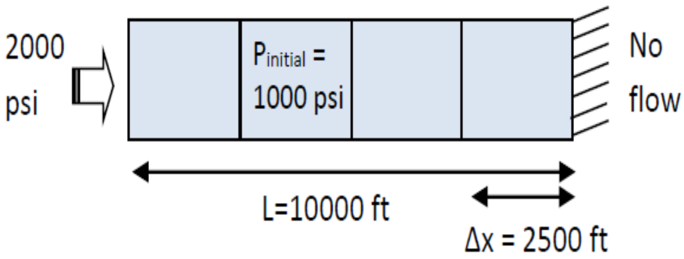
Example 1. Explicit Solution to 1D flow

Consider a 1D reservoir with the following reservoir and fluid properties:

$\phi = 0.2$
k = 50 mD
μ =1 cp
B_w = 1 rb/stb (assume constant)
$c_t = 10^{-6}$ psi^{-1}.
L= 10,000 ft (Reservoir Length)
A = 200,000 ft^2 (Cross Sectional Area)



The initial condition is P = 1000 psi. The boundary conditions are P = 2000 psi at x = 0 and no flow (q = 0) at x = L. Determine the pressure field in the reservoir using 4 uniform blocks. Use a time step of Δt = 1.0 days.

<u>Solution:</u>
The pressure is governed by the 1D diffusivity equation with the following boundary conditions:

$$\frac{1}{\alpha}\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}; \quad \alpha = \left(\frac{k}{\mu \phi c_t}\right)$$

$$IC: \mathrm{p}(x,0) = 1000\ psi$$
$$BC1: \mathrm{p}(0,t) = 2000\ psi$$
$$BC2: \frac{\partial p}{\partial x}(L,t) = 0$$

The explicit finite difference solution is given by the formula:

$$P_i^{n+1} = P_i^n + \frac{\alpha \Delta t}{(\Delta x)^2}\left[P_{i-1}^n - 2P_i^n + P_{i+1}^n\right]$$

Where the dimensionless diffusivity is written as:

$$\eta = \alpha \frac{\Delta t}{(\Delta x)^2} = \left(\frac{k}{\mu \phi c_t}\right)\frac{\Delta t}{(\Delta x)^2} = \frac{50\text{ mD}}{(1\text{ cp})(0.2)(1.0\times 10^{-6}\text{ psi}^{-1})}\frac{1.0\text{ days}}{(2500\text{ ft})^2} = 40\frac{\text{mD-days-psi}}{\text{cp-ft}^2} = 0.2532$$

Note that we are guaranteed stability with the explicit method since $\eta<0.5$.

<u>At t = 0 days</u>, the pressure is uniform at 1000 psi. $P^0 = [1000\ 1000\ 1000\ 1000]$ psi

<u>At t = 1.0 days (n = 1)</u>, we solve explicitly for the pressures

$$P_1^1 = P_1^0 + \eta\left[P_0^0 - 2P_1^0 + P_2^0\right] = 1000 + 0.2532\left[P_0^0 - 2(1000) + 1000\right] =$$
$$P_2^1 = P_2^0 + \eta\left[P_1^0 - 2P_2^0 + P_3^0\right] = 1000 + 0.2532\left[1000 - 2(1000) + 1000\right] =$$
$$P_3^1 = P_3^0 + \eta\left[P_2^0 - 2P_3^0 + P_4^0\right] = 1000 + 0.2532\left[1000 - 2(1000) + 1000\right] =$$
$$P_4^1 = P_4^0 + \eta\left[P_3^0 - 2P_4^0 + P_5^0\right] = 1000 + 0.2532\left[1000 - 2(1000) + P_5^0\right] =$$

There is no "block #0" or "block #5". We must use our boundary conditions:

