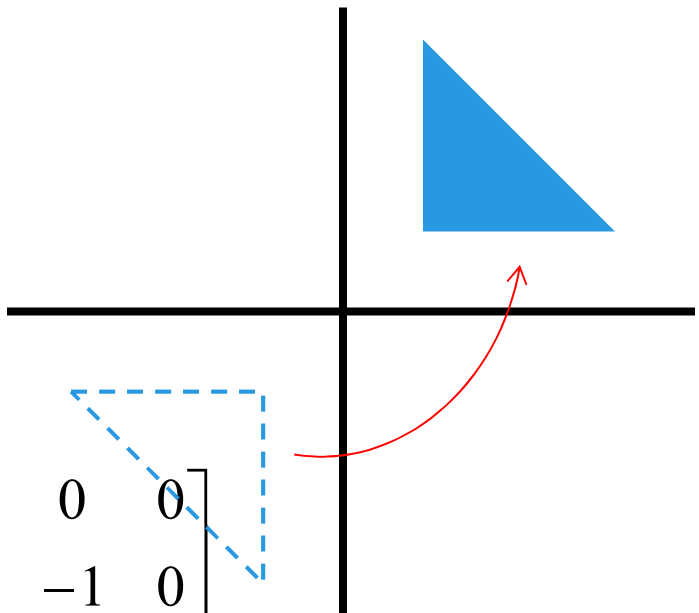


$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation keeps y values but flips the x values

# 2D REFLECTION

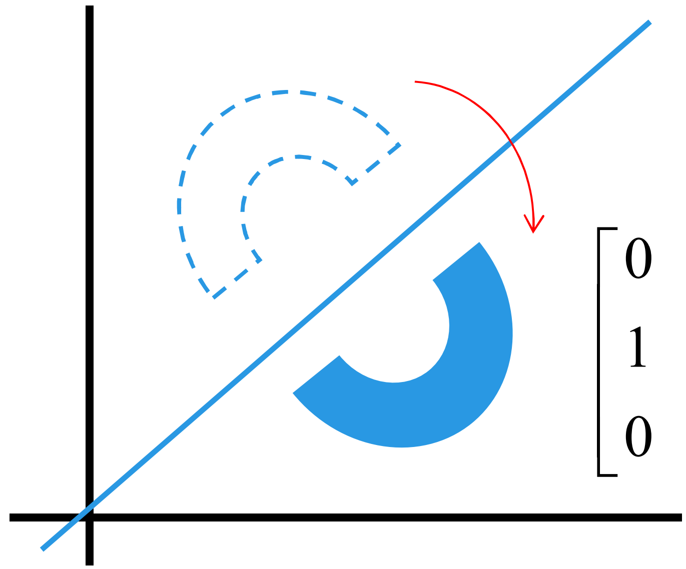
Reflection relative to the coordinate origin



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation flips both x values and y values by Reflecting relative to the coordinate origin

Reflection axis as the diagonal line x=y



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# 2D Reflection

Elements of the reflection matrix can be set to values other than ±1.

Reflection parameter:

- >1 – shifts the mirror image of a point farther from the reflection axis.
- <1 – brings the mirror image of a point closer to the reflection axis.

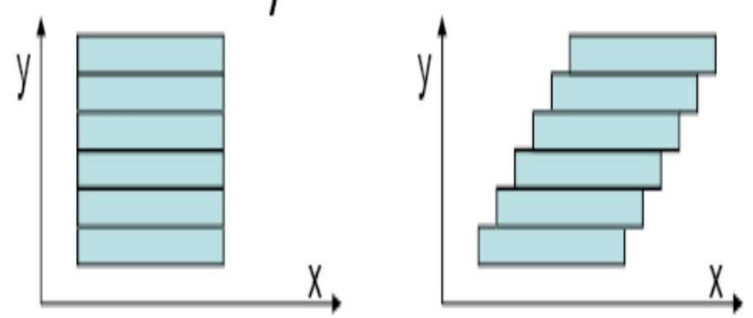
Thus, a reflected object can also be enlarged, reduced or distorted.

# 2D Shear

Transformation that distort the shape of an object.

