Assignment 5

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Abstract—**This is a simple document explaining how to prove the congruence of two triangles.**

Download all and latex-tikz codes from

svn co https://github.com/gadepall/school/trunk/ncert/geometry/figs

1 Problem

$\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and the vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
a) $\triangle ABD \cong \triangle ACD$
b) $\triangle ABP \cong \triangle ACP$
c) AP bisects $\angle A$ as well as $\angle D$
d) AP is the parpendicular bisector of BC

2 Explanation

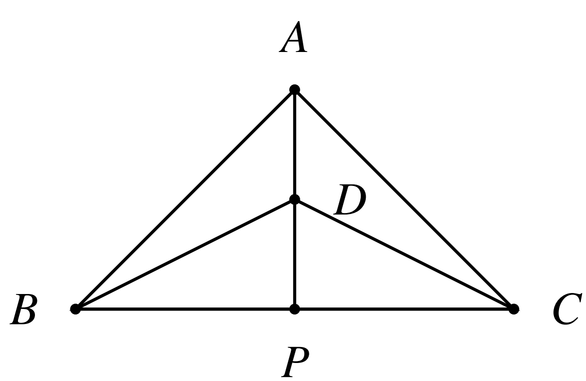

Fig. 0: Iso-sceles Triangles by Latex-Tikz

The above problem statement is depicted in the figure 0 where the vertices are: A, B and C for $\triangle ABC$ and D, B and C for $\triangle DBC$. For $\triangle ABC$ the sides AB, BC and CA are represented by the vectors $\mathbf{A} - \mathbf{B}$, $\mathbf{B} - \mathbf{C}$ and $\mathbf{C} - \mathbf{A}$.

From the problem statement we get that:

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{A} - \mathbf{C}\| \tag{2.0.1}$$

$$\|\mathbf{D} - \mathbf{B}\| = \|\mathbf{D} - \mathbf{C}\| \tag{2.0.2}$$

$$\mathbf{A} - \mathbf{D} = k_2 \times (\mathbf{A} - \mathbf{P}) \tag{2.0.3}$$

Now, let $\mathbf{B} - \mathbf{P} = \mathbf{P} = \mathrm{k} \times (\mathbf{B} - \mathbf{C})$

As for $\triangle DBC$, $\|\mathbf{D} - \mathbf{B}\| = \|\mathbf{D} - \mathbf{C}\|$, so we can say that $\angle DBP = \angle DCP$ and according to the triangular law for vector addition, $\mathbf{B} - \mathbf{D} = (\mathbf{B} - \mathbf{C}) + (\mathbf{C} - \mathbf{D})$

$$\angle DBP = \angle DCP \tag{2.0.4}$$

$$\implies \frac{(\mathbf{B} - \mathbf{D})^T \mathbf{P}}{\|\mathbf{B} - \mathbf{D}\| \, \|\mathbf{P}\|} = \tag{2.0.5}$$

$$\frac{(\mathbf{C} - \mathbf{D})^T ((\mathbf{B} - \mathbf{C}) - \mathbf{P})}{\|\mathbf{C} - \mathbf{D}\| \, \|((\mathbf{B} - \mathbf{C}) - \mathbf{P})\|} \tag{2.0.6}$$

$$\implies \frac{((\mathbf{B} - \mathbf{C}) + (\mathbf{C} - \mathbf{D}))^T \mathbf{P}}{k} = \tag{2.0.7}$$

$$\frac{(\mathbf{C} - \mathbf{D})^T ((\mathbf{B} - \mathbf{C}) - \mathbf{P})}{1 - k} \tag{2.0.8}$$

$$\implies (\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) + (\mathbf{C} - \mathbf{D})^T (\mathbf{B} - \mathbf{C}) = \tag{2.0.9}$$

$$(\mathbf{C} - \mathbf{D})^T (\mathbf{B} - \mathbf{C}) \tag{2.0.10}$$

$$\implies (\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{2.0.11}$$

$$\implies \|\mathbf{B} - \mathbf{C}\| = 0 \tag{2.0.12}$$

In a similar way, applying tangent law we can get that :

$$\angle DBP = \angle DCP \tag{2.0.13}$$

$$\implies \frac{\|(\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D})\|}{\|\mathbf{P}\|} = \frac{\|(\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D})\|}{\|(\mathbf{B}-\mathbf{C})-\mathbf{P}\|} \tag{2.0.14}$$

$$\implies k \times \|\mathbf{B}-\mathbf{C}\| = (1-k) \times \|\mathbf{B}-\mathbf{C}\| \tag{2.0.15}$$

$$\implies k = \frac{1}{2} \tag{2.0.16}$$

Let $\angle DPB = \phi$, so

$$\cos\phi = \frac{\|\mathbf{B}-\mathbf{P}\|}{\|\mathbf{B}-\mathbf{D}\|} \tag{2.0.17}$$

$$\implies \cos\phi = \frac{k\,\|\mathbf{B}-\mathbf{C}\|}{\|\mathbf{B}-\mathbf{D}\|} \tag{2.0.18}$$

$$\implies \cos\phi = 0 \tag{2.0.19}$$

$$\implies \phi = 90^\circ \tag{2.0.20}$$

But we know that

$$\cos\phi = \frac{\mathbf{P}^T((\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D}))}{\|\mathbf{P}\|\,\|((\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D}))\|} \tag{2.0.21}$$

$$\cos\phi = \frac{(\mathbf{B}-\mathbf{C})^T((\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D}))}{\|\mathbf{B}-\mathbf{C}\|\,\|((\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D}))\|} \tag{2.0.22}$$

