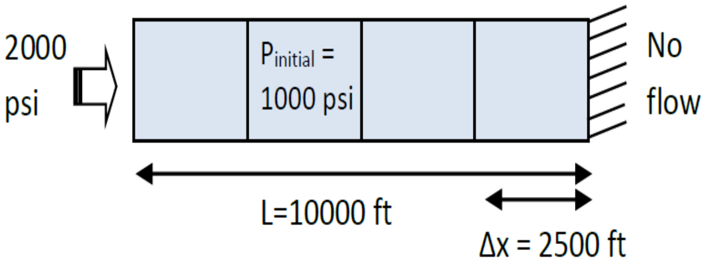
Example 4. Crank-Nicholson Solution to 1D flow in terms of transmissibility (flow rate units)

Consider a 1D reservoir with the following reservoir and fluid properties:

$\phi = 0.2$
k = 50 mD
μ =1 cp
B_W=1 rb/stb (assume constant)
$c_t = 10^{-6}$ psi^{-1}.
L= 10,000 ft (reservoir length)
A = 200,000 ft^2 (Cross Sectional Area)



The initial condition is P = 1000 psi. The boundary conditions are P = 2000 psi at x = 0 and no flow (q = 0) at x = L. Determine the pressure field in the reservoir using 4 uniform blocks. Use a time step of Δt = 1 days.

Solution:
The pressure is governed by the 1D diffusivity equation with the following boundary conditions:

$$\frac{1}{\alpha}\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}; \quad \alpha = \left(\frac{k}{\mu\phi c_t}\right)$$

$$IC: \mathrm{p}(x,0) = 1000\ psi$$

$$BC1: \mathrm{p}(0,t) = 2000\ psi$$

$$BC2: \frac{\partial p}{\partial x}(L,t) = 0$$

The Crank-Nicholson solution is a hybrid of the explicit and implicit methods with θ = ½.

$$\left((1-\theta)\mathbf{T} + \frac{\mathbf{B}}{\Delta t}\right)\mathbf{P}^{n+1} = \left(\frac{\mathbf{B}}{\Delta t} - \theta\mathbf{T}\right)\mathbf{P}^{n} + \mathbf{Q}$$

Where:

$$T = \frac{kA}{\mu B_w \Delta x} = \frac{50 \cdot 200000}{1 \cdot 1 \cdot 2500} = 4000 \frac{mD - ft}{cp}$$

$$B_i = V_i \phi c_t = \left(200000\, ft^2\right)\left(2500\, ft\right)\left(0.2\right)\left(1E-6\, psi^{-1}\right) = 100 \frac{ft^3}{psi}$$

$$Q_1 = 2TP_{in} = 2 \cdot 4000 \cdot 2000 = 1.6E7 \frac{mD - ft - psi}{cp}$$

Plugging in the numbers we get:

$$\begin{bmatrix} 138 & -12.7 & 0 & 0 \\ -12.7 & 125.3 & -12.7 & 0 \\ 0 & -12.7 & 125.3 & -12.7 \\ 0 & 0 & -12.7 & 112.7 \end{bmatrix} \begin{pmatrix} P_1^{n+1} \\ P_2^{n+1} \\ P_3^{n+1} \\ P_4^{n+1} \end{pmatrix} = \begin{pmatrix} 62 & 12.7 & 0 & 0 \\ 12.7 & 74.7 & 12.7 & 0 \\ 0 & 12.7 & 74.7 & 12.7 \\ 0 & 0 & 12.7 & 87.3 \end{pmatrix} \begin{pmatrix} P_1^n \\ P_2^n \\ P_3^n \\ P_4^n \end{pmatrix} + \begin{pmatrix} 1.6E7 \\ 0 \\ 0 \\ 0 \end{pmatrix} \times 6.33E-03$$

Table 4.1 summarizes the solution for the C-N method at various times and compares it to the explicit, implicit, CMG, and analytical solutions.

Table 4.1 Summary of block pressures for example 4 using various solution techniques

	Method	**Block #1 (psi) 1250 ft**	**Block #2 (psi) 3750 ft**	**Block #3 (psi) 6250 ft**	**Block #4 (psi) 8750 ft**
Initial		1000	1000	1000	1000
1 days	Explicit	1506.4	1000	1000	1000
	Implicit	1295.1	1051.1	1008.9	1001.8
	CMG	1294.9	1051.1	1008.9	1001.8
	C-N	1370.5	1037.8	1003.9	1000.4
	Analytical	1482.3	1035.0	1000.4	1000.0
2 days	Explicit	1628.1	1128.2	1000	1000
	Implicit	1472.5	1117.9	1026.9	1006.9
	CMG	1472.2	1117.8	1026.8	1006.9
	C-N	1547.8	1117.5	1018.3	1002.8
	Analytical	1619.3	1136.1	1013.0	1000.5
3 days	Explicit	1689.9	1222.3	1032.5	1000
	Implicit	1582.9	1184.9	1051.6	1015.9
	CMG	1582.6	1184.8	1051.5	1015.9
	C-N	1642	1196.5	1043.9	1009.2
	Analytical	1685	1223.6	1042.5	1004.8
∞		2000	2000	2000	2000

The results are not surprising; the mixed (Crank-Nicholson) method gives an answer "in between" the explicit and implicit methods. Moreover, we expect the answer to be more accurate because it is an $O(\Delta t^2)$ method. However, it does require more computational effort per time step than either explicit or implicit method. Alternatively, we could have used smaller time steps in the explicit/implicit methods instead to get more accuracy.

