Assignment 5

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***Abstract*—This is a simple document explaining how to prove the congruence of two triangles.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

$\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and the vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
a) $\triangle ABD \cong \triangle ACD$
b) $\triangle ABP \cong \triangle ACP$
c) AP bisects $\angle A$ as well as $\angle D$
d) AP is the parpendicular bisector of BC

2 Explanation

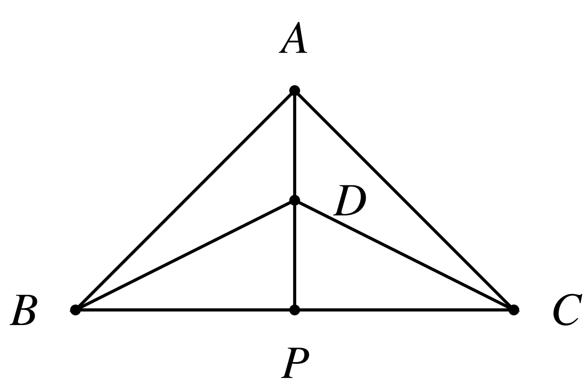

Fig. 0: Iso-sceles Triangles by Latex-Tikz

The above problem statement is depicted in the figure 0 where the vertices are: A, B and C for $\triangle ABC$ and D, B and C for $\triangle DBC$. For $\triangle ABC$ the sides AB, BC and CA are represented by the vectors $\mathbf{A}-\mathbf{B}$, $\mathbf{B}-\mathbf{C}$ and $\mathbf{C}-\mathbf{A}$ and for $\triangle DBC$ the sides DB, BC and CD are represented by $\mathbf{D}-\mathbf{B}$, $\mathbf{B}-\mathbf{C}$ and $\mathbf{C}-\mathbf{D}$.

From the problem statement we get that:

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \tag{2.0.1}$$

$$\|\mathbf{D}-\mathbf{B}\| = \|\mathbf{D}-\mathbf{C}\| \tag{2.0.2}$$

$$\mathbf{P}-\mathbf{C} = K_2(\mathbf{B}-\mathbf{C}) \tag{2.0.3}$$

$$\mathbf{P}-\mathbf{B} = K_1(\mathbf{B}-\mathbf{C}) \tag{2.0.4}$$

As the $\triangle ABC$ is an iso-scelen triangle,

$$\angle ABC = \angle ACB = \gamma \tag{2.0.5}$$

Similarly, as the $\triangle DBC$ is an iso-scelen triangle,

$$\angle DBC = \angle DCB \tag{2.0.6}$$

Let consider $\angle APB = \phi_1$, $\angle APC = \phi_2$ and $\phi_1 + \phi_2 = 180\ ^\circ$. From the triangular law of vector addition, we can also get that:

$$\mathbf{A}-\mathbf{B} = (\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{B}) \tag{2.0.7}$$

$$\mathbf{A}-\mathbf{C} = (\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{C}) \tag{2.0.8}$$

$$\mathbf{D}-\mathbf{B} = (\mathbf{D}-\mathbf{P}) + (\mathbf{P}-\mathbf{B}) \tag{2.0.9}$$

$$\mathbf{D}-\mathbf{C} = (\mathbf{D}-\mathbf{P}) + (\mathbf{P}-\mathbf{C}) \tag{2.0.10}$$

Now squaring both side of equation 2,

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \tag{2.0.11}$$

$$\text{or, } \|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}-\mathbf{C}\|^2 \tag{2.0.12}$$

$$\text{or, } \|(\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{B})\|^2 = \|(\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{C})\|^2 \tag{2.0.13}$$

$$\text{or,} ((\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{B}))^T((\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{B})) = \tag{2.0.14}$$

$$((\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{C}))^T((\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{C})) \tag{2.0.15}$$

$$\text{or,} ((\mathbf{A}-\mathbf{P})^T + K_1(\mathbf{B}-\mathbf{C})^T)((\mathbf{A}-\mathbf{P}) + K_1(\mathbf{B}-\mathbf{C})) = \tag{2.0.16}$$

