Supervised Learning

G V V Sharma*

Abstract—**This manual introduces various methods in supervised learning.**

1 LEAST SQUARES METHOD

1.1 The Steinhart–Hart equation is a model of the resistance of a thermistor at different temperatures. The equation is given by

$$\frac{1}{T} = w_1 + w_2 \ln(R) + w_3 [\ln(R)]^3 \tag{1.1}$$

The equation of the line between two points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{A} - \mathbf{B}) \tag{1.2}$$

Alternatively, it can be expressed as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{1.3}$$

where $\mathbf{n}$ is the solution of

$$(\mathbf{A} - \mathbf{B})^T \mathbf{n} = 0 \tag{1.4}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

1.2 In $\triangle ABC$,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.5}$$

and the equations of the medians through $\mathbf{B}$ and $\mathbf{C}$ are respectively

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{1.6}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4 \tag{1.7}$$

Find the area of $\triangle ABC$.
Solution: The centroid $\mathbf{O}$ is the solution of (1.6),(1.7) and is obtained as the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \tag{1.8}$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix} 1 & 1 & 5 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.9}$$

Thus,

$$\mathbf{O} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.10}$$

Let AD be the median through $\mathbf{A}$. Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \tag{1.11}$$

$$\implies \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} \tag{1.12}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 1 \end{pmatrix} \tag{1.13}$$

From (1.7) and (1.13),

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} = 5 \tag{1.14}$$

$$\implies 5 + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 12 \tag{1.15}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 7 \tag{1.16}$$

From (1.16) and (1.7), $\mathbf{C}$ can be obtained by solving

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \qquad (1.17)$$

using the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 7 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.18)$$

$$\implies \mathbf{C} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad (1.19)$$

From (1.12),

$$\mathbf{B} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \qquad (1.20)$$

Thus,

$$\frac{1}{2}\begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 7 & 4 \\ 2 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 9 \qquad (1.21)$$

2 Orthogonality

2.1 $\mathbf{u}^T\mathbf{x} = 0 \implies \mathbf{u} \perp \mathbf{x}$. Show that

$$\mathbf{u}^T\mathbf{x} = \mathbf{P}^T\mathbf{x} = 0 \implies \mathbf{P} = \alpha\mathbf{u} \qquad (2.1)$$

2.2 The foot of the perpendicular drawn from the origin on the line

$$AB : \mathbf{u}^T\mathbf{x} = \lambda \qquad (2.2)$$

where

$$\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad (2.3)$$

is $\mathbf{P}$. The line meets the x-axis at $\mathbf{A}$ and y-axis at $\mathbf{B}$. Show that $\mathbf{P} = \alpha\mathbf{u}$ and find α.
Solution: From (2.2),

$$\mathbf{u}^T\mathbf{A} = \mathbf{u}^T\mathbf{B} = \lambda \qquad (2.4)$$

$$\implies \mathbf{u}^T(\mathbf{A} - \mathbf{B}) = 0 \qquad (2.5)$$

Since $OP \perp AB$,

$$\mathbf{P}^T(\mathbf{A} - \mathbf{B}) = 0 \qquad (2.6)$$

Thus, from (2.1),

$$\mathbf{P} = \alpha\mathbf{u} \qquad (2.7)$$

Since $\mathbf{P}$ lies on (2.2),

$$\mathbf{u}^T\mathbf{P} = \alpha\mathbf{u}^T\mathbf{u} = \lambda \qquad (2.8)$$

$$\implies \alpha = \frac{\lambda}{\mathbf{u}^T\mathbf{u}} = \frac{\lambda}{10}. \qquad (2.9)$$

2.3 Find $\mathbf{A}$.
Solution: Let

$$\mathbf{A} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (2.10)$$

From (2.2),

$$\mathbf{u}^T\mathbf{A} = a\begin{pmatrix} 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \qquad (2.11)$$

$$\implies a = \frac{\lambda}{3} \qquad (2.12)$$

$$\text{and } \mathbf{A} = \frac{\lambda}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (2.13)$$

