

Linear Classification

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CONTENTS

1	The Gaussian Distribution	1
2	CDF and PDF	1
3	Detection & Estimation	2
4	Bayes Classifier	3
5	Probability of Error	3
6	Linear Discriminant Analysis	3
7	Optimum Classifier	4

Abstract—This manual provides an introduction to linear methods in regression.

1 THE GAUSSIAN DISTRIBUTION

1.1 Generate a Gaussian random number with 0 mean and unit variance.

Solution: Open a text editor and type the following program.

```
#!/usr/bin/env python

#This program generates a Gaussian random
#no with 0 mean and unit variance

#Importing numpy
import numpy as np

print (np.random.normal(0,1))
```

Save the file as gaussian_no.py and run the program.

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1.2 The mean of a random variable X is defined as

$$E[X] = \frac{1}{N} \sum_{i=1}^N X_i \quad (1.1)$$

and its variance as

$$\text{var}[X] = E[X - E[X]]^2 \quad (1.2)$$

Verify that the program in 1.1 actually generates a Gaussian random variable with 0 mean and unit variance.

Solution: Use the header in the previous program, type the following code and execute.

```
#This program generates a Gaussian random
#no with 0 mean and unit variance

#Importing numpy
import numpy as np

simlen = int(1e5) #No of samples

n = np.random.normal(0,1,simlen)#Random
vector

mean = np.sum(n)/simlen #Mean value

print (mean)

var = np.sum(np.square(n - mean*np.ones
((1,simlen))))/simlen

print (var)
```

1.3 Using the previous program, verify your results for different values of the mean and variance.

2 CDF AND PDF

2.1 A Gaussian random variable X with mean 0 and unit variance can be expressed as $X \sim$

$\mathcal{N}(0, 1)$. Its cumulative distribution function (CDF) is defined as

$$F_X(x) = \Pr(X < x), \quad (2.1)$$

Plot $F_X(x)$.

Solution: The following code yields Fig. 2.1.

```
#Importing numpy, scipy, mpmath and pyplot
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-4,4,30)#points on the x axis
simlen = int(1e5) #number of samples
err = [] #declaring probability list
n = np.random.normal(0,1,simlen)

for i in range(0,30):
    err_ind = np.nonzero(n < x[i]) #
        checking probability condition
    err_n = np.size(err_ind) #
        computing the probability
    err.append(err_n/simlen) #storing
        the probability values in a list

plt.plot(x.T,err)#plotting the CDF
plt.grid() #creating the grid
plt.xlabel('$x$')
plt.ylabel('$F_X(x)$')
plt.show() #opening the plot window
```