$$((\mathbf{A}-\mathbf{P})^T + K_2(\mathbf{B}-\mathbf{C})^T)((\mathbf{A}-\mathbf{P}) + K_2(\mathbf{B}-\mathbf{C})) \tag{2.0.17}$$

$$\text{or,} (K_1 - K_2)((\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{P}) \tag{2.0.18}$$

$$+(\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{C}) + (K_1 + K_2)\,\|\mathbf{B}-\mathbf{C}\|^2) = 0 \tag{2.0.19}$$

So, there are 2 cases, either $(K_1 - K_2) = 0$ or

$$((\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{P}) \tag{2.0.20}$$

$$+(\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{C}) + (K_1 + K_2)\,\|\mathbf{B}-\mathbf{C}\|^2) = 0 \tag{2.0.21}$$

$$\text{or,} (K_1 + K_2)\,\|\mathbf{B}-\mathbf{C}\|^2 = -(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{P}) \tag{2.0.22}$$

$$-(\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{C}) \tag{2.0.23}$$

$$\text{or,} (K_1 + K_2)\frac{\|\mathbf{B}-\mathbf{C}\|}{\|\mathbf{A}-\mathbf{P}\|} = \frac{-(\mathbf{P}-\mathbf{B})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{P}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.24}$$

$$-\frac{(\mathbf{P}-\mathbf{C})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{P}-\mathbf{C}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.25}$$

$$\text{or,} (K_1 + K_2)\frac{\|\mathbf{B}-\mathbf{C}\|}{\|\mathbf{A}-\mathbf{P}\|} = -\cos\phi_1 - \cos\phi_2 \tag{2.0.26}$$

$$\text{or,} (K_1 + K_2)\frac{\|\mathbf{B}-\mathbf{C}\|}{\|\mathbf{A}-\mathbf{P}\|} = -\cos\phi_1 + \cos\phi_1 \tag{2.0.27}$$

$$\text{or,} (K_1 + K_2)\frac{\|\mathbf{B}-\mathbf{C}\|}{\|\mathbf{A}-\mathbf{P}\|} = 0 \tag{2.0.28}$$

$$\implies K_1 = -K_2 \tag{2.0.29}$$

So, for the first case, that is, when $K_1 = K_2$:

$$\cos\theta_1 = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.30}$$

$$\text{or, } \cos\theta_1 = \frac{((\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{B}))^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.31}$$

Similarly,

$$\cos\theta_2 = \frac{(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{C}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.32}$$

$$\text{or, } \cos\theta_2 = \frac{((\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{C}))^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.33}$$

$$\text{or, } \cos\theta_2 = \frac{((\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{B}))^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.34}$$

So, we can say that, $\theta_1 = \theta_2$ Now for $\triangle DBC$,

$$\cos\alpha = \frac{(\mathbf{D}-\mathbf{B})^T(\mathbf{D}-\mathbf{P})}{||\mathbf{D}-\mathbf{B}||\,||\mathbf{D}-\mathbf{P}||} \tag{2.0.35}$$

$$\text{or,} \cos\alpha = \frac{((\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{B}))^T(\mathbf{D}-\mathbf{P})}{||\mathbf{D}-\mathbf{B}||\,||\mathbf{D}-\mathbf{P}||} \tag{2.0.36}$$

Similarly,

$$\cos\beta = \frac{(\mathbf{D}-\mathbf{C})^T(\mathbf{D}-\mathbf{P})}{||\mathbf{D}-\mathbf{C}||\,||\mathbf{D}-\mathbf{P}||} \tag{2.0.37}$$

$$\text{or,} \cos\beta = \frac{((\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{C}))^T(\mathbf{D}-\mathbf{P})}{||\mathbf{D}-\mathbf{B}||\,||\mathbf{D}-\mathbf{P}||} \tag{2.0.38}$$

$$\text{or,} \cos\beta = \frac{((\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{B}))^T(\mathbf{D}-\mathbf{P})}{||\mathbf{D}-\mathbf{B}||\,||\mathbf{D}-\mathbf{P}||} \tag{2.0.39}$$

