Assignment 5

Jayati Dutta

***Abstract*—This is a simple document explaining how to prove the congruence of two triangles.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

$\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and the vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
a) $\triangle ABD \cong \triangle ACD$
b) $\triangle ABP \cong \triangle ACP$
c) AP bisects $\angle A$ as well as $\angle D$
d) AP is the parpendicular bisector of BC

2 Explanation

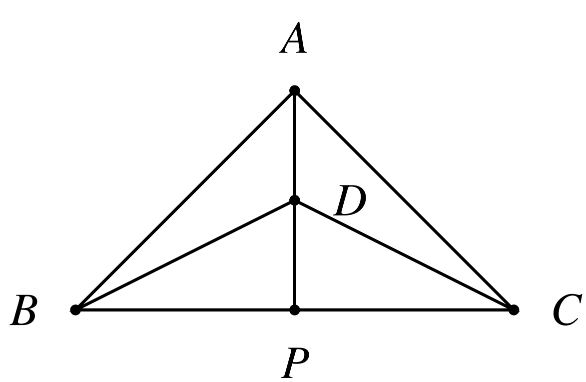

Fig. 0: Iso-sceles Triangles by Latex-Tikz

The above problem statement is depicted in the figure 0 where the vertices are: A, B and C for $\triangle ABC$ and D, B and C for $\triangle DBC$. For $\triangle ABC$ the sides AB, BC and CA are represented by the vectors $\mathbf{A}-\mathbf{B}$, $\mathbf{B}-\mathbf{C}$ and $\mathbf{C}-\mathbf{A}$ and for $\triangle DBC$ the sides DB, BC and CD are represented by $\mathbf{D}-\mathbf{B}$, $\mathbf{B}-\mathbf{C}$ and $\mathbf{C}-\mathbf{D}$.

From the problem statement we get that:

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \tag{2.0.1}$$

$$\|\mathbf{D}-\mathbf{B}\| = \|\mathbf{D}-\mathbf{C}\| \tag{2.0.2}$$

$$\mathbf{P}-\mathbf{C} = K_2(\mathbf{B}-\mathbf{C}) \tag{2.0.3}$$

$$\mathbf{P}-\mathbf{B} = K_1(\mathbf{B}-\mathbf{C}) \tag{2.0.4}$$

As the $\triangle ABC$ is an iso-scelen triangle,

$$\angle ABC = \angle ACB \tag{2.0.5}$$

Similarly, as the $\triangle DBC$ is an iso-scelen triangle,

$$\angle DBC = \angle DCB \tag{2.0.6}$$

From the triangular law of vector addition, we can also get that:

$$\mathbf{A}-\mathbf{B} = (\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{B}) \tag{2.0.7}$$

$$\mathbf{A}-\mathbf{C} = (\mathbf{A}-\mathbf{P}) + (\mathbf{P}-\mathbf{C}) \tag{2.0.8}$$

$$\mathbf{D}-\mathbf{B} = (\mathbf{D}-\mathbf{P}) + (\mathbf{P}-\mathbf{B}) \tag{2.0.9}$$

$$\mathbf{D}-\mathbf{C} = (\mathbf{D}-\mathbf{P}) + (\mathbf{P}-\mathbf{C}) \tag{2.0.10}$$

Now squaring both side of equation 2,

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \tag{2.0.11}$$
$$\text{or,}\ \|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}-\mathbf{C}\|^2 \tag{2.0.12}$$
$$\text{or,}\ \|(\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B})\|^2 = \|(\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{C})\|^2 \tag{2.0.13}$$
$$\text{or,}((\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B}))^T((\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B})) = \tag{2.0.14}$$
$$((\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{C}))^T((\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{C})) \tag{2.0.15}$$
$$\text{or,}((\mathbf{A}-\mathbf{P})^T+K_1(\mathbf{B}-\mathbf{C})^T)((\mathbf{A}-\mathbf{P})+K_1(\mathbf{B}-\mathbf{C})) = \tag{2.0.16}$$
$$((\mathbf{A}-\mathbf{P})^T+K_2(\mathbf{B}-\mathbf{C})^T)((\mathbf{A}-\mathbf{P})+K_2(\mathbf{B}-\mathbf{C})) \tag{2.0.17}$$
$$\text{or,}(K_1-K_2) \tag{2.0.18}$$
$$((\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{P}) \tag{2.0.19}$$
$$+(\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{C})+(K_1+K_2)\,\|B-C\|^2) = 0 \tag{2.0.20}$$

