Assignment 19

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Abstract

This is a simple document explaining how to express any matrix in square root form and how to express any matrix in Jordon form.

Download all and latex-tikz codes from

svn co https://github.com/gadepall/school/trunk/ncert/geometry/figs

1 Problem

Use the result of Exercise 15 (that is, if $\mathbf{A} = \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2$ then $\mathbf{A}^2 = \mathbf{I} + \mathbf{N}$) to prove that if c is a non-zero complex number and $\mathbf{N}$ is a nilpotent complex matrix, then $(c\mathbf{I} + \mathbf{N})$ has a square root. Now use the Jordon form to prove that every non-singular complex $n \times n$ matrix has a square root.

2 Solution

Given	If $\mathbf{A} = \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2$ then $\mathbf{A}^2 = \mathbf{I} + \mathbf{N}$ that is, $\mathbf{I} + \mathbf{N}$ has a square root, where $\mathbf{N}$ is a nilpotent matrix.
To prove	
1	$(c\mathbf{I} + \mathbf{N})$ has a square root,that is $(c\mathbf{I} + \mathbf{N}) = \mathbf{X}^2$ where c is a non-zero complex number, $\mathbf{N}$ is a nilpotent complex matrix and $\mathbf{X}$ is any matrix.
2	Every non-singular complex $n \times n$ matrix $(\mathbf{B})$ has a square root, that is, $\mathbf{B} = \mathbf{Y}^2$ where $\mathbf{Y}$ is any matrix.
Proof 1	Let us consider, $c \neq 0$ and $c \in \mathbf{C}$ then, there exists $\frac{1}{c} \in \mathbf{C}$ Given $\mathbf{N}$ is a nilpotent, $\implies \frac{1}{c}\mathbf{N}$ is also nilpotent. From the problem statement, we get that $(\mathbf{I} + \frac{1}{c}\mathbf{N})$ has a square root, that is, $(\mathbf{I} + \frac{1}{c}\mathbf{N}) = \mathbf{M}^2$ where $\mathbf{M}^2$ is any matrix. so, $(\mathbf{I} + \frac{1}{c}\mathbf{N}) = \mathbf{M}^2$ Multiplying both side with c, we get $c(\mathbf{I} + \frac{1}{c}\mathbf{N}) = c\mathbf{M}^2$ $(c\mathbf{I} + \mathbf{N}) = c\mathbf{M}^2$ $\implies (c\mathbf{I} + \mathbf{N}) = (\sqrt{c}\mathbf{M})^2$ $\implies (c\mathbf{I} + \mathbf{N}) = \mathbf{X}^2$ where $\mathbf{X} = (\sqrt{c}\mathbf{M})$

Conclusion	Hence it is proved that $(c\mathbf{I}+\mathbf{N})$ has a square root
Proof 2	Let **B** is any non-singular complex matrix. So, $\mathbf{B}\mathbf{B}^{-1} = \mathbf{I}$ As per the Jordon's Theorem, every square matrix B is similar to a Jordon matrix **J**, that is, $\mathbf{B} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$ Now, let consider a $n \times n$ nilpotent shift matrix **N** $\mathbf{N} = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}$ **I** is a $n \times n$ identity matrix, so $c\mathbf{I}+\mathbf{N} = \begin{pmatrix} c & 1 & 0 & ... & 0 \\ 0 & c & 1 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & 1 \\ 0 & 0 & 0 & ... & c \end{pmatrix}$ and this is a Jordon form. So, we can consider $\mathbf{J} = c\mathbf{I}+\mathbf{N}$ and as $c\mathbf{I}+\mathbf{N} = \mathbf{X}^2 \implies \mathbf{J} = \mathbf{X}^2$ $\implies \mathbf{B} = \mathbf{P}\mathbf{X}^2\mathbf{P}^{-1}$ $\implies \mathbf{B} = \mathbf{P}\mathbf{X}\mathbf{P}^{-1}\mathbf{P}\mathbf{X}\mathbf{P}^{-1}$ $\implies \mathbf{B} = (\mathbf{P}\mathbf{X}\mathbf{P}^{-1})^2$ $\implies \mathbf{B} = \mathbf{Y}^2$, where $\mathbf{Y} = \mathbf{P}\mathbf{X}\mathbf{P}^{-1}$ This implies that **B** has a square root.
Conclusion	Hence it is proved that every non-singular complex $n \times n$ matrix has a square root.

TABLE 0: **Solution summary**