Linear Classification

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Contents

***Abstract*—This manual provides an introduction to linear lethods in regression.**

1 The Gaussian Distribution

1.1 Generate a Gaussian random number with 0 mean and unit variance.
Solution: Open a text editor and type the following program.

```
#!/usr/bin/env python

#This program generates a Gaussian random
    no with 0 mean and unit variance

#Importing numpy
import numpy as np

print (np.random.normal(0,1))
```

Save the file as gaussian_no.py and run the program.

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1.2 The mean of a random variable X is defined as

$$E[X] = \frac{1}{N}\sum_{i=1}^{N} X_i \tag{1.1}$$

and its variance as

$$\text{var}[X] = E[X - E[X]]^2 \tag{1.2}$$

Verify that the program in 1.1 actually generates a Gaussian random variable with 0 mean and unit variance.
Solution: Use the header in the previous program, type the following code and execute.

```
#This program generates a Gaussian random
    no with 0 mean and unit variance

#Importing numpy
import numpy as np


simlen = int(1e5) #No of samples

n = np.random.normal(0,1,simlen)#Random
    vector

mean = np.sum(n)/simlen #Mean value

print (mean)

var = np.sum(np.square(n - mean*np.ones
    ((1,simlen))))/simlen

print (var)
```

1.3 Using the previous program, verify you results for different values of the mean and variance.

2 CDF and PDF

2.1 A Gaussian random variable X with mean 0 and unit variance can be expressed as $X \sim$

$\mathcal{N}(0,1)$. Its cumulative distribution function (CDF) is defined as

$$F_X(x) = \Pr(X < x), \tag{2.1}$$

Plot $F_X(x)$.
Solution: The following code yields Fig. 2.1.

```
#Importing numpy, scipy, mpmath and pyplot
import numpy as np
import matplotlib.pyplot as plt




x = np.linspace(−4,4,30)#points on the x axis
simlen = int(1e5) #number of samples
err = [] #declaring probability list
n = np.random.normal(0,1,simlen)

for i in range(0,30):
        err_ind = np.nonzero(n < x[i]) #
            checking probability condition
        err_n = np.size(err_ind) #
            computing the probability
        err.append(err_n/simlen) #storing
            the probability values in a list


plt.plot(x.T,err)#plotting the CDF
plt.grid() #creating the grid
plt.xlabel('$x$')
plt.ylabel('$F_X(x)$')
plt.show() #opening the plot window
```

2.2 List the properties of $F_X(x)$ based on Fig. 2.1.
2.3 Let

$$p_X(x_i) = \frac{F_X(x_i) - F_X(x_{i-1})}{h}, i = 1, 2, \dots h \tag{2.2}$$

for $x_i = x_{i-1} + h, x_1 = -4$. Plot $p_X(x_i)$. On the same graph, plot

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -4 < x < 4 \tag{2.3}$$

Solution: The following code yields the graph in Fig. 2.3

```
https://github.com/gadepall/EE1390/raw/
    master/manuals/supervised/linear_class/
    codes/1.4.py
```