Slide to another shape

Internal layers are shifted w.r.t. each other




2 common shearing transformation
- Shift coordinate *x* values
- Shift coordinate *y* values

# 2D SHEAR

An x-direction shear relative to the x axis is produced with the transformation matrix

$$\begin{pmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

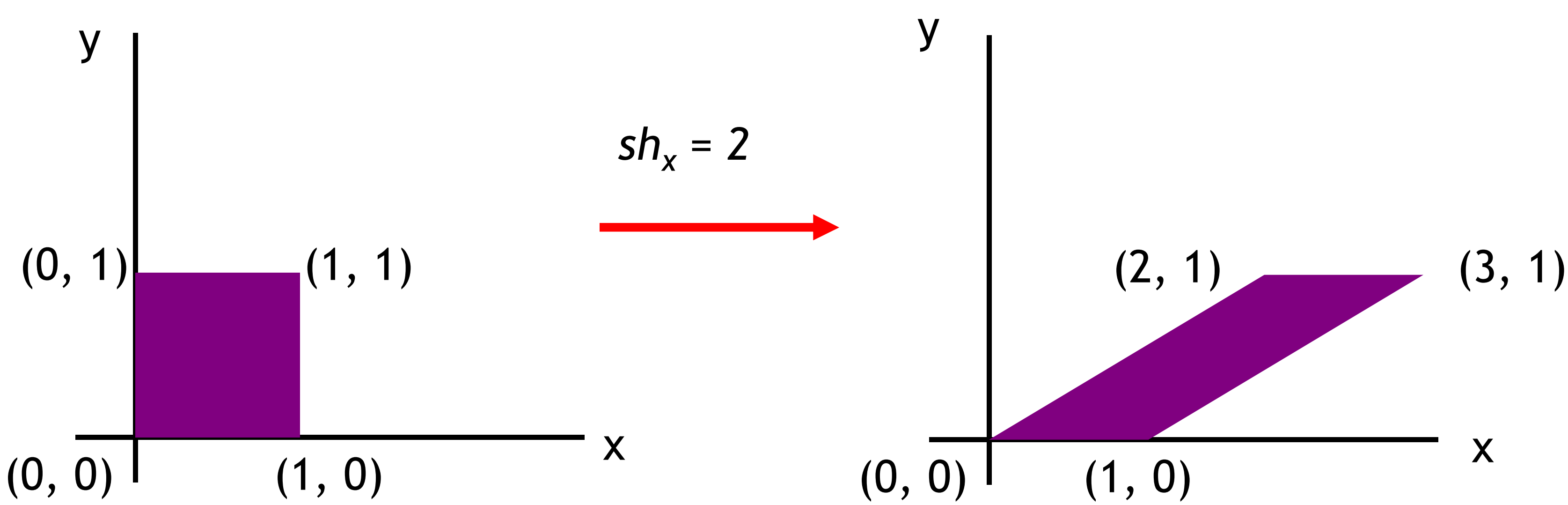
which transforms coordinate positions as

$$x' = x + sh_x \cdot y, \qquad y' = y$$

# 2D Shear

Any real number can be assigned to the shear parameter $sh_x$.

A coordinate position ($x$, $y$) is then shifted horizontally by an amount proportional to its perpendicular distance ( $y$ value) from the $x$ axis.



