Challenge Problem 2

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1 PROBLEM

The center of the conic section:

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0 \qquad (1.0.1)$$

2 EXPLANATION

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \qquad (2.0.1)$$

and

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad (2.0.2)$$

$$f = 5 \qquad (2.0.3)$$

Now, from the eigen values of $\lambda = 0, 4$ we got the eigen vectors. For $\lambda_1 = 0$, the eigen vector of $\mathbf{V}$ is $\mathbf{p_1}$ and for $\lambda_2 = 4$, the eigen vector of $\mathbf{V}$ is $\mathbf{p_2}$ and those are:

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \qquad (2.0.4)$$

and

$$\mathbf{p_2} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \qquad (2.0.5)$$

Now, $\mathbf{P}$ should be chosen in such a way that $\mathbf{P}^T = \mathbf{P}^{-1}$ and $\mathbf{P}^T V \mathbf{P} = \mathbf{D}$. but here it is not satisfying these quations. Here, $\mathbf{P}^{-1} = \frac{1}{4}\mathbf{P}^T$. So, $\mathbf{P}$ should be selected as :

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad (2.0.6)$$

If I did it so, then $|\mathbf{P}| = 1$ and $\mathbf{P}^{-1} = \mathbf{P}^T$ and $\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D}$. So,

$$\mathbf{p_1} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \qquad (2.0.7)$$

and

$$\mathbf{p_2} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \qquad (2.0.8)$$

Now, η can be calculated as :

$$\eta = 2\mathbf{p_1}^T \mathbf{u} \qquad (2.0.9)$$

$$\eta = 2 \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad (2.0.10)$$

$$\eta = 3 - 2\sqrt{3} \qquad (2.0.11)$$

Now,

$$\mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T = \begin{pmatrix} 2.884 & -2.2 \end{pmatrix} \qquad (2.0.12)$$

$$\frac{\eta}{2}\mathbf{p_1} - \mathbf{u} = \frac{3 - 2\sqrt{3}}{2} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad (2.0.13)$$

$$\frac{\eta}{2}\mathbf{p_1} - \mathbf{u} = \begin{pmatrix} -3.116 \\ 1.8 \end{pmatrix} \qquad (2.0.14)$$

Now considering the augmented matrix,

$$\begin{pmatrix} 2.884 & -2.2 & -5 \\ 3 & -\sqrt{3} & -3.116 \\ -\sqrt{3} & 1 & 1.8 \end{pmatrix} \xleftarrow{R_2 \leftarrow R_2/\sqrt{3}} \qquad (2.0.15)$$

$$\begin{pmatrix} 2.884 & -2.2 & -5 \\ \sqrt{3} & -1 & -1.8 \\ -\sqrt{3} & 1 & 1.8 \end{pmatrix} \qquad (2.0.16)$$

$$\xleftarrow{R_3 \leftarrow R_2 + R_3} \qquad (2.0.17)$$

$$\begin{pmatrix} 2.884 & -2.2 & -5 \\ \sqrt{3} & -1 & -1.8 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.0.18)$$

So,

$$\begin{pmatrix} 2.884 & -2.2 \\ \sqrt{3} & -1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -5 \\ -1.8 \end{pmatrix} \qquad (2.0.19)$$

$$\begin{pmatrix} 2.884 & -2.2 \\ \sqrt{3} & -1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -5 \\ -1.8 \end{pmatrix} \qquad (2.0.20)$$

$$(2.0.21)$$

From there we can get $\mathbf{c} = \begin{pmatrix} 1.12 \\ 3.74 \end{pmatrix}$. The vertex of

parabola will be at (1.12, 3.74).