

Assignment 7

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Abstract—This is a simple document explaining how to determine the foot of the perpendicular of any point on x-axis to a plane using Singular Value Decomposition.

Download all and latex-tikz codes from

svn co <https://github.com/gadepall/school/trunk/ncert/geometry/figs>

1 PROBLEM

Find the foot of the perpendicular from X-axis to the plane $3y - 4z + 7 = 0$ using SVD.

2 EXPLANATION

Let the given point on X-axis is $\begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}$. Let us consider orthogonal vectors \mathbf{m}_1 and \mathbf{m}_2 to the normal vector \mathbf{n} . Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $\mathbf{m}^T \mathbf{n} = 0$.

$$\mathbf{m}^T \mathbf{n} = 0$$

$$\begin{aligned} \Rightarrow (a \ b \ c) \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} &= 0 \\ \Rightarrow 0a + 3b - 4c &= 0 \end{aligned} \quad (2.0.1)$$

Now, for $a=0$ and $b=4$, $c=3$ and for $a=1$, $b=0$, $c=0$. So,

$$m_1 = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \quad (2.0.2)$$

$$m_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.3)$$

Now,

$$M = \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \quad (2.0.4)$$

Now, we will solve $Mx = b$

$$M\mathbf{x} = \mathbf{b}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} \quad (2.0.5)$$

Now the equation 2.0.5 will be solved by using SVD. \mathbf{M} can be expressed as: $\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T$. Now,

$$\mathbf{M}^T \mathbf{M} = \begin{pmatrix} 0 & 4 & 3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \quad (2.0.6)$$

$$\Rightarrow \mathbf{M}^T \mathbf{M} = \begin{pmatrix} 25 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.7)$$

Now, to get the eigen values of $\mathbf{M}^T \mathbf{M}$,

$$|\mathbf{M}^T \mathbf{M} - \lambda I| = 0 \quad (2.0.8)$$

$$\Rightarrow \begin{vmatrix} 25 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \quad (2.0.9)$$

$$\Rightarrow \lambda_1 = 25 \quad (2.0.10)$$

$$\lambda_2 = 1 \quad (2.0.11)$$

Now,

$$S = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} \quad (2.0.12)$$

$$S = \begin{pmatrix} 5 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.13)$$

So, the pseudo inverse of the diagonal matrix is:

$$S_+ = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.14)$$

Now, the eigen vector of $\mathbf{M}^T\mathbf{M}$ for $\lambda_1 = 25$,

$$\begin{pmatrix} 25-25 & 0 \\ 0 & 1-25 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.15)$$

$$\text{or, } \begin{pmatrix} 0 & 0 \\ 0 & -24 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.16)$$

$$\Rightarrow v_1 = 1 \quad (2.0.17)$$

$$v_2 = 0 \quad (2.0.18)$$

Now, the eigen vector of $\mathbf{M}^T\mathbf{M}$ for $\lambda_2 = 1$,

$$\begin{pmatrix} 25-1 & 0 \\ 0 & 1-1 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.19)$$

$$\text{or, } \begin{pmatrix} 24 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.20)$$

$$\Rightarrow v_3 = 0 \quad (2.0.21)$$

$$v_4 = 1 \quad (2.0.22)$$

So,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.23)$$

Now,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 & 3 \\ 1 & 0 & 0 \end{pmatrix} \quad (2.0.24)$$

$$\Rightarrow \mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 12 \\ 0 & 12 & 9 \end{pmatrix} \quad (2.0.25)$$

Now, to get the eigen values of $\mathbf{M}\mathbf{M}^T$,

$$|\mathbf{M}\mathbf{M}^T - \lambda I| = 0 \quad (2.0.26)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 16-\lambda & 12 \\ 0 & 12 & 9-\lambda \end{vmatrix} = 0 \quad (2.0.27)$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 25\lambda) = 0 \quad (2.0.28)$$

$$\Rightarrow \lambda_1 = 0 \quad (2.0.29)$$

$$\lambda_2 = 1 \quad (2.0.30)$$

$$\lambda_3 = 25 \quad (2.0.31)$$

Now, the eigen vector of $\mathbf{M}\mathbf{M}^T$ for $\lambda_1 = 0$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 12 \\ 0 & 12 & 9 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.32)$$

$$(2.0.33)$$

To get the row-reduced echelon form,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 12 \\ 0 & 12 & 9 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{4}]{R_3 \leftarrow \frac{R_3}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 4 & 3 \end{pmatrix} \quad (2.0.34)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.35)$$

So,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.36)$$

$$\Rightarrow \mathbf{u}_1 = 0 \quad (2.0.37)$$

$$\mathbf{u}_2 = -3 \quad (2.0.38)$$

$$\mathbf{u}_3 = 4 \quad (2.0.39)$$

Now, the eigen vector of $\mathbf{M}\mathbf{M}^T$ for $\lambda_2 = 1$,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 15 & 12 \\ 0 & 12 & 8 \end{pmatrix} \begin{pmatrix} u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.40)$$

$$(2.0.41)$$

To get the row-reduced echelon form,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 15 & 12 \\ 0 & 12 & 8 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{3}]{R_3 \leftarrow \frac{R_3}{4}} \begin{pmatrix} 0 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.42)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 0 & 5 & 4 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.43)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-2}} \begin{pmatrix} 0 & 5 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.44)$$

$$\xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} 0 & 5 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.45)$$

