Fuzzy membership function

This is used to convert the crisp input provided to the fuzzy inference system.

- Fuzzy logic itself is not fuzzy, rather it deals with the fuzziness in the data.
- And this fuzziness in the data is best described by the fuzzy membership function.

A fuzzy inference system is the core part of any fuzzy logic system. Fuzzification is the first step in Fuzzy Inference System.

Formally, a membership function for a fuzzy set A on the universe of discourse X is defined as μ_A: **X → [0, 1]**,

where each element of X is mapped to a value between 0 and 1.

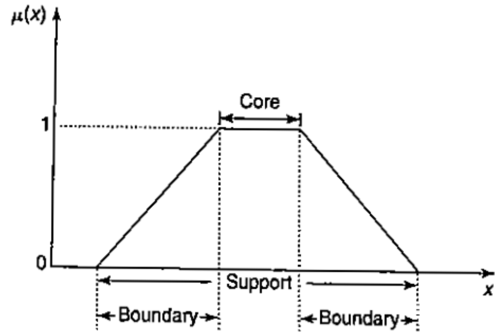
- This value, called **membership value** or **degree of membership**, quantifies the grade of membership of the element in X to the fuzzy set A.

  Here, **X** is the universal set and **A** is the fuzzy set derived from **X**.

The fuzzy membership function is the graphical way of visualizing the degree of membership of any value in a given fuzzy set. In the graph, X-axis represents the universe of discourse and the Y-axis represents the degree of membership in the range [0, 1].

Features membership function

: The three main basic features involved in characrerizing membership function are the following:



1) **Core:** The core of a membership function for some fuzzy set $\underline{A}$ is defined as that region of the universe that is characterised by complete membership in set $\underline{A}$. The core has elements x of the universe such that:

$$\mu_{\underline{A}}(x) = 1$$

The core of a fuzzy set may be empty set.

2) **Support:** For a fuzzy set $\underline{A}$ is defined as that region of the universe that is characterised by a nonzero membership in set $\underline{A}$. The support comprises elements x of the universe such that:

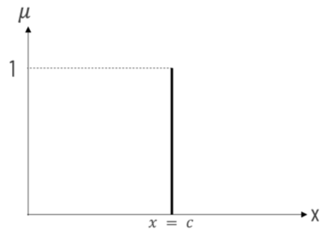
$$\mu_{\underline{A}}(x) > 0$$

3) **Boundary:** For a fuzzy set $\underline{A}$ is defined as that region of the universe containing elements that have a nonzero but not complete membership in set $\underline{A}$. The boundary comprises those elements of x of the universe such that:

$$0 < \mu_{\underline{A}}(x) < 0$$

Singleton membership function

The Singleton membership function assigns membership value 1 to a particular value of x and assigns value 0 to the rest of all. It is represented by the impulse function as shown.



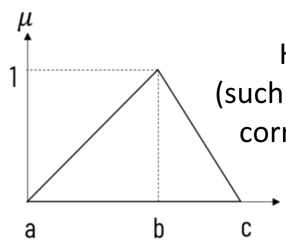
Singleton membership function Mathematically it is formulated as,

$$\mu(x) = \begin{cases} 1, & if\ x = c \\ 0, & otherwise \end{cases}$$

Triangular membership function

This is one of the most widely accepted and used membership functions (MF) in fuzzy controller design. The triangle which fuzzifies the input can be defined by three parameters a, b and c, where c defines the base and b defines the height of the triangle.

Trivial case:



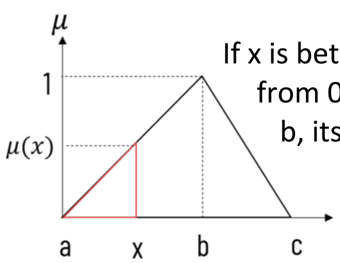
Here, in the diagram, X-axis represents the input from the process (such as air conditioner, washing machine, etc.) and the Y axis represents the corresponding fuzzy value.

If input x = b, then it is having full membership in the given set. So, μ(x) = 1, if x = b

And if the input is less than a or greater than b, then it does belong to the fuzzy set at all, and its membership value will be 0

$$\mu(x) = 0, \quad x < a \text{ or } x > c$$

x is between a and b:

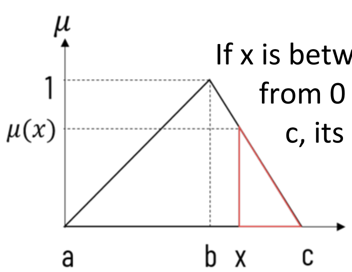


If x is between a and b, as shown in the figure, its membership value varies from 0 to 1. If it is near a, its membership value is close to 0, and if x is near b, its membership value gets close to 1.

