Assignment 5

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Abstract—**This is a simple document explaining how to prove the congruence of two triangles.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

$\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and the vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
a) $\triangle ABD \cong \triangle ACD$
b) $\triangle ABP \cong \triangle ACP$
c) AP bisects $\angle A$ as well as $\angle D$
d) AP is the parpendicular bisector of BC

2 Explanation

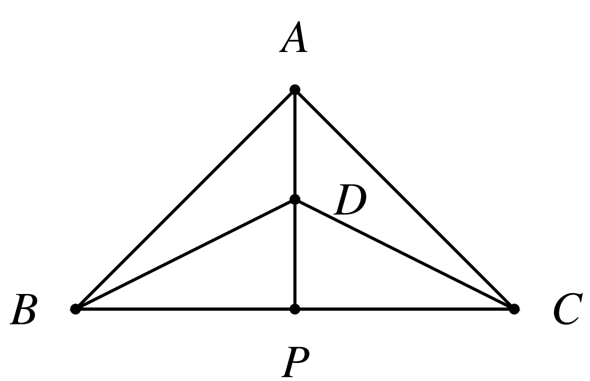

Fig. 0: Iso-sceles Triangles by Latex-Tikz

The above problem statement is depicted in the figure 0 where the vertices are: A, B and C for $\triangle ABC$ and D, B and C for $\triangle DBC$. For $\triangle ABC$ the sides AB, BC and CA are represented by the vectors **AB** , **BC** and **CA**.

From the problem statement we get that:

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \implies \|\mathbf{AB}\| = \|\mathbf{AC}\| \tag{2.0.1}$$

$$\|\mathbf{D}-\mathbf{B}\| = \|\mathbf{D}-\mathbf{C}\| \implies \|\mathbf{DB}\| = \|\mathbf{DC}\| \tag{2.0.2}$$

$$\mathbf{AD} = k_2 \times \mathbf{AP} \tag{2.0.3}$$

Now, let **BP**= **P** = k ×**BC**

As for $\triangle DBC$, $\|\mathbf{DB}\| = \|\mathbf{DC}\|$, so we can say that $\angle DBP = \angle DCP$ and according to the triangular law for vector addition, **BD** = **BC** + **CD**

$$\angle DBP = \angle DCP \tag{2.0.4}$$
$$\implies \frac{\mathbf{BD}^T\mathbf{P}}{\|\mathbf{BD}\|\,\|\mathbf{P}\|} = \frac{\mathbf{CD}^T(\mathbf{BC}-\mathbf{P})}{\|\mathbf{CD}\|\,\|(\mathbf{BC}-\mathbf{P})\|} \tag{2.0.5}$$
$$\implies \frac{(\mathbf{BC}+\mathbf{CD})^T\mathbf{P}}{k} = \frac{\mathbf{CD}^T(\mathbf{BC}-\mathbf{P})}{1-k} \tag{2.0.6}$$
$$\implies \mathbf{BC}^T\mathbf{BC} + \mathbf{CD}^T\mathbf{BC} = \mathbf{CD}^T\mathbf{BC} \tag{2.0.7}$$
$$\implies \mathbf{BC}^T\mathbf{BC} = 0 \tag{2.0.8}$$
$$\implies \|\mathbf{BC}\| = 0 \tag{2.0.9}$$

In a similar way, applying tangent law we can get that :

$$\angle DBP = \angle DCP \tag{2.0.10}$$
$$\implies \frac{\|\mathbf{AP}-\mathbf{AD}\|}{\|\mathbf{P}\|} = \frac{\|\mathbf{AP}-\mathbf{AD}\|}{\|\mathbf{BC}-\mathbf{P}\|} \tag{2.0.11}$$
$$\implies k \times \|\mathbf{BC}\| = (1-k) \times \|\mathbf{BC}\| \tag{2.0.12}$$
$$\implies k = \frac{1}{2} \tag{2.0.13}$$

Let $\angle DPB = \phi$, so

$$\cos\phi = \frac{\|\mathbf{BP}\|}{\|\mathbf{BD}\|} \tag{2.0.14}$$
$$\implies \cos\phi = \frac{k\,\|\mathbf{BC}\|}{\|\mathbf{BD}\|} \tag{2.0.15}$$
$$\implies \cos\phi = 0 \tag{2.0.16}$$
$$\implies \phi = 90^\circ \tag{2.0.17}$$

But we know that

$$\cos\phi = \frac{\mathbf{P}^T(\mathbf{AP}-\mathbf{AD})}{\|\mathbf{P}\|\,\|(\mathbf{AP}-\mathbf{AD})\|} \tag{2.0.18}$$

$$\cos\phi = \frac{\mathbf{BC}^T(\mathbf{AP}-\mathbf{AD})}{\|\mathbf{BC}\|\,\|(\mathbf{AP}-\mathbf{AD})\|} \tag{2.0.19}$$

