Assignment 5

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***Abstract*—This is a simple document explaining how to prove the congruence of two triangles.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

$\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and the vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
a) $\triangle ABD \cong \triangle ACD$
b) $\triangle ABP \cong \triangle ACP$
c) AP bisects $\angle A$ as well as $\angle D$
d) AP is the parpendicular bisector of BC

2 Explanation

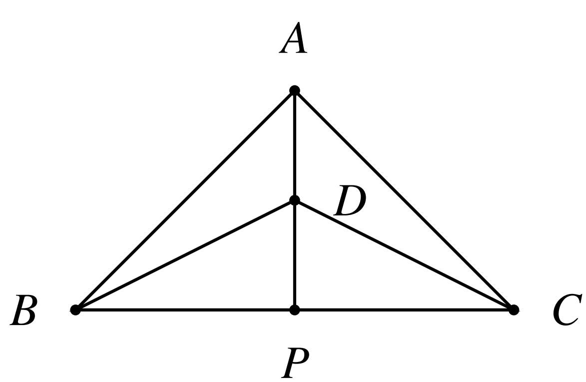

Fig. 0: Iso-sceles Triangles by Latex-Tikz

The above problem statement is depicted in the figure 0 where the vertices are: A, B and C for $\triangle ABC$ and D, B and C for $\triangle DBC$. For $\triangle ABC$ the sides AB, BC and CA are represented by the vectors $\mathbf{A}-\mathbf{B}$, $\mathbf{B}-\mathbf{C}$ and $\mathbf{C}-\mathbf{A}$ and for $\triangle DBC$ the sides DB, BC and CD are represented by $\mathbf{D}-\mathbf{B}$, $\mathbf{B}-\mathbf{C}$ and $\mathbf{C}-\mathbf{D}$.

From the problem statement we get that:

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \tag{2.0.1}$$

$$\|\mathbf{D}-\mathbf{B}\| = \|\mathbf{D}-\mathbf{C}\| \tag{2.0.2}$$

$$\mathbf{P}-\mathbf{C} = K_2(\mathbf{B}-\mathbf{C}) \tag{2.0.3}$$

$$\mathbf{P}-\mathbf{B} = K_1(\mathbf{B}-\mathbf{C}) \tag{2.0.4}$$

$$\begin{aligned}
&\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \\
&\implies \|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}-\mathbf{C}\|^2 \\
&\implies \|(\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B})\|^2 = \|(\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{C})\|^2 \\
&\implies \|\mathbf{A}-\mathbf{P}\|^2 + \|\mathbf{P}-\mathbf{B}\|^2 + 2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}) = \\
&\quad \|\mathbf{A}-\mathbf{P}\|^2 + \|\mathbf{P}-\mathbf{C}\|^2 + 2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{C}) \\
&\implies \|\mathbf{P}-\mathbf{B}\|^2 + 2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}) = \\
&\quad \|\mathbf{P}-\mathbf{C}\|^2 + 2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{C}) \\
&\implies \|\mathbf{P}-\mathbf{B}\|^2 - \|\mathbf{P}-\mathbf{C}\|^2 \\
&\quad + 2(\mathbf{A}-\mathbf{P})^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) = 0 \\
&\implies ((\mathbf{P}-\mathbf{B})+(\mathbf{P}-\mathbf{C}))^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) \\
&\quad + 2(\mathbf{A}-\mathbf{P})^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) = 0 \\
&\implies (\mathbf{B}+\mathbf{C}-2\mathbf{P})^T(\mathbf{B}-\mathbf{C}) \\
&\quad + 2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}-\mathbf{P}+\mathbf{C}) = 0 \\
&\implies (\mathbf{B}+\mathbf{C}-2\mathbf{P})^T(\mathbf{B}-\mathbf{C}) + 2(\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies ((\mathbf{B}+\mathbf{C}-2\mathbf{P})^T - 2(\mathbf{A}-\mathbf{P})^T)(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies ((\mathbf{B}+\mathbf{C}-2\mathbf{P}) - (\mathbf{2A}-\mathbf{2P}))^T(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies (\mathbf{B}+\mathbf{C}-\mathbf{2A})^T(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies (\mathbf{B}-\mathbf{C})^T(\mathbf{B}+\mathbf{C}-\mathbf{2A}) = 0
\end{aligned} \tag{2.0.5}$$

Now, from 2.0.5

$$\begin{aligned}
&(\mathbf{B}-\mathbf{C})^T\left(\left(\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}}\right)-\mathbf{A}\right) = 0 \\
&\implies \left(\left(\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}}\right)-\mathbf{A}\right) \perp (\mathbf{B}-\mathbf{C})
\end{aligned} \tag{2.0.6}$$

So, we can conclude that $\mathbf{A}-\mathbf{P}$ bisects $\mathbf{B}-\mathbf{C}$

perpendicularly. Similarly,

$$
\begin{aligned}
&\|\mathbf{D}-\mathbf{B}\| = \|\mathbf{D}-\mathbf{C}\| \\
&\implies \|\mathbf{D}-\mathbf{B}\|^2 = \|\mathbf{D}-\mathbf{C}\|^2 \\
&\implies \|(\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{B})\|^2 = \|(\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{C})\|^2 \\
&\implies \|\mathbf{D}-\mathbf{P}\|^2 + \|\mathbf{P}-\mathbf{B}\|^2 + 2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}) = \\
&\|\mathbf{D}-\mathbf{P}\|^2 + \|\mathbf{P}-\mathbf{C}\|^2 + 2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{C}) \\
&\implies \|\mathbf{P}-\mathbf{B}\|^2 + 2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}) = \\
&\|\mathbf{P}-\mathbf{C}\|^2 + 2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{C}) \\
&\implies \|\mathbf{P}-\mathbf{B}\|^2 - \|\mathbf{P}-\mathbf{C}\|^2 \\
&+ 2(\mathbf{D}-\mathbf{P})^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) = 0 \\
&\implies ((\mathbf{P}-\mathbf{B})+(\mathbf{P}-\mathbf{C}))^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) \\
&+ 2(\mathbf{D}-\mathbf{P})^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) = 0 \\
&\implies (\mathbf{B}+\mathbf{C}-2\mathbf{P})^T(\mathbf{B}-\mathbf{C}) \\
&+ 2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}-\mathbf{P}+\mathbf{C}) = 0 \\
&\implies (\mathbf{B}+\mathbf{C}-2\mathbf{P})^T(\mathbf{B}-\mathbf{C}) + 2(\mathbf{D}-\mathbf{P})^T(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies ((\mathbf{B}+\mathbf{C}-2\mathbf{P})^T - 2(\mathbf{D}-\mathbf{P})^T)(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies ((\mathbf{B}+\mathbf{C}-2\mathbf{P}) - (\mathbf{2D}-\mathbf{2P}))^T(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies (\mathbf{B}+\mathbf{C}-\mathbf{2D})^T(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies (\mathbf{B}-\mathbf{C})^T(\mathbf{B}+\mathbf{C}-\mathbf{2D}) = 0
\end{aligned} \tag{2.0.7}
$$

Similarly, 2.0.7 dividing by 2, we get:

$$
\begin{aligned}
&(\mathbf{B}-\mathbf{C})^T\left(\left(\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}}\right)-\mathbf{D}\right) = 0 \\
&\implies \left(\left(\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}}\right)-\mathbf{D}\right) \perp (\mathbf{B}-\mathbf{C})
\end{aligned} \tag{2.0.8}
$$

So, we can conclude that $\mathbf{D}-\mathbf{P}$ bisects $\mathbf{B}-\mathbf{C}$ perpendicularly.