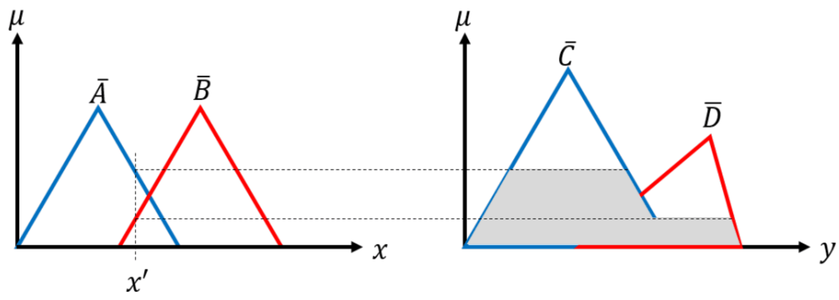
Defuzzification

- It converts the crisp input into a fuzzy value.
- **Defuzzification** converts the fuzzy output of the fuzzy inference engine into a crisp value so that it can be fed to the controller.
- The fuzzy results generated cannot be used in an application, where a decision has to be taken only on crisp values.
- A controller can only understand the crisp output. So it is necessary to convert the fuzzy output into a crisp value.
- There is no systematic procedure for choosing a good defuzzification strategy.
- The selection of defuzzification procedure depends on the properties of the application.

Rule base:

Consider the following two rules in the fuzzy rule base.

R_1: If x is A then y is C
R_2: If x is B then y is D
A pictorial representation of the above rule base is shown in the following figures



What is the **crisp output** for an input say x' ?

Defuzzification methods:

Lambda Cut Method
Maxima Methods

- Height method
- First of maxima (FoM)
- Last of maxima (LoM)
- Mean of maxima (MoM)

Weighted average method
Centroid methods

- Center of gravity method (CoG)
- Center of sum method (CoS)
- Center of area method (CoA)

Lambda Cut Method

- The lambda cut method of defuzzification converts a fuzzy set into a crisp (non-fuzzy) set by selecting all elements whose **membership value is greater than or equal to a specific threshold, called lambda(λ).**
- In other-words, $A_\lambda = \{x | \mu A(x) \geq \lambda\}$
- This Lambda-cut set A_λ is also called alpha-cut set.
- A1 = {(x1, 0.9), (x2, 0.5), (x3, 0.2), (x4, 0.3)}
- Then $A_{0.6}$ = {(x1, 1), (x2, 0), (x3, 0), (x4, 0)} = {x1}

Two fuzzy sets P and Q are defined on x as follows.

$\mu(x)$	x_1	x_2	x_3	x_4	x_5
P	0.1	0.2	0.7	0.5	0.4
Q	0.9	0.6	0.3	0.2	0.8

Find the following :

(a) $P_{0.2}$, $Q_{0.3}$

(b) $(P \cup Q)_{0.6}$

(c) $(P \cup \overline{P})_{0.8}$

(d) $(P \cap Q)_{0.4}$

The Lambda-cut method for a fuzzy set can also be extended to fuzzy relation also.

Example: For a fuzzy relation R

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}$$

We are to find λ-cut relations for the following values of $\lambda = 0, 0.2, 0.9, 0.5$

$$R_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and}$$

$$R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } R_{0.5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Maxima Method: Max- Membership Method

- A category of methods used in fuzzy logic to convert a fuzzy set into a single, precise ("crisp") output value **by selecting the element(s) with the highest degree of membership.**
- **First of Maxima (FOM):** This method selects the **smallest** value in the domain that has the maximum membership degree.
- **Last of Maxima (LOM):** This method selects the **largest** value in the domain that has the maximum membership degree.
- **Mean of Maxima (MOM):** This method calculates the **average (mean)** of all the values in the domain that share the maximum membership degree. This is commonly known as the middle-of-maxima method.