# 2D Shear

We can generate *x*-direction shears relative to other reference lines with

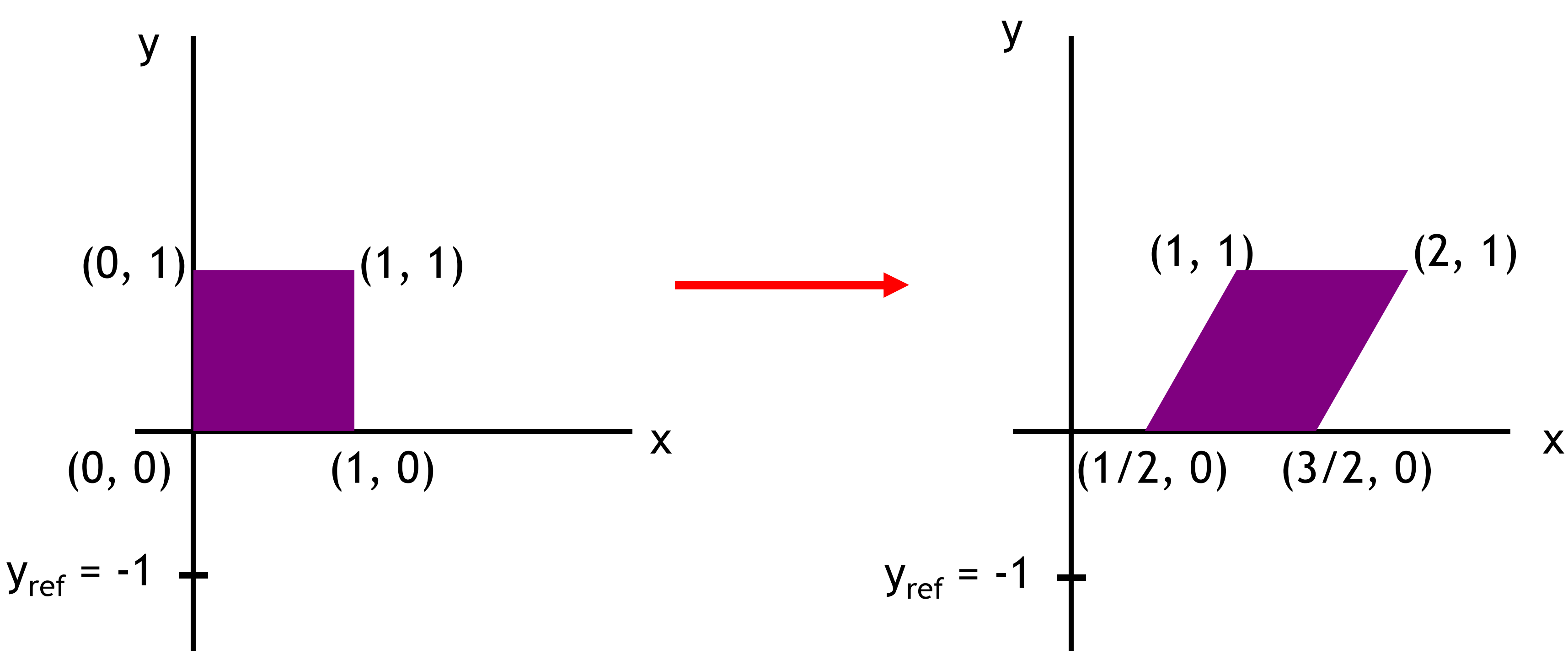
$$\begin{pmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, coordinate positions are transformed as

$$x' = x + sh_x( y - y_{ref}) , \qquad y' = y$$

# EXAMPLE

$sh_x$=0.5 and $y_{ref}$ = -1



# 2D SHEAR

A *y*-direction shears relative to other reference lines can generate with

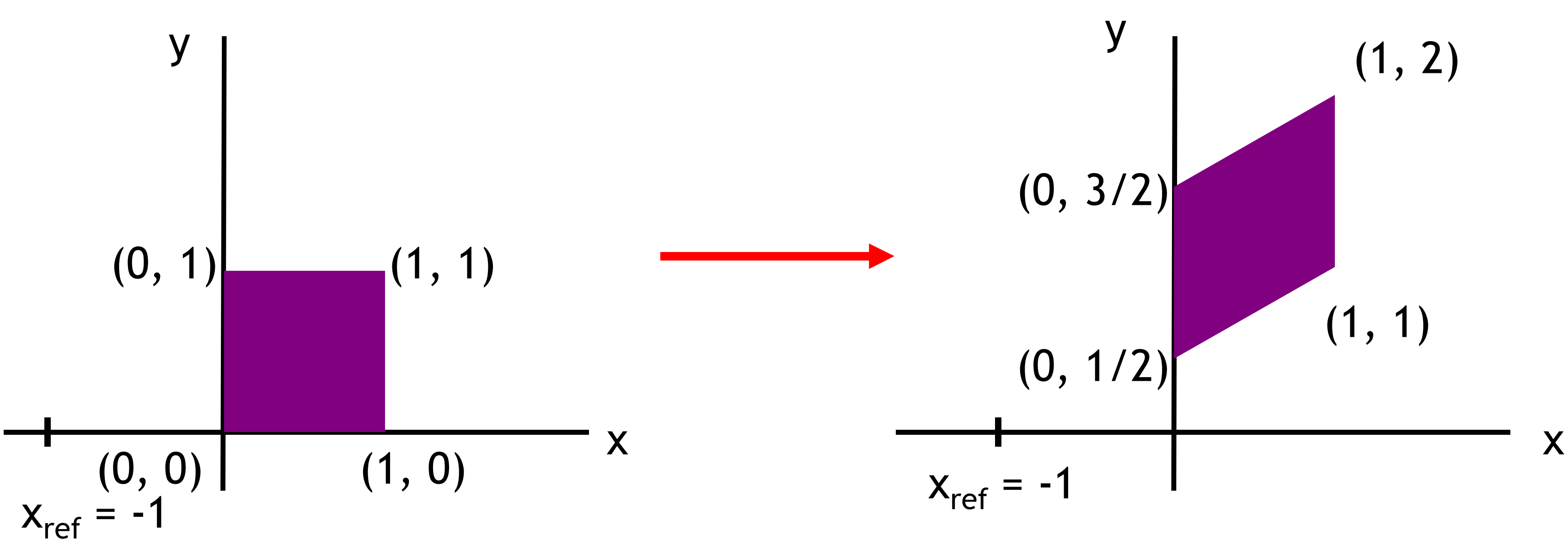
$$\begin{pmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{pmatrix}$$