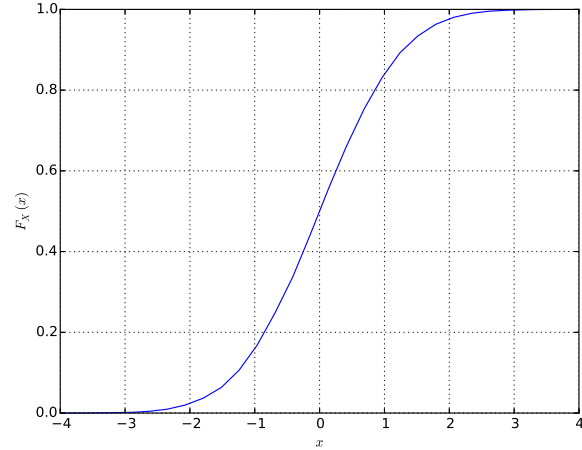


Fig. 2.1: CDF of X

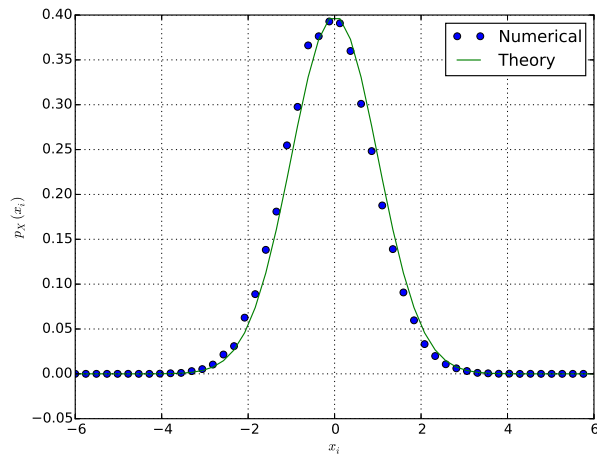


Fig. 2.3: The PDF of X

2.2 List the properties of $F_X(x)$ based on Fig. 2.1.

2.3 Let

$$p_X(x_i) = \frac{F_X(x_i) - F_X(x_{i-1})}{h}, i = 1, 2, \dots, h \quad (2.2)$$

for $x_i = x_{i-1} + h, x_1 = -4$. Plot $p_X(x_i)$. On the same graph, plot

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -4 < x < 4 \quad (2.3)$$

Solution: The following code yields the graph in Fig. 2.3

```
https://github.com/gadepall/EE1390/raw/master/manuals/supervised/linear\_class/codes/1.4.py
```

Thus, the PDF is the derivative of the CDF. For $X \sim \mathcal{N}(0, 1)$, the PDF is

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty \quad (2.4)$$

2.4 For $X \sim \mathcal{N}(\mu, \sigma^2)$,

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty \quad (2.5)$$

Plot $p_X(x)$ for different values of μ and σ in the same graph. Comment.

3 DETECTION & ESTIMATION

3.1 Use the following code

https://raw.githubusercontent.com/gadepall/EE1390/master/manuals/supervised/linear_class/codes/2.3.py

to generate a scatterplot of X .

- 3.2 Suppose you wanted to classify X into two groups. How would you do so by looking at the scatterplot?

4 BAYES CLASSIFIER

- 4.1 Let

$$x = A(2s - 1) + n \quad (4.1)$$

where $s \in \{0, 1\}$, $n \sim \mathcal{N}(0, 1)$.

- 4.2 Show that

$$x|0 \sim \mathcal{N}(-A, 1) \quad (4.2)$$

$$x|1 \sim \mathcal{N}(A, 1) \quad (4.3)$$

Solution: From the given information, for $s = 0$,

$$x|0 = -A + n \quad (4.4)$$

$$\Rightarrow E[x|0] = -A \quad (4.5)$$

$$\text{and } E[(x + A)^2 | 0] = E[n^2] = 1 \quad (4.6)$$

Similar approach can be used for $x|1$.

- 4.3 Find

$$p_{X|0}(x) \text{ and } p_{X|1}(x) \quad (4.7)$$

Solution:

$$p_X(x|0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+A)^2}{2}}, \quad -\infty < x < \infty \quad (4.8)$$

$$p_X(x|1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-A)^2}{2}}, \quad -\infty < x < \infty \quad (4.9)$$

- 4.4 Show that e^{-x} is monotonically decreasing.

- 4.5 Find

$$p_X(x|1) \underset{0}{\geq} p_X(x|0) \quad (4.10)$$

Solution: The given condition can be expressed as

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-A)^2}{2}} \underset{0}{\geq} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+A)^2}{2}} \quad (4.11)$$

$$\Rightarrow -\frac{(x-A)^2}{2} \underset{0}{\geq} -\frac{(x+A)^2}{2} \quad (4.12)$$

$$\Rightarrow x \underset{0}{\geq} 0 \quad (4.13)$$

after simplification.

- 4.6 Show that

$$p_X(0|x) \underset{1}{\geq} p_X(1|x) \quad (4.14)$$

$$\Rightarrow p_X(x|0) \underset{1}{\geq} p_X(x|1) \quad (4.15)$$

if

$$p(0) = p(1) \quad (4.16)$$

Solution: Since

$$p_X(1|x) \underset{0}{\geq} p_X(0|x) \quad (4.17)$$

$$\Rightarrow \frac{p_X(x|1)p(1)}{p(x)} \underset{0}{\geq} \frac{p_X(x|0)p(0)}{p(x)}, \quad (4.18)$$

the result follows.

5 PROBABILITY OF ERROR

- 5.1 Find the probability of error given that $s = 0$.

