Challenge Problem 3

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1 Problem

Prove that - Normal matrices are unitarily diagonalizable.

2 Explanation

Let A be a normal matrix, then we have to prove A is unitary diagonalizable.
Definition:
A is normal if $AA^* = A^*A$
Definition:
A is unitary diagonalizable if there is a unitary matrix U and diagonal matrix D such that $UAU^* = D$.
Proof:
As A is normal, so $AA^* = A^*A$. Now, by mathematical induction first we will consider the orthonormal vectors for n=2.
Consider an eigen vector U of A corresponds to the eigen value λ and U is unit vector. Now V is considered in such a way that U, V forms an orthonormal basis in C^2.
As U is an eigen vector , so $AU = \lambda U$. Now,

$$AU = U\begin{pmatrix}\lambda & U^{-1}AV \\ 0 & \end{pmatrix} \tag{2.0.1}$$

$$AU = U\begin{pmatrix}\lambda & \alpha \\ 0 & \beta\end{pmatrix} \tag{2.0.2}$$

$$A = U\begin{pmatrix}\lambda & \alpha \\ 0 & \beta\end{pmatrix}U^* \tag{2.0.3}$$

where $U^{-1}AV = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ Since A is normal,

$$A^*A = U\begin{pmatrix}\bar{\lambda} & 0 \\ \bar{\alpha} & \bar{\beta}\end{pmatrix}U^*U\begin{pmatrix}\lambda & \alpha \\ 0 & \beta\end{pmatrix}U^* \tag{2.0.4}$$

$$\implies A^*A = U\begin{pmatrix}\bar{\lambda}\lambda & \alpha\bar{\lambda} \\ \lambda\bar{\alpha} & \alpha\bar{\alpha} + \bar{\beta}\beta\end{pmatrix}U^* \tag{2.0.5}$$

Where $\bar{\lambda} = \lambda^*$ is the conjugate of λ. Similarly,

$$AA^* = U\begin{pmatrix}\bar{\lambda}\lambda & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \bar{\beta}\beta\end{pmatrix}U^* \tag{2.0.6}$$

As $AA^* = A^*A$

$$\alpha\bar{\alpha} = 0 \tag{2.0.7}$$

$$\implies \|\alpha\| = 0 \tag{2.0.8}$$

$$\implies \alpha = 0 \tag{2.0.9}$$

$$\implies A = U\begin{pmatrix}\lambda & 0 \\ 0 & \beta\end{pmatrix}U^* \tag{2.0.10}$$

$$\implies A = UDU^* \tag{2.0.11}$$

So, A is unitary diagonalizable when D is a 2x2 matrix. Now, assume that the result holds for (n-1). we can claim that there is y^* such that

$$AU = U\begin{pmatrix}\lambda & y^* \\ 0 & B\end{pmatrix} \tag{2.0.12}$$

where $B \in C^{(n-1)\times(n-1)}$, we decompose the matrix A into blocks and compute the products of AA^* and A^*A as follows:

$$AA^* = U\begin{pmatrix}\lambda & y^* \\ 0 & B\end{pmatrix}\begin{pmatrix}\bar{\lambda} & 0 \\ \bar{y^*} & \bar{B}\end{pmatrix}U^* \tag{2.0.13}$$

$$\implies AA^* = U\begin{pmatrix}\bar{\lambda}\lambda + y^*\bar{y^*} & y^*\bar{B} \\ B\bar{y^*} & \bar{B}B\end{pmatrix}U^* \tag{2.0.14}$$

$$\implies AA^* = U\begin{pmatrix}\|\lambda\|^2 + \|y^*\|^2 & y^*\bar{B} \\ B\bar{y^*} & \bar{B}B\end{pmatrix}U^* \tag{2.0.15}$$

Where $\bar{y^*}$ is the conjugate transpose of y^*. Now,

$$A^*A = U\begin{pmatrix}\bar{\lambda} & 0 \\ \bar{y^*} & \bar{B}\end{pmatrix}\begin{pmatrix}\lambda & y^* \\ 0 & B\end{pmatrix}U^* \tag{2.0.16}$$

$$\implies A^*A = U\begin{pmatrix}\bar{\lambda}\lambda & y^*\bar{\lambda} \\ \lambda\bar{y^*} & \bar{B}B + \bar{y^*}y^*\end{pmatrix}U^* \tag{2.0.17}$$

$$\implies A^*A = U\begin{pmatrix}\|\lambda\|^2 & y^*\bar{\lambda} \\ \lambda\bar{y^*} & \bar{B}B + \|y^*\|^2\end{pmatrix}U^* \tag{2.0.18}$$

As $AA^* = A^*A$,

$$y^*\overline{y^*} = 0 \tag{2.0.19}$$

$$\implies \|y^*\|^2 = 0 \tag{2.0.20}$$

$$\implies y^* = 0 \tag{2.0.21}$$

$$\implies A = U\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}U^* \tag{2.0.22}$$

$$\implies A = UDU^* \tag{2.0.23}$$

As $\bar{B} = B^*$ and $B^*B = BB^*$, so B is also normal and $B \in C^{(n-1)\times(n-1)}$. So it must be diagonal by mathematical induction hypothesis. Let,

$$B = MD_1M^* \tag{2.0.24}$$

where M is unitary matrix and D_1 is diagonal matrix and both are in $C^{(n-1)\times(n-1)}$. Now,

$$AU = U\begin{pmatrix} \lambda & 0 \\ 0 & MD_1M^* \end{pmatrix} \tag{2.0.25}$$

$$AU = U\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & D_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & M^* \end{pmatrix} \tag{2.0.26}$$

$$AU\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} = U\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & D_1 \end{pmatrix} \tag{2.0.27}$$

$$AW = W\begin{pmatrix} \lambda & 0 \\ 0 & D_1 \end{pmatrix} \tag{2.0.28}$$

$$A = W\begin{pmatrix} \lambda & 0 \\ 0 & D_1 \end{pmatrix}W^* \tag{2.0.29}$$

where W is also a unitary matrix and $W = U\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$ This implies that if A is normal then A is unitary diagonalizable.