Derivation of R_0 in SIR and SEIR models over a network with 2 vertices

Jayanth Kumar N

Indian Institute of Science Education and Research, Pune

June 15, 2017

1 The Next generation operator method

R_0 is defined as the spectral radius of the 'next generation operator'. To find the next generation operator, we need to first identify the infected and non-infected compartments. Suppose there are n compartments of which m are infected. Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ where each x_i denotes the number of individuals in i^{th} compartment. Let, $F_i(x)$, denote the rate of apperance of new infections in compartment i. $V_i^-(x)$ be the rate of transfer of individuals into compartment i by all other means and $V_i^+(x)$ be the rate of transfer of individuals out of compartment i.

$$\frac{dx_i}{dt} = F_i(x) - V_i(x) = f(x_i)$$

where $V_i(x) = V_i^+(x) - V_i^-(x)$.

We then construct $\mathcal{F}$ and $\mathcal{V}$ matrices by taking the partial derivatives of the F_i with respect to x_i and similarly for $\mathcal{V}$ by taking partial derivatives of V_i. We define R_0 to be the spectral radius of the $\mathcal{F}\mathcal{V}^{-1}$.
Spectral radius of square matrix It is the largest absolute value of the matrix's eigen value.

1.1 Assumptions

1. If $\bar{x} \geq 0$, then $F_i, V_i^+, V_i^- \geq 0 \forall i$
2. If $\bar{x} = 0$, then $V_i^- = 0$
3. $F_i = 0$ if $i \geq m$
4. If $\bar{x} \in X_s$, where X_s is set of all Disease Free states. Then $F_i, V_i^+ = 0$
5. It is assumed that a disease free equilibrium exists and it is a locally asymptotically stable solution of the disease free model.Thus if x_0 denotes a disease free equilibrium of the system, then if $\mathcal{F}(x)$ is set to zero, then all eigenvalues of $Df(x_0)$ have negative real parts.

1.2 SIR model with birth and death

The following is the derivation of R_0 for SIR model with birth and death.
Equations

$$\frac{dS_1}{dt} = \lambda(S_1 + I_1 + R_1) - \mu S_1 - \beta_1 S_1 I_1 + \epsilon S_2 - \epsilon S_1$$

$$\frac{dI_1}{dt} = -\mu I_1 + \beta_1 S_1 I_1 + \epsilon I_2 - \epsilon I_1 - \gamma I_1$$

$$\frac{dR_1}{dt} = -\mu R_1 + \epsilon R_2 - \epsilon R_1 + \gamma I_1$$

Similarlly for City 2

- **For single city in isolation with no transfer rates**
  $\epsilon = 0$, $n = 3$ and $m = 1$
  F_i is the rate of apperance of infection at compartment i, It is enough to consider the I compartment, Since only I contributes for the infection.

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I$$
$$F = \beta SI, V = -\gamma I - \mu I$$

$$\frac{dF}{dI} = \beta S, \frac{dV}{dI} = -\gamma - \mu$$

at Disease Free equilibrium $S1 = N_0$ *where* N_0 *is the initial number of people*

$$\frac{dF}{dI}|_{DFE} = \beta N_0, \frac{dV}{dI}|_{DFE} = -\gamma - \mu$$

$$\mathcal{F} = \frac{dF}{dI}|_{DFE}, \mathcal{V} = \frac{dV}{dI}|_{DFE}$$

Since R_0 equals the spectral radius of $\mathcal{F}\mathcal{V}^{-1}$, We find that

$$R_0 = \frac{\beta N}{(\gamma + \mu)}$$

- **For two cities with transfer rates**
  $n = 6$ and $m = 2$ where I_1 and I_2 are the two infected compartments.

$$F_1 = \beta_1 S_1 I_1, F_2 = \beta_2 S_2 I_2$$
$$V_1 = -\gamma I_1 - \mu I_1 - \epsilon I_1 + \epsilon I2, V_2 = -\gamma I_2 - \mu I_2 - \epsilon I_2 + \epsilon I1$$

Now,

$$\mathcal{F} = \begin{bmatrix} \beta_1 N_1 & 0 \\ 0 & \beta_2 N_2 \end{bmatrix}$$

$$\mathcal{V} = \begin{bmatrix} -\gamma - \mu - \epsilon & \epsilon \\ \epsilon & -\gamma - \mu - \epsilon \end{bmatrix}$$