So,

$$\begin{pmatrix} 0 & 5 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.46)$$

$$\Rightarrow \mathbf{u}_4 = 1 \quad (2.0.47)$$

$$\mathbf{u}_5 = 0 \quad (2.0.48)$$

$$\mathbf{u}_6 = 0 \quad (2.0.49)$$

Similarly, the eigen vector of \mathbf{MM}^T for $\lambda_3 = 25$,

$$\begin{pmatrix} -24 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.50)$$

$$\Rightarrow \mathbf{u}_7 = 0 \quad (2.0.51)$$

$$\mathbf{u}_8 = 4 \quad (2.0.52)$$

$$\mathbf{u}_9 = 3 \quad (2.0.53)$$

Now,

$$\mathbf{U} = \begin{pmatrix} 0 & 1 & 0 \\ -3 & 0 & 4 \\ 4 & 0 & 3 \end{pmatrix} \quad (2.0.54)$$

$$\Rightarrow \mathbf{U}^T = \begin{pmatrix} 0 & -3 & 4 \\ 1 & 0 & 0 \\ 0 & 4 & 3 \end{pmatrix} \quad (2.0.55)$$

$$(2.0.56)$$

$$\mathbf{X} = \mathbf{VS}_+ \mathbf{U}^T \mathbf{b} \quad (2.0.57)$$

Now,

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \quad (2.0.58)$$

$$\mathbf{S}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} 0 \\ p \end{pmatrix} \quad (2.0.59)$$

$$\mathbf{VS}_+ \mathbf{U}^T \mathbf{b} = \begin{pmatrix} 0 \\ p \end{pmatrix} \quad (2.0.60)$$

$$\mathbf{X} = \begin{pmatrix} 0 \\ p \end{pmatrix} \quad (2.0.61)$$

So, $\mathbf{X} = \begin{pmatrix} 0 \\ p \end{pmatrix}$.

Verifying the solution of 2.0.61 using,

$$\mathbf{M}^T \mathbf{M} \mathbf{X} = \mathbf{M}^T \mathbf{b} \quad (2.0.62)$$

$$\Rightarrow \begin{pmatrix} 25 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 0 \\ p \end{pmatrix} \quad (2.0.63)$$

Now, taking the augmented matrix,

$$\begin{pmatrix} 25 & 0 & 0 \\ 0 & 1 & p \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{25}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p \end{pmatrix} \quad (2.0.64)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 0 \\ p \end{pmatrix} \quad (2.0.65)$$

So we can conclude that the solution is verified.

Now, the plane $3y - 4z + 7 = 0$ is parallel to x-axis. Another plane that is perpendicular to the

above plane and passing through x-axis is $4y + 3z = 0$ which intersects the above plane at a straight $4y + 3z = -\frac{7}{25}$ and the point p will be on that line of the plane as the perpendicular distance of the plane $3y - 4z + 7 = 0$ from x-axis is $\frac{7}{25}$.

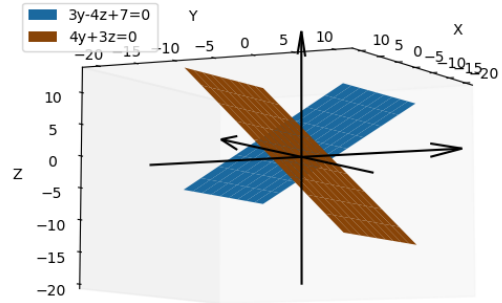


Fig. 0: Perpendicular surfaces

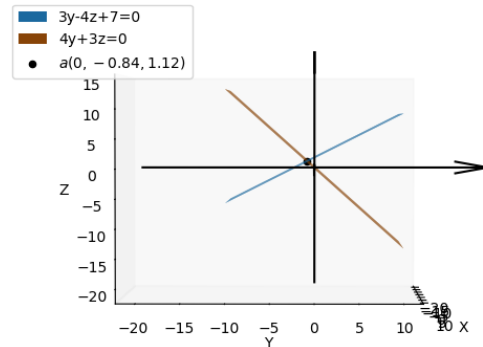


Fig. 0: Perpendicular surfaces

Now,

$$3y - 4z + 7 = 0 \quad (2.0.66)$$

$$4y + 3z = 0 \quad (2.0.67)$$

From these 2 equations we get

$$\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \end{pmatrix} \quad (2.0.68)$$

Now, the augmented matrix is $\begin{pmatrix} 3 & -4 & -7 \\ 4 & 3 & 0 \\ 0 & & \end{pmatrix}$

$$\begin{pmatrix} 3 & -4 & -7 \\ 4 & 3 & 0 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{R_1}{3}]{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & -\frac{4}{3} & -\frac{7}{3} \\ 1 & \frac{3}{4} & 0 \end{pmatrix} \quad (2.0.69)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\frac{4}{3} & -\frac{7}{3} \\ 0 & \frac{25}{12} & \frac{7}{3} \end{pmatrix} \quad (2.0.70)$$

$$\implies z = \frac{28}{25} \quad (2.0.71)$$

$$y = -\frac{21}{25} \quad (2.0.72)$$

In this case, the co-ordinate of the point on the plane $3y - 4z + 7 = 0$ does not depend on any value of x . This fact is verified through the plot 0.