Now, coordinate positions are transformed as

$$x' = x, \qquad y' = y + sh_y(x - x_{ref})$$

# EXAMPLE

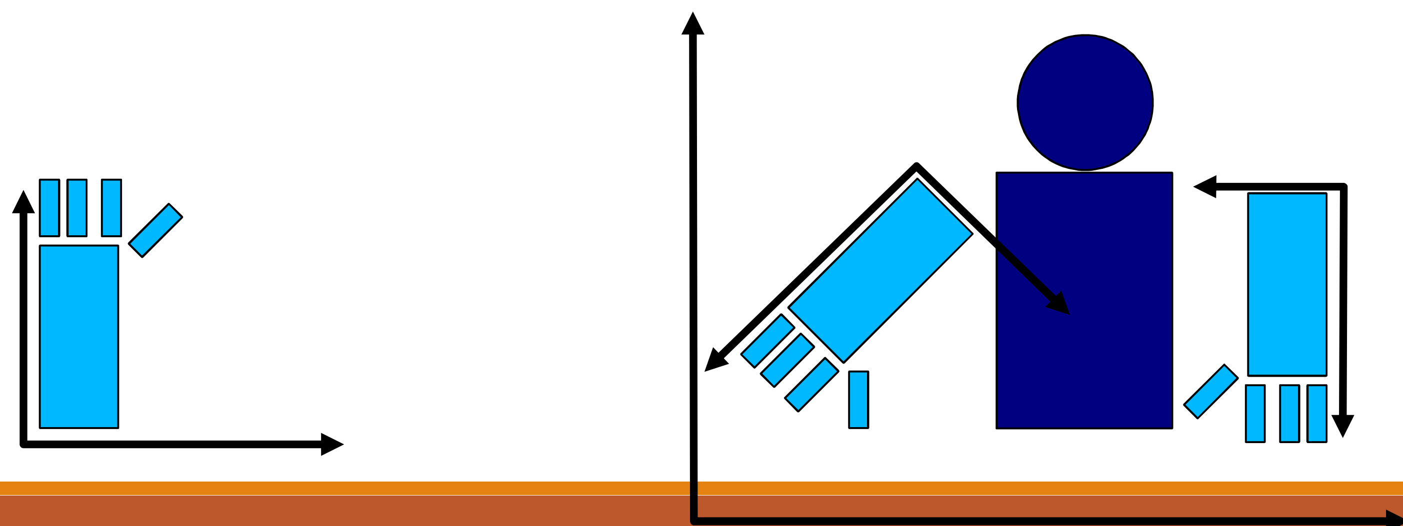
$sh_y$=0.5 and $x_{ref}$ = -1



# Transformations Between the Coordinate Systems

Between different systems: Polar coordinates to cartesian coordinates

Between two cartesian coordinate systems. For example, relative coordinates or window to viewport transformation.



# Transformations Between the Coordinate Systems

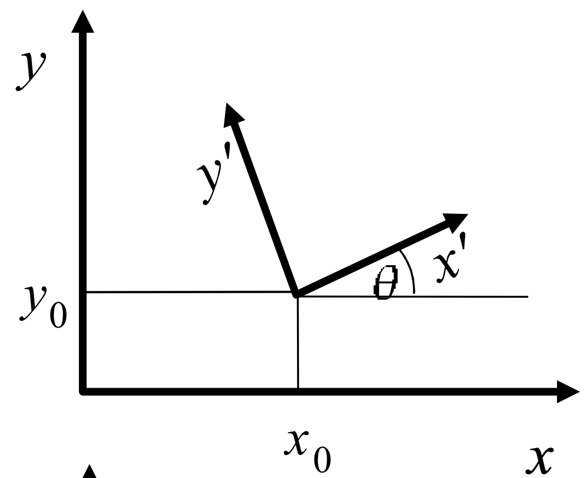
How to transform from *x,y* to *x',y'* ?