Suppose, a fuzzy set **Young** is defined as follows:

Young = {(15,0.5), (20,0.8), (25,0.8), (30,0.5), (35,0.3) }

Then the crisp value of **Young** using MoM method is

$x^* = \frac{20+25}{2} = 22.5$

Thus, a person of 22.5 years old is treated as young!

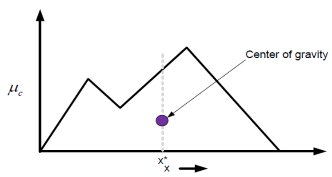
Centroid Method

- **Center of Gravity(COG):**

1. The basic principle in CoG method is to find the point x^* where a vertical line would slice the aggregate into two equal masses.
2. Mathematically, the CoG can be expressed as follows :

$$x^* = \frac{\int x.\mu_C(x)dx}{\int \mu_C(x)dx}$$

3. Graphically,



Note:

1. x^* is the x-coordinate of center of gravity.
2. $\int \mu_C(x)dx$ denotes the area of the region bounded by the curve μ_C.
3. If μ_C is defined with a discrete membership function, then CoG can be stated as :

$$x^* = \frac{\sum_{i=1}^{n} x_i.\mu_C(x_i)}{\sum_{i=1}^{n} \mu_C(x_i)} \ ;$$

4. Here, x_i is a sample element and n represents the number of samples in fuzzy set C.

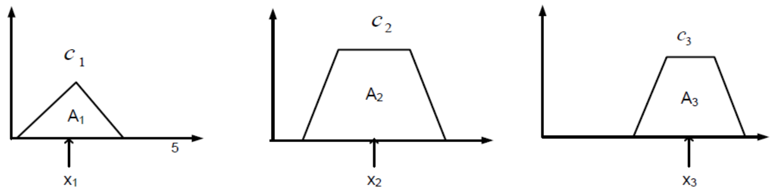
- **Center of Sum (COS)**

  If the output fuzzy set $C = C_1 \cup C_2 \cup \ldots . C_n$, then the crisp value according to CoS is defined as

$$x^* = \frac{\sum_{i=1}^{n} x_i . A_{c_i}}{\sum_{i=1}^{n} A_{c_i}}$$

  Here, A_{c_i} denotes the area of the region bounded by the fuzzy set C_i and x_i is the geometric center of the area A_{c_i}.

  Graphically,



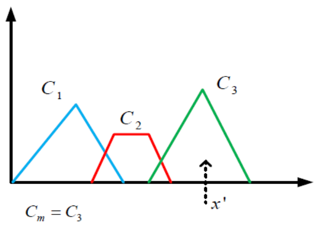
- **Center of largest Area (COA)**

  If the fuzzy set has two subregions, then the center of gravity of the subregion with the largest area can be used to calculate the defuzzified value.

  Mathematically, $x^* = \frac{\int \mu_{cm}(x).x' dx}{\int \mu_{cm}(x) dx}$;

  Here, C_m is the region with largest area, x' is the center of gravity of C_m.

  Graphically,



Weighted Average Method

- The weighted average (WA) defuzzification method finds a crisp output by calculating the weighted average of the centers of the output fuzzy sets, where each set is weighted by its maximum membership value.

$$x^* = \frac{\sum(\mu_{max_i} \times C_i)}{\sum \mu_{max_i}}$$

- x^* is the crisp defuzzified output.
- μ_{max_i} is the maximum membership value of the i-th fuzzy set.
- C_i is the center of the i-th fuzzy set (the crisp value where its membership is maximal).