We can compute the fuzzy value of x using a similar triangle rule,

$\mu(x) = (x - a) / (b - a), \quad a \le x \le b$

x is between b and c:



If x is between b and c, as shown in the figure, its membership value varies from 0 to 1. If it is near b, its membership value is close to 1, and if x is near c, its membership value gets close to 0.

We can compute the fuzzy value of x using a similar triangle rule,

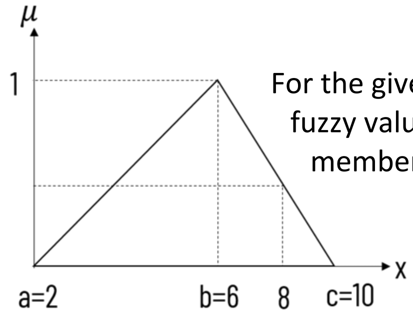
$\mu(x) = (c - x) / (c - b), \quad b \le x \le c$

Combine all together: We can combine all the above scenarios in single equation as,

$$\mu_{triangle}(x;a,b,c) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{c-x}{c-b}, & b \le x \le c \\ 0, & c \le x \end{cases}$$

$$= \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Example: Triangular membership function: Determine μ, corresponding to x = 8.0



For the given values of a, b and c, we have to compute the fuzzy value corresponding to x = 8. Using the equation of the triangular membership function,

$$\mu_{triangle}(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

$$= \max\left(\min\left(\frac{x-2}{6-2}, \frac{10-x}{10-6}\right), 0\right)$$

$$= \max\left(\min\left(\frac{x-2}{4}, \frac{10-x}{4}\right), 0\right)$$

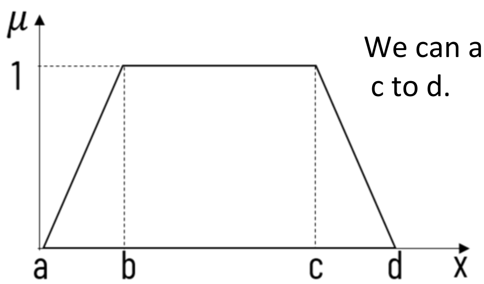
$We\ put\ x = 8.0$

$$= \max\left(\min\left(\frac{3}{2}, \frac{1}{2}\right), 0\right) = \frac{1}{2} = 0.5$$

Thus, x = 8 will be mapped to a fuzzy value of 0.5 using the given triangle fuzzy membership function

Trapezoidal membership function

The trapezoidal membership function is defined by four parameters: a, b, c and d. Span b to c represents the highest membership value that element can take. And if x is between (a, b) or (c, d), then it will have a membership value between 0 and 1.



We can apply the triangle MF if elements are in between a to b or c to d.

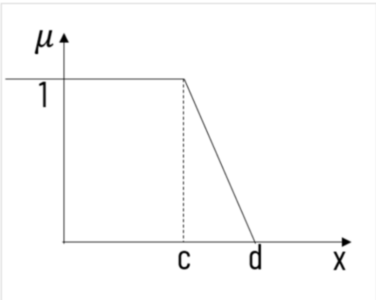
It is quite obvious to combine all together as,

$$
\mu_{trapezoidal}(x;a,b,c,d) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \dfrac{d-x}{d-c}, & c \le x \le d \\ 0, & d \le x \end{cases}
$$

$$
= \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)
$$

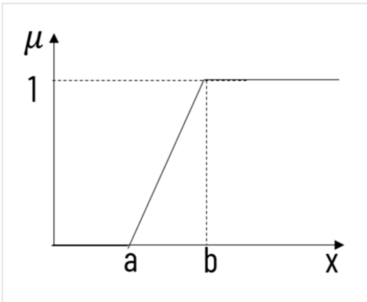
There are two special forms of trapezoidal function based on the openness of function. They are known as R-function (Open right) and L-function (Left open). Shape and parameters of both functions are depicted here:

