Assignment 5

Jayati Dutta

***Abstract*—This is a simple document explaining how to prove the congruence of two triangles.**

Download all and latex-tikz codes from

svn co https://github.com/gadepall/school/trunk/ncert/geometry/figs

1 Problem

$\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and the vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
a) $\triangle ABD \cong \triangle ACD$
b) $\triangle ABP \cong \triangle ACP$
c) AP bisects $\angle A$ as well as $\angle D$
d) AP is the parpendicular bisector of BC

2 Explanation

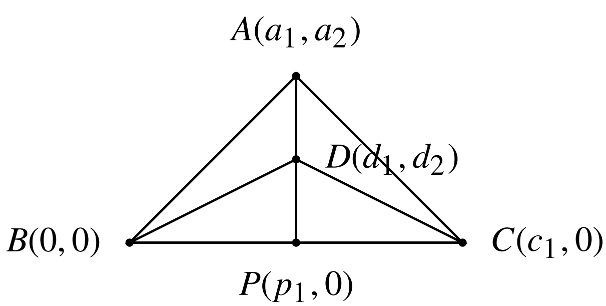

Fig. 0: Iso-sceles Triangles by Latex-Tikz

The above problem statement is depicted in the figure 0 where the vertices are: $\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} p_1 \\ 0 \end{pmatrix}$

From the problem statement we get that:

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{A} - \mathbf{C}\| \implies \|\mathbf{AB}\| = \|\mathbf{AC}\| \quad (2.0.1)$$

$$\|\mathbf{D} - \mathbf{B}\| = \|\mathbf{D} - \mathbf{C}\| \implies \|\mathbf{DB}\| = \|\mathbf{DC}\| \quad (2.0.2)$$

Now, let $\mathbf{P} = \mathrm{k}\mathbf{C}$ as $\mathbf{P}$ is on $\mathbf{BC}$. So,

$$\begin{pmatrix} p_1 \\ 0 \end{pmatrix} = k \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \implies p_1 = kc_1 \quad (2.0.3)$$

Now, squaring both side of the equation 2,

$$\|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A} - \mathbf{C}\|^2 \quad (2.0.4)$$
$$\implies a_1^2 + a_2^2 = (a_1 - c_1)^2 + a_2^2 \quad (2.0.5)$$
$$\implies c_1^2 = 2a_1c_1 \quad (2.0.6)$$
$$\implies c_1(c_1 - 2a_1) = 0 \quad (2.0.7)$$

So, either $c_1 = 0$ or $c_1 = 2a_1$. As $c_1 = 0$ is not possible, so $c_1 = 2a_1$.

Similarly, squaring both side of the equation 2,

$$\|\mathbf{D} - \mathbf{B}\|^2 = \|\mathbf{D} - \mathbf{C}\|^2 \quad (2.0.8)$$
$$\implies d_1^2 + d_2^2 = (d_1 - c_1)^2 + d_2^2 \quad (2.0.9)$$
$$\implies c_1^2 = 2d_1c_1 \quad (2.0.10)$$
$$\implies c_1(c_1 - 2d_1) = 0 \quad (2.0.11)$$

So, either $c_1 = 0$ or $c_1 = 2d_1$. As $c_1 = 0$ is not possible, so $c_1 = 2d_1$.

So, it is observed that $d_1 = a_1$. So we can say that $\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} a_1 \\ d_2 \end{pmatrix}$ and as $\mathbf{AD}$ is extended to intersect $\mathbf{BC}$ at $\mathbf{P}$, so $\mathbf{P} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$.

Now, $\|\mathbf{BP}\| = p_1 = a_1$ and $\|\mathbf{CP}\| = \sqrt{(2a_1 - a_1)^2} = a_1$.

For $\triangle DBC$ being iso-scelen, $\angle DBC = \angle DCB$.

$$\angle DBC = \angle DCB \quad (2.0.12)$$
$$\implies \frac{\mathbf{CD}^T\mathbf{CP}}{\|\mathbf{CD}\|\,\|\mathbf{CP}\|} = \frac{\mathbf{BD}^T\mathbf{BP}}{\|\mathbf{BD}\|\,\|\mathbf{BP}\|} \quad (2.0.13)$$
$$\implies k\,\|\mathbf{CP}\| = (1-k)\,\|\mathbf{BP}\| \quad (2.0.14)$$
$$\implies k \times a_1 = (1-k) \times a - 1 \quad (2.0.15)$$
$$\implies k = \frac{1}{2} \quad (2.0.16)$$

Now, let the angle between $\mathbf{AB}$ and $\mathbf{AP}$ is θ_1 and the angle between $\mathbf{AP}$ and $\mathbf{AC}$ is θ_2.

$$\cos\theta_1 = \frac{\mathbf{AB}^T\mathbf{AP}}{\|\mathbf{AB}\|\,\|\mathbf{AP}\|} \tag{2.0.17}$$

$$\implies \cos\theta_1 = \frac{a_1^2 - ka_1c_1 + a_2^2}{\|\mathbf{AB}\|\,\|\mathbf{AP}\|} \tag{2.0.18}$$

$$\implies \cos\theta_1 = \frac{a_1^2 - 2ka_1^2 + a_2^2}{\|\mathbf{AC}\|\,\|\mathbf{AP}\|} \tag{2.0.19}$$

Now, putting the value of k = $\frac{1}{2}$,

$$\cos\theta_1 = \frac{a_2^2}{\|\mathbf{AC}\|\,\|\mathbf{AP}\|} \tag{2.0.20}$$

Similarly,

$$\cos\theta_2 = \frac{\mathbf{AC}^T\mathbf{AP}}{\|\mathbf{AC}\|\,\|\mathbf{AP}\|} \tag{2.0.21}$$

$$\implies \cos\theta_2 = \frac{-a_1^2 + 2ka_1^2 + a_2^2}{\|\mathbf{AC}\|\,\|\mathbf{AP}\|} \tag{2.0.22}$$

And now, putting the value of k = $\frac{1}{2}$,

$$\cos\theta_2 = \frac{a_2^2}{\|\mathbf{AC}\|\,\|\mathbf{AP}\|} \tag{2.0.23}$$

So, it can be easily observed that $\theta_1 = \theta_2$ (considering the principle value) and this proves that $\mathbf{AP}$ bisects $\angle A$.

Now, let consider the angle between $\mathbf{DP}$ and $\mathbf{BP}$ is ϕ.

$$\cos\phi = \frac{\mathbf{PD}^T\mathbf{PB}}{\|\mathbf{PD}\|\,\|\mathbf{PB}\|} \tag{2.0.24}$$

$$\implies \cos\phi = \frac{p_1(p_1 - d_1)}{\|\mathbf{PD}\|\,\|\mathbf{PB}\|} \tag{2.0.25}$$

$$\implies \cos\phi = \frac{p_1(2kd_1 - d_1)}{\|\mathbf{PD}\|\,\|\mathbf{PB}\|} \tag{2.0.26}$$

$$\implies \cos\phi = \frac{p_1(d_1 - d_1)}{\|\mathbf{PD}\|\,\|\mathbf{PB}\|} \tag{2.0.27}$$

$$\implies \cos\phi = 0 \tag{2.0.28}$$

$$\implies \phi = 90^\circ \tag{2.0.29}$$

so, we can say that $\mathbf{DP}$ is perpendicular to $\mathbf{BC}$ and this indicates that $\mathbf{AP}$ is the perpendicular bisector of $\mathbf{BC}$.