Assignment 11

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***Abstract*—This is a simple document explaining how to form a basis of a vector space and how to get the coordinates of the vector.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

In C^3, let $\alpha_{\mathbf{1}} = (1,0,-i)$, $\alpha_{\mathbf{2}} = (1+i, 1-i, 1)$, $\alpha_{\mathbf{3}}$ = (i,i,i). Prove that these vectors form a basis for C^3. What are the coordinates of the vector (a,b,c) in the basis?

2 Explanation

Now,

$$C_1\alpha_{\mathbf{1}} + C_2\alpha_{\mathbf{2}} + C_3\alpha_{\mathbf{3}} = \mathbf{0} \tag{2.0.1}$$

$$\implies C_1\begin{pmatrix}1\\0\\-i\end{pmatrix} + C_2\begin{pmatrix}1+i\\1-i\\1\end{pmatrix} + C_3\begin{pmatrix}i\\i\\i\end{pmatrix} = \mathbf{0} \tag{2.0.2}$$

So,

$$\begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\-i & 1 & i\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.0.3}$$

Considering the co-efficient matrix A:

$$\begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\-i & 1 & i\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3+iR_1} \begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & i & i-1\end{pmatrix}$$
$$\xleftrightarrow{R_3\leftarrow R_3/i} \begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & 1 & 1+i\end{pmatrix} \xleftrightarrow{R_3\leftarrow (1-i)R_3}$$
$$\begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & 1-i & 2\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3-R_2} \begin{pmatrix}1 & 1+i & i\\0 & 1-i & i\\0 & 0 & 2-i\end{pmatrix}$$
$$\xleftrightarrow{R_2\leftarrow \frac{1+i}{1-i}R_2} \begin{pmatrix}1 & 1+i & i\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1-R_2}$$
$$\begin{pmatrix}1 & 0 & i+1\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix} \tag{2.0.4}$$

Now let

$$R = \begin{pmatrix}1 & 0 & i+1\\0 & 1+i & -1\\0 & 0 & 2-i\end{pmatrix} \tag{2.0.5}$$

Where R is the row reduced form of matrix A. So $\alpha_{\mathbf{1}}$,$\alpha_{\mathbf{2}}$ and $\alpha_{\mathbf{3}}$ are linearly independent which implies that these 3 vectors form a basis of vector space C^3.

Now, consider a vector $\beta = \begin{pmatrix}a\\b\\c\end{pmatrix}$ and let the coordinates are $x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ such that

$$Ax = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.6}$$

Now, considering the augmented matrix:

$$\begin{pmatrix} 1 & 1+i & i & a \\ 0 & 1-i & i & b \\ -i & 1 & i & c \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + iR_1}$$

$$\begin{pmatrix} 1 & 1+i & i & a \\ 0 & 1-i & i & b \\ 0 & i & i-1 & c+ia \end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/i} \begin{pmatrix} 1 & 1+i & i & a \\ 0 & 1-i & i & b \\ 0 & 1 & 1+i & a-ic \end{pmatrix} \xleftrightarrow{R_3 \leftarrow (1-i)R_3}$$

$$\begin{pmatrix} 1 & 1+i & i & a \\ 0 & 1-i & i & b \\ 0 & 1-i & 2 & (a-c)-i(a+c) \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2}$$

$$\begin{pmatrix} 1 & 1+i & i & a \\ 0 & 1-i & i & b \\ 0 & 0 & 2-i & (a-c-b)-i(a+c) \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i} R_2} \begin{pmatrix} 1 & 1+i & i & a \\ 0 & 1+i & -1 & ib \\ 0 & 0 & 2-i & (a-c-b)-i(a+c) \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$

$$\begin{pmatrix} 1 & 0 & i+1 & a-ib \\ 0 & 1+i & -1 & ib \\ 0 & 0 & 2-i & (a-c-b)-i(a+c) \end{pmatrix}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + R_3]{R_3 \leftarrow \frac{R_3}{2-i}}$$

$$\begin{pmatrix} 1 & 0 & 3 & (2a-b-c)-i(a+b+c) \\ 0 & 1+i & -1 & ib \\ 0 & 0 & 1 & \frac{(a-c-b)-i(a+c)}{2-i} \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_3}$$

$$\begin{pmatrix} 1 & 0 & 3 & (2a-b-c)-i(a+b+c) \\ 0 & 1+i & 0 & ib + \frac{(a-c-b)-i(a+c)}{2-i} \\ 0 & 0 & 1 & \frac{(a-c-b)-i(a+c)}{2-i} \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{(a+b-2c)(1-2i)}{5} \\ 0 & 1 & 0 & \frac{(3a-2b-c)-i(a-4b+3c)}{5(1+i)} \\ 0 & 0 & 1 & \frac{(3a-2b-c)-i(a-4b+3c)}{5} \end{pmatrix} \tag{2.0.7}$$

So,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{(a+b-2c)(1-2i)}{5} \\ \frac{(3a-2b-c)-i(a-4b+3c)}{5(1+i)} \\ \frac{(3a-2b-c)-i(a-4b+3c)}{5} \end{pmatrix} \tag{2.0.8}$$

$$\implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{(a+b-2c)(1-2i)}{5} \\ \frac{(3a-2b-c)-i(a-4b+3c)}{5(1+i)} \\ \frac{(3a-2b-c)-i(a-4b+3c)}{5} \end{pmatrix} \tag{2.0.9}$$