Unit - 4

Estimation :- let $x_i, i = 1, 2, \ldots n$ is sample any function that we have to observed on the sample is called statistic

Eg:- mean $(\bar{x}) = \frac{1}{n} \Sigma x_i$

variance$(s) = \frac{1}{n} \Sigma (x_i - \bar{x})^2$

If these statistic used to find the approximate value of population then these statistic are called estimator and the value of estimator is called estimated value of statistic.

Eg:- If the sample mean $(\bar{x})$ we use to calculate the population mean (σ) then we say sample mean $(\bar{x})$ is estimator of population mean (σ).

Properties of Estimator :-

(i) Unbiased (ii) Consistent

(iii) Efficient (iv) Sufficient

Unbiased Estimator :-

If the expectation of estimator (T)

$E(T) = \theta$ then T is called

unbiased estimator of θ.

Eg :- $\boxed{E(\bar{x}) = \mu}$

$\Rightarrow$ Sample mean is unbiased estimator of Population mean.

Note :- $\boxed{E(s^2) = \frac{(n-1)}{n} \cdot \sigma^2}$

$\because E(s^2) \neq \sigma^2$

$\Rightarrow$ Sample variance is not unbiased

unbiased estimator of population variance.

But $\boxed{E(S^2) = \sigma^2}$

where $S^2 = \left(\frac{1}{n-1}\right) \Sigma (x_i - \bar{x})^2$

is unbiased estimater of σ^2.

NOTE :- If T is an unbiased estimator of $\theta \Rightarrow T^2$ is baised with θ^2.

Consistent :- let T is an estimator of θ then T is called Consistent if $T \to \theta$ as $n \to \infty$ under Probability that is

$P(|T-\theta| < \epsilon) \to 1$ as $n \to \infty$

sufficient condition for Consistent Estimator :-

(i) $E(T_n) \to r(\theta)$ as $n \to \infty$

(ii) $V(T_n) \to 0$ as $n \to \infty$

Note :- If T_n is a consistent estimater of $\gamma(\theta)$ then $f(T_n)$ is also consistent estimater of $f(\gamma(\theta))$

Note :- Sample mean $(\bar{x})$ is consistent ~~estimeator~~ estimater of μ in Normal distribution $N(\mu, \sigma^2)$

Note :- But for Cauchy distribution, sample mean $(\bar{x})$ is not consistent but sample median is consistent estimater of μ.

PTO

Note :- $\because \bar{x} = \frac{1}{n} \Sigma x_i$

$$s^2 = \frac{1}{n} \Sigma (x_i - \bar{x})^2$$

$$\Rightarrow E(\bar{x}) = \frac{1}{n} E(x_1 + x_2 + \cdots + x_n)$$

$$= \frac{1}{n} \{E(x_1) + E(x_2) + \cdots + E(x_n)\}$$

$$= \frac{1}{n} \{\mu + \mu + \cdots + \mu\}$$

$$= \frac{\cancel{n} \cdot \mu}{\cancel{n}} = \mu$$

$\Rightarrow \boxed{E(\bar{x}) = \mu} \longrightarrow$ unbiased.

$$\because V(\bar{x}) = V(\frac{1}{n} \Sigma x_i)$$

$$= \frac{1}{n^2} \{V(\Sigma x_i)\}$$

$$= \frac{1}{n^2} \{V(x_1) + V(x_2) + \cdots + V(x_n)\}$$

$$= \frac{1}{n^2} \{\sigma^2 + \sigma^2 + \cdots + \sigma^2\}$$

$$= \frac{\cancel{n} \cdot \sigma^2}{n^{\cancel{2}}} = \frac{\sigma^2}{n}$$

$$\Rightarrow \boxed{V(\bar{x}) = \frac{\sigma^2}{n}}$$

$$\Rightarrow \text{as } n \to \infty \quad \boxed{V(\bar{x}) \to 0} \Rightarrow \text{Consistent}$$

Note:- let $x_i \sim N(\mu, 1)$

then $t = \frac{1}{n}\sum x_i^2$ is unbiased estimator of (μ^2+1).