2.4 Find the ratio $BP : PA$.
Solution: Let

$$\frac{BP}{PA} = k \qquad (2.14)$$

Then,

$$k\mathbf{A} + \mathbf{B} = (k + 1)\mathbf{P} \qquad (2.15)$$

$$\implies k\mathbf{A}^T\mathbf{A} + \mathbf{A}^T\mathbf{B} = (k + 1)\mathbf{P}^T\mathbf{A} \qquad (2.16)$$

$$\implies ka^2 = \alpha(k + 1)\lambda \qquad (2.17)$$

using (2.7), (2.10), (2.2) and $\mathbf{A} \perp \mathbf{B}$. Substituting from (2.9) and (2.12),

$$\implies k\frac{\lambda^2}{9} = (k + 1)\frac{\lambda^2}{10} \qquad (2.18)$$

$$\implies k = 9 \qquad (2.19)$$

3 Locus

3.1 The line through

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad (3.1)$$

intersects the coordinate axes at $\mathbf{P}$ and $\mathbf{Q}$. $\mathbf{O}$ is the origin and rectangle $OPRQ$ is completed as shown in Fig. (**??**),

3.2 Show that

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \tag{3.2}$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \tag{3.3}$$

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \tag{3.4}$$

3.3 Show that

$$(\mathbf{A} - \mathbf{P})^T \mathbf{n} = 0$$
$$(\mathbf{A} - \mathbf{Q})^T \mathbf{n} = 0 \tag{3.5}$$
$$(\mathbf{P} - \mathbf{Q})^T \mathbf{n} = 0$$

Solution: Trivial using (1.3) and (1.4).

3.4 Show that

$$(2\mathbf{A} - \mathbf{R})^T \mathbf{n} = 0 \tag{3.6}$$

$$\mathbf{R}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{n} = 0 \tag{3.7}$$

Solution: From (3.5) and (3.4)

$$[2\mathbf{A} - (\mathbf{P} + \mathbf{Q})]^T \mathbf{n} = 0 \tag{3.8}$$

resulting in (3.6). From (3.5) and (3.2),(3.3), (3.7) is obtained.

3.5 Show that

$$\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0. \tag{3.9}$$

3.6 Find the locus of $\mathbf{R}$.

Solution: For $\mathbf{n}$ to be unique in (3.6),(3.7),

$$(2\mathbf{A} - \mathbf{R}) = k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R}$$

$$\implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R})$$

$$= k\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R}$$

$$= k\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \tag{3.10}$$

where k is some constant. Thus, the desired locus is

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R}) = 0 \tag{3.11}$$

$$\implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - 2\mathbf{A}^T \mathbf{R} = 0 \tag{3.12}$$

4 Conics

4.1 The equation of a quadratic curve is given by

$$Ax_1^2 + Bx_1 x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \tag{4.1}$$

Show that (4.1) can be expressed as

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \tag{4.2}$$

Find the matrix V and vector $\mathbf{u}$.

4.2 The tangent to (4.1) at a point $\mathbf{p}$ on the curve is given by

$$\begin{pmatrix} \mathbf{p}^T & 1 \end{pmatrix} \begin{pmatrix} V & \mathbf{u} \\ \mathbf{u}^T & F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \tag{4.3}$$

Show that (4.3) can be expressed as

$$\left(\mathbf{p}^T V + \mathbf{u}^T \right) \mathbf{x} + \mathbf{p}^T \mathbf{u} + F = 0 \tag{4.4}$$

4.3 Classify the various conic sections based on (4.2).
Solution:

5 Circle

5.1 Find the centre and radius of the circle

$$C_1 : \mathbf{x}^T \mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{5.1}$$

Solution: let $\mathbf{c}$ be the centre of the circle. Then

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \tag{5.2}$$
$$\implies (\mathbf{x} - \mathbf{c})^T (\mathbf{x} - \mathbf{c}) = r^2 \tag{5.3}$$
$$\implies \mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} = r^2 - \mathbf{c}^T \mathbf{c} \tag{5.4}$$

Comparing with (5.1),

$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.5}$$

$$r^2 - \mathbf{c}^T \mathbf{c} = 1 \implies r = \sqrt{2} \tag{5.6}$$

5.2 Find the tangent to the circle C_1 at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solution: From (4.3), the tangent T is given by

$$\left[\begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix} \right] \mathbf{x} - \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{5.7}$$

$$\implies T : \mathbf{n}^T \mathbf{x} = 3 \tag{5.8}$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.9}$$

5.3 The tangent T in (5.8) cuts off a chord AB from a circle C_2 whose centre is

$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \qquad (5.10)$$

Find $\mathbf{A} + \mathbf{B}$.