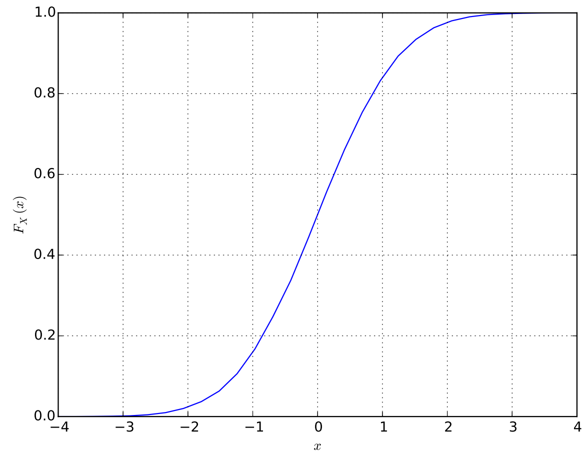

Fig. 2.1: CDF of X

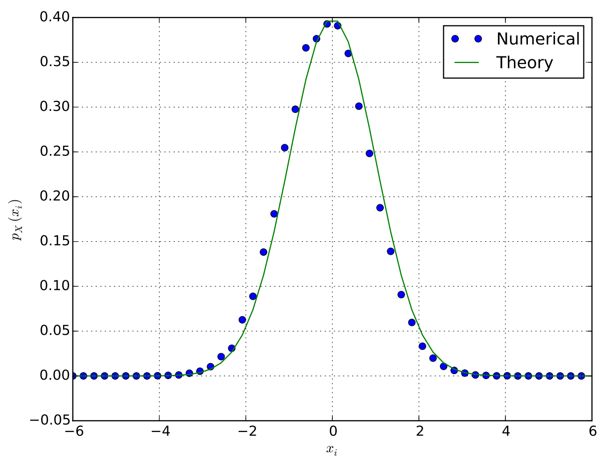

Fig. 2.3: The PDF of X

Thus, the PDF is the derivative of the CDF. For $X \sim \mathcal{N}(0,1)$, the PDF is

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty \tag{2.4}$$

2.4 For $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$,

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty \tag{2.5}$$

Plot $p_X(x)$ for different values of μ and σ in the same graph. Comment.

3 Detection & Estimation

3.1 Use the following code

```
https://raw.githubusercontent.com/gadepall/EE1390/master/manuals/supervised/linear_class/codes/2.3.py
```

to generate a scatterplot of X.

3.2 Suppose you wanted to classify X into two groups. How would you do so by looking at the scatterplot?

4 Bayes Classifier

4.1 Let

$$x = A(2s-1) + n \tag{4.1}$$

where $s \in \{0,1\}, n \sim \mathcal{N}(0,1)$.

4.2 Show that

$$x|0 \sim \mathcal{N}(-A,1) \tag{4.2}$$
$$x|1 \sim \mathcal{N}(A,1) \tag{4.3}$$

Solution: From the given information, for $s = 0$,

$$x|0 = -A + n \tag{4.4}$$
$$\implies E[x|0] = -A \tag{4.5}$$
$$\text{and } E\left[(x+A)^2|0\right] = E\left[n^2\right] = 1 \tag{4.6}$$

Similar approach can be used for $x|1$.

4.3 Find

$$p_{X|0}(x) \text{ and } p_{X|1}(x) \tag{4.7}$$

Solution:

$$p_X(x|0) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x+A)^2}{2}}, \quad -\infty < x < \infty \tag{4.8}$$
$$p_X(x|1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-A)^2}{2}}, \quad -\infty < x < \infty \tag{4.9}$$

4.4 Show that e^{-x} is monotonically decreasing.

4.5 Find

$$p_X(x|1) \underset{0}{\overset{1}{\gtrless}} p_X(x|0) \tag{4.10}$$

Solution: The given condition can be expressed as

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-A)^2}{2}} \underset{0}{\overset{1}{\gtrless}} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x+A)^2}{2}} \tag{4.11}$$
$$\implies -\frac{(x-A)^2}{2} \underset{0}{\overset{1}{\gtrless}} -\frac{(x+A)^2}{2} \tag{4.12}$$
$$\implies x \underset{0}{\overset{1}{\gtrless}} 0 \tag{4.13}$$

after simplification.

4.6 Show that

$$p_X(0|x) \underset{1}{\overset{0}{\gtrless}} p_X(1|x) \tag{4.14}$$
$$\implies p_X(x|0) \underset{1}{\overset{0}{\gtrless}} p_X(x|1) \tag{4.15}$$

if

$$p(0) = p(1) \tag{4.16}$$

Solution: Since

$$p_X(1|x) \underset{0}{\overset{1}{\gtrless}} p_X(0|x) \tag{4.17}$$
$$\implies \frac{p_X(x|1)p(1)}{p(x)} \underset{0}{\overset{1}{\gtrless}} \frac{p_X(x|0)p(0)}{p(x)}, \tag{4.18}$$

the result follows.