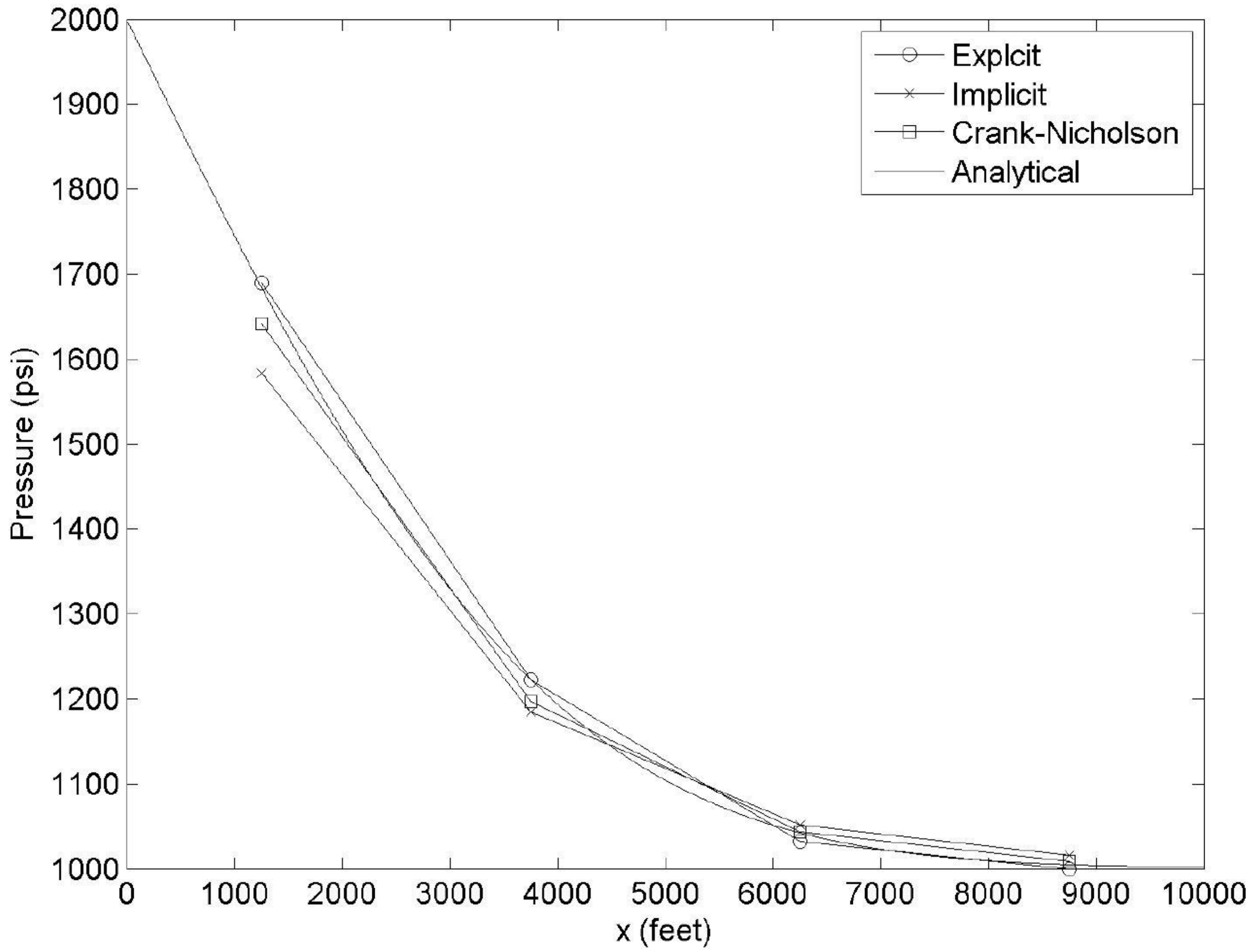


Figure 4.1 Comparison of the explicit, implicit, and C-N methods (4 grid blocks, Δt= 1 day) to the analytical solution after 3 days. As expected, none of the three numerical approaches perfectly match the analytical solution, but all would converge to it if more (an infinite number) grids were employed.