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Challenge Problem 2

Jayati Dutta

Download all and latex-tikz codes from

svn co https://github.com/gadepall/school/trunk/ ncert/geometry/figs

1 Problem

The center of the conic section:

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0 (1.0.1)$$

2 EXPLANATION

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \tag{2.0.1}$$

and

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.0.2}$$

$$f = 5 \tag{2.0.3}$$

Now, from the eigen values of $\lambda = 0$, 4 we got the eigen vectors. For $\lambda_1 = 0$, the eigen vector of **V** is $\mathbf{p_1}$ and for $\lambda_2 = 4$, the eigen vector of **V** is $\mathbf{p_2}$ and those are:

$$\mathbf{p_1} = \begin{pmatrix} 1\\\sqrt{3} \end{pmatrix} \tag{2.0.4}$$

and

$$\mathbf{p_2} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \tag{2.0.5}$$

Now, **P** should be chosen in such a way that $\mathbf{P}^T = \mathbf{P}^{-1}$ and $\mathbf{P}^T V \mathbf{P} = \mathbf{D}$. but here it is not satisfying these quations. Here, $\mathbf{P}^{-1} = \frac{1}{4} \mathbf{P}^T$. Now, in this case we also observed that,

$$4(\mathbf{Vc} + \mathbf{u}) = \eta \mathbf{p_1} \tag{2.0.6}$$

But this can be written as:

$$\mathbf{Vc} + \mathbf{u} = \frac{\eta}{2} \left(\frac{1}{2} \mathbf{p_1} \right) \tag{2.0.7}$$

So, P should be selected as:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$
 (2.0.8)

If I did it so, then $|\mathbf{P}|=1$ and $\mathbf{P}^{-1}=\mathbf{P}^{T}$ and $\mathbf{P}^{T}\mathbf{V}\mathbf{P}=\mathbf{D}$. So,

$$\mathbf{p_1} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{2.0.9}$$

and

$$\mathbf{p_2} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \tag{2.0.10}$$

Now, η can be calculated as :

$$\eta = 2\mathbf{p_1}^T \mathbf{u} \tag{2.0.11}$$

$$\eta = 2 \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.0.12}$$

$$\eta = 3 - 2\sqrt{3} \tag{2.0.13}$$

Now,

$$\mathbf{u}^{T} + \frac{\eta}{2} \mathbf{p}_{1}^{T} = \begin{pmatrix} \frac{15 - 2\sqrt{3}}{4} & \frac{-14 + 3\sqrt{3}}{4} \end{pmatrix}$$
 (2.0.14)

$$\frac{\eta}{2}\mathbf{p_1} - \mathbf{u} = \frac{3 - 2\sqrt{3}}{2} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$
 (2.0.15)

$$\frac{\eta}{2}\mathbf{p_1} - \mathbf{u} = \begin{pmatrix} \frac{-9 - 2\sqrt{3}}{4} \\ \frac{2 + 3\sqrt{3}}{4} \end{pmatrix}$$
 (2.0.16)

Now considering the augmented matrix,

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5\\ 3 & -\sqrt{3} & \frac{-9-2\sqrt{3}}{4}\\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \stackrel{R_2 \leftarrow R_2/\sqrt{3}}{\longleftrightarrow} (2.0.17)$$

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5\\ \sqrt{3} & -1 & \frac{-(2+3\sqrt{3})}{4}\\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2+3\sqrt{3}\\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -6R_2+R_2\\ R_3 \leftarrow R_3 + R_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5\\ \sqrt{3} & -1 & \frac{-(2+3\sqrt{3})}{4}\\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} (2.0.18)$$

$$\underset{R_2 \leftarrow R_2/\sqrt{3}}{\overset{R_3 \leftarrow R_2 + R_3}{\longleftrightarrow}} \quad (2.0.19)$$

$$\begin{pmatrix} 2.88397 & -2.20096 & -5\\ 1 & -0.57735 & -1.03923\\ 0 & 0 & 0 \end{pmatrix} (2.0.20)$$

$$\stackrel{R_1 \leftarrow R_1/2.88397}{\longleftrightarrow} \begin{pmatrix} 1 & -0.76317 & -1.7337 \\ 1 & -0.57735 & -1.03923 \\ 0 & 0 & 0 \end{pmatrix} (2.0.21)$$

$$\stackrel{R_2 \leftarrow R_2 - R_1}{\longleftrightarrow} \begin{pmatrix} 1 & -0.76317 & -1.7337 \\ 0 & 0.18582 & 0.69447 \\ 0 & 0 & 0 \end{pmatrix} (2.0.22)$$

So,

$$\begin{pmatrix} 1 & -0.76317 \\ 0 & 0.18582 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -1.7337 \\ 0.69447 \end{pmatrix}$$
 (2.0.23)

$$\implies c_1 = 1.118 \tag{2.0.24}$$

$$c_2 = 3.737 \qquad (2.0.25)$$

From there we can get $\mathbf{c} = \begin{pmatrix} 1.12 \\ 3.74 \end{pmatrix}$. The vertex of parabola will be at (1.12, 3.74).

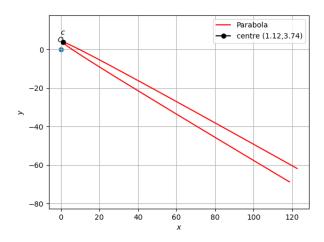


Fig. 0: Parabola

2.1. Verification of the above problem using python code.

Solution: The following Python code generates Fig. 0

codes/check parab.py