Date: / /

3). When $s \le t$,

$$W_t = W_s + (W_t - W_s)$$

$$E[W_s W_t] = E[W_s(W_s + W_t - W_s)] = E[W_s^2] + E[W_s(W_t - W_s)]$$

As $W_v - W_u$ is independent of $\{W_x : x \le u\}$, $W_t - W_s = 0$

If $E[W_t] = 0$, $Var(W_t) = t$, $\therefore E[W_s^2] = s$.

$$E[W_s W_t] = s, \min(s, t).$$

When $s > t$, just reverse s and t as $E[W_s W_t] = E[W_t W_s]$

$$\therefore E[W_s W_t] = t = \min(s, t).$$

4). For any $s < t$, $W_t - W_s \sim N(0, t-s)$

$$E[W_t - W_s] = E[W_t] - E[W_s] = 0.$$

$$Var[W_t - W_s] = Var[W_t] + Var[W_s] - 2Cov(W_t, W_s)$$

$$Cov(W_t, W_s) = \min(s, t) = s$$

$$\therefore Var(W_t - W_s) = t + s - 2s = t - s.$$

$\therefore$ Mean $= 0$ and Variance $= t - s$.

5). $W_t = W_s + (W_t - W_s)$

$$E[W_t | F_s] = E[W_s + (W_t - W_s) | F_s] = \underbrace{E[W_s | F_s]}_{= W_s} + E[W_t - W_s | F_s]$$

The increment $W_t - W_s$ is independent of F_s and has mean 0. $\therefore E[W_t - W_s / F_s] = E[W_t - W_s] = 0$

$$\therefore E[W_t / F_s] = W_s.$$

As W_t is F_t measurable and integrable since $W_t \sim N(0, t)$, brownian motion is martingale