Now, we can conclude that

$$\frac{(\mathbf{B}-\mathbf{C})^T((\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D}))}{\|\mathbf{B}-\mathbf{C}\|\,\|((\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D}))\|} = 0 \tag{2.0.23}$$

$$\implies (\mathbf{B}-\mathbf{C})^T((\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D})) = 0 \tag{2.0.24}$$

In a similar way,

$$\cos\phi = \frac{\mathbf{P}^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{P}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.25}$$

$$\implies \cos\phi = \frac{(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{B}-\mathbf{C}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.26}$$

$$\tag{2.0.27}$$

Now,

$$\frac{(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{B}-\mathbf{C}\|\,\|\mathbf{A}-\mathbf{P}\|} = 0 \tag{2.0.28}$$

$$\implies (\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{P}) = 0 \tag{2.0.29}$$

So we can say that, AP is a perpendicular bisector of BC as $\mathbf{P} = \frac{1}{2}(\mathbf{B}-\mathbf{C})$.

Now, let $\angle BAP = \theta_1$ and $\angle CAP = \theta_2$

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.30}$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.31}$$

$$\text{or,} \cos\theta_2 = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.32}$$

$$\tag{2.0.33}$$

According to the vector triangular law for $\triangle ABC$, $\mathbf{B}-\mathbf{A} = (\mathbf{B}-\mathbf{C}) + (\mathbf{C}-\mathbf{A})$

$$\cos\theta_1 = \frac{((\mathbf{B}-\mathbf{C})^T + (\mathbf{C}-\mathbf{A})^T)(\mathbf{A}-\mathbf{P})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.34}$$

$$\implies \cos\theta_1 = \frac{(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{P}\|} + \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.35}$$

$$\implies \cos\theta_1 = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.36}$$

$$\implies \cos\theta_1 = \cos\theta_2 \tag{2.0.37}$$

$$\implies \theta_1 = \theta_2 \tag{2.0.38}$$

Similarly, let $\angle BDP = \alpha$ and $\angle CDP = \beta$

$$\cos\alpha = -\frac{(\mathbf{B}-\mathbf{D})^T(\mathbf{P}-\mathbf{D})}{\|\mathbf{B}-\mathbf{D}\|\,\|\mathbf{P}-\mathbf{D}\|} \tag{2.0.39}$$

$$\text{or,} \cos\alpha = -\frac{(\mathbf{B}-\mathbf{D})^T((\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D}))}{\|\mathbf{B}-\mathbf{D}\|\,\|\mathbf{P}-\mathbf{D}\|} \tag{2.0.40}$$

$$\cos\beta = -\frac{(\mathbf{C}-\mathbf{D})^T(\mathbf{P}-\mathbf{D})}{\|\mathbf{C}-\mathbf{D}\|\,\|\mathbf{P}-\mathbf{D}\|} \tag{2.0.41}$$

$$\text{or,} \cos\beta = -\frac{(\mathbf{C}-\mathbf{D})^T((\mathbf{A}-\mathbf{P})-(\mathbf{A}-\mathbf{D}))}{\|\mathbf{B}-\mathbf{D}\|\,\|\mathbf{P}-\mathbf{D}\|} \tag{2.0.42}$$

According to the vector triangular law for $\triangle DBC$,

$$\mathbf{B}-\mathbf{D} = (\mathbf{B}-\mathbf{C}) + (\mathbf{C}-\mathbf{D})$$

$$\cos\alpha = \frac{((\mathbf{B}-\mathbf{C})^T + (\mathbf{C}-\mathbf{D})^T)((\mathbf{A}-\mathbf{D}) - (\mathbf{A}-\mathbf{P}))}{\|\mathbf{B}-\mathbf{D}\|\,\|\mathbf{P}-\mathbf{D}\|} \tag{2.0.43}$$

$$\implies \cos\alpha = \frac{(\mathbf{B}-\mathbf{C})^T((\mathbf{A}-\mathbf{D}) - (\mathbf{A}-\mathbf{P}))}{\|\mathbf{B}-\mathbf{D}\|\,\|\mathbf{P}-\mathbf{D}\|} \tag{2.0.44}$$

$$+\frac{(\mathbf{C}-\mathbf{D})^T((\mathbf{A}-\mathbf{D}) - (\mathbf{A}-\mathbf{P}))}{\|\mathbf{B}-\mathbf{D}\|\,\|\mathbf{P}-\mathbf{D}\|} \tag{2.0.45}$$

$$\implies \cos\alpha = -\frac{(\mathbf{C}-\mathbf{D})^T((\mathbf{A}-\mathbf{P}) - (\mathbf{A}-\mathbf{D}))}{\|\mathbf{B}-\mathbf{D}\|\,\|\mathbf{P}-\mathbf{D}\|} \tag{2.0.46}$$

$$\implies \cos\alpha = \cos\beta \tag{2.0.47}$$

$$\implies \alpha = \beta \tag{2.0.48}$$

So, we can conclude that AP bisects $\angle A$ as well as $\angle D$.