Assignment 11

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***Abstract*—This is a simple document explaining how to form a basis of a vector space and how to get the coordinates of the vector.**

Download all and latex-tikz codes from

svn co https://github.com/gadepall/school/trunk/ncert/geometry/figs

1 Problem

In C^3, let $\alpha_{\mathbf{1}} = (1, 0, -i)$, $\alpha_{\mathbf{2}} = (1+i, 1-i, 1)$, $\alpha_{\mathbf{3}}$ = (i, i, i). Prove that these vectors form a basis for C^3. What are the coordinates of the vector (a,b,c) in the basis?

2 Explanation

Now,

$$C_1\alpha_{\mathbf{1}} + C_2\alpha_{\mathbf{2}} + C_3\alpha_{\mathbf{3}} = \mathbf{0} \tag{2.0.1}$$

$$\implies C_1(1, 0, -i) + C_2(1+i, 1-i, 1) + C_3(i, i, i) = \mathbf{0} \tag{2.0.2}$$

So,

$$\begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ -i & 1 & i \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.3}$$

Considering the co-efficient matrix A:

$$\begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ -i & 1 & i \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + iR_1} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & i & i-1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3/i} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & 1 & 1+i \end{pmatrix} \xleftrightarrow{R_3 \leftarrow (1-i)R_3} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & 1-i & 2 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & 0 & 2-i \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i}R_2} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & i+1 \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \tag{2.0.4}$$

Now let

$$R = \begin{pmatrix} 1 & 0 & i+1 \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \tag{2.0.5}$$

Where R is the row reduced form of matrix A. Now,

$$\begin{pmatrix} 1 & 0 & i+1 \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.6}$$

$$\implies (2-i)C_3 = 0 \tag{2.0.7}$$

$$\implies C_3 = 0 \tag{2.0.8}$$

$$C_1 + (i-1)C_3 = 0 \tag{2.0.9}$$

$$\implies C_1 = 0 \tag{2.0.10}$$

$$C_2 = 0 \tag{2.0.11}$$

As $C_1, C_2, C_3 \neq 0$, so $\alpha_{\mathbf{1}}$,$\alpha_{\mathbf{2}}$ and $\alpha_{\mathbf{3}}$ are linearly independent and as the dimension of the vector space C^3 is 3. So we can say that these 3 vectors span the vector space C^3 which implies that these 3 vectors form a basis of vector space C^3.

Now, consider a vector $\beta = (a, b, c)$ such that

$$\beta R = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \tag{2.0.12}$$

Where $\rho_1 = \begin{pmatrix} 1 & 0 & i-1 \end{pmatrix}$, $\rho_2 = \begin{pmatrix} 0 & 1+i & -1 \end{pmatrix}$ and $\rho_3 = \begin{pmatrix} 0 & 0 & 2-i \end{pmatrix}$ are the rows of the matrix R.

Now,

$$\beta R = \begin{pmatrix} a & b & c \end{pmatrix} QA \tag{2.0.13}$$

When matrix A is row reduced to the matrix R, if same operations are simultaneously operated on the Identity matrix I, then we can get matrix Q. So,

$$Q = \begin{pmatrix} 1 & -i & 0 \\ 0 & i & 0 \\ 1-i & -1 & -i-1 \end{pmatrix} \tag{2.0.14}$$

Now,

$$x_i = \begin{pmatrix} a & b & c \end{pmatrix} Q_i \tag{2.0.15}$$

where x_i represents the coordinate of i^{th} vector. So,

$$x_1 = \begin{pmatrix} a & b & c \end{pmatrix} Q_1 \tag{2.0.16}$$

$$\implies x_1 = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1-i \end{pmatrix} \tag{2.0.17}$$

$$\implies x_1 = a + c(1-i) \tag{2.0.18}$$

$$\implies x_1 = (a+c) - ic \tag{2.0.19}$$

Similarly,

$$x_2 = \begin{pmatrix} a & b & c \end{pmatrix} Q_2 \tag{2.0.20}$$

$$\implies x_2 = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -i \\ i \\ -1 \end{pmatrix} \tag{2.0.21}$$

$$\implies x_2 = -c + i(b-a) \tag{2.0.22}$$

$$x_3 = \begin{pmatrix} a & b & c \end{pmatrix} Q_3 \tag{2.0.23}$$

$$\implies x_3 = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -i-1 \end{pmatrix} \tag{2.0.24}$$

$$\implies x_3 = -c - ic \tag{2.0.25}$$