Solution: The desired probability is given by

$$P_{e|0} = \Pr(x|0 > 0) = \Pr(-A + n > 0) \quad (5.1)$$

$$= \Pr(n > A) \quad (5.2)$$

- 5.2 The Q -function is defined as

$$Q(A) = \Pr(n > A), \quad A > 0 \quad (5.3)$$

Plot $Q(A)$ from 0 to 10 dB.

Solution:

https://raw.githubusercontent.com/gadepall/EE1390/master/manuals/supervised/linear_class/codes/bpsk_ber.py

6 LINEAR DISCRIMINANT ANALYSIS

- 6.1 The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \quad (6.1)$$

where $\boldsymbol{\mu}$ is the mean vector, $\Sigma = E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T]$ is the covariance matrix and $|\Sigma|$ is the determinant of Σ .

- 6.2 For

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (6.2)$$

show that

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left[-\frac{1}{2(1-\rho^2)} \times \left\{ \frac{(x_1 - \mu_1)^2}{\sigma_1^2} + \frac{(x_2 - \mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1 - \mu_1)(x_2 - \mu_2)}{\sigma_1\sigma_2} \right\} \right] \quad (6.3)$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \quad (6.4)$$

6.3 Let

$$\mathbf{s}_0 = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (6.5)$$

$$\mathbf{s}_1 = \begin{pmatrix} 0 \\ a \end{pmatrix} \quad (6.6)$$

If

$$\mathbf{x} = \mathbf{s} + \mathbf{n} \quad (6.7)$$

where $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1\}$ and $\mathbf{n} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$, show that

$$\mathbf{x}|0 = \begin{pmatrix} a + n_1 \\ n_2 \end{pmatrix}, \quad (6.8)$$

and

$$\mathbf{x}|1 = \begin{pmatrix} n_1 \\ a + n_2 \end{pmatrix}, \quad (6.9)$$

6.4 Obtain a scatterplot of s_0 and s_1 and guess the decision region.

Solution:

https://raw.githubusercontent.com/gadepall/EE1390/master/manuals/supervised/linear_class/codes/bfsk_scatter.py

6.5 Find

$$p_{\mathbf{x}|\mathbf{s}_0}(\mathbf{x}) \quad (6.10)$$

$$p_{\mathbf{x}|\mathbf{s}_1}(\mathbf{x}) \quad (6.11)$$

6.6 How will you decide between \mathbf{s}_0 and \mathbf{s}_1 if you have \mathbf{x} ?

Let

$$C(\mathbf{g}_k, \mathbf{g}_l) = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \quad (7.2)$$

where \mathbf{g}_i are different classes of output data. Thus C is a *correctness* metric.

7.2 Show that

$$\max_{\mathbf{g} \in \mathbf{G}} E[C(\mathbf{G}, f(\mathbf{X}))] = \max_{\mathbf{g} \in \mathbf{G}} p(\mathbf{g}|\mathbf{X} = \mathbf{x}) \quad (7.3)$$

Solution: In the above,

$$\begin{aligned} \max_{\mathbf{g} \in \mathbf{G}} E[C(\mathbf{G}, f(\mathbf{X}))] \\ = \max_{\mathbf{g} \in \mathbf{G}} E_{\mathbf{X}}[E_{\mathbf{G}}\{C(\mathbf{G}, f(\mathbf{x}))\}] \end{aligned} \quad (7.4)$$

$$= \max_{\mathbf{g} \in \mathbf{G}} \sum_{k=1}^K C\{\mathbf{g}_k, \mathbf{g}\} p(\mathbf{g}_k|\mathbf{X} = \mathbf{x}) \quad (7.5)$$

From (7.2), the above expression simplifies to

$$\max_{\mathbf{g} \in \mathbf{G}} E[C(\mathbf{G}, f(\mathbf{X}))] = \max_{\mathbf{g} \in \mathbf{G}} p(\mathbf{g}|\mathbf{X} = \mathbf{x}) \quad (7.6)$$

7 OPTIMUM CLASSIFIER

7.1 Let (\mathbf{X}, \mathbf{G}) be an input/output dataset, whose relation f is unknown. Also

$$\mathbf{g} \in \mathbf{G} = \{\mathbf{g}_k\}_{k=1}^K \quad (7.1)$$