Assignment 19

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Abstract

This is a simple document explaining how to express any matrix in square root form and how to express any matrix in Jordon form.

Download all and latex-tikz codes from

svn co https://github.com/gadepall/school/trunk/ncert/geometry/figs

1 Problem

Use the result of Exercise 15 (that is, if $\mathbf{A} = \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2$ then $\mathbf{A}^2 = \mathbf{I} + \mathbf{N}$) to prove that if c is a non-zero complex number and $\mathbf{N}$ is a nilpotent complex matrix, then $(c\mathbf{I} + \mathbf{N})$ has a square root. Now use the Jordon form to prove that every non-singular complex $n \times n$ matrix has a square root.

2 Solution

Given	If $\mathbf{A} = \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2$ then $\mathbf{A}^2 = \mathbf{I} + \mathbf{N}$ that is, $\mathbf{I} + \mathbf{N}$ has a square root, where $\mathbf{N}$ is a nilpotent matrix.
To prove	
1	$(c\mathbf{I} + \mathbf{N})$ has a square root,that is $(c\mathbf{I} + \mathbf{N}) = \mathbf{X}^2$ where c is a non-zero complex number, $\mathbf{N}$ is a nilpotent complex matrix and $\mathbf{X}$ is any matrix.
2	Every non-singular complex $n \times n$ matrix $(\mathbf{B})$ has a square root, that is, $\mathbf{B} = \mathbf{Y}^2$ where $\mathbf{Y}$ is any matrix.
Proof 1	As given in the problem statement, $\mathbf{A}^2 = \mathbf{I} + \mathbf{N}$ $\implies \mathbf{A}^2 = c\mathbf{I} - c\mathbf{I} + \mathbf{I} + \mathbf{N}$ $\implies \mathbf{A}^2 + c\mathbf{I} - \mathbf{I} = c\mathbf{I} + \mathbf{N}$ $\implies \mathbf{A}^2 + (c-1)\mathbf{I} = c\mathbf{I} + \mathbf{N}$
Conclusion	
Proof 2	Let $\mathbf{B}$ is any non-singular complex matrix. So, $\mathbf{B}\mathbf{B}^{-1} = \mathbf{I}$ As per the Jordon's Theorem, every square matrix B is similar to a Jordon matrix $\mathbf{J}$, that is, $\mathbf{B} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$ Now, let consider a $n \times n$ nilpotent shift matrix $\mathbf{N}$ $\mathbf{N} =$

	$\begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}$ $\mathbf{I}$ is a $n \times n$ identity matrix, so $c\mathbf{I} + \mathbf{N} =$ $\begin{pmatrix} c & 1 & 0 & ... & 0 \\ 0 & c & 1 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & 1 \\ 0 & 0 & 0 & ... & c \end{pmatrix}$ and this is a Jordon form. So, we can consider $\mathbf{J} = c\mathbf{I} + \mathbf{N}$ and as $c\mathbf{I} + \mathbf{N} = \mathbf{X}^2$ $\implies \mathbf{B} = \mathbf{P}\mathbf{X}^2\mathbf{P}^{-1}$ $\implies \mathbf{B} = \mathbf{P}\mathbf{X}\mathbf{P}^{-1}\mathbf{P}\mathbf{X}\mathbf{P}^{-1}$ $\implies \mathbf{B} = (\mathbf{P}\mathbf{X}\mathbf{P}^{-1})^2$ $\implies \mathbf{B} = \mathbf{Y}^2$, where $\mathbf{Y} = \mathbf{P}\mathbf{X}\mathbf{P}^{-1}$ This implies that $\mathbf{B}$ has a square root.
Conclusion	Hence it is proved that every non-singular complex $n \times n$ matrix has a square root.

TABLE 0: **Solution summary**