As $(\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{C}) \neq 0$, $(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{P}) \neq 0$ and $K_1+K_2 > 0$, so we can say that $K_1 = K_2$.

$$K_1 = K_2 \implies \mathbf{P}-\mathbf{C} = \mathbf{P}-\mathbf{B} \tag{2.0.21}$$

So, it can be concluded that $\mathbf{A}-\mathbf{P}$ bisects $\mathbf{B}-\mathbf{C}$. Now,

$$\cos\theta_1 = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.22}$$
$$\text{or,}\cos\theta_1 = \frac{((\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B}))^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.23}$$

Similarly,

$$\cos\theta_2 = \frac{(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{C}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.24}$$
$$\text{or,}\cos\theta_2 = \frac{((\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{C}))^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.25}$$
$$\text{or,}\cos\theta_2 = \frac{((\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B}))^T(\mathbf{A}-\mathbf{P})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{P}\|} \tag{2.0.26}$$

So, we can say that, $\theta_1 = \theta_2$ Now for $\triangle DBC$,

$$\cos\alpha = \frac{(\mathbf{D}-\mathbf{B})^T(\mathbf{D}-\mathbf{P})}{\|\mathbf{D}-\mathbf{B}\|\,\|\mathbf{D}-\mathbf{P}\|} \tag{2.0.27}$$
$$\text{or,}\cos\alpha = \frac{((\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{B}))^T(\mathbf{D}-\mathbf{P})}{\|\mathbf{D}-\mathbf{B}\|\,\|\mathbf{D}-\mathbf{P}\|} \tag{2.0.28}$$

Similarly,

$$\cos\beta = \frac{(\mathbf{D}-\mathbf{C})^T(\mathbf{D}-\mathbf{P})}{\|\mathbf{D}-\mathbf{C}\|\,\|\mathbf{D}-\mathbf{P}\|} \tag{2.0.29}$$
$$\text{or,}\cos\beta = \frac{((\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{C}))^T(\mathbf{D}-\mathbf{P})}{\|\mathbf{D}-\mathbf{B}\|\,\|\mathbf{D}-\mathbf{P}\|} \tag{2.0.30}$$
$$\text{or,}\cos\beta = \frac{((\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{B}))^T(\mathbf{D}-\mathbf{P})}{\|\mathbf{D}-\mathbf{B}\|\,\|\mathbf{D}-\mathbf{P}\|} \tag{2.0.31}$$

So, we can conclude that $\alpha = \beta$. These imply that $\mathbf{A}-\mathbf{P}$ bisects $\angle A$ as well as $\angle D$.

Now, for $\triangle ABP$,

$$\angle BAP + \angle ABP + \angle APB = 180^\circ \tag{2.0.32}$$

Now, for $\triangle ACP$,

$$\angle CAP + \angle ACP + \angle APC = 180^\circ \tag{2.0.33}$$

But we know that, for $\triangle ABP$ and $\triangle ACP$:

$$\angle BAP = \angle CAP \tag{2.0.34}$$
$$\angle ABP = \angle ACP \tag{2.0.35}$$
$$\implies \angle APB = \angle APC \tag{2.0.36}$$

Now,

$$\angle APB + \angle APC = 180^\circ \tag{2.0.37}$$
$$\implies 2\angle APB = 180^\circ \tag{2.0.38}$$
$$\implies \angle APB = 90^\circ \tag{2.0.39}$$
$$\tag{2.0.40}$$

Hence, it is proved that $\mathbf{A}-\mathbf{P}$ bisects $\mathbf{B}-\mathbf{C}$ perpendicularly.