Assignment 7

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***Abstract*—This is a simple document explaining how to determine the foot of the perpendicular of any point on x-axis to a plane using Singular Value Decomposition.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

Find the foot of the perpendicular from X-axis to the plane $3y - 4z + 7 = 0$ using SVD.

2 Explanation

Let the given point on X-axis is $\begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}$. Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$. Let $\mathbf{m}$=$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $\mathbf{m}^T\mathbf{n}$=0.

$$\mathbf{m}^T\mathbf{n} = 0$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = 0$$

$$\implies 0a + 3b - 4c = 0 \tag{2.0.1}$$

Now, for a=0 and b=4, c=3 and for a=1, b=0, c=0. So,

$$m_1 = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \tag{2.0.2}$$

$$m_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.3}$$

Now,

$$M = \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \tag{2.0.4}$$

Now, we will solve $Mx = b$

$$\mathbf{Mx} = \mathbf{b}$$

$$\implies \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} \tag{2.0.5}$$

Now the equation 2.0.5 will be solved by using SVD. $\mathbf{M}$ can be expressed as: $\mathbf{M} = \mathbf{USV}^T$. Now,

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 0 & 4 & 3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \tag{2.0.6}$$

$$\implies \mathbf{M}^T\mathbf{M} = \begin{pmatrix} 25 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.7}$$

Now, to get the eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda I\right| = 0 \tag{2.0.8}$$

$$\implies \begin{vmatrix} 25 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \tag{2.0.9}$$

$$\implies \lambda_1 = 25 \tag{2.0.10}$$

$$\lambda_2 = 1 \tag{2.0.11}$$

Now,

$$S = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

$$S = \begin{pmatrix} 5 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

So, the pseudo inverse of the diagonal matrix is:

$$S_+ = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.14}$$

Now, the eigen vector of $\mathbf{M}^T\mathbf{M}$ for $\lambda_1 = 25$,

$$\begin{pmatrix}25-25 & 0\\ 0 & 1-25\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.15}$$

$$\text{or,} \begin{pmatrix}0 & 0\\ 0 & -24\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.16}$$

$$\implies v_1 = 1 \tag{2.0.17}$$

$$v_2 = 0 \tag{2.0.18}$$

Now, the eigen vector of $\mathbf{M}^T\mathbf{M}$ for $\lambda_2 = 1$,

$$\begin{pmatrix}25-1 & 0\\ 0 & 1-1\end{pmatrix}\begin{pmatrix}v_3\\ v_4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.19}$$

$$\text{or,} \begin{pmatrix}24 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_3\\ v_4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.20}$$

$$\implies v_3 = 0 \tag{2.0.21}$$

$$v_4 = 1 \tag{2.0.22}$$

So,

$$\mathbf{V} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{2.0.23}$$

Now,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix}0 & 1\\ 4 & 0\\ 3 & 0\end{pmatrix}\begin{pmatrix}0 & 4 & 3\\ 1 & 0 & 0\end{pmatrix} \tag{2.0.24}$$

$$\implies \mathbf{M}\mathbf{M}^T = \begin{pmatrix}1 & 0 & 0\\ 0 & 16 & 12\\ 0 & 12 & 9\end{pmatrix} \tag{2.0.25}$$

Now, to get the eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda I\right| = 0 \tag{2.0.26}$$

$$\implies \begin{vmatrix}1-\lambda & 0 & 0\\ 0 & 16-\lambda & 12\\ 0 & 12 & 9-\lambda\end{vmatrix} = 0 \tag{2.0.27}$$

$$\implies (1-\lambda)(\lambda^2 - 25\lambda) = 0 \tag{2.0.28}$$

$$\implies \lambda_1 = 0 \tag{2.0.29}$$

$$\lambda_2 = 1 \tag{2.0.30}$$

$$\lambda_3 = 25 \tag{2.0.31}$$

Now, the eigen vector of $\mathbf{M}\mathbf{M}^T$ for $\lambda_1 = 0$,

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 16 & 12\\ 0 & 12 & 9\end{pmatrix}\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.0.32}$$

(2.0.33)