Sol^n:- $\because E(x_i) = \mu$ and

$V(x_i) = 1$

$$\Rightarrow E\left(\frac{1}{n}\sum x_i^2\right) = \frac{1}{n}\sum E(x_i^2) \quad \text{--- (1)}$$

$$\because V(X) = E(X^2) - (E(X))^2$$

$$\Rightarrow V(X_i) + (E(X_i))^2 = E(X_i^2)$$

$$\Rightarrow \sum V(x_i) + \sum (E(x_i))^2 = \sum E(x_i^2)$$

$$\sum 1 + \sum \mu^2 = \sum E(x_i^2)$$

$$n + n\cdot\mu^2 = \sum E(x_i^2)$$

$\Rightarrow \frac{1}{n} \Sigma E(x_i^2) = \mu^2+1$

$\Rightarrow E(\frac{1}{n} \Sigma x_i^2) = \mu^2+1$

$\Rightarrow (\mu^2+1)$ is unbiased estimator of $\frac{1}{n} \Sigma x_i^2$

<u>Proved</u>

<u>Que</u>:- let $x_1, x_2, x_3, x_4, x_5 \sim N(\mu, \sigma^2)$

let $t_1 = \dfrac{x_1+x_2+x_3+x_4+x_5}{5}$

$t_2 = \dfrac{x_1+x_2}{2} + x_3$

$t_3 = \dfrac{2x_1+x_2+\lambda x_3}{3}$

(i) if t_3 is unbiased estimator of μ then find λ.

(ii) Test t_1 and t_2 are unbiased or not

(iii) Test t_1 and t_2 and t_3 are consistent or not.

Sol^n:- (i) $\because$ $E(t_1) = \frac{1}{5}\{E(X_1)+E(X_2)+E(X_3)$
$+E(X_4)+E(X_5)\}$

$= \frac{1}{5}\cdot 5\mu = \mu$

$\Rightarrow$ $\boxed{t_1 \text{ is unbiased for } \mu}$

(ii) $E(t_2) = 2\mu \Rightarrow$ Not unbiased.

(iii) $E(t_3) = \mu \Rightarrow \lambda = 0$

(iv) $\because$ $V(t_1) = \frac{1}{25}\{V(X_1)+V(X_2)+V(X_3)+V(X_4)$
$+V(X_5)\}$

$= \frac{1}{25}\cdot 5\cdot\sigma^2 = \frac{\sigma^2}{5}$

$V(t_2) = \frac{3}{2}\sigma^2$ and $V(t_3) = \frac{5}{9}\sigma^2$

Note:- least variance $\Rightarrow$ Best Estimator

Q If $X_1, X_2, \cdots X_3$ is a random size of 3 from a population with mean μ and variance σ^2. let

$$T_1 = X_1 + X_2 - X_3, \quad T_2 = 2X_1 + 3X_3 - 4X_2$$

$$T_3 = \frac{1}{3}(\lambda X_1 + X_2 + X_3)$$

(i) Test the unbiasedness of T_1, T_2, T_3.

(ii) Is T_3 is consistent estimator.

(iii) Find the best Estimator.

Sol^n :- $\because$ $E(T_1) = \mu$

$E(T_2) = \mu$

For $E(T_3) = \mu \Rightarrow \boxed{\lambda = 1}$

$\because T_3 = \frac{X_1 + X_2 + X_3}{3}$ = mean $\Rightarrow$ Consistent

Maximum Likelyhood Estimator :-

M.L. funtion :-

$$L = L(\theta) = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= f(x_1, \theta) \cdot f(x_2, \theta) \cdots \cdot f(x_n, \theta)$$

We have to maximize the M.L. funtion for which we have

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 L}{\partial \theta^2} < 0 \quad \text{—} (1)$$

$$\Rightarrow \frac{1}{L}\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial}{\partial \theta}(\log L) = 0$$

Eqn (1) we can write

$$\frac{\partial}{\partial \theta}(\log L) = 0, \quad \frac{\partial^2}{\partial \theta^2}(\log L) < 0$$

Cramer-Rao Theorem :- If the probability $p \to 1$ as $n \to \infty$ then $\frac{\partial}{\partial \theta}(\log L) = 0$ has a convergent soln that is M.L. estimator's are consistent.

Note :- MLE are always consistent but may or may not be unbaised.

eg:- if $x_i \sim N(\mu, \sigma^2)$ then

$MLE(\mu) = \bar{x}$ which is both consistent and unbaised but

$MLE(\sigma^2) = s^2$ which is consistent but not unbaised.

Note !- If T is MLE of θ then $f(T)$ is also MLE of $f(\theta)$ for any one-one-onto funtion f.

Note : MLE's are most efficient estimators

Note :- $\bar{x}$ is most effcient estimator of μ.