Solution: Let the radius of C_2 be r. From the given information,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{C}) = r^2 \qquad (5.11)$$
$$(\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = r^2 \qquad (5.12)$$

Subtracting (5.12) from (5.11),

$$\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B} - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \qquad (5.13)$$
$$\implies (\mathbf{A} + \mathbf{B})^T (\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0$$
$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T (\mathbf{A} - \mathbf{B}) = 0 \qquad (5.14)$$

$\because \mathbf{A}, \mathbf{B}$ lie on T, from (5.8),

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = 3 \qquad (5.15)$$
$$\implies \mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0, \qquad (5.16)$$

From (5.14) and (5.16)

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \qquad (5.17)$$
$$\implies \mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C} = k\mathbf{n}^T\mathbf{n} \qquad (5.18)$$
$$\implies \frac{\mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C}}{\mathbf{n}^T\mathbf{n}} = k \qquad (5.19)$$
$$\implies k = 2 \qquad (5.20)$$

using (5.15). Substituting in (5.17)

$$\mathbf{A} + \mathbf{B} = 2 (\mathbf{n} + \mathbf{C}) \qquad (5.21)$$

5.4 If $AB = 4$, find $\mathbf{A}^T\mathbf{B}$.

Solution: From the given information,

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \qquad (5.22)$$

resulting in

$$\|\mathbf{A} + \mathbf{B}\|^2 - \|\mathbf{A} - \mathbf{B}\|^2 = 4\|\mathbf{n} + \mathbf{C}\|^2 - 4^2 \quad (5.23)$$
$$\implies \mathbf{A}^T\mathbf{B} = \|\mathbf{n} + \mathbf{C}\|^2 - 4 = 17 \qquad (5.24)$$

using (5.21) and simplifying.

5.5 Show that

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 8 - r^2 \qquad (5.25)$$

Solution:

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \qquad (5.26)$$
$$\implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) = 4^2 \qquad (5.27)$$

From (5.27),

$$[(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})]^T [(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})] = 4^2 \qquad (5.28)$$

which can be expressed as

$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 + 2 (\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 4^2 \qquad (5.29)$$

Upon substituting from (5.12) and (5.11) and simplifying, (5.25) is obtained.

5.6 Find r.

Solution: (5.25) can be expressed as

$$\mathbf{A}^T\mathbf{B} - \mathbf{C}^T (\mathbf{A} + \mathbf{B}) + \mathbf{C}^T\mathbf{C} = 8 - r^2 \qquad (5.30)$$
$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T (\mathbf{A} + \mathbf{B}) - \mathbf{C}^T\mathbf{C} = r^2 \qquad (5.31)$$
$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T (2\mathbf{n} + \mathbf{C}) = r^2 \qquad (5.32)$$
$$\implies r = \sqrt{6}. \qquad (5.33)$$

6 PARABOLA

6.1 Find the tangent at $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ to the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + 6 = 0 \qquad (6.1)$$

Solution: Substituting

$$\mathbf{p} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad (6.2)$$

in (4.4), the desired equation is

$$\left[\begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -1 \end{pmatrix} \right] \mathbf{x}$$
$$+ \frac{1}{2} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 6 = 0 \qquad (6.3)$$

resulting in

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 5 \qquad (6.4)$$

6.2 The line in (6.4) touches the circle

$$\mathbf{x}^T\mathbf{x} + 4 \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} + c = 0 \qquad (6.5)$$

Find c.

