Assignment 14

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***Abstract*—This is a simple document explaining how to describe a linear functional on a vector space for certain conditions.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

In R^3, let $\alpha_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $\alpha_2 = \begin{pmatrix}0\\1\\-2\end{pmatrix}$ and $\alpha_3 = \begin{pmatrix}-1\\-1\\0\end{pmatrix}$. Describe explicitly a linear functional f on R^3 such that $f(\alpha_1) = f(\alpha_2) = 0$ but $f(\alpha_3) \neq 0$

2 Explanation

Let us consider $\alpha = \begin{pmatrix}a\\b\\c\end{pmatrix}$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2 = \alpha \tag{2.0.1}$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2 = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.2}$$

The coefficient matrix is :

$$A = \begin{pmatrix}1 & 0 & -1\\0 & 1 & -1\\1 & -2 & 0\end{pmatrix} \tag{2.0.3}$$

So,

$$A\mathbf{x} = \alpha \tag{2.0.4}$$

$$\implies x = A^{-1}\alpha \tag{2.0.5}$$

$$\tag{2.0.6}$$

Now to get A^{-1}, we will consider Gauss-Jordon theorem. So, we will take $(A|I)$, where I is a 3×3 identity matrix.

$$\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\1 & -2 & 0 & 0 & 0 & 1\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\0 & -2 & 1 & -1 & 0 & 1\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + 2R_2} \begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\0 & 0 & -1 & -1 & 2 & 1\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3/(-1)} \begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\0 & 0 & 1 & 1 & -2 & -1\end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 + R_3]{R_2 \leftarrow R_2 + R_3} \begin{pmatrix}1 & 0 & 0 & 2 & -2 & -1\\0 & 1 & 0 & 1 & -1 & -1\\0 & 0 & 1 & 1 & -2 & -1\end{pmatrix} \tag{2.0.7}$$

Now, we can say that

$$A^{-1} = \begin{pmatrix}2 & -2 & -1\\1 & -1 & -1\\1 & -2 & -1\end{pmatrix} \tag{2.0.8}$$

As

$$\mathbf{x} = A^{-1}\alpha \tag{2.0.9}$$

$$\implies \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = A^{-1}\begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.10}$$

$$\implies \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2 & -2 & -1\\1 & -1 & -1\\1 & -2 & -1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.11}$$

Now, let us consider $B_1 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $B_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ and $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. So,

$$x_1 = \mathbf{B_1}^T \alpha \tag{2.0.12}$$
$$x_2 = \mathbf{B_2}^T \alpha \tag{2.0.13}$$
$$x_3 = \mathbf{B_3}^T \alpha \tag{2.0.14}$$

Now, as f is a linear functional on R^3,

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2 \tag{2.0.15}$$
$$\implies f(\alpha) = f(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2) \tag{2.0.16}$$
$$\implies f(\alpha) = x_1 f(\alpha_1) + x_2 f(\alpha_2) + x_3 f(\alpha_3) \tag{2.0.17}$$
$$\implies f(\alpha) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \tag{2.0.18}$$
$$\implies f(\alpha) = \mathbf{x}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \tag{2.0.19}$$

As mentioned in the problem statement, $f(\alpha_1) = f(\alpha_2) = 0$ and $f(\alpha_3) \neq 0$.

Now,

$$f(\alpha) = \mathbf{x}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \tag{2.0.20}$$
$$\implies f(\alpha) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \tag{2.0.21}$$
$$\implies f(\alpha) = x_3 f(\alpha_3) \tag{2.0.22}$$
$$\implies f(\alpha) = f(\alpha_3) \mathbf{B_3}^T \alpha \tag{2.0.23}$$
$$\implies f(a, b, c) = f(\alpha_3) \begin{pmatrix} 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.0.24}$$

So, the function can be defined as:

$$f(x, y, z) = f(\alpha_3) \begin{pmatrix} 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.0.25}$$
$$\textbf{or,} f(\mathbf{x}) = f(\alpha_3) \mathbf{B_3}^T \mathbf{x} \tag{2.0.26}$$

Using this defined function, it can be verified that:

$$f(\alpha_1) = f(\alpha_3) \begin{pmatrix} 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.0.27}$$
$$\implies f(1, 0, 1) = f(\alpha_3) \begin{pmatrix} 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{2.0.28}$$
$$\implies f(1, 0, 1) = 0 \tag{2.0.29}$$

Similarly,

$$f(\alpha_2) = f(\alpha_3) \begin{pmatrix} 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.0.30}$$
$$\implies f(0, 1, -2) = f(\alpha_3) \begin{pmatrix} 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \tag{2.0.31}$$
$$\implies f(0, 1, -2) = 0 \tag{2.0.32}$$

and

$$f(\alpha_3) = f(\alpha_3) \begin{pmatrix} 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.0.33}$$
$$\implies f(-1, -1, 0) = f(\alpha_3) \begin{pmatrix} 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \tag{2.0.34}$$
$$\implies f(-1, -1, 0) \neq 0 \tag{2.0.35}$$

Hence, the above problem statement is verified.