NOTE :- <u>variance of MLE</u>

$$V(\hat{\theta}) = \frac{1}{\left\{E\left(-\frac{\partial}{\partial \theta^2} \log L\right)\right.}$$

Note :- sufficient estimators are MLE.

ex :- In random sampling from normal population $N(\mu, \sigma^2)$. find MLE for

(i) for μ, if σ^2 is know

(ii) for σ^2, if μ is given

(iii) simultaneous estimater μ and σ^2.

soln :- let $X \sim N(\mu, \sigma^2)$ then pdf is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$\Rightarrow$ ~~likely~~ likelihood function is defined by

$$L = \prod_{i=1}^{n} \left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} \right\}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot \prod_{i=1}^{n} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$\Rightarrow$ $L = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$

$\Rightarrow \log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$ —— (1)

(i) If σ^2 is given, we get the likelihood equation for estimating mean μ as

$\frac{\partial}{\partial\mu}(\log L) = 0$

$\Rightarrow -\frac{1}{2\sigma^2}\left(2\cdot\sum(x_i-\mu)(-1)\right) = 0$

$\Rightarrow \sum(x_i-\mu) = 0$

$\Rightarrow \sum x_i - \sum\mu = 0$

$\Rightarrow \sum x_i = n\cdot\mu$

$\Rightarrow \mu = \frac{1}{n}\sum x_i = \bar{x}$

$\Rightarrow$ $\boxed{\text{MLE for } \mu = \bar{x} \text{ (sample mean)}}$

(ii) If μ is given then we have to find MLE for σ^2 :

$$\frac{\partial}{\partial \sigma^2}(\log L) = 0$$

$$\Rightarrow -\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2} \Sigma (x_i - \mu)^2 = 0$$

$$\Rightarrow -n + \frac{1}{\sigma^2} \Sigma (x_i - \mu)^2 = 0$$

$$\Rightarrow \frac{1}{n} \Sigma (x_i - \mu)^2 = \sigma^2$$

$$\Rightarrow \text{MLE for } \sigma^2 = \frac{1}{n} \Sigma (x_i - \mu)^2$$

(iii) Simultenous MLE are given by

$$\frac{\partial}{\partial \mu} \log L = 0 \quad , \quad \frac{\partial}{\partial \sigma^2} (\log L) = 0$$

$$\Rightarrow \hat{\mu} = \bar{x} \quad , \quad \sigma^2 = \frac{1}{n} \Sigma (x_i - \mu)^2$$

$$= \frac{1}{n} \Sigma (x_i - \bar{x})^2$$

$$\boxed{\hat{\sigma}^2 = s^2}$$

Que: (i) Find MLE for the parameter λ in Poisson distribution of sample size n. Also find its variance.

(ii) Also show that, sample mean $\bar{x}$ is sufficient estimator for λ in Poisson distribution.

Soln:- ∵ pdf of Poisson distribution with parameter λ is defined by

$$P(X=x) = f(x,\lambda) = \frac{e^{-\lambda}\cdot \lambda^{x}}{x!}, \quad x = 0, 1, \ldots$$

⇒ Likelihood function of random sample $x_1, x_2, \ldots x_n$ is

$$L = \prod_{i=1}^{n} f(x_i,\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\cdot \lambda^{x_i}}{x_i!}$$

$$= \frac{e^{-\lambda n}\cdot \lambda^{\Sigma x_i}}{x_1!\, x_2! \cdots x_n!}$$

$$\Rightarrow \log L = -\lambda n + n\cdot\bar{x}(\log\lambda) - \Sigma \log(x_i)!$$

$$\Rightarrow \frac{\partial}{\partial\lambda}(\log L) = -n + \frac{n\bar{x}}{\lambda} = 0$$

$$\Rightarrow \boxed{\lambda = \bar{x}}$$

The variance of MLE is defined by

$$V(\hat{\lambda}) = \frac{1}{E\left(-\frac{\partial^2}{\partial \lambda^2}(\log L)\right)}$$

$$= \frac{1}{E\left(-\frac{\partial}{\partial \lambda}\left(-n + \frac{n\bar{x}}{\lambda}\right)\right)}$$

$$= \frac{1}{E\left(\frac{n\bar{x}}{\lambda^2}\right)} = \frac{1}{\frac{n}{\lambda^2} E(\bar{x})}$$

$$= \frac{1}{\frac{n}{\lambda^2} \cdot \lambda} = \frac{\lambda}{n}$$

$$\boxed{V(\hat{\lambda}) = \frac{\lambda}{n}}$$

Central Limit Theorem (CLT) :-

If X_i, $i = 1, 2, \ldots n$ be independent r.v. such that $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$ then R.V. $S_n = X_1 + X_2 + \cdots + X_n$ is assymptotically normal with μ and S.D. σ

where $\mu = \sum_{i=1}^{n} \mu_i$, $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$