1) At $x = 0$, we have a constant pressure boundary $P=P_{in}=2000$ psi

$$P_{in} = \frac{P_0 + P_1}{2} \Rightarrow P_0 = 2P_{in} - P_1$$

2) At $x = L$, we have a "no-flow" condition

$$\frac{P_4 - P_5}{\Delta x} = 0 \Rightarrow P_5 = P_4$$

Substitute these values into the pressure equations:

$$P_1^1 = P_1^0 + \eta\left[\left(2P_{in} - P_1^0\right) - 2P_1^0 + P_2^0\right] = 1000 + 0.2532\left[\left(2\cdot 2000 - 1000\right) - 2\left(1000\right) + 1000\right] = 1506.4 \text{ psi}$$

$$P_2^1 = P_2^0 + \eta\left[P_1^0 - 2P_2^0 + P_3^0\right] = 1000 + 0.2532\left[1000 - 2\left(1000\right) + 1000\right] = 1000 \text{ psi}$$

$$P_3^1 = P_3^0 + \eta\left[P_2^0 - 2P_3^0 + P_4^0\right] = 1000 + 0.2532\left[1000 - 2\left(1000\right) + 1000\right] = 1000 \text{ psi}$$

$$P_4^1 = P_4^0 + \eta\left[P_3^0 - 2P_4^0 + P_4^0\right] = 1000 + 0.2532\left[1000 - 2\left(1000\right) + 1000\right] = 1000 \text{ psi}$$

<u>At t = 2.0 days (n = 2)</u>

$$P_1^2 = P_1^1 + \eta\left[\left(2P_{in} - P_1^1\right) - 2P_1^1 + P_2^1\right] = 1506.4 + 0.2532\left[\left(2\cdot 2000 - 1506.4\right) - 2\left(1506.4\right) + 1000\right] = 1628.1 \text{ psi}$$

$$P_2^2 = P_2^1 + \eta\left[P_1^1 - 2P_2^1 + P_3^1\right] = 1000 + 0.2532\left[1506.4 - 2\left(1000\right) + 1000\right] = 1128.2 \text{ psi}$$

$$P_3^2 = P_3^1 + \eta\left[P_2^1 - 2P_3^1 + P_4^1\right] = 1000 + 0.2532\left[1000 - 2\left(1000\right) + 1000\right] = 1000 \text{ psi}$$

$$P_4^2 = P_4^1 + \eta\left[P_3^1 - 2P_4^1 + P_4^1\right] = 1000 + 0.2532\left[1000 - 2\left(1000\right) + 1000\right] = 1000 \text{ psi}$$

<u>At t = 3.0 days (n = 3)</u>

$$P_1^3 = P_1^2 + \eta\left[\left(2P_{in} - P_1^2\right) - 2P_1^2 + P_2^2\right] = 1628.1 + 0.2532\left[\left(2\cdot 2000 - 1628.1\right) - 2\left(1628.1\right) + 1128.2\right] = 1689.9 \text{ psi}$$

$$P_2^3 = P_2^2 + \eta\left[P_1^2 - 2P_2^2 + P_3^2\right] = 1128.2 + 0.2532\left[1628.1 - 2\left(1128.2\right) + 1000\right] = 1223.3 \text{ psi}$$

$$P_3^3 = P_3^2 + \eta\left[P_2^2 - 2P_3^2 + P_4^2\right] = 1000 + 0.2532\left[1128.2 - 2\left(1000\right) + 1000\right] = 1032.5 \text{ psi}$$

$$P_4^3 = P_4^2 + \eta\left[P_3^2 - 2P_4^2 + P_4^2\right] = 1000 + 0.2532\left[1000 - 2\left(1000\right) + 1000\right] = 1000 \text{ psi}$$

At t=∞, the reservoir comes to equilibrium at P = 2000 psi in all blocks

Table 1.1 Summary of block pressures using the explicit method for example 1.

	Block #1 (psi) 1250 ft	**Block #2 (psi) 3750 ft**	**Block #3 (psi) 6250 ft**	**Block #4 (psi) 8750 ft**
0 days	1000	1000	1000	1000
1 days	1506.4	1000	1000	1000
2 days	1628.1	1128.2	1000	1000
3 days	1689.9	1222.3	1032.5	1000
∞	2000	2000	2000	2000

The solution makes physical sense. The pressure in the reservoir increases over time as a result of the dirichlet (constant pressure) boundary condition of $P = 2000$ psi, until the reservoir comes to steady state at 2000 psi. Pressure increases fastest in block #1 because it is near the boundary and slowest in block #4 because it is far from the boundary.