Solution: Comparing (4.2) and (6.5),

$$V = I,$$

$$\mathbf{u} = 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{6.6}$$

Comparing (4.4) and (6.4),

$$\mathbf{p} + 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{6.7}$$

$$\implies \mathbf{p} = - \begin{pmatrix} 6 \\ 7 \end{pmatrix} \tag{6.8}$$

and

$$c + \mathbf{p}^T \mathbf{u} = 5 \tag{6.9}$$

$$\implies c = 5 + 2 \begin{pmatrix} 6 & 7 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{6.10}$$

$$= 95 \tag{6.11}$$

7 ELLIPSE

7.1 Express the following equation in the form given in (4.1)

$$E : 5x_1^2 - 6x_1 x_2 + 5x_2^2 + 22x_1 - 26x_2 + 29 = 0 \tag{7.1}$$

Solution: (7.1) can be expressed as

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + 29 = 0 \tag{7.2}$$

where

$$V = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 11 \\ -13 \end{pmatrix} \tag{7.3}$$

7.2 Find $\mathbf{c}$ and K such that

$$(\mathbf{x} - \mathbf{c})^T V (\mathbf{x} - \mathbf{c}) = K \tag{7.4}$$

Solution: (7.4) can be expressed as

$$\mathbf{x}^T V \mathbf{x} - 2\mathbf{c}^T V \mathbf{x} + \mathbf{c}^T V \mathbf{c} - K = 0. \tag{7.5}$$

Comparing with (7.2),

$$V\mathbf{c} = -\mathbf{u} \tag{7.6}$$

$$\mathbf{c}^T V \mathbf{c} - K = 29 \tag{7.7}$$

$$\implies \mathbf{c} = -V^{-1}\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \tag{7.8}$$

and $K = 8$ \hfill (7.9)

7.3 Show that (7.4) can be expressed as

$$\mathbf{y}^T D \mathbf{y} = 1 \tag{7.10}$$

Solution: Let

$$PDP^T = V \tag{7.11}$$

For

$$\mathbf{y} = \frac{P^T (\mathbf{x} - \mathbf{c})}{\sqrt{K}}, \tag{7.12}$$

(7.4) transforms to (7.10).

7.4 If

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{7.13}$$

$$P = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \tag{7.14}$$

show that $(\lambda_1, \mathbf{P}_1)$ and $(\lambda_2, \mathbf{P}_2)$ satisfy

$$V\mathbf{y} = \lambda \mathbf{y} \tag{7.15}$$

if

$$P^T P = I \tag{7.16}$$

7.5 Find the length of the semi-major and semi-minor axes of E.

Solution: The values are given by

$$\sqrt{\frac{K}{\lambda}} \tag{7.17}$$

obtained by solving for λ in (7.15). Thus,

$$|\lambda I - V| = 0 \tag{7.18}$$

$$\implies \begin{vmatrix} \lambda - 5 & 3 \\ 3 & \lambda - 5 \end{vmatrix} = 0 \tag{7.19}$$

$$\implies \lambda^2 - 10\lambda + 16 = 0 \tag{7.20}$$

$$\implies \lambda = 2, 8 \tag{7.21}$$

Thus, the length of the semi-major axis is 2 and that of the semi-minor axis is 1.

7.6 Find $\mathbf{P}_1$ and $\mathbf{P}_2$.

Solution: From (7.15)

$$V\mathbf{P}_1 = \lambda_1 \mathbf{y} \tag{7.22}$$

$$\implies (V - \lambda I) \mathbf{y} = 0 \tag{7.23}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{y} = 0 \tag{7.24}$$

$$\implies \mathbf{n}_1^T \mathbf{y} = 0 \tag{7.25}$$

where

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{7.26}$$

Similarly, $\mathbf{P}_2$ is a point on

$$\mathbf{n}_2^T \mathbf{y} = 0 \tag{7.27}$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7.28}$$

7.7 Find the equation of the major axis for E.
Solution: The major axis for (7.10) is the line

$$\mathbf{y} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{7.29}$$

From (7.12), letting

$$P\mathbf{y} = \frac{(\mathbf{x} - \mathbf{c})}{\sqrt{K}}, \tag{7.30}$$

$$\implies \lambda_1 \mathbf{P}_1 = \frac{\mathbf{x} - \mathbf{c}}{\sqrt{K}} \tag{7.31}$$

since

$$P\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{P}_1 \text{ and } P^T P = I \tag{7.32}$$

