Probability

Probability means possibility. It defines the likelihood of occurrence of an event. Probability is the numerical assessment of likelihood on a scale from 0 (impossibility) to 1 (absolute certainty).

There are three possible states of expectation– 'certainty', 'impossibility' and 'uncertainty'.

Probability theory is the study of uncertainty.

Experiment (Trial) is a procedure that has a well defined set of possible out comes.

Sample Space

The set of all possible outcomes of a statistical experiment is called the sample space and is represented by the symbol S.

1. Experiment :- Flip a coin.

   Sample Space $S = \{H, T\}$.

   element/member of sample space/ Sample point

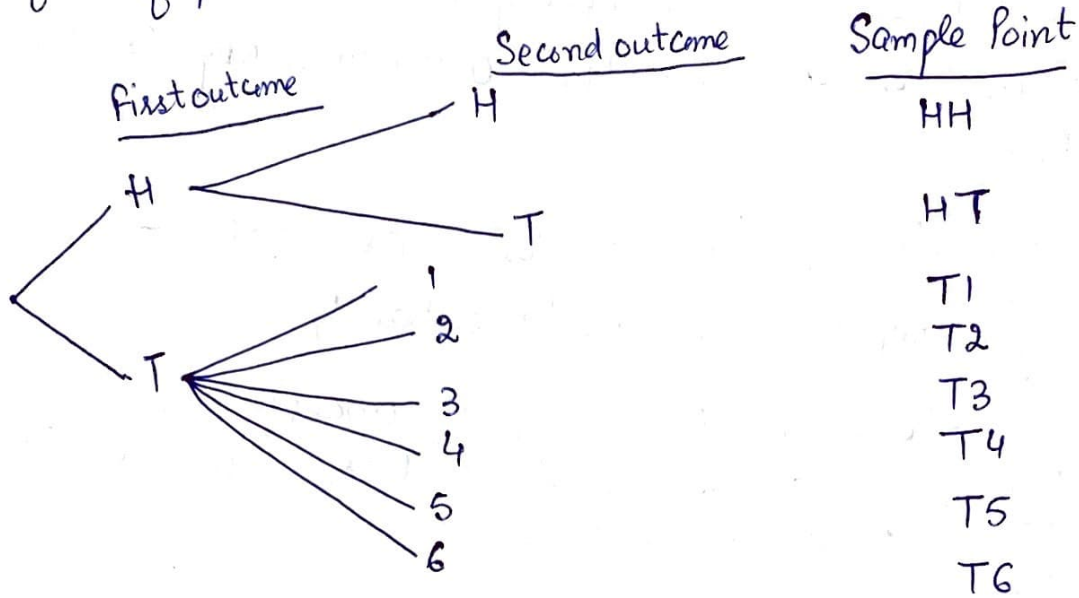
2. Experiment :- Toss a fair die (unbiased die).

   $S_1 = \{1, 2, 3, 4, 5, 6\}$. (If we are interested in the number that shows on the top face)