Now, we can conclude that

$$\frac{\mathbf{BC}^T(\mathbf{AP}-\mathbf{AD})}{\|\mathbf{BC}\|\,\|(\mathbf{AP}-\mathbf{AD})\|} = 0 \tag{2.0.20}$$

$$\implies \mathbf{BC}^T(\mathbf{AP}-\mathbf{AD}) = 0 \tag{2.0.21}$$

In a similar way,

$$\cos\phi = \frac{\mathbf{P}^T\mathbf{AP}}{\|\mathbf{P}\|\,\|\mathbf{AP}\|} \tag{2.0.22}$$

$$\implies \cos\phi = \frac{\mathbf{BC}^T\mathbf{AP}}{\|\mathbf{BC}\|\,\|\mathbf{AP}\|} \tag{2.0.23}$$

$$\tag{2.0.24}$$

Now,

$$\frac{\mathbf{BC}^T\mathbf{AP}}{\|\mathbf{BC}\|\,\|\mathbf{AP}\|} = 0 \tag{2.0.25}$$

$$\implies \mathbf{BC}^T\mathbf{AP} = 0 \tag{2.0.26}$$

So we can say that, AP is a perpendicular bisector of BC as $\mathbf{P} = \frac{1}{2}\mathbf{BC}$.

Now, let $\angle BAP = \theta_1$ and $\angle CAP = \theta_2$

$$\cos\theta_1 = \frac{\mathbf{BA}^T\mathbf{AP}}{\|\mathbf{BA}\|\,\|\mathbf{AP}\|} \tag{2.0.27}$$

$$\cos\theta_2 = \frac{\mathbf{CA}^T\mathbf{AP}}{\|\mathbf{CA}\|\,\|\mathbf{AP}\|} \tag{2.0.28}$$

$$\text{or}, \cos\theta_2 = \frac{\mathbf{CA}^T\mathbf{AP}}{\|\mathbf{BA}\|\,\|\mathbf{AP}\|} \tag{2.0.29}$$

$$\tag{2.0.30}$$

According to the vector triangular law for $\triangle ABC$, $\mathbf{BA} = \mathbf{BC} + \mathbf{CA}$

$$\cos\theta_1 = \frac{(\mathbf{BC}^T + \mathbf{CA}^T)\mathbf{AP}}{\|\mathbf{BA}\|\,\|\mathbf{AP}\|} \tag{2.0.31}$$

$$\implies \cos\theta_1 = \frac{\mathbf{BC}^T\mathbf{AP}}{\|\mathbf{BA}\|\,\|\mathbf{AP}\|} + \frac{\mathbf{CA}^T\mathbf{AP}}{\|\mathbf{BA}\|\,\|\mathbf{AP}\|} \tag{2.0.32}$$

$$\implies \cos\theta_1 = \frac{\mathbf{CA}^T\mathbf{AP}}{\|\mathbf{BA}\|\,\|\mathbf{AP}\|} \tag{2.0.33}$$

$$\implies \cos\theta_1 = \cos\theta_2 \tag{2.0.34}$$

$$\implies \theta_1 = \theta_2 \tag{2.0.35}$$

Similarly, let $\angle BDP = \alpha$ and $\angle CDP = \beta$

$$\cos\alpha = -\frac{\mathbf{BD}^T\mathbf{PD}}{\|\mathbf{BD}\|\,\|\mathbf{PD}\|} \tag{2.0.36}$$

$$\text{or}, \cos\alpha = -\frac{\mathbf{BD}^T(\mathbf{AP}-\mathbf{AD})}{\|\mathbf{BD}\|\,\|\mathbf{PD}\|} \tag{2.0.37}$$

$$\cos\beta = -\frac{\mathbf{CD}^T\mathbf{PD}}{\|\mathbf{CD}\|\,\|\mathbf{PD}\|} \tag{2.0.38}$$

$$\text{or}, \cos\beta = -\frac{\mathbf{CD}^T(\mathbf{AP}-\mathbf{AD})}{\|\mathbf{BD}\|\,\|\mathbf{PD}\|} \tag{2.0.39}$$

According to the vector triangular law for $\triangle DBC$, $\mathbf{BD} = \mathbf{BC} + \mathbf{CD}$

$$\cos\alpha = \frac{(\mathbf{BC}^T + \mathbf{CD}^T)(\mathbf{AD}-\mathbf{AP})}{\|\mathbf{BD}\|\,\|\mathbf{PD}\|} \tag{2.0.40}$$

$$\implies \cos\alpha = \frac{\mathbf{BC}^T(\mathbf{AD}-\mathbf{AP})}{\|\mathbf{BD}\|\,\|\mathbf{PD}\|} + \frac{\mathbf{CD}^T(\mathbf{AD}-\mathbf{AP})}{\|\mathbf{BD}\|\,\|\mathbf{PD}\|} \tag{2.0.41}$$

$$\implies \cos\alpha = -\frac{\mathbf{CD}^T(\mathbf{AP}-\mathbf{AD})}{\|\mathbf{BD}\|\,\|\mathbf{PD}\|} \tag{2.0.42}$$

$$\implies \cos\alpha = \cos\beta \tag{2.0.43}$$

$$\implies \alpha = \beta \tag{2.0.44}$$

So, we can conclude that AP bisects $\angle A$ as well as $\angle D$.