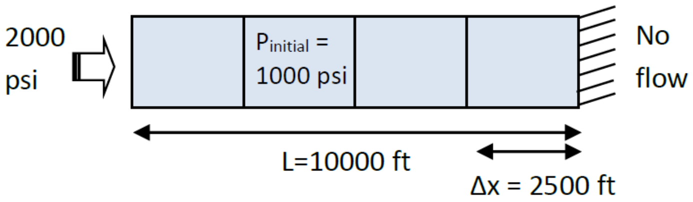
Lecture 17

Example 2. Implicit Solution to 1D flow

Consider a 1D reservoir with the following reservoir and fluid properties:

$\phi = 0.2$
k = 50 mD
μ =1 cp
B_w = 1 rb/stb (assume constant)
$c_t = 10^{-6}$ psi^{-1}.
L= 10,000 ft (reservoir length)
A = 200,000 ft^2 (Cross Sectional Area)



The initial condition is P = 1000 psi. The boundary conditions are P = 2000 psi at x = 0 and no flow (q = 0) at $x = L$. Determine the pressure field in the reservoir using 4 uniform blocks. Use a time step of Δt = 1.0 days.

Solution:
The pressure is governed by the 1D diffusivity equation with the following boundary conditions:

$$\frac{1}{\alpha}\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}; \quad \alpha = \left(\frac{k}{\mu\phi c_t}\right)$$

$$IC: \mathrm{p}(x,0) = 1000\ psi$$

$$BC1: \mathrm{p}(0,t) = 2000\ psi$$

$$BC2: \frac{\partial p}{\partial x}(L,t) = 0$$

The implicit finite difference solution is given by the formula:

$$-\eta P_{i-1}^{n+1} + (1+2\eta)P_i^{n+1} - \eta P_{i+1}^{n+1} = P_i^n$$

Where the dimensionless diffusivity is written as:

$$\eta = \alpha \frac{\Delta t}{(\Delta x)^2} = \left(\frac{k}{\mu \phi c_t}\right)\frac{\Delta t}{(\Delta x)^2} = \frac{50 \text{ mD}}{(1 \text{ cp})(0.2)(1.0\times 10^{-6} \text{ psi}^{-1})}\frac{1.0 \text{ days}}{(2500 \text{ ft})^2} = 40\frac{\text{mD-days-psi}}{\text{cp-ft}^2} = 0.2532$$

The equations for each block can now be written as a system of linear equations

$$-\eta P_0^1 + (1+2\eta)P_1^1 - \eta P_2^1 = P_1^0$$
$$-\eta P_1^1 + (1+2\eta)P_2^1 - \eta P_3^1 = P_2^0$$
$$-\eta P_2^1 + (1+2\eta)P_3^1 - \eta P_4^1 = P_3^0$$
$$-\eta P_3^1 + (1+2\eta)P_4^1 - \eta P_5^1 = P_4^0$$

Once again we have to use boundary conditions to eliminate the blocks outside of the domain:

1) At $x = 0$, we have a constant pressure boundary $P = P_{in} = 2000$ psi

$$P_{in} = \frac{P_0 + P_1}{2} \Rightarrow P_0 = 2P_{in} - P_1$$

2) At $x = L$, we have a no-flow condition

$$\frac{P_4 - P_5}{\Delta x} = 0 \Rightarrow P_5 = P_4$$

Substituting the boundary conditions into the equations and writing them in matrix form:

$$\begin{pmatrix} 1+3\eta & -\eta & 0 & 0 \\ -\eta & 1+2\eta & -\eta & 0 \\ 0 & -\eta & 1+2\eta & -\eta \\ 0 & 0 & -\eta & 1+\eta \end{pmatrix} \begin{pmatrix} P_1^1 \\ P_2^1 \\ P_3^1 \\ P_4^1 \end{pmatrix} = \begin{pmatrix} P_1^0 \\ P_2^0 \\ P_3^0 \\ P_4^0 \end{pmatrix} + \begin{pmatrix} 2\eta P_{in} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

