Assignment 7

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***Abstract*—This is a simple document explaining how to determine the foot of the perpendicular of any point on x-axis to a plane using Singular Value Decomposition.**

Download all and latex-tikz codes from

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svn co https://github.com/gadepall/school/trunk/
    ncert/geometry/figs
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1 Problem

Find the foot of the perpendicular from X-axis to the plane $3y - 4z + 7 = 0$ using SVD.

2 Explanation

Let the given point on X-axis is $\begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}$. Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$. Let $\mathbf{m}$=$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $\mathbf{m}^T\mathbf{n}$=0.

$$\mathbf{m}^T\mathbf{n} = 0$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = 0$$

$$\implies 0a + 3b - 4c = 0 \quad (2.0.1)$$

Now, for a=0 and b=4, c=3 and for a=1, b=0, c=0. So,

$$m_1 = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \quad (2.0.2)$$

$$m_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.3)$$

Now,

$$M = \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \quad (2.0.4)$$

Now, we will solve $\mathbf{Mx} = \mathbf{b}$ Now let a point is taken on x-axis, that is, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, so $\mathbf{b}$ will be:

$$\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ -\frac{7}{4} \end{pmatrix} \quad (2.0.5)$$

$$\mathbf{Mx} = \mathbf{b}$$

$$\implies \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ -\frac{7}{4} \end{pmatrix} \quad (2.0.6)$$

Now the equation 2.0.6 will be solved by using SVD. $\mathbf{M}$ can be expressed as: $\mathbf{M} = \mathbf{USV}^T$. Now,

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 0 & 4 & 3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix} \quad (2.0.7)$$

$$\implies \mathbf{M}^T\mathbf{M} = \begin{pmatrix} 25 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.8)$$

Now, to get the eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda I\right| = 0 \quad (2.0.9)$$

$$\implies \begin{vmatrix} 25 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \quad (2.0.10)$$

$$\implies \lambda_1 = 25 \quad (2.0.11)$$

$$\lambda_2 = 1 \quad (2.0.12)$$

Now,

$$S = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} \quad (2.0.13)$$

$$S = \begin{pmatrix} 5 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.14)$$

So, the pseudo inverse of the diagonal matrix is:

$$S_+ = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.15)$$

Now, the eigen vector of $\mathbf{M}^T\mathbf{M}$ for $\lambda_1 = 25$,

$$\begin{pmatrix} 25-25 & 0 \\ 0 & 1-25 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.16}$$

$$\text{or,} \begin{pmatrix} 0 & 0 \\ 0 & -24 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.17}$$

$$\implies v_1 = 1 \tag{2.0.18}$$

$$v_2 = 0 \tag{2.0.19}$$

Now, the eigen vector of $\mathbf{M}^T\mathbf{M}$ for $\lambda_2 = 1$,

$$\begin{pmatrix} 25-1 & 0 \\ 0 & 1-1 \end{pmatrix}\begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.20}$$

$$\text{or,} \begin{pmatrix} 24 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.21}$$

$$\implies v_3 = 0 \tag{2.0.22}$$

$$v_4 = 1 \tag{2.0.23}$$

So,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.24}$$

Now,

$$\mathbf{MM}^T = \begin{pmatrix} 0 & 1 \\ 4 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 4 & 3 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.0.25}$$

$$\implies \mathbf{MM}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 12 \\ 0 & 12 & 9 \end{pmatrix} \tag{2.0.26}$$

Now, to get the eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda I\right| = 0 \tag{2.0.27}$$

$$\implies \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 16-\lambda & 12 \\ 0 & 12 & 9-\lambda \end{vmatrix} = 0 \tag{2.0.28}$$

$$\implies (1-\lambda)(\lambda^2 - 25\lambda) = 0 \tag{2.0.29}$$

$$\implies \lambda_1 = 0 \tag{2.0.30}$$

$$\lambda_2 = 1 \tag{2.0.31}$$

$$\lambda_3 = 25 \tag{2.0.32}$$

Now, the eigen vector of $\mathbf{MM}^T$ for $\lambda_1 = 0$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 12 \\ 0 & 12 & 9 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.33}$$

$$\tag{2.0.34}$$

To get the row-reduced echelon form,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 12 \\ 0 & 12 & 9 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{4}]{R_3 \leftarrow \frac{R_3}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 4 & 3 \end{pmatrix} \tag{2.0.35}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.36}$$

So,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.37}$$

$$\implies \mathbf{u_1} = 0 \tag{2.0.38}$$

$$\mathbf{u_2} = -3 \tag{2.0.39}$$

$$\mathbf{u_3} = 4 \tag{2.0.40}$$

Now, the eigen vector of $\mathbf{MM}^T$ for $\lambda_2 = 1$,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 15 & 12 \\ 0 & 12 & 8 \end{pmatrix}\begin{pmatrix} u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.41}$$

$$\tag{2.0.42}$$

To get the row-reduced echelon form,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 15 & 12 \\ 0 & 12 & 8 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{3}]{R_3 \leftarrow \frac{R_3}{4}} \begin{pmatrix} 0 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.43}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 0 & 5 & 4 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.44}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-2}} \begin{pmatrix} 0 & 5 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.45}$$

$$\xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} 0 & 5 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.46}$$