Also, from (7.25),

$$\lambda_1 \mathbf{n}_1^T \mathbf{P}_1 = \frac{\mathbf{n}_1 (\mathbf{x} - \mathbf{c})}{\sqrt{K}} = 0 \tag{7.33}$$

resulting in

$$\mathbf{n}_1^T \mathbf{x} = \mathbf{n}_1^T \mathbf{c} = -3 \tag{7.34}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{7.35}$$

which is the major axis of E.
7.8 Find the minor axis of E.
7.9 Let $\mathbf{F}_1, \mathbf{F}_2$ be such that

$$\|\mathbf{x} - \mathbf{F}_1\| + \|\mathbf{x} - \mathbf{F}_2\| = 2k \tag{7.36}$$

Find $\mathbf{F}_1, \mathbf{F}_2$ and k.

8 HYPERBOLA

8.1 Tangents are drawn to the hyperbola

$$\mathbf{x}^T V \mathbf{x} = 36 \tag{8.1}$$

where

$$V = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$. If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \tag{8.3}$$

find the equation of PQ.
Solution: The equations of the two tangents are obtained using (4.4) as

$$\mathbf{P}^T V \mathbf{x} = 36 \tag{8.4}$$

$$\mathbf{Q}^T V \mathbf{x} = 36. \tag{8.5}$$

Since both pass through $\mathbf{T}$

$$\mathbf{P}^T V \mathbf{T} = 36 \implies \mathbf{P}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{8.6}$$

$$\mathbf{Q}^T V \mathbf{T} = 36 \implies \mathbf{Q}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{8.7}$$

Thus, $\mathbf{P}, \mathbf{Q}$ satisfy

$$\begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{x} = -36 \tag{8.8}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \tag{8.9}$$

which is the equation of PQ.
8.2 In $\triangle PTQ$, find the equation of the altitude $TD \perp PQ$.
Solution: Since

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{8.10}$$

using (1.3) and (8.9), the equation of TD is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{T}) = 0 \tag{8.11}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{8.12}$$

8.3 Find D.
Solution: From (8.9) and (8.12),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{8.13}$$

$$\implies \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{8.14}$$

8.4 Show that the equation of PQ can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda \mathbf{m} \tag{8.15}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.16}$$

8.5 Show that for $\mathbf{V}^T = \mathbf{V}$,

$$(\mathbf{D} + \lambda\mathbf{m})^T V (\mathbf{D} + \lambda\mathbf{m}) + F = 0 \qquad (8.17)$$

can be expressed as

$$\lambda^2\mathbf{m}^T V\mathbf{m} + 2\lambda\mathbf{m}^T V\mathbf{D} + \mathbf{D}^T V\mathbf{D} + F = 0 \quad (8.18)$$

8.6 Find $\mathbf{P}$ and $\mathbf{Q}$.
Solution: From (8.15) and (8.1) (8.18) is obtained. Substituting from (8.16), (8.2) and (8.14)

$$\mathbf{m}^T V\mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \qquad (8.19)$$

$$\mathbf{m}^T V\mathbf{D} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -12 \end{pmatrix} = 0 \quad (8.20)$$

$$\mathbf{D}^T V\mathbf{D} = \begin{pmatrix} 0 & -12 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -12 \end{pmatrix} = -144$$
$$(8.21)$$

Substituting in (8.18)

$$4\lambda^2 - 144 = 36 \qquad (8.22)$$
$$\implies \lambda = \pm 3\sqrt{5} \qquad (8.23)$$

Substituting in (8.15),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3\begin{pmatrix} \sqrt{5} \\ -4 \end{pmatrix} \qquad (8.24)$$

$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3\begin{pmatrix} \sqrt{5} \\ 4 \end{pmatrix} \qquad (8.25)$$

8.7 Find the area of $\triangle PTQ$.
Solution: Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \qquad (8.26)$$
$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \qquad (8.27)$$

the desired area is

$$\frac{1}{2}PQ \times TD = 45\sqrt{5} \qquad (8.28)$$

8.8 Repeat the previous exercise using determinants.