The Eigen values of $\mathcal{F}\mathcal{V}^{-1}$ was calculated using mathematica,

The spectral radius defined as the absolute value of the largest eigen value turns out to be

$$R_0 = \frac{X + \sqrt{X^2 - 4\beta_1 \beta 2 N_1 N_2 Y}}{2Y}$$

where $X = (\beta_1 N_1 + \beta 2 N_2))(\epsilon + \mu + \gamma)$ and $Y = (2\epsilon\gamma + \gamma^2 + 2\epsilon\mu + 2\mu\gamma + \mu^2)$

1.3 SEIR Model

Equations

$$\frac{dS_1}{dt} = \lambda(S_1 + E_1 + I_1 + R_1) - \mu S_1 - \beta_1 S_1 I_1 + \epsilon S_2 - \epsilon S_1$$

$$\frac{dE_1}{dt} = -\mu E_1 + \beta_1 S_1 I_1 + \epsilon E_2 - \epsilon E_1 - \sigma E_1$$

$$\frac{dI_1}{dt} = \sigma E_1 - \gamma I_1 - \mu I_1$$

$$\frac{dR_1}{dt} = -\mu R_1 + \epsilon R_2 - \epsilon R_1 + \gamma I_1$$

Similarlly for City 2

- **For single city in isolation with no transfer rates**
  $\epsilon = 0$, $n = 4$ and $m = 2$, E_1 and I_1 are considered as infective compartments

$$F_1 = \beta_1 S_1 I_1, F_2 = 0$$

$$V_1 = -\mu E_1 - \sigma_1, V_2 = \sigma E_1 - \gamma I_1 - \mu I_1$$

Now, calculating $\mathcal{F}$ and $\mathcal{V}$ at DFE $S_1 = N_1$

$$\mathcal{F} = \begin{bmatrix} 0 & \beta_1 N_1 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{V} = \begin{bmatrix} \text{-}\sigma - \mu & 0 \\ \sigma & \text{-}\gamma - \mu \end{bmatrix}$$

The Spectral radius of $\mathcal{F}\mathcal{V}^{-1}$ was calculated using mathematica.

$$R_0 = \frac{\beta_1 N_1 \sigma}{(\gamma + \mu)(\mu + \sigma)}$$

For two cities with transfer rates
$n = 8$ and $m = 4$, E_1,I_1,E_1 and I_1 are considered as infective compartments.

$$F_1 = \beta_1 S_1 I_1, F_2 = 0$$

$$F_3 = \beta_2 S_2 I_2, F_4 = 0$$

$$V_1 = -\mu E_1 + \epsilon E_2 - \epsilon E_1 - \sigma_1, V_2 = \sigma E_1 - \gamma I_1 - \mu I_1$$

$$V_3 = -\mu E_2 + \epsilon E_1 - \epsilon E_2 - \sigma_2, V_4 = \sigma E_2 - \gamma I_2 - \mu I_2$$

Now, calculating $\mathcal{F}$ and $\mathcal{V}$ at DFE $S_1 = N_1 and S_2 = I_2$

$$\mathcal{F} = \begin{bmatrix} 0 & \beta_1 N_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 N_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{V} = \begin{bmatrix} \text{-}\sigma - \mu - \epsilon & 0 & \epsilon & 0 \\ \sigma & \text{-}\gamma - \mu & 0 & 0 \\ \epsilon & 0 & \text{-}\sigma - \epsilon - \mu & 0 \\ 0 & 0 & \sigma & \text{-}\mu - \gamma \end{bmatrix}$$

The Spectral radius of $\mathcal{F}\mathcal{V}^{-1}$ was calculated using mathematica.

$$R_0 = \frac{X + \sqrt{X^2 - 4\beta_1\beta 2 N_1 N_2 \sigma^2 Y}}{2(\gamma + \mu)Y}$$

where $X = (\beta_1 N_1 \sigma + \beta 2 N_2 \sigma))(\epsilon + \mu + \sigma)$ and $Y = (2\epsilon\mu + \mu^2 + 2\epsilon\sigma + 2\mu\sigma + \sigma^2)$