

# Assignment 12

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**Abstract**—This is a simple document explaining how to form a basis of a vector space and how to get the coordinates of the vector.

Download all and latex-tikz codes from

svn co <https://github.com/gadepall/school/trunk/ncert/geometry/figs>

## 1 PROBLEM

In  $C^3$ , let  $\alpha_1 = (1, 0, -i)$ ,  $\alpha_2 = (1 + i, 1 - i, 1)$ ,  $\alpha_3 = (i, i, i)$ . Prove that these vectors form a basis for  $C^3$ . What are the coordinates of the vector  $(a, b, c)$  in the basis?

## 2 EXPLANATION

Now,

$$C_1\alpha_1 + C_2\alpha_2 + C_3\alpha_3 = \mathbf{0} \quad (2.0.1)$$

$$\Rightarrow C_1 \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + C_2 \begin{pmatrix} 1+i \\ 1-i \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} i \\ i \\ i \end{pmatrix} = \mathbf{0} \quad (2.0.2)$$

So,

$$\begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ -i & 1 & i \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.3)$$

Considering the co-efficient matrix A:

$$\begin{aligned} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ -i & 1 & i \end{pmatrix} &\xleftrightarrow{R_3 \leftarrow R_3 + iR_1} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & i & i-1 \end{pmatrix} \\ &\xleftrightarrow{R_3 \leftarrow R_3/i} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & 1 & 1+i \end{pmatrix} \xleftrightarrow{R_3 \leftarrow (1-i)R_3} \\ &\begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & 1-i & 2 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & 0 & 2-i \end{pmatrix} \\ &\xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i}R_2} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \\ &\begin{pmatrix} 1 & 0 & i+1 \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \quad (2.0.4) \end{aligned}$$

Now let

$$R = \begin{pmatrix} 1 & 0 & i+1 \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \quad (2.0.5)$$

Where  $R$  is the row reduced form of matrix  $A$ . So  $\alpha_1, \alpha_2$  and  $\alpha_3$  are linearly independent which implies that these 3 vectors form a basis of vector space  $C^3$ .

Now, consider a vector  $\beta = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$  and let the

coordinates are  $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$  such that

$$Ax = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.0.6)$$

$$\Rightarrow x = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.0.7)$$

Let us consider a matrix  $(A|I)$  where  $I$  is a  $3 \times 3$  identity matrix. Now, applying the Gauss-Jordon

theorem we can get  $A^{-1}$

$$\begin{aligned}
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ -i & 1 & i & 0 & 0 & 1 \end{pmatrix} \\
 & \xleftrightarrow{R_3 \leftarrow R_3 + iR_1} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & i & i-1 & i & 0 & 1 \end{pmatrix} \\
 & \xleftrightarrow{R_3 \leftarrow R_3 / i} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1 & 1+i & 1 & 0 & -i \end{pmatrix} \\
 & \xleftrightarrow{R_3 \leftarrow (1-i)R_3} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1-i & 2 & 1-i & 0 & -i-1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix} \\
 & \xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i}R_2} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix} \\
 & \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \\
 & \begin{pmatrix} 1 & 0 & i+1 & 1 & -i & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix} \\
 & \xleftrightarrow{R_3 \leftarrow \frac{R_3}{2-i}} \\
 & \xleftrightarrow{R_1 \leftarrow R_1 + R_3} \\
 & \begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \\
 & \xleftrightarrow{R_2 \leftarrow R_2 + R_3} \\
 & \begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & 0 & \frac{3-i}{5} & \frac{-2+4i}{5} & -\frac{3i+1}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \\
 & \xleftrightarrow{R_2 \leftarrow \frac{R_2}{1+i}} \\
 & \begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1 & 0 & \frac{3-i}{5(1+i)} & \frac{-2+4i}{5(1+i)} & -\frac{3i+1}{5(1+i)} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \\
 & \xleftrightarrow{R_1 \leftarrow R_1 - 3R_3} \\
 & \begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \quad (2.0.8)
 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \quad (2.0.9)$$

$$\Rightarrow (I|A^{-1}) = \begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \quad (2.0.10)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \quad (2.0.11)$$

So,

$$x = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.0.12)$$

$$\Rightarrow x = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.0.13)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.0.14)$$