CS 180 - Algorithms and Complexity
Homework 1

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Discussion 1D

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Solution 1. The following is the solution to exercise 2 on page 22 of the textbook.

In the instance of the Stable Matching Problem in which there exists $m \in M$ and $w \in W$ such that m is ranked first on the preference list of w and w is ranked first on the preference list of m, the pair (m, w) always belongs to every stable matching S.
Hence, the statement is **true**.

The reason being that m would always propose to w first and since m is ranked first on w's preference list she would accept the proposal even if she is engaged to some m'. Furthermore, she would reject all other proposals from then on as the other men would be lower on her preference list than m.

Solution 2. The following is the solution to exercise 3 on page 22 of the textbook.

For the following set of TV shows (2 each), there does not exist a stable matching according to the given criterion.

For Network A, let the ratings of their shows be 1 and 10.
For Network B, let the ratings of their shows be 7 and 15.

We get the following 4 schedules each of which is not stable.

1. S = (1,10) and T = (7,15). With this scheduling, B gets both time slots and A gets 0 time slots. However, if A changes its schedule to (10,1), it can get 1 slot. Hence, the schedule is unstable.
2. S = (10,1) and T = (7,15). With this schedule, both B and A get 1 time slot. However, if B changes its schedule to (15,7), it can get 2 slots. Hence, the schedule is not stable.
3. S = (1, 10) and T = (15,7). Again, both B and A get one slot each. However, if B changes the schedule to (7,15), it can get both slots. Hence, the schedule is not stable.

4. S = (1, 10) and T = (7,15). B gets 2 slots while A gets 0 slots. However, there exists a schedule S' = (10,1) such that A now gets 1 slot. Hence, this schedule is not stable.

Thus, there does not always exist a stable pair of schedules.

Solution 3. The following is the solution to exercise 8 on page 27 of the textbook.

Consider the following example,
$M = \{m1,\ m2,\ m3\}$, $W = \{w1,\ w2,\ w3\}$
Let their priorities be the following:

$m1$	$m2$	$m3$	$w1$	$w2$	$w3$(T)	$w3$(F)
$w3$	$w1$	$w3$	$m1$	$m1$	$m2$	$m2$
$w1$	$w3$	$w1$	$m2$	$m2$	$m1$	$m3$
$w2$	$w2$	$w2$	$m3$	$m3$	$m3$	$m1$

For $w3$'s true preference list (indicated with (T)), the G-S algorithm produces the following matches:
$(m1, w3)$, $(m2,\ w1)$, $(m3,\ w2)$

With her false preference list, the G-S algorithms runs as follows:
$(m1,\ w3)$
$(m2,\ w1)$
$(m3,\ w3)$ & $(m1)$ ($w3$ breaks up with $m1$)
$(m1,\ w1)$ & $(m2)$ ($w1$ breaks up with $m2$)
$(m2,\ w3)$ & $m3$ ($w3$ breaks up with $m3$)
$(m3,\ w2)$

At the end, the algorithm produces the following matches:
$(m1,\ w1)$, $(m2,\ w3)$, $(m3,\ w2)$

Thus, by lying on her preference list, $w3$ gets a better match.

Solution 4. The following is the solution to exercise 4 on page 67 of the textbook.

$$2^{\sqrt{\log n}} \leq n \log^3 n \leq n^{\frac{4}{3}} \leq n^{\log n} \leq 2^n \leq 2^{n^2} \leq 2^{2^n}$$

Solution 5.a. To prove by induction that the sum of the first n positive integers is $\frac{(n)(n+1)}{2}$.

Proof. For $n = 1$,

$$\begin{aligned} \text{LHS} &= 1 \\ \text{RHS} &= \sum_{i=1}^{1} i \\ &= 1 \end{aligned}$$

Hence, it holds for the base case.

Lets assume that it holds for $n = k$ i.e.

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

For $n = k + 1$,

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + (k+1) \\
&= \frac{k(k+1)}{2} + k + 1 && \text{(by inductive hypothesis)} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}.
\end{aligned}$$

Thus, our inductive hypothesis was true and holds for all n. □

Solution 5.b. To prove by induction that $1 \times 2 + 2 \times 3 + ...\ n\ (n + 1)$ is $\frac{(n)(n+1)(n+2)}{3}$.

Proof. For $n = 1$,

$$\begin{aligned}
\text{LHS} &= 1 \times 2 \\
&= 2 \\
\text{RHS} &= \frac{n(n+1)(n+2)}{3} \\
&= \frac{1(2)(3)}{3} \\
&= 2
\end{aligned}$$

Hence, it holds for the base case.

Lets assume that it holds for $n = k$ i.e.

$$1 \times 2 + 2 \times 3 + ...k(k+1) = \frac{(k)(k+1)(k+2)}{3}$$

For $n = k + 1$,

$$\begin{aligned} 1 \times 2 + 2 \times 3 + ...k(k+1) + (k+1)(k+2) &= \frac{(k)(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{(by IH)} \\ &= (k+1)(k+2)\left(\frac{k+3}{3}\right) \\ &= \frac{(k+1)(k+2)(k+3)}{3} \\ &= \frac{(k+1)((k+1)+1)((k+1)+2)}{3} \end{aligned}$$

Thus, our inductive hypothesis was true and holds for all n. $\square$

Solution 6. Let the number of steps be n. Let the first egg be thrown from every p step starting from the first step. Once the first egg breaks, the second egg can be thrown from the last p steps to find the exact step where an egg breaks. Therefore, the total number of tries with two eggs would be $\frac{n}{p} + p - 1$. (We just check the preceding p - 1 floors and not repeat the floor where the first egg broke.)

For n = 200, we need to minimize the following equation:

$$y = \frac{200}{p} + p - 1 \tag{1}$$

$\implies$ We need to set its first derivative equal to 0.

$$\begin{aligned} -\frac{200}{p^2} + 1 &= 0 \\ p^2 &= 200 \\ p &= \sqrt{200} \end{aligned}$$

Thus, $2\sqrt{200} - 1$ tries are needed in the worst case. We take $\sqrt{200}$ as 14 as that gives us the minimum value.

For the general case of n steps, we minimize equation (1) where we replace 200 with n.

$$\begin{aligned} -\frac{n}{p^2} + 1 &= 0 \\ p^2 &= n \\ p &= \sqrt{n} \end{aligned}$$

Thus, $2\sqrt{n} - 1$ tries are required in the worst case. We round $\sqrt{n}$ to the nearest ones.

(When $p = \sqrt{200}$ or $\sqrt{n}$ is used in the second derivative of equation (1), it gives a positive value indicating that we have indeed found a minimum.)