Assignment 11

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***Abstract*—This is a simple document explaining how to form a basis of a vector space and how to get the coordinates of the vector.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

In C^3, let $\alpha_{\mathbf{1}} = (1, 0, -i)$, $\alpha_{\mathbf{2}} = (1+i, 1-i, 1)$, $\alpha_{\mathbf{3}}$ = (i, i, i). Prove that these vectors form a basis for C^3. What are the coordinates of the vector (a,b,c) in the basis?

2 Explanation

Now,

$$C_1\alpha_{\mathbf{1}} + C_2\alpha_{\mathbf{2}} + C_3\alpha_{\mathbf{3}} = \mathbf{0} \tag{2.0.1}$$

$$\implies C_1\begin{pmatrix}1\\0\\-i\end{pmatrix} + C_2\begin{pmatrix}1+i\\1-i\\1\end{pmatrix} + C_3\begin{pmatrix}i\\i\\i\end{pmatrix} = \mathbf{0} \tag{2.0.2}$$

So,

$$\begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\-i & 1 & i\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.0.3}$$

Considering the co-efficient matrix A:

$$\begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\-i & 1 & i\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3+iR_1} \begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & i & i-1\end{pmatrix}$$
$$\xleftrightarrow{R_3\leftarrow R_3/i} \begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & 1 & 1+i\end{pmatrix} \xleftrightarrow{R_3\leftarrow(1-i)R_3}$$
$$\begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & 1-i & 2\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3-R_2} \begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & 0 & 2-i\end{pmatrix}$$
$$\xleftrightarrow{R_2\leftarrow \frac{1+i}{1-i}R_2} \begin{pmatrix}1 & 1+i & i\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1-R_2}$$
$$\begin{pmatrix}1 & 0 & i+1\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix} \tag{2.0.4}$$

Now let

$$R = \begin{pmatrix}1 & 0 & i+1\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix} \tag{2.0.5}$$

Where R is the row reduced form of matrix A. So $\alpha_{\mathbf{1}}$,$\alpha_{\mathbf{2}}$ and $\alpha_{\mathbf{3}}$ are linearly independent which implies that these 3 vectors form a basis of vector space C^3.

Now, consider a vector $\beta = \begin{pmatrix}a\\b\\c\end{pmatrix}$ and let the coordinates are $x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ such that

$$Ax = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.6}$$

$$\implies x = A^{-1}\begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.7}$$

Let us consider a matrix $(A|I)$ where I is a 3x3 identity matrix. Now, applying the Gauss-Jordon theorem we can get A^{-1}

$$
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ -i & 1 & i & 0 & 0 & 1 \end{pmatrix}
\xleftrightarrow{R_3 \leftarrow R_3 + iR_1}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & i & i-1 & i & 0 & 1 \end{pmatrix}
\xleftrightarrow{R_3 \leftarrow R_3/i}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1 & 1+i & 1 & 0 & -i \end{pmatrix}
\xleftrightarrow{R_3 \leftarrow (1-i)R_3}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1-i & 2 & 1-i & 0 & -i-1 \end{pmatrix}
\xleftrightarrow{R_3 \leftarrow R_3 - R_2}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix}
\xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i} R_2}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix}
\xleftrightarrow{R_1 \leftarrow R_1 - R_2}
\begin{pmatrix} 1 & 0 & i+1 & 1 & -i & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix}
\xleftrightarrow[R_1 \leftarrow R_1 + R_3]{R_3 \leftarrow \frac{R_3}{2-i}}
\begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}
\xleftrightarrow{R_2 \leftarrow R_2 + R_3}
\begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & 0 & \frac{3-i}{5} & \frac{-2+4i}{5} & -\frac{3i+1}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}
\xleftrightarrow{R_2 \leftarrow \frac{R_2}{1+i}}
\begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1 & 0 & \frac{3-i}{5(1+i)} & \frac{-2+4i}{5(1+i)} & -\frac{3i+1}{5(1+i)} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}
\xleftrightarrow{R_1 \leftarrow R_1 - 3R_3}
\tag{2.0.8}
$$

$$
\begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \tag{2.0.9}
$$

$$
\implies (I|A^{-1}) = \begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \tag{2.0.10}
$$

$$
\implies A^{-1} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \tag{2.0.11}
$$

So,

$$
x = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.0.12}
$$

$$
\implies x = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.0.13}
$$

$$
\implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.0.14}
$$