If we are interested only in whether the number is even or odd, then

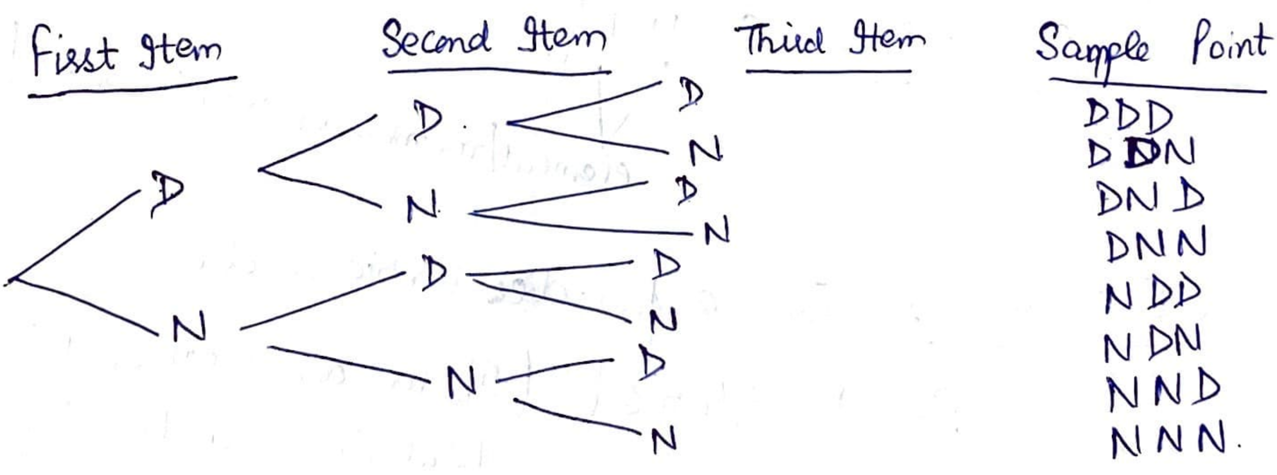
$S_2 = \{\text{even}, \text{odd}\}$.

Ex An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once.



$S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$.

Ex Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, D, or nondefective, N.



$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$.

Ex: Write the sample space if the possible outcomes of an experiment are the set of cities in the world with a population over 1 million.

$S = \{x \mid x \text{ is a city with a population over 1 million}\}$

(Statement/ Rule method)

Basic Terminology

1. ## Random Experiment

If in each trial of an experiment conducted under identical conditions, the outcome is not unique, but may be any of the possible outcomes, then such an experiment is called a random experiment.

Ex:- Tossing a coin, throwing a die, selecting a card from a pack of playing cards, selecting a family out of a given group of families etc.

2. Outcome : The result of a random experiment is called an out come. Ex:- H or T, when a fair die is tossed.

3. ## Trial and Event

Any particular performance of a random experiment is called a trial and out come or combination of outcomes are termed as events.

Ex : (i) If a coin is tossed repeatedly, the result is not unique. We may get any of the two faces, head or tail. Thus, tossing a coin is a random experiment or trial and getting of a head or tail is an event.

(ii) Experiment:- Throwing a die.
Outcomes :- 1, 2, 3, 4, 5, 6
Events : 'Odd number', 'even number' or 'Getting a point greater than 4'.

Event

An event A is the set of outcomes or in other words, a subset of sample space.

Ex: $A = \{3, 6\}$ if we are interested in the event A that the outcome when a die is tossed is divisible by 3.

Favourable Events or cases

The number of cases favourable to an event in an experiment, is the number of outcomes which ensures the happening of the event.

Ex: (i) In drawing a card from a pack of cards the number of cases favourable to drawing of an ace is 4, for drawing a spade is 13 and for drawing a red card is 26.

(ii) In throwing of two dice, the number of cases favourable to getting the sum 5 is

$(1,4), (4,1), (2,3), (3,2)$ i.e. 4.

Complement of an event

The complement of an event A with respect of S is the subset of all elements of S that are not in A.

Notation: A^c or A'.

$$A' = \{x \mid x \notin A\}.$$

Ex: $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{3, 6\}$ $\qquad A^c = \{1, 2, 4, 5\}$.

Ex: Let R be the event that a red card is selected from an ordinary deck of 52 playing cards and let S be the entire deck.

Then, R' is the event that the card selected from the deck is not a red card but a black card.

Ex: Consider the sample space

$S = \{$book, cell phone, mp3, paper, stationery, laptop$\}$.

Let $A = \{$book, stationery, laptop, paper$\}$.

Then, $A' = \{$cell phone, mp3$\}$.

Intersection of two events

The intersection of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

Ex: $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$, $B = \{4, 5, 6\}$

A ↓ even number

B ↓ number greater than 3

$A \cap B = \{4, 6\} \rightarrow$ an even number greater than 3.

Ex: Let E be the event that a person selected at random in a classroom is majoring in engineering and let F be the event that the person is female.