At $t = 1$ days, the solution can be found by solving the system of equations:

$$\begin{pmatrix} 1+3(0.2532) & -0.2532 & 0 & 0 \\ -0.2532 & 1+2(0.2532) & -(0.2532) & 0 \\ 0 & -0.2532 & 1+2(0.2532) & -0.2532 \\ 0 & 0 & -0.2532 & 1+0.2532 \end{pmatrix} \begin{pmatrix} P_1^1 \\ P_2^1 \\ P_3^1 \\ P_4^1 \end{pmatrix} = \begin{pmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \end{pmatrix} + \begin{pmatrix} 2\cdot 0.2532\cdot 2000 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Which has the solution $P^1 = [1295.1; 1051.1; 1009.9; 1001.8]$ psi

At $t = 2$ days, use the solution at the previous time step. Note that the matrix remains the same, but the "right hand side (RHS)" vector changes from the previous timestep

$$\begin{pmatrix} 1+3(0.2532) & -0.2532 & 0 & 0 \\ -0.2532 & 1+2(0.2532) & -(0.2532) & 0 \\ 0 & -0.2532 & 1+2(0.2532) & -0.2532 \\ 0 & 0 & -0.2532 & 1+0.2532 \end{pmatrix} \begin{pmatrix} P_1^2 \\ P_2^2 \\ P_3^2 \\ P_4^2 \end{pmatrix} = \begin{pmatrix} 1295.1 \\ 1051.1 \\ 1009.9 \\ 1001.8 \end{pmatrix} + \begin{pmatrix} 2\cdot 0.2532\cdot 2000 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Which has the solution P^2 = [1472.5; 1117.9; 1026.9; 1006.9] psi

At t = 0.03 days, use the solution at the previous time step. Note that the η matrix remains the same.

$$\begin{pmatrix} 1+3(0.2532) & -0.2532 & 0 & 0 \\ -0.2532 & 1+2(0.2532) & -(0.2532) & 0 \\ 0 & -0.2532 & 1+2(0.2532) & -0.2532 \\ 0 & 0 & -0.2532 & 1+0.2532 \end{pmatrix} \begin{pmatrix} P_1^3 \\ P_2^3 \\ P_3^3 \\ P_4^3 \end{pmatrix} = \begin{pmatrix} 1472.5 \\ 1117.9 \\ 1026.9 \\ 1006.9 \end{pmatrix} + \begin{pmatrix} 2 \cdot 0.2532 \cdot 2000 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Which has the solution P^3 = [1582.9; 1184.9; 1051.6; 1015.9] psi

Eventually, the solution reaches steady state and P^{ss} = [2000; 2000; 2000; 2000] psi. Table 2.1 summarizes the results obtained using the implicit method as shown above and compares them to the results for the same problem. As expected, the results are nearly identical. The very slight differences are caused by the fact that CMG does not allow for dirichlet boundary conditions, so one was fabricated by placing a constant bottomhole pressure well near the edge and introducing a slight amount of additional error. In a sense, the "hand" solution is slightly more accurate than CMG, but both have noticeable discrepancies compared to the "true" (analytical solution).

Table 2.1 Comparison of numerical solution by hand (example 2) to the same problem in CMG

	Method	**Block #1 (psi) 1250 ft**	**Block #2 (psi) 3750 ft**	**Block #3 (psi) 6250 ft**	**Block #4 (psi) 8750 ft**
Initial		1000	1000	1000	1000
1 day	Implicit	1295.1	1051.1	1008.9	1001.8
	CMG	1294.9	1051.1	1008.9	1001.8
2 days	Implicit	1472.5	1117.9	1026.9	1006.9
	CMG	1472.2	1117.8	1026.8	1006.9
3 days	Implicit	1582.9	1184.9	1051.6	1015.9
	CMG	1582.6	1184.8	1051.5	1015.9
∞		2000	2000	2000	2000