Assignment 13

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***Abstract*—This is a simple document explaining how a linear operator of a vector space depends upon an ordered basis of that vector space.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

Let V be a two-dimensional vector space over the field F and let B be an ordered basis for V. If T is a linear operator on V and

$$[T]_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.0.1}$$

Prove that

$$T^2 - (a+d)T + (ad-bc)I = 0 \tag{1.0.2}$$

2 Explanation

Here T is a linear operator on V and B is an ordered basis of V. Now, consider the transform $(T^2 - (a+d)T + (ad-bc)I)$ with respect to basis B:

$$[T^2 - (a+d)T + (ad-bc)I]_B \tag{2.0.1}$$

$$= [T^2]_B - [(a+d)T]_B + [(ad-bc)I]_B \tag{2.0.2}$$

$$= [TT]_B - (a+d)[T]_B + (ad-bc)I_B \tag{2.0.3}$$

$$= [T]_B[T]_B - (a+d)[T]_B + (ad-bc)I \tag{2.0.4}$$

Now,if U and T are two linear operator on V then $[UT]_B = [U]_B[T]_B$, and here T is linear operator so $[T^2]_B = [T]_B[T]_B$.

Now,

$$([T]_B)^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.0.5}$$

$$\implies ([T]_B)^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{pmatrix} \tag{2.0.6}$$

$$(a+d)[T]_B = (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.0.7}$$

$$\textbf{or,}(a+d)[T]_B = \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} \tag{2.0.8}$$

$$(ad-bc)I = (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.9}$$

$$\textbf{or,}(ad-bc)I = \begin{pmatrix} (ad-bc) & 0 \\ 0 & (ad-bc) \end{pmatrix} \tag{2.0.10}$$

Now,

$$\begin{aligned} &[T]_B[T]_B - (a+d)[T]_B + (ad-bc)I \\ &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{pmatrix} \\ &- \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} + \begin{pmatrix} (ad-bc) & 0 \\ 0 & (ad-bc) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{2.0.11}$$

So,

$$[T^2 - (a+d)T + (ad-bc)I]_B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_B \tag{2.0.12}$$

$$\implies T^2 - (a+d)T + (ad-bc)I = 0 \tag{2.0.13}$$

Hence, the above problem statement is proved.