R-function: it has a = b = -∞



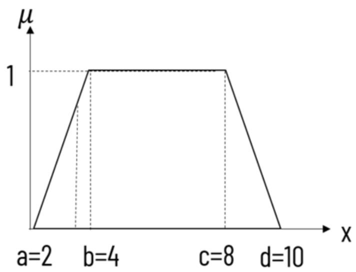
R-function

L-function: It has c = d = +∞



L-function

Example: Trapezoidal membership function Determine μ, corresponding to x = 3.5



$$
\mu_{trapezoidal}(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)
$$

$$
= \max\left(\min\left(\frac{x-2}{4-2}, 1, \frac{10-x}{10-8}\right), 0\right)
$$

$$
= \max\left(\min\left(\frac{x-2}{2}, 1, \frac{10-x}{2}\right), 0\right)
$$

$We\ put\ x = 3.5$

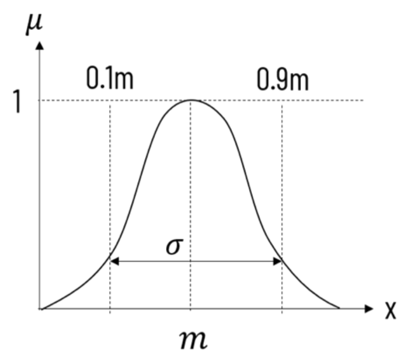
$$
= \max\left(\min\left(\frac{1.5}{2}, 1, \frac{6.3}{2}\right), 0\right)
$$

$$
= \max(0.75, 0) = 0.75
$$

Gaussian membership function

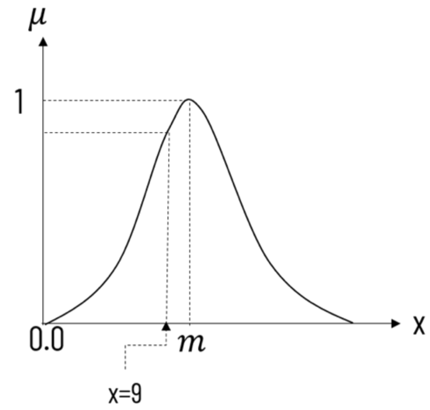
A Gaussian MF is specified by two parameters {m, σ} and can be defined as follows.

In this function, m represents the mean / center of the gaussian curve and σ represents the spread of the curve. This is a more natural way of representing the data distribution, but due to mathematical complexity, it is not much used for fuzzification.



$$
\mu_{gaussian}(x;m,\sigma) = e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}
$$

Example: Gaussian membership function: Determine μ corresponding to x = 9, m = 10 and σ = 3.0



$$\mu_{gaussian}(x; m, \sigma) = e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$$Take\ m = 10.0\ and\ \sigma = 3.0$$

$$\mu_{gaussian} = e^{-\frac{1}{2}\left(\frac{x-10}{3}\right)^2}$$

$$Put\ x = 9$$

$$\mu_{gaussian} = e^{-\frac{1}{2}\left(\frac{9-10}{3}\right)^2} = 0.9459$$

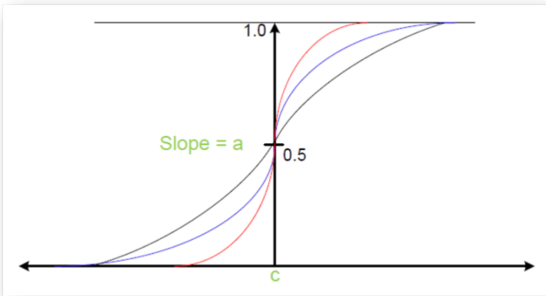
Sigmoid Membership function

Sigmoid functions are widely used in classification tasks in machine learning. Specifically, it is used in logistic regression and neural networks, where it suppresses the input and maps it between 0 and 1.

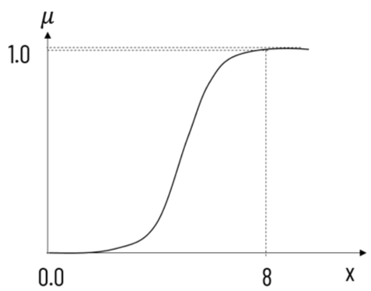
It is controlled by parameters a and c. Where *a* controls the slope at the crossover point *x = c* Mathematically, it is defined as

$$\mu_{sigmoid}(x; a, c) = \frac{1}{1 + e^{-a(x-c)}}$$

Graphically, we can represent it as,



Example: Sigmoid function: Determine μ corresponding to x = 8



$$\mu_{sigmoid}(x; a, c) = \frac{1}{1+e^{-a(x-b)}}$$

$$Take\ b = 6, a = 2$$

$$\mu_{sigmoid} = \frac{1}{1+e^{-2(x-6)}}$$

$$Put\ x = 8$$

By using the equation of the sigmoid function

$$\mu_{sigmoid} = \frac{1}{1+e^{-2(8-6)}} = \frac{1}{1+e^{-4}} = 0.98$$

membership