So, we can conclude that $\alpha = \beta$. These imply that $\mathbf{A}-\mathbf{P}$ bisects $\angle A$ as well as $\angle D$. For $K_1 = K_2$, $\mathbf{P}-\mathbf{C} = \mathbf{P}-\mathbf{B}$

$$\cos\phi_1 = \frac{(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{B})}{||\mathbf{P}-\mathbf{B}||\,||\mathbf{A}-\mathbf{P}||} \tag{2.0.40}$$

$$\text{or,} \cos\phi_1 = \frac{((\mathbf{A}-\mathbf{B})+(\mathbf{B}-\mathbf{P}))^T(\mathbf{P}-\mathbf{B})}{||\mathbf{P}-\mathbf{B}||\,||\mathbf{A}-\mathbf{P}||} \tag{2.0.41}$$

$$\text{or,} \cos\phi_1 = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{P}-\mathbf{B})+(\mathbf{B}-\mathbf{P})^T(\mathbf{P}-\mathbf{B})}{||\mathbf{P}-\mathbf{B}||\,||\mathbf{A}-\mathbf{P}||} \tag{2.0.42}$$

$$\text{or,} \cos\phi_1 = \cos\gamma - \frac{||\mathbf{B}-\mathbf{P}||}{||\mathbf{A}-\mathbf{P}||} \tag{2.0.43}$$

$$\tag{2.0.44}$$

Similarly,

$$\cos\phi_2 = \cos\gamma - \frac{||\mathbf{P}-\mathbf{C}||}{||\mathbf{A}-\mathbf{P}||} \tag{2.0.45}$$

$$\text{or,} \cos\phi_2 = \cos\gamma - \frac{||\mathbf{P}-\mathbf{B}||}{||\mathbf{A}-\mathbf{P}||} \tag{2.0.46}$$

$$\text{or,} \cos\phi_2 = \cos\gamma - \frac{||\mathbf{B}-\mathbf{P}||}{||\mathbf{A}-\mathbf{P}||} \tag{2.0.47}$$

$$\tag{2.0.48}$$

From here it can be concluded

$$\cos\phi_1 = \cos\phi_2 \tag{2.0.49}$$

$$\text{or,}\phi_1 = \phi_2 \tag{2.0.50}$$

$$\text{or,}\phi_1 + \phi_2 = 180^\circ \tag{2.0.51}$$

$$\implies \phi_1 = 90^\circ \tag{2.0.52}$$

For the second case, that is, when $K_1 = -K_2$, $\mathbf{P}-\mathbf{C} = \mathbf{B}-\mathbf{P}$

$$\cos\phi_1 = \cos\gamma - \frac{||\mathbf{B}-\mathbf{P}||}{||\mathbf{A}-\mathbf{P}||} \tag{2.0.53}$$

$$\cos\phi_2 = \cos\gamma - \frac{||\mathbf{P}-\mathbf{C}||}{||\mathbf{A}-\mathbf{P}||} \tag{2.0.54}$$

$$\implies \cos\phi_2 = \cos\gamma - \frac{||\mathbf{B}-\mathbf{P}||}{||\mathbf{A}-\mathbf{P}||} \tag{2.0.55}$$

$$\tag{2.0.56}$$

Which also implies that

$$\cos\phi_1 = \cos\phi_2 \tag{2.0.57}$$

$$\text{or,}\phi_1 = \phi_2 \tag{2.0.58}$$

$$\text{or,}\phi_1 + \phi_2 = 180^\circ \tag{2.0.59}$$

$$\implies \phi_1 = 90^\circ \tag{2.0.60}$$

Hence, it is proved that AP bisects BC perpendicularly.