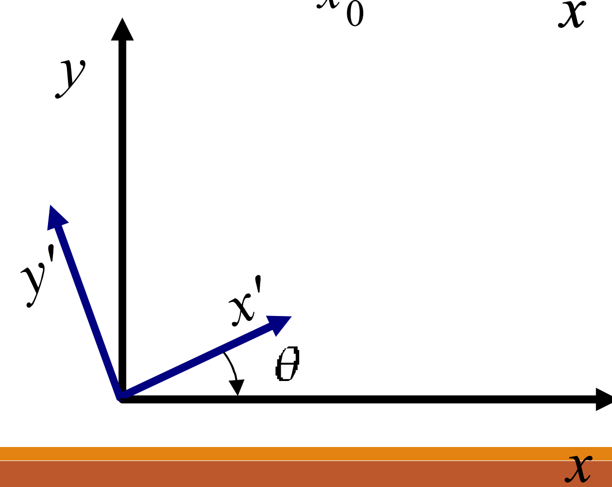
*Superimpose x',y' to x,y*

*Transformation:*

- *Translate so that* $(x_0, y_0)$ *moves to (0,0) of x,y*
- *Rotate x' axis onto x axis*

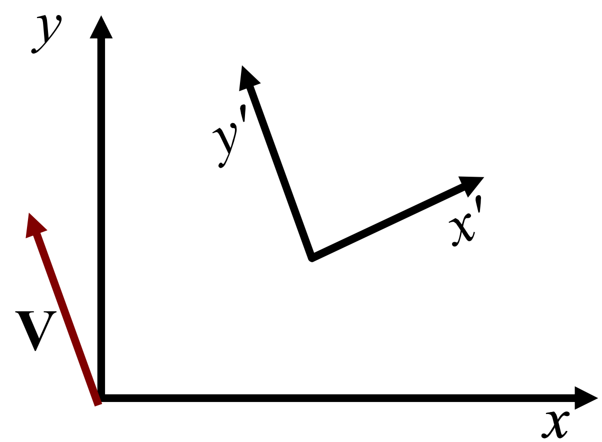
$$R(-\theta)\cdot T(-x_0, -y_0)$$



# Transformations Between the Coordinate Systems

Alternate method for rotation:
Specify a vector **V** for positive $y'$ axis:




unit vector in the $y'$ direction :

$$\mathbf{v} = \frac{\mathbf{V}}{|\mathbf{V}|} = (v_x, v_y)$$

unit vector in the $x'$ direction, rotate $\mathbf{v}$ clockwise $90^\circ$

$$\mathbf{u} = (v_y, -v_x) = (u_x, u_y)$$

# Transformations Between the Coordinate Systems

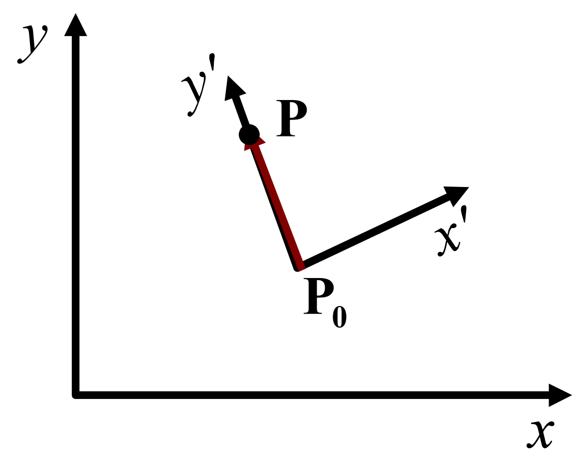
Elements of any rotation matrix can be expressed as elements of a set of orthogonal unit vectors:

$$\mathbf{R} = \begin{bmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} v_y & -v_x & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Choose the directions for v relative to position P0 .

The components of v calculated as

$$\mathbf{v} = \frac{\mathbf{P} - \mathbf{P_0}}{|\mathbf{P} - \mathbf{P_0}|}$$




U is obtained as perpendicular to v

# Affine Transformations

- An affine transformation is an important class of linear 2-D geometric transformations which maps variables (*e.g.* pixel **intensity values** located at position (x,y) in an input image) into new variables (*e.g.* in an output image (x',y') by applying a linear combination **of translation, rotation, scaling** and/or shearing (*i.e.* non-uniform scaling in some directions) operations.

- Coordinate transformations of the form:

$$x' = a_{xx}x + a_{xy}y + b_x$$
$$y' = a_{yx}x + a_{yy}y + b_y$$

Translation, rotation, scaling, reflection, shear. Any affine transformation can be expressed as the combination of these.

# Summary

Homogeneous coordinates

Reflection and shearing – w.r.t origin and fixed point

Transformation between systems

Affine transformations