# Assignment 14

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***Abstract*—This is a simple document explaining how to describe a linear functional on a vector space for certain conditions.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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## 1 Problem

In $R^3$, let $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ and $\alpha_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$. Describe explicitly a linear functional f on $R^3$ such that $f(\alpha_1) = f(\alpha_2) = 0$ but $f(\alpha_3) \neq 0$

## 2 Explanation

Let us consider $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2 = \alpha \tag{2.0.1}$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.0.2}$$

The coefficient matrix is :

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \tag{2.0.3}$$

So,

$$A\mathbf{x} = \alpha \tag{2.0.4}$$

$$\implies x = A^{-1}\alpha \tag{2.0.5}$$

$$\tag{2.0.6}$$

Now to get $A^{-1}$, we will consider Gauss-Jordon theorem. So, we will take $(A|I)$, where $I$ is a $3 \times 3$ identity matrix.

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1}$$
$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + 2R_2}$$
$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3/(-1)}$$
$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 + R_3]{R_2 \leftarrow R_2 + R_3}$$
$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \tag{2.0.7}$$

Now, we can say that

$$A^{-1} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \tag{2.0.8}$$

As

$$\mathbf{x} = A^{-1}\alpha \tag{2.0.9}$$

$$\implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.0.10}$$

$$\implies \mathbf{x} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.0.11}$$

Now, as $f$ is a linear functional on $R^3$,

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2 \quad (2.0.12)$$

$$\implies f(\alpha) = f(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2) \quad (2.0.13)$$

$$\implies f(\alpha) = x_1 f(\alpha_1) + x_2 f(\alpha_2) + x_3 f(\alpha_3) \quad (2.0.14)$$

$$\implies f(\alpha) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \quad (2.0.15)$$

$$\implies f(\alpha) = \mathbf{x}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \quad (2.0.16)$$

As mentioned in the problem statement, $f(\alpha_1) = f(\alpha_2) = 0$ and $f(\alpha_3) \neq 0$.

Now,

$$f(\alpha) = \mathbf{x}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \quad (2.0.17)$$

$$\implies f(\alpha) = \mathbf{x}^T \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \quad (2.0.18)$$

$$\implies f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \quad (2.0.19)$$

$$\implies f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} f(\alpha_3) \\ -2f(\alpha_3) \\ -f(\alpha_3) \end{pmatrix} \quad (2.0.20)$$

$$\implies f(a, b, c) = f(\alpha_3) \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \quad (2.0.21)$$

So, the function can be defined as:

$$f(a, b, c) = f(\alpha_3) \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \quad (2.0.22)$$

$$\textbf{or,} f(\alpha) = f(\alpha_3)(a - 2b - c) \quad (2.0.23)$$