Example

Given

- **Fuzzy Set 1:**
  - Center (C_1): 3
  - Maximum Membership (μ_{max1}): 0.7
- **Fuzzy Set 2:**
  - Center (C_2): 7
  - Maximum Membership (μ_{max2}): 1.0

$$x^* = \frac{(0.7 \times 3) + (1.0 \times 7)}{0.7 + 1.0} = \frac{2.1 + 7}{1.7} = \frac{9.1}{1.7} \approx 5.35$$

Numerical Questions using Lambda cut method

1. Consider two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$, both defined on X, given as follows:

$\mu(x_i;X)$	x_1	x_2	x_3	x_4	x_5
$\underset{\sim}{A}$	0.2	0.3	0.4	0.7	0.1
$\underset{\sim}{B}$	0.4	0.5	0.6	0.8	0.9

Express the following λ-cut sets using Zadeh's notation:

(a) $(\overline{\underset{\sim}{A}})_{0.7}$; (b) $(\underset{\sim}{B})_{0.2}$; (c) $(\underset{\sim}{A} \cup \underset{\sim}{B})_{0.6}$;
(d) $(\underset{\sim}{A} \cap \underset{\sim}{B})_{0.5}$; (e) $(\underset{\sim}{A} \cup \overline{\underset{\sim}{A}})_{0.7}$; (f) $(\underset{\sim}{B} \cap \overline{\underset{\sim}{B}})_{0.3}$;
(g) $(\overline{\underset{\sim}{A} \cap \underset{\sim}{B}})_{0.6}$; (h) $(\overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}})_{0.8}$

Solution: The two fuzzy sets given are

$$\underset{\sim}{A} = \left\{\frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.1}{x_5}\right\}$$

$$\underset{\sim}{B} = \left\{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5}\right\}$$

We now find the λ-cut set:

$$\boxed{A_\lambda = \{x|\mu_{\underset{\sim}{A}}(x) \geq \lambda\}}$$

(a) $(\overline{\underset{\sim}{A}})_{0.7} = 1 - \mu_{\underset{\sim}{A}}(x)$

$$= \left\{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5}\right\}$$

$$(\overline{A})_{0.7} = \{x_1, x_2, x_5\}$$

(b) $$\underset{\sim}{B} = \left\{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5}\right\}$$

$$(B)_{0.2} = \{x_1, x_2, x_3, x_4, x_5\}$$

(c) $(\underset{\sim}{A} \cup \underset{\sim}{B}) = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)]$

$$= \left\{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5}\right\}$$

$$(A \cup B)_{0.6} = \{x_3, x_4, x_5\}$$

(d) $(\underset{\sim}{A} \cap \underset{\sim}{B}) = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)]$

$$= \left\{\frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.1}{x_5}\right\}$$

$$(A \cap B)_{0.5} = \{x_4\}$$

(e) $(\underset{\sim}{A} \cup \overline{\underset{\sim}{A}}) = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\overline{\underset{\sim}{A}}}(x)]$

$$= \left\{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.7}{x_4} + \frac{0.9}{x_5}\right\}$$

$$(A \cup \overline{A})_{0.7} = \{x_1, x_2, x_4, x_5\}$$

(f) $(\underset{\sim}{B} \cap \overline{\underset{\sim}{B}}) = \min[\mu_{\underset{\sim}{B}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)]$

$$= \left\{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.2}{x_4} + \frac{0.1}{x_5}\right\}$$

$(B \cap \overline{B})_{0.3} = \{x_1, x_2, x_3\}$

(g) $(\overline{\underset{\sim}{A} \cap \underset{\sim}{B}}) = 1 - \mu_{(\underset{\sim}{A} \cap \underset{\sim}{B})}$

$$= \left\{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5}\right\}$$

$(\overline{A \cap B})_{0.6} = \{x_1, x_2, x_3, x_5\}$

(h) $(\overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}) = \max[\mu_{\overline{\underset{\sim}{A}}}(x), \mu_{\overline{\underset{\sim}{B}}}(x)]$

$$= \left\{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5}\right\}$$

$(\overline{A} \cup \overline{B})_{0.8} = \{x_1, x_5\}$

2. Using Zadeh's notation, determine the λ-cut sets for the given fuzzy sets:

$$\underset{\sim}{S}_1 = \left\{\frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1.0}{80} + \frac{1.0}{100}\right\}$$

$$\underset{\sim}{S}_2 = \left\{\frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} + \frac{.95}{80} + \frac{1.0}{100}\right\}$$