So,

$$\begin{pmatrix} 0 & 5 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.47}$$

$$\implies \mathbf{u_4} = 1 \tag{2.0.48}$$

$$\mathbf{u_5} = 0 \tag{2.0.49}$$

$$\mathbf{u_6} = 0 \tag{2.0.50}$$

Similarly, the eigen vector of $\mathbf{MM}^T$ for $\lambda_3 = 25$,

$$\begin{pmatrix} -24 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.51}$$

$$\implies \mathbf{u_7} = 0 \tag{2.0.52}$$

$$\mathbf{u_8} = 4 \tag{2.0.53}$$

$$\mathbf{u_9} = 3 \tag{2.0.54}$$

Now,

$$\mathbf{U} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{3}{5} & 0 & \frac{4}{5} \\ \frac{4}{5} & 0 & \frac{3}{5} \end{pmatrix} \tag{2.0.55}$$

$$\implies \mathbf{U}^T = \begin{pmatrix} 0 & -\frac{3}{5} & \frac{4}{5} \\ 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix} \tag{2.0.56}$$

$$\tag{2.0.57}$$

$$\mathbf{X} = \mathbf{VS_+U}^T\mathbf{b} \tag{2.0.58}$$

Now,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{7}{5} \\ 0 \\ -\frac{21}{20} \end{pmatrix} \tag{2.0.59}$$

$$\mathbf{S_+U}^T\mathbf{b} = \begin{pmatrix} -\frac{7}{25} \\ 0 \end{pmatrix} \tag{2.0.60}$$

$$\mathbf{VS_+U}^T\mathbf{b} = \begin{pmatrix} -\frac{7}{25} \\ 0 \end{pmatrix} \tag{2.0.61}$$

$$\mathbf{X} = \begin{pmatrix} -\frac{7}{25} \\ 0 \end{pmatrix} \tag{2.0.62}$$

So, $\mathbf{X}$=$\begin{pmatrix} -\frac{7}{25} \\ 0 \end{pmatrix}$.
Verifying the solution of 2.0.62 using,

$$\mathbf{M}^T\mathbf{MX} = \mathbf{M}^T\mathbf{b} \tag{2.0.63}$$

$$\implies \begin{pmatrix} 25 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} -\frac{21}{4} \\ 0 \end{pmatrix} \tag{2.0.64}$$

Now, taking the augmented matrix,

$$\begin{pmatrix} 25 & 0 & -\frac{21}{4} \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{25}} \begin{pmatrix} 1 & 0 & -\frac{21}{100} \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.65}$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} -\frac{21}{100} \\ 0 \end{pmatrix} \tag{2.0.66}$$

Now, the plane $3y - 4z + 7 = 0$ is parallel to x-axis. Another plane that is perpendicular to the above plane and passing through x-axis is $4y+3z = 0$ which intersects the above plane at a straight $4y + 3z = -\frac{7}{25}$ and the point p will be on that line of the plane as the perpendicular distance of the plane $3y - 4z + 7 = 0$ from x-axis is $\frac{7}{25}$.

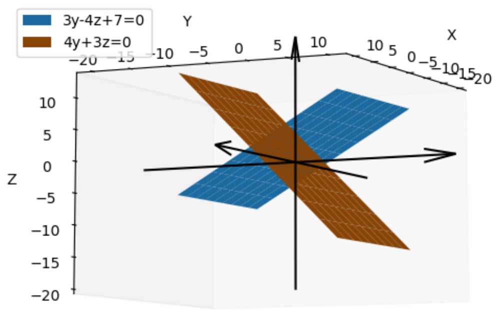

Fig. 0: Perpendicular surfaces

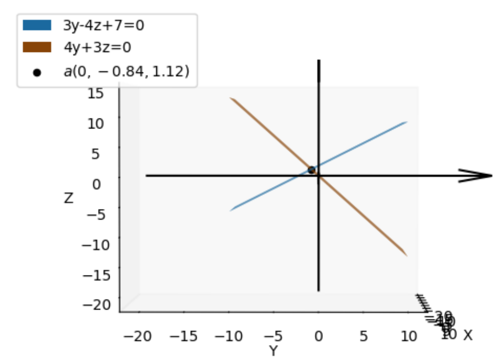

Fig. 0: Perpendicular surfaces

Now,

$$3y - 4z + 7 = 0 \tag{2.0.67}$$

$$4y + 3z = 0 \tag{2.0.68}$$

From these 2 equations we get

$$\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \end{pmatrix} \tag{2.0.69}$$

Now, the augmented matrix is $\begin{pmatrix} 3 & -4 & -7 \\ 4 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 3 & -4 & -7 \\ 4 & 3 & 0 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{R_1}{3}]{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & -\frac{4}{3} & -\frac{7}{3} \\ 1 & \frac{3}{4} & 0 \end{pmatrix} \tag{2.0.70}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\frac{4}{3} & -\frac{7}{3} \\ 0 & \frac{25}{12} & \frac{7}{3} \end{pmatrix} \tag{2.0.71}$$

$$\implies z = \frac{28}{25} \tag{2.0.72}$$

$$y = -\frac{21}{25} \tag{2.0.73}$$

In this case, the co-ordinate of the point on the plane $3y - 4z + 7 = 0$ does not depend on any value of x. This fact is verified through the plot 0.