5 Probability of Error

5.1 Find the probability of error given that $s = 0$.
Solution: The desired probability is given by

$$P_{e|0} = \Pr(x|0 > 0) = \Pr(-A+n > 0) \tag{5.1}$$
$$= \Pr(n > A) \tag{5.2}$$

5.2 The Q-function is defined as

$$Q(A) = \Pr(n > A), \quad A > 0 \tag{5.3}$$

Plot $Q(A)$ from 0 to 10 dB.
Solution:

```
https://raw.githubusercontent.com/gadepall/EE1390/master/manuals/supervised/linear_class/codes/bpsk_ber.py
```

6 Linear Discriminant Analysis

6.1 The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^k|\boldsymbol{\Sigma}|}}\exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} \tag{6.1}$$

where $\boldsymbol{\mu}$ is the mean vector, $\boldsymbol{\Sigma} = E\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T\right]$ is the covariance matrix and $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$.

6.2 For

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{6.2}$$

show that

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right\}\right] \tag{6.3}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix} \tag{6.4}$$

6.3 Let

$$\mathbf{s}_0 = \begin{pmatrix}a\\0\end{pmatrix} \tag{6.5}$$

$$\mathbf{s}_1 = \begin{pmatrix}0\\a\end{pmatrix} \tag{6.6}$$

If

$$\mathbf{x} = \mathbf{s} + \mathbf{n} \tag{6.7}$$

where $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1\}$ and $\mathbf{n} \sim \mathcal{N}\left(0, \sigma^2\mathbf{I}\right)$, show that

$$\mathbf{x}|0 = \begin{pmatrix}a+n_1\\ n_2\end{pmatrix}, \tag{6.8}$$

and

$$\mathbf{x}|1 = \begin{pmatrix}n_1\\ a+n_2\end{pmatrix}, \tag{6.9}$$

6.4 Obtain a scatterplot of s_0 and s_1 and guess the decision region.
Solution:

```
https://raw.githubusercontent.com/gadepall/
    EE1390/master/manuals/supervised/
    linear_class/codes/bfsk_scatter.py
```

6.5 Find

$$p_{\mathbf{x}|\mathbf{s}_0}(\mathbf{x}) \tag{6.10}$$

$$p_{\mathbf{x}|\mathbf{s}_1}(\mathbf{x}) \tag{6.11}$$

6.6 How will you decide between $\mathbf{s}_0$ and $\mathbf{s}_1$ if you have $\mathbf{x}$?

7 Optimum Classifier

7.1 Let $(\mathbf{X}, \mathbf{G})$ be an input/output dataset, whose relation f is unknown. Also

$$\mathbf{g} \in \mathbf{G} = \{\mathbf{g}_k\}_{k=1}^K \tag{7.1}$$

Let

$$C(\mathbf{g}_k, \mathbf{g}_l) = \begin{cases}1 & k = l\\ 0 & k \neq l\end{cases} \tag{7.2}$$

where $\mathbf{g}_i$ are different classes of output data. Thus C is a *correctness* metric.

7.2 Show that

$$\max_{\mathbf{g}\in\mathbf{G}} E\left[C(\mathbf{G}, f(\mathbf{X}))\right] = \max_{\mathbf{g}\in\mathbf{G}} p(\mathbf{g}|\mathbf{X} = \mathbf{x}) \tag{7.3}$$

Solution: In the above,

$$\max_{\mathbf{g}\in\mathbf{G}} E\left[C(\mathbf{G}, f(\mathbf{X}))\right] = \max_{\mathbf{g}\in\mathbf{G}} E_{\mathbf{X}}\left[E_{\mathbf{G}}\left\{C(\mathbf{G}, f(\mathbf{x}))\right\}\right] \tag{7.4}$$

$$= \max_{\mathbf{g}\in\mathbf{G}} \sum_{k=1}^{K} C\{\mathbf{g}_k, \mathbf{g}\}\, p(\mathbf{g}_k|\mathbf{X} = \mathbf{x}) \tag{7.5}$$

From (7.2), the above expression simplifies to

$$\max_{\mathbf{g}\in\mathbf{G}} E\left[C(\mathbf{G}, f(\mathbf{X}))\right] = \max_{\mathbf{g}\in\mathbf{G}} p(\mathbf{g}|\mathbf{X} = \mathbf{x}) \tag{7.6}$$