Express the following for $\lambda = 0.5$:

(a) $(\underset{\sim}{S}_1 \cup \underset{\sim}{S}_2)$; (b) $(\underset{\sim}{S}_1 \cap \underset{\sim}{S}_2)$; (c) $\overline{\underset{\sim}{S}_1}$; (d) $\overline{\underset{\sim}{S}_2}$;

(e) $(\overline{\underset{\sim}{S}_1 \cup \underset{\sim}{S}_2})$; (f) $(\overline{\underset{\sim}{S}_1 \cap \underset{\sim}{S}_2})$

Solution: The two fuzzy sets given are

$$\underset{\sim}{S}_1 = \left\{\frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1.0}{80} + \frac{1.0}{100}\right\}$$

$$\underset{\sim}{S}_2 = \left\{\frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} + \frac{.95}{80} + \frac{1.0}{100}\right\}$$

The λ-cut set is obtained using

$$A_\lambda = \{x \mid \mu_{\underset{\sim}{A}}(x) \geq \lambda\}$$

Here $\lambda = 0.5$.

(a) $(\underset{\sim}{S}_1 \cup \underset{\sim}{S}_2) = \max[\mu_{\underset{\sim}{S}_1}(x), \mu_{\underset{\sim}{S}_2}(x)]$

$$= \left\{\frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} + \frac{1.0}{80} + \frac{1.0}{100}\right\}$$

$(S_1 \cup S_2)_{0.5} = \{20, 40, 60, 80, 100\}$

(b) $(\underset{\sim}{S}_1 \cap \underset{\sim}{S}_2) = \min[\mu_{\underset{\sim}{S}_1}(x), \mu_{\underset{\sim}{S}_2}(x)]$

$$= \left\{\frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} + \frac{.95}{80} + \frac{1.0}{100}\right\}$$

$(S_1 \cap S_2)_{0.5} = \{40, 60, 80, 100\}$

(c) $\overline{\underset{\sim}{S}_1} = 1 - \mu_{\underset{\sim}{S}_1}(x)$

$$= \left\{\frac{1}{0} + \frac{0.5}{20} + \frac{0.35}{40} + \frac{0.15}{60} + \frac{0}{80} + \frac{0}{100}\right\}$$

$(\overline{S_1})_{0.5} = \{0, 20\}$

(d) $\overline{\underset{\sim}{S}_2} = 1 - \mu_{\underset{\sim}{S}_2}(x)$

$$= \left\{\frac{1}{0} + \frac{0.55}{20} + \frac{0.4}{40} + \frac{0.2}{60} + \frac{0.05}{80} + \frac{0}{100}\right\}$$

$(\overline{S_2})_{0.5} = \{0, 20\}$

(e) $(\overline{\underset{\sim}{S}_1 \cup \underset{\sim}{S}_2}) = 1 - \mu_{\underset{\sim}{S}_1 \cup \underset{\sim}{S}_2(x)}$

$$= \left\{\frac{1}{0} + \frac{0.5}{20} + \frac{0.35}{40} + \frac{0.15}{60} + \frac{0}{80} + \frac{0}{100}\right\}$$

$(\overline{S_1 \cup S_2})_{0.5} = \{0, 20\}$

(f) $(\overline{\underset{\sim}{S}_1 \cap \underset{\sim}{S}_2}) = 1 - \mu_{\underset{\sim}{S}_1 \cap \underset{\sim}{S}_2(x)}$

$$= \left\{\frac{1}{0} + \frac{0.55}{20} + \frac{0.4}{40} + \frac{0.2}{60} + \frac{0.05}{80} + \frac{0}{100}\right\}$$

$(\overline{S_1 \cap S_2})_{0.5} = \{0, 20\}$