Challenge Problem 2

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Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

The center of the conic section:

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0 \tag{1.0.1}$$

2 Explanation

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \tag{2.0.1}$$

and

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.0.2}$$

$$f = 5 \tag{2.0.3}$$

Now, from the eigen values of λ =0, 4 we got the eigen vectors. For λ_1=0, the eigen vector of $\mathbf{V}$ is $\mathbf{p_1}$ and for λ_2=4, the eigen vector of $\mathbf{V}$ is $\mathbf{p_2}$ and those are:

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{2.0.4}$$

and

$$\mathbf{p_2} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \tag{2.0.5}$$

Now, $\mathbf{P}$ should be chosen in such a way that $\mathbf{P}^T = \mathbf{P}^{-1}$ and $\mathbf{P}^T V \mathbf{P} = \mathbf{D}$. but here it is not satisfying these quations. Here, $\mathbf{P}^{-1} = \frac{1}{4}\mathbf{P}^T$. Now, in this case we also observed that,

$$4(\mathbf{Vc} + \mathbf{u}) = \eta \mathbf{p_1} \tag{2.0.6}$$

But this can be written as:

$$\mathbf{Vc} + \mathbf{u} = \frac{\eta}{2}\left(\frac{1}{2}\mathbf{p_1}\right) \tag{2.0.7}$$

So, $\mathbf{P}$ should be selected as :

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \tag{2.0.8}$$

If I did it so, then $\left|\mathbf{P}\right|$=1 and $\mathbf{P}^{-1}$=$\mathbf{P}^T$ and $\mathbf{P}^T\mathbf{V}\mathbf{P}$=$\mathbf{D}$. So,

$$\mathbf{p_1} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{2.0.9}$$

and

$$\mathbf{p_2} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \tag{2.0.10}$$

Now, η can be calculated as :

$$\eta = 2\mathbf{p_1}^T\mathbf{u} \tag{2.0.11}$$

$$\eta = 2\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.0.12}$$

$$\eta = 3 - 2\sqrt{3} \tag{2.0.13}$$

Now,

$$\mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T = \begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} \end{pmatrix} \tag{2.0.14}$$

$$\frac{\eta}{2}\mathbf{p_1} - \mathbf{u} = \frac{3-2\sqrt{3}}{2}\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{2.0.15}$$

$$\frac{\eta}{2}\mathbf{p_1} - \mathbf{u} = \begin{pmatrix} \frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4} \end{pmatrix} \tag{2.0.16}$$

Now considering the augmented matrix,

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ 3 & -\sqrt{3} & \frac{-9-2\sqrt{3}}{4} \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/\sqrt{3}} \tag{2.0.17}$$

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ \sqrt{3} & -1 & \frac{-(2+3\sqrt{3})}{4} \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \tag{2.0.18}$$

$$\xleftrightarrow[R_2 \leftarrow R_2/\sqrt{3}]{R_3 \leftarrow R_2+R_3} \tag{2.0.19}$$

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ 1 & -\frac{1}{\sqrt{3}} & -\frac{-(2+3\sqrt{3})}{4\sqrt{3}} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.20}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/\frac{15-2\sqrt{3}}{4}} \begin{pmatrix} 1 & \frac{-14+3\sqrt{3}}{15-2\sqrt{3}} & -\frac{5}{15-2\sqrt{3}} \\ 1 & -\frac{1}{\sqrt{3}} & -\frac{-(2+3\sqrt{3})}{4\sqrt{3}} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.21}$$

$$\xleftrightarrow{R_2 \leftarrow R_2-R_1} \begin{pmatrix} 1 & \frac{-14+3\sqrt{3}}{15-2\sqrt{3}} & -\frac{5}{15-2\sqrt{3}} \\ 0 & -\frac{24-16\sqrt{3}}{15\sqrt{3}-6} & \frac{-12+39\sqrt{3}}{4\sqrt{3}(15-2\sqrt{3})} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.22}$$

So,

$$\begin{pmatrix} 1 & \frac{-14+3\sqrt{3}}{15-2\sqrt{3}} \\ 0 & -\frac{24-16\sqrt{3}}{15\sqrt{3}-6} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -\frac{5}{15-2\sqrt{3}} \\ \frac{-12+39\sqrt{3}}{4\sqrt{3}(15-2\sqrt{3})} \end{pmatrix} \tag{2.0.23}$$

$$\implies c_2 = -\frac{-12+39\sqrt{3}}{4(24-16\sqrt{3})} \tag{2.0.24}$$

$$\implies c_2 = 3.74 \tag{2.0.25}$$

$$c_1 = 1.12 \tag{2.0.26}$$

From there we can get $\mathbf{c}=\begin{pmatrix} 1.12 \\ 3.74 \end{pmatrix}$. The vertex of parabola will be at (1.12, 3.74).

2.1. Verification of the above problem using python code.

Solution: The following Python code generates Fig. 0

```
codes/check_parab.py
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So the solution is matching with the given plot, hence it is verified.

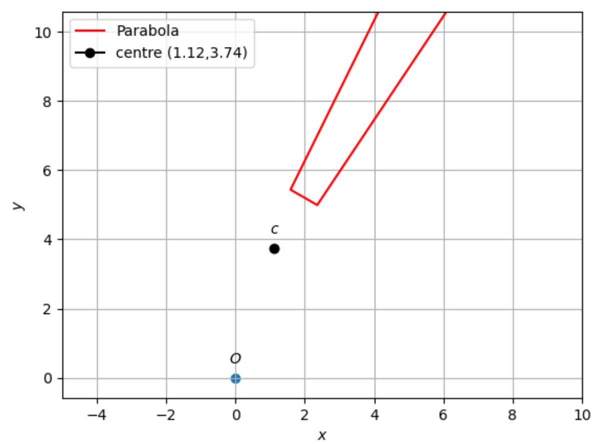

Fig. 0: Parabola