Assignment 11

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***Abstract*—This is a simple document explaining how to form a basis of a vector space and how to get the coordinates of the vector.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

In C^3, let $\alpha_1 = (1, 0, -i)$, $\alpha_2 = (1+i, 1-i, 1)$, α_3 = (i, i, i). Prove that these vectors form a basis for C^3. What are the coordinates of the vector (a,b,c) in the basis?

2 Explanation

Now,

$$C_1\alpha_1 + C_2\alpha_2 + C_3\alpha_3 = \mathbf{0} \tag{2.0.1}$$

$$\implies C_1\begin{pmatrix}1\\0\\-i\end{pmatrix} + C_2\begin{pmatrix}1+i\\1-i\\1\end{pmatrix} + C_3\begin{pmatrix}i\\i\\i\end{pmatrix} = \mathbf{0} \tag{2.0.2}$$

So,

$$\begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\-i & 1 & i\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.0.3}$$

Considering the co-efficient matrix A:

$$\begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\-i & 1 & i\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3+iR_1} \begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & i & i-1\end{pmatrix}$$
$$\xleftrightarrow{R_3\leftarrow R_3/i} \begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & 1 & 1+i\end{pmatrix} \xleftrightarrow{R_3\leftarrow (1-i)R_3}$$
$$\begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & 1-i & 2\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3-R_2} \begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & 0 & 2-i\end{pmatrix}$$
$$\xleftrightarrow{R_2\leftarrow \frac{1+i}{1-i}R_2} \begin{pmatrix}1 & 1+i & i\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1-R_2}$$
$$\begin{pmatrix}1 & 0 & i+1\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix} \tag{2.0.4}$$

Now let

$$R = \begin{pmatrix}1 & 0 & i+1\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix} \tag{2.0.5}$$

Where R is the row reduced form of matrix A. Now,

$$\begin{pmatrix}1 & 0 & i+1\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.0.6}$$

$$\implies (2-i)C_3 = 0 \tag{2.0.7}$$
$$\implies C_3 = 0 \tag{2.0.8}$$
$$C_1 + (i+1)C_3 = 0 \tag{2.0.9}$$
$$\implies C_1 = 0 \tag{2.0.10}$$
$$C_2 = 0 \tag{2.0.11}$$

As $C_1, C_2, C_3 \neq 0$, so α_1,α_2 and α_3 are linearly independent which implies that these 3 vectors form a basis of vector space C^3.

Now, consider a vector $\beta = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that

$$\beta^T R = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \tag{2.0.12}$$

Where $\rho_1 = \begin{pmatrix} 1 \\ 0 \\ i+1 \end{pmatrix}$, $\rho_2 = \begin{pmatrix} 0 \\ 1+i \\ -1 \end{pmatrix}$ and $\rho_3 = \begin{pmatrix} 0 \\ 0 \\ 2-i \end{pmatrix}$ are the rows of the matrix R.

Now,

$$\beta^T R = \begin{pmatrix} a & b & c \end{pmatrix} QA \tag{2.0.13}$$

When matrix A is row reduced to the matrix R, if same operations are simultaneously operated on the Identity matrix I, then we can get matrix Q. So,

$$Q = \begin{pmatrix} 1 & -i & 0 \\ 0 & i & 0 \\ 1-i & -1 & -i-1 \end{pmatrix} \tag{2.0.14}$$

Now,

$$x_i = \begin{pmatrix} a & b & c \end{pmatrix} Q_i \tag{2.0.15}$$

where x_i represents the coordinate of i^{th} vector. So,

$$\beta^T R = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} A \tag{2.0.16}$$

$$\implies \beta^T R = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{2.0.17}$$

$$\implies \beta^T R = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 \tag{2.0.18}$$

Now,

$$x_1 = \begin{pmatrix} a & b & c \end{pmatrix} Q_1 \tag{2.0.19}$$

$$\implies x_1 = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1-i \end{pmatrix} \tag{2.0.20}$$

$$\implies x_1 = a + c(1-i) \tag{2.0.21}$$

$$\implies x_1 = (a+c) - ic \tag{2.0.22}$$

Similarly,

$$x_2 = \begin{pmatrix} a & b & c \end{pmatrix} Q_2 \tag{2.0.23}$$

$$\implies x_2 = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -i \\ i \\ -1 \end{pmatrix} \tag{2.0.24}$$

$$\implies x_2 = -c + i(b-a) \tag{2.0.25}$$

$$x_3 = \begin{pmatrix} a & b & c \end{pmatrix} Q_3 \tag{2.0.26}$$

$$\implies x_3 = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -i-1 \end{pmatrix} \tag{2.0.27}$$

$$\implies x_3 = -c - ic \tag{2.0.28}$$