Challenge Problem 3

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1 Problem

Prove that - Complex-valued circulant matrices are simultaneously orthogonally diagonalizable.In other words, there exists an n×n matrix P such that for every n×n circulant matrix A,the matrix P^HAP is diagonal. Here, P^H is the conjugate transpose/Hermitian (sometimes confusingly called the adjoint) of P. What are P and P^{-1}?

2 Explanation

Let us consider A be a nxn circulant matrix, that is,

$$A = \begin{pmatrix} C_0 & C_1 & C_2 & ... & C_{n-1} \\ C_{n-1} & C_0 & C_1 & ... & C_{n-2} \\ . & ... & .. & .. & . \\ . & ... & .. & .. & . \\ . & ... & .. & .. & . \\ C_1 & . & .. & ... & C_0 \end{pmatrix} \tag{2.0.1}$$

Where element of the matrix is considered as complex. So, considering the Hermitian transpose of the matrix A we can get that:

$$A^H = \begin{pmatrix} C_0^* & C_1^* & C_2^* & ... & C_{n-1}^* \\ C_{n-1}^* & C_0^* & C_1^* & ... & C_{n-2}^* \\ . & ... & .. & .. & . \\ . & ... & .. & .. & . \\ . & ... & .. & .. & . \\ C_1^* & . & .. & ... & C_0^* \end{pmatrix}^T \tag{2.0.2}$$

$$\implies A^H = \begin{pmatrix} C_0^* & C_{n-1}^* & C_{n-2}^* & ... & C_1^* \\ C_1^* & C_0^* & .. & ... & .. \\ . & ... & .. & .. & . \\ . & ... & .. & .. & . \\ . & ... & .. & .. & . \\ C_{n-1}^* & . & .. & ... & C_0^* \end{pmatrix} \tag{2.0.3}$$

It is observed that A^H is also a circulant matrix. One of the important properties of circulant matrices is that, all circulant matrices can commute to each other. So, matrix A can commute to its Hermitian transpose matrix A^H, that is,

$$AA^H = A^HA \tag{2.0.4}$$

This implies that matrix A is a normal matrix, that is why matrix A has a full set of mutually orthogonal eigen vectors. As all the circulant matrices can commute to each other, so they have same set of eigen vectors but different eigen values.

Let, $x^{(0)}$ is one eigen vector and

$$x^{(0)} = \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix} \tag{2.0.5}$$

$$\implies Ax^{(0)} = A \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix} \tag{2.0.6}$$

$$\implies Ax^{(0)} = (C_0 + C_1 + C_2 + ... + C_{n-1})x^{(0)} \tag{2.0.7}$$

$$\implies Ax^{(0)} = \lambda_0 x^{(0)} \tag{2.0.8}$$

The eigen vectors can also be written as :

$$\omega_n^i = e^{\frac{2\pi i}{n}} \tag{2.0.9}$$

So, the k^{th} eigen vector of a circulant matrix is :

$$x^{(k)} = \begin{pmatrix} \omega_n^{0k} \\ \omega_n^{1k} \\ \omega_n^{2k} \\ . \\ . \\ \omega_n^{(n-1)k} \end{pmatrix} \tag{2.0.10}$$

Now, let us consider a matrix P whose columns are the eigen vectors, such that:

$$P = \begin{pmatrix} x^{(0)} & x^{(1)} & ... & x^{(n-1)} \end{pmatrix} \tag{2.0.11}$$

These eigen vectors are mutually orthogonal.

If any vector is multiplied by the matrix P, then DFT operation can be performed. The full form of

DFT is Discrete Fourier Transform which is mainly used for numerical calculation in Digital Signal Processing. The DFT transforms N discrete-time samples to N discrete-frequency samples. If $x[n]$ is the discrete time samples and $X[k]$ is the discrete frequency samples,then the relation between them will be:

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-\frac{2\pi nk}{N}\right) \tag{2.0.12}$$

$$\implies X[k] = \sum_{n=0}^{N-1} x[n]\omega_N^{-nk} \tag{2.0.13}$$

Where N is the length of the sequence.Now, lets consider the length of the sequence is 2, that is, N = 2. So P will be :

$$P = \begin{pmatrix} x^{(0)} & x^{(1)} \end{pmatrix} \tag{2.0.14}$$

and

$$x^{(k)} = \begin{pmatrix} \omega_n^{0k} \\ \omega_n^{1k} \end{pmatrix} \tag{2.0.15}$$

So,

$$x^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.16}$$

$$x^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.0.17}$$

Now,

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.0.18}$$

Similarly, it can be also observed that when the length of sequence is 4x4, the P matrix is:

$$P = \begin{pmatrix} x^{(0)} & x^{(1)} & x^{(2)} & x^{(3)} \end{pmatrix} \tag{2.0.19}$$

and

$$x^{(k)} = \begin{pmatrix} \omega_n^{0k} \\ \omega_n^{1k} \\ \omega_n^{2k} \\ \omega_n^{3k} \end{pmatrix} \tag{2.0.20}$$

So,

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \tag{2.0.21}$$

If we calculate $X[k]$ using 2.0.13, we will get same value. So P is a DFT matrix and its columns are orthogonal. P is symmetric and

$$\frac{1}{n} P^H P = I \tag{2.0.22}$$

$$\implies \frac{1}{n} P^H = P^{-1} \tag{2.0.23}$$

As P is considered complex matrix . Let's consider $F = \frac{1}{\sqrt{n}} P$, so

$$\frac{1}{\sqrt{n}} P^H \frac{1}{\sqrt{n}} P = I \tag{2.0.24}$$

$$\implies F^H F = I \tag{2.0.25}$$

$$\implies F^H = F^{-1} \tag{2.0.26}$$

So, F is a unitary matrix, that implies that P is also unitary matrix. As the circulant matrix A is normal, so it is unitarily diagonalizable, so

$$D = P^{-1} A P \tag{2.0.27}$$

$$\implies D = \frac{1}{n}(P^H A P) \tag{2.0.28}$$

Now, for calculating eigen value let us consider an eigen vector $x^{(k)}$ such that

$$A x^{(k)} = y \tag{2.0.29}$$

Then the l^{th} component is

$$y_l = \sum_{j=0}^{n-1} C_{j-l} \omega_n^{jk} \tag{2.0.30}$$

$$\implies y_l = \omega_n^{lk} \sum_{j=0}^{n-1} C_{j-l} \omega_n^{(j-l)k} \tag{2.0.31}$$

But as C_j and ω_n^j both are periodic, so

$$\sum_{j=0}^{n-1} C_{j-l} \omega_n^{(j-l)k} = \sum_{j=0}^{n-1} C_j \omega_n^{jk} = \alpha_k \tag{2.0.32}$$

and $\omega_n^{lk} = x^{(k)}$ So,

$$A x^{(k)} = \alpha_k x^{(k)} \tag{2.0.33}$$

Where

$$\alpha_k = \sum_{j=0}^{n-1} C_j \omega_n^{jk} \tag{2.0.34}$$

α_k is the k^{th} eigen value. If α is a vector of all eigen values then $\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ . \\ . \\ \alpha_{(n-1)} \end{pmatrix}$ So, $\alpha = PC$ where C is the first row of A matrix, that is , the eigen values of A are the DFT of the first row of the matrix A.