Challenge Problem 3

Jayati Dutta

1 Problem

Prove that - Normal matrices are unitarily diagonalizable.

2 Explanation

Let A be a normal matrix, then we have to prove A is unitary diagonalizable.

Definition:
A is normal if $AA^* = A^*A$

Definition:
A is unitary diagonalizable if there is a unitary matrix U and diagonal matrix D such that $UAU^* = D$.

Proof:
As A is normal, so $AA^* = A^*A$. Now, by mathematical induction first we will consider the orthonormal vectors for n=2.

Consider an eigen vector U of A corresponds to the eigen value λ and U is unit vector. Now V is considered in such a way that U, V forms an orthonormal basis in C^2.

As U is an eigen vector , so $AU = \lambda U$. Now,

$$AU = U\begin{pmatrix} \lambda & U^{-1}AV \\ 0 & \end{pmatrix} \qquad (2.0.1)$$

$$AU = U\begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix} \qquad (2.0.2)$$

$$A = U\begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}U^* \qquad (2.0.3)$$

where $U^{-1}AV = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ Since A is normal,

$$A^*A = U\begin{pmatrix} \bar{\lambda} & 0 \\ \bar{\alpha} & \bar{\beta} \end{pmatrix}U^*U\begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}U^* \qquad (2.0.4)$$

$$\implies A^*A = U\begin{pmatrix} \bar{\lambda}\lambda & \alpha\bar{\lambda} \\ \lambda\bar{\alpha} & \alpha\bar{\alpha} + \bar{\beta}\beta \end{pmatrix}U^* \qquad (2.0.5)$$

Where $\bar{\lambda} = \lambda^*$ is the conjugate transpose of λ. Similarly,

$$AA^* = U\begin{pmatrix} \bar{\lambda}\lambda & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \bar{\beta}\beta \end{pmatrix}U^* \qquad (2.0.6)$$

As $AA^* = A^*A$

$$\alpha\bar{\alpha} = 0 \qquad (2.0.7)$$

$$\implies \|\alpha\| = 0 \qquad (2.0.8)$$

$$\implies \alpha = 0 \qquad (2.0.9)$$

$$\implies A = U\begin{pmatrix} \lambda & 0 \\ 0 & \beta \end{pmatrix}U^* \qquad (2.0.10)$$

$$\implies A = UDU^* \qquad (2.0.11)$$

So, A is unitary diagonalizable when D is a 2x2 matrix. Now, assume that the result holds for (n-1). we can claim that there is y^* such that

$$AU = U\begin{pmatrix} \lambda & y^* \\ 0 & B \end{pmatrix} \qquad (2.0.12)$$

where $B \in C^{(n-1)\times(n-1)}$, we decompose the matrix A into blocks and compute the products of AA^* and A^*A as follows:

$$AA^* = U\begin{pmatrix} \lambda & y^* \\ 0 & B \end{pmatrix}\begin{pmatrix} \bar{\lambda} & 0 \\ \bar{y}^* & \bar{B} \end{pmatrix} \qquad (2.0.13)$$

$$\implies AA^* = U\begin{pmatrix} \bar{\lambda}\lambda + y^*\bar{y}^* & y^*\bar{B} \\ B\bar{y}^* & \bar{B}B \end{pmatrix}U^* \qquad (2.0.14)$$

$$\implies AA^* = U\begin{pmatrix} \|\lambda\|^2 + \|y^*\|^2 & y^*\bar{B} \\ B\bar{y}^* & \bar{B}B \end{pmatrix}U^* \qquad (2.0.15)$$

Now,

$$A^*A = U\begin{pmatrix} \bar{\lambda} & 0 \\ \bar{y}^* & \bar{B} \end{pmatrix}\begin{pmatrix} \lambda & y^* \\ 0 & B \end{pmatrix} \qquad (2.0.16)$$

$$\implies A^*A = U\begin{pmatrix} \bar{\lambda}\lambda & y^*\bar{\lambda} \\ \lambda\bar{y}^* & \bar{B}B + \bar{y}^*y^* \end{pmatrix}U^* \qquad (2.0.17)$$

$$\implies A^*A = U\begin{pmatrix} \|\lambda\|^2 & y^*\bar{\lambda} \\ \lambda\bar{y}^* & \bar{B}B + \|y^*\|^2 \end{pmatrix}U^* \qquad (2.0.18)$$

As $AA^* = A^*A$,

$$y^* \bar{y}^* = 0 \qquad (2.0.19)$$

$$\implies \|y^*\|^2 = 0 \qquad (2.0.20)$$

$$\implies y^* = 0 \qquad (2.0.21)$$

$$\implies A = U \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} U^* \qquad (2.0.22)$$

$$\implies A = UDU^* \qquad (2.0.23)$$

As $\bar{B} = B^*$ and $B^*B = BB^*$, so B is also normal and $B \in C^{(n-1)\times(n-1)}$. So it must be diagonal by mathematical induction hypothesis. Let,

$$B = MD_1 M^* \qquad (2.0.24)$$

where M is unitary matrix and D_1 is diagonal matrix and both are in $C^{(n-1)\times(n-1)}$. Now,

$$AU = U \begin{pmatrix} \lambda & 0 \\ 0 & MD_1 M^* \end{pmatrix} \qquad (2.0.25)$$

$$AU = U \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & M^* \end{pmatrix} \qquad (2.0.26)$$

$$AU \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} = U \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & D_1 \end{pmatrix} \qquad (2.0.27)$$

$$AW = W \begin{pmatrix} \lambda & 0 \\ 0 & D_1 \end{pmatrix} \qquad (2.0.28)$$

$$A = W \begin{pmatrix} \lambda & 0 \\ 0 & D_1 \end{pmatrix} W^* \qquad (2.0.29)$$

where W is also a unitary matrix and $W = U \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$

This implies that if A is normal then A is unitary diagonalizable.