Then, $E \cap F$ is the event of all female engineering students in the classroom.

Let $V = \{a, e, i, o, u\}$ and $C = \{l, r, s, t\}$

$V \cap C = \phi$

V are C are mutually exclusive or disjoint.

Mutually Exclusive or disjoint events

Two events A are B are mutually exclusive or disjoint if $A \cap B = \phi$, that is, if A and B have no elements in common.

In other words, A and B are mutually exclusive iff they cannot occur simultaneously.

Union of two events

The union of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

Ex: In die-tossing experiment, if

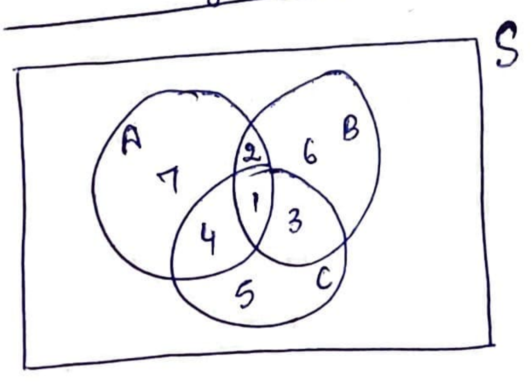
$A = \{2, 4, 6\}$, $B = \{4, 5, 6\}$,

then, $A \cup B = \{2, 4, 5, 6\}$

Ex: Let $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$,

then $A \cup B = \{a, b, c, d, e\}$.

Venn diagrams



$A \cap B$ = Regions 1 and 2

$B \cap C$ = Regions 1 and 3

$A \cup C$ = Regions 1, 2, 3, 4, 5 and 7

$A \cap B'$ = Regions 4 and 7.

$A \cap B \cap C$ = Region 1

$(A \cup B) \cap C'$ = Regions 2, 6, 7.

Certain (Sure) Event

An event associated with a random experiment is called a certain event if it always occurs whenever the experiment is performed.

Ex: Getting a number less than 7, when we throw a die.

Impossible event

An event associated with a random experiment is called an ~~certain event if it always occurs~~ impossible event if never occurs whenever the experiment is performed.

Ex: Getting a number 7 on a six-sided die.

Mutually Exhaustive Events

Two or more events associated with a random experiment are exhaustive if their union is the sample space.

Equally likely Events

The random events that have an equal chance of occurring are termed equally likely events.

Ex: (i) In a random toss of an unbiased coin, head and tail are equally likely events.

(ii) In throwing an unbiased die, all the six faces are equally likely to come.

Independent Events ~~Sevel~~

Several events are said to be independent if their occurrence is not dependent on any other event.

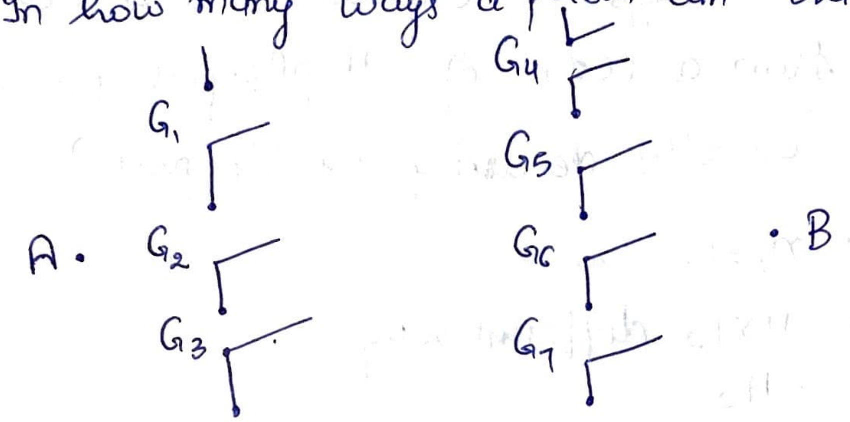
Ex :- (i) When a die is thrown twice, the result of the first throw does not affect the result of the second throw.