To get the row-reduced echelon form,

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 16 & 12\\ 0 & 12 & 9\end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{4}]{R_3 \leftarrow \frac{R_3}{3}} \begin{pmatrix}1 & 0 & 0\\ 0 & 4 & 3\\ 0 & 4 & 3\end{pmatrix} \tag{2.0.34}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix}1 & 0 & 0\\ 0 & 4 & 3\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.35}$$

So,

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 4 & 3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.0.36}$$

$$\implies \mathbf{u_1} = 0 \tag{2.0.37}$$

$$\mathbf{u_2} = -3 \tag{2.0.38}$$

$$\mathbf{u_3} = 4 \tag{2.0.39}$$

Now, the eigen vector of $\mathbf{M}\mathbf{M}^T$ for $\lambda_2 = 1$,

$$\begin{pmatrix}0 & 0 & 0\\ 0 & 15 & 12\\ 0 & 12 & 8\end{pmatrix}\begin{pmatrix}u_4\\ u_5\\ u_6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.0.40}$$

(2.0.41)

To get the row-reduced echelon form,

$$\begin{pmatrix}0 & 0 & 0\\ 0 & 15 & 12\\ 0 & 12 & 8\end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{3}]{R_3 \leftarrow \frac{R_3}{4}} \begin{pmatrix}0 & 5 & 4\\ 0 & 3 & 2\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.42}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix}0 & 5 & 4\\ 0 & -2 & -2\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.43}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-2}} \begin{pmatrix}0 & 5 & 4\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.44}$$

$$\xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix}0 & 5 & 4\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.45}$$

So,

$$\begin{pmatrix}0 & 5 & 4\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}u_4\\ u_5\\ u_6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.0.46}$$

$$\implies \mathbf{u_4} = 1 \tag{2.0.47}$$

$$\mathbf{u_5} = 0 \tag{2.0.48}$$

$$\mathbf{u_6} = 0 \tag{2.0.49}$$

Similarly, the eigen vector of $\mathbf{MM}^T$ for $\lambda_3 = 25$,

$$\begin{pmatrix} -24 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.50}$$

$$\implies \mathbf{u_7} = 0 \tag{2.0.51}$$

$$\mathbf{u_8} = 4 \tag{2.0.52}$$

$$\mathbf{u_9} = 3 \tag{2.0.53}$$

Now,

$$\mathbf{U} = \begin{pmatrix} 0 & 1 & 0 \\ -3 & 0 & 4 \\ 4 & 0 & 3 \end{pmatrix} \tag{2.0.54}$$

$$\implies \mathbf{U}^T = \begin{pmatrix} 0 & -3 & 4 \\ 1 & 0 & 0 \\ 0 & 4 & 3 \end{pmatrix} \tag{2.0.55}$$

$$\tag{2.0.56}$$

$$\mathbf{X} = \mathbf{VS_+U}^T\mathbf{b} \tag{2.0.57}$$

Now,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \tag{2.0.58}$$

$$\mathbf{S_+U}^T\mathbf{b} = \begin{pmatrix} 0 \\ p \end{pmatrix} \tag{2.0.59}$$

$$\mathbf{VS_+U}^T\mathbf{b} = \begin{pmatrix} 0 \\ p \end{pmatrix} \tag{2.0.60}$$

$$\mathbf{X} = \begin{pmatrix} 0 \\ p \end{pmatrix} \tag{2.0.61}$$

So, $\mathbf{X}=\begin{pmatrix} 0 \\ p \end{pmatrix}$.
Verifying the solution of 2.0.61 using,

$$\mathbf{M}^T\mathbf{MX} = \mathbf{M}^T\mathbf{b} \tag{2.0.62}$$

$$\implies \begin{pmatrix} 25 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 0 \\ p \end{pmatrix} \tag{2.0.63}$$

Now, taking the augmented matrix,

$$\begin{pmatrix} 25 & 0 & 0 \\ 0 & 1 & p \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{25}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p \end{pmatrix} \tag{2.0.64}$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 0 \\ p \end{pmatrix} \tag{2.0.65}$$

So we can conclude that the solution is verified.