Challenge Problem 5

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1 Problem

Challenging problem: Intersection of transforms, matrices, signal processing, probability:

https://github.com/abhishekt711/EE5609/blob/master/Assignment20/Assignment_20.pdf

This problem provides an opportunity to link all the above. What is the link?

2 Explanation

The transition probability of Markov chain is expressed as:
P_{ij} = P[next state S_j at time t=1|current state S_i at t=0]
and the Transition Probability Matrix (TPM) is given as:

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \tag{2.0.1}$$

In this problem, $P_{11} = P_{22} = P_{33} = 0 \implies$ Any state cannot retain in its own state after 1 step execution, that is , for n=1. These P_{11}, P_{22}, P_{33} are theretaintion probabilities.

The summamtion of each row of this TPM, P is 1 which is similar to a Binary Symmetric Channel Matrix which is very important concept in Information Theory.

Now, while considering 2nd step execution, the TPM for n=2 is:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \tag{2.0.2}$$

From here we get that after 2nd execution, the retaintion probability is $\frac{1}{2}$ for all the 3 states, that is,
$P_{11}^{(2)}$ = P[next state S_1 at time t=2|current state S_1 at t=0] = $\frac{1}{2}$.

The transition probabilities from any state to other states are equal, that is, $P_{21} = P_{23}$,$P_{31} = P_{32}$ and $P_{12} = P_{13}$.

The transition probability from one state to another state of a current state is the retaintion probability of the next state, that is, $P_{21}^{(n)} = P_{23}^{(n)} = P_{22}^{(n+1)}$, for all the 3 states.

In each period , that is, for n=0,1,2,..., the retaintion probabilities($P_{22}, P_{22}^{(2)}, P_{22}^{(3)}, ...$) for all 3 states are same, that is, $P_{11}^{(2)} = P_{22}^{(2)} = P_{33}^{(2)}$.

π is the stationary distribution and $\pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$ such that $\pi\mathbf{P} = \pi \implies$ The markov chain is in steady state with period 1 and π is the state probabilities vector which is fixed for the entire Markov chain with the values $\pi = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$. It means the Markov chain is in steady state conditionand all the 3 states have equal share in the process.

If we consider $n = n_1$th execution, then the output of the state 1 can be defined as:

$$Y_1[n_1] = \frac{1}{2}Y_2[n_1 - 1] + \frac{1}{2}Y_3[n_1 - 1] \tag{2.0.3}$$

Where Y_1, Y_2, Y_3 denote the output of state 1, state 2, state 3. Similarly,

$$Y_2[n_1] = \frac{1}{2}Y_1[n_1 - 1] + \frac{1}{2}Y_3[n_1 - 1] \tag{2.0.4}$$

$$Y_3[n_1] = \frac{1}{2}Y_1[n_1 - 1] + \frac{1}{2}Y_2[n_1 - 1] \tag{2.0.5}$$

It works as a Finite state machine. Markov chain can be represented by Finite State Machine(FSM) as the states of time t+1 depends on the states of time t. But in the Markov chain the transition of the states are probabilistic while in FSM it is deterministic.