A plot of the numerical (explicit) solution to the problem is shown in figure 1.1a, 1.1b, and 1.1c using 4, 8, and 12 blocks, respectively. A plot of the actual (analytical) solution is given in figure 1.1d. Recall the analytical solution for this problem is given by:

$$p(x,t) = p_{B1} - \frac{4p_{init}}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{\frac{\alpha(2n+1)^n \pi^2 t}{4L^2}} \cos \frac{(2n+1)\pi x}{2L}$$

The numerical solution demonstrates the same trends as the analytical solution, but is not as smooth because only a finite # of grid blocks are used. The numerical solution is particularly poor for only 4 grid blocks, but we could have increased accuracy by using more grids and smaller time steps as long as the stability criterion ($\eta < 0.5$) was still met. Figure 1.1 demonstrates improved accuracy with more grids and smaller timesteps.

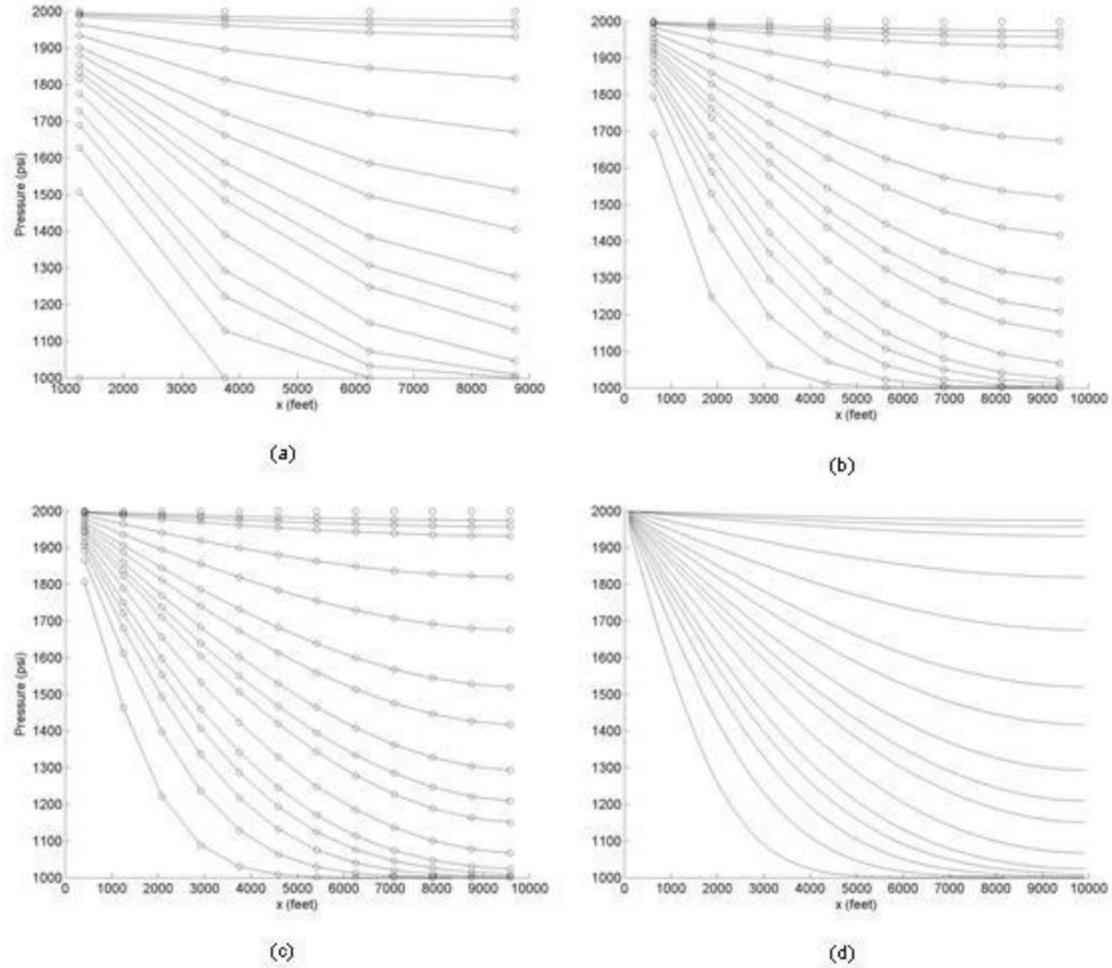


Figure 1.1. Solutions to pressure vs. reservoir distance at various times. (a) 4 grids (Δx = 2500 feet), Δt = 1 day; (b) 8 grids (Δx = 1250 feet), Δt = 0.1 day; (c) 12 grids (Δx = 833 feet), Δt = 0.01 day; (d) analytical solution. For the numerical solutions, the points are the grid pressures and the continuous lines are linearly interpolated values.