Assignment 6

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Abstract—**This is a simple document explaining how to determine a conic section from a given second degree equation.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

What conics do the following equation represents? When possible, find the center and the equation reffered to the center.

$$55x^2 - 120xy + 20y^2 + 64x - 48y = 0 \tag{1.0.1}$$

2 Explanation

The general equation of second degree can be represented as:

$$\mathbf{X}^T\mathbf{V}\mathbf{X} + 2\mathbf{u}^T\mathbf{X} + f = 0 \tag{2.0.1}$$

The above 1.0.1 can also be written as:

$$\mathbf{X}^T\begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix}\mathbf{X} + 2\begin{pmatrix} 32 & -24 \end{pmatrix}\mathbf{X} + 0 = 0 \tag{2.0.2}$$

So,

$$\mathbf{V} = \begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix} \tag{2.0.3}$$

and

$$\mathbf{u} = \begin{pmatrix} 32 \\ -24 \end{pmatrix} \tag{2.0.4}$$

$$f = 0 \tag{2.0.5}$$

Now,

$$\det\mathbf{V} = \begin{vmatrix} 55 & -60 \\ -60 & 20 \end{vmatrix} \tag{2.0.6}$$

$$\implies \det\mathbf{V} = -2500 < 0 \tag{2.0.7}$$

As $\det\mathbf{V} < 0$, so we can say that the above conic section 1.0.1 is hyperbola.Now,

$$\mathbf{V}^{-1} = \frac{1}{-2500}\begin{pmatrix} 20 & 60 \\ 60 & 55 \end{pmatrix} \tag{2.0.8}$$

The center of this hyperbola will be:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.9}$$

$$\implies \mathbf{c} = \frac{1}{2500}\begin{pmatrix} 20 & 60 \\ 60 & 55 \end{pmatrix}\begin{pmatrix} 32 \\ -24 \end{pmatrix} \tag{2.0.10}$$

$$\implies \mathbf{c} = \begin{pmatrix} -\frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \tag{2.0.11}$$

$$\tag{2.0.12}$$

Now the characteristic equation of $\mathbf{V}$ is obtained as:

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\implies \begin{vmatrix} 55-\lambda & -60 \\ -60 & 20-\lambda \end{vmatrix} = 0 \tag{2.0.14}$$

$$\implies \lambda^2 - 75\lambda - 2500 = 0 \tag{2.0.15}$$

The eigen values are given by:

$$\lambda_1 = 100 \tag{2.0.16}$$

$$\lambda_2 = -25 \tag{2.0.17}$$

The eigen vector $\mathbf{P}$ is defined as:

$$\mathbf{V}\mathbf{P} = \lambda\mathbf{P} \tag{2.0.18}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{P} = \mathbf{0} \tag{2.0.19}$$

For λ_1=100,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} -45 & -60 \\ -60 & -80 \end{pmatrix} \tag{2.0.20}$$

By row reduction,

$$\begin{pmatrix} -45 & -60 \\ -60 & -80 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/(-5)]{R_2 \leftarrow R_2/(-5)} \tag{2.0.21}$$

$$\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/3]{R_2 \leftarrow R_2/4} \tag{2.0.22}$$

$$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \tag{2.0.23}$$

So,

$$(\mathbf{V}-\lambda_1\mathbf{I})\mathbf{P_1} = \mathbf{0} \tag{2.0.24}$$

$$\implies \begin{pmatrix}3 & 4\\0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.25}$$

$$\implies \mathbf{P_1} = \begin{pmatrix}-\frac{4}{3}\\1\end{pmatrix} \tag{2.0.26}$$

Similarly, For λ_2=100,

$$(\mathbf{V}-\lambda_2\mathbf{I}) = \begin{pmatrix}80 & -60\\-60 & 45\end{pmatrix} \tag{2.0.27}$$

By row reduction,

$$\begin{pmatrix}80 & -60\\-60 & 45\end{pmatrix} \xleftrightarrow[R_1\leftarrow R_1/5]{R_2\leftarrow R_2/5} \tag{2.0.28}$$

$$\begin{pmatrix}16 & -12\\-12 & 9\end{pmatrix} \xleftrightarrow[R_1\leftarrow R_1/4]{R_2\leftarrow R_2/(-3)} \tag{2.0.29}$$

$$\begin{pmatrix}4 & -3\\4 & -3\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-R_1} \begin{pmatrix}4 & -3\\0 & 0\end{pmatrix} \tag{2.0.30}$$

So,

$$(\mathbf{V}-\lambda_2\mathbf{I})\mathbf{P_2} = \mathbf{0} \tag{2.0.31}$$

$$\implies \begin{pmatrix}4 & -3\\0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.32}$$

$$\implies \mathbf{P_2} = \begin{pmatrix}1\\\frac{4}{3}\end{pmatrix} \tag{2.0.33}$$

By eigen decomposition $\mathbf{V}$ can also be written as:

$$\mathbf{V} = \mathbf{PDP}^T \tag{2.0.34}$$

where

$$\mathbf{P} = \begin{pmatrix}\mathbf{P_1} & \mathbf{P_2}\end{pmatrix} \tag{2.0.35}$$

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} \tag{2.0.36}$$

So,

$$\mathbf{P} = \begin{pmatrix}-\frac{4}{3} & 1\\1 & \frac{4}{3}\end{pmatrix} \tag{2.0.37}$$

$$\mathbf{D} = \begin{pmatrix}100 & 0\\0 & -25\end{pmatrix} \tag{2.0.38}$$

and

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 16 > 0 \tag{2.0.39}$$

So, the axes are:

$$a = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} = \frac{2}{5} \tag{2.0.40}$$

$$b = \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \frac{4}{5} \tag{2.0.41}$$

Now, the equation 1.0.1 can be written as:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.0.42}$$

where,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x}-\mathbf{c}) \tag{2.0.43}$$

So,

$$\mathbf{y}^T\begin{pmatrix}100 & 0\\0 & -25\end{pmatrix}\mathbf{y} = 16 \tag{2.0.44}$$

$$\implies \mathbf{y}^T\begin{pmatrix}100 & 0\\0 & -25\end{pmatrix}\mathbf{y} - 16 = 0 \tag{2.0.45}$$

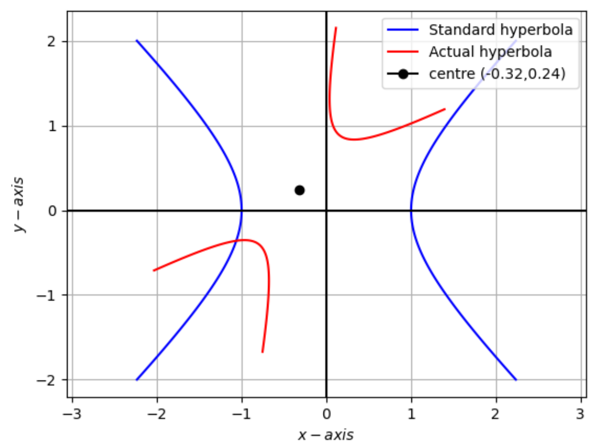

Fig. 0: Comparison of the Standerd and Actual Hyperbola

2.1. Verification of the above problem using python code.
Solution: The following Python code generates Fig. 0

codes/hyperbola_3.py