(ii) In tossing a coin, the event of getting a head in the first toss is independent of getting a head in second toss.

Counting Sample Points

Rule 1: If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Ex: In how many ways a person can travel from point A to B?



Level I   Level II

G1 – G4	G2 – G4	G3 – G4
G1 – G5	G2 – G5	G3 – G5
G1 – G6	G2 – G6	G3 – G6
G1 – G7	G2 – G7	G3 – G7

Possible ways = 12.

Here $n_1 = 3$, $n_2 = 4$

$\therefore$ Possible ways $= 3 \times 4 = 12$.

Ex: How many sample points are there in the sample space when a pair of dice is thrown once?

Sol: $n_1 = 6$, $n_2 = 6$

$n_1 n_2 = 36$ possible ways.

Sample points = 36.

Ex: If a 22 member club needs to elect a chair and a treasurer, how many different ways can be these two be elected?

Sol: $n_1 = 22$, $n_2 = 21$

$n_1 n_2 = 22 \times 21 = 462$ different ways.

Ex: In how many ways a captain and a vice captain be selected from a team of 11 players where every candidate is equally deserving for the post?

Sol: $n_1 = 11$, $n_2 = 10$

$n_1 \times n_2 = 11 \times 10$ different ways.

$= 110$

Rule 2: If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of these first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

Ex: Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Sol: Since $n_1 = 2$, $n_2 = 4$, $n_3 = 3$, $n_4 = 5$, there are

$2 \times 4 \times 3 \times 5 = 120$ different ways to order the parts.

Q:- A local telephone number is seven digit sequence in which the 1st digit cannot be either 0 or 1. How many such telephone numbers can be formed?

Sol:- 8 10 10 10 10 10 10

80,00000 different numbers can be formed.

* If repetition is not allowed.

8 9 8 7 6 5 4

4,83,840 different numbers.

Q:- How many even four digit numbers can be formed from the digits 0, 1, 2, 5, 6 and 9 if each digit can be used only once.

Sol:- Even four digit number

_ _ _ _

For unit digit, we have only three choices, 0, 2, 6.
Also, the thousands position cannot be 0.
Hence, we consider the unit digit in two parts, 0 or not 0.

If the unit position is 0,

$$\underset{3}{\downarrow}\quad \underset{4}{\downarrow}\quad \underset{5}{\downarrow}\quad \overset{0}{\underset{1\text{ choice}}{\downarrow}}$$

∴ Total $3 \times 4 \times 5 \times 1 = 60$ ~~choice~~ even four digit numbers.

If the unit position is not 0,

$$\underset{4}{\underline{\ }}\ \ \underset{4}{\underline{\ }}\ \ \underset{3}{\underline{\ }}\ \ \underset{2\text{ choices}}{\overset{2016}{\underline{\ }}}$$

$\therefore$ Total $4 \times 4 \times 3 \times 2 = 96$ even four digit numbers.

Since the above two cases are mutually exclusive, the total number of even four digit numbers can be calculated as $60 + 96 = 156$.

Permutation and Combination

> Number of permutations of n objects taken r at a time
> $$^nP_r = \frac{n!}{(n-r)!}$$

* A permutation is an arrangement of all or part of a set of objects.

Consider 3 letters a, b, c

Permutations: abc, acb, bac, bca, cab, cba.

* The number of permutations of n objects is $n!$

The number of permutations of four letters a, b, c, d will be $4! = 24$.

Now, the number of permutations by taking two letters at a time from 4.

ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc

i.e. $^4P_2 = \frac{4!}{2!} = 4 \cdot 3 = 12$.

Ex: In one year, three awards (research, teaching and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Sol: $^{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = 25 \cdot 24 \cdot 23 = 13,800.$

Ex: A president and a treasurer are to be choosen from a student club consisting of 50 people. How many different choices of officers are possible if

(a) there are no restrictions.

Sol: $^{50}P_2 = \frac{50!}{48!} = 50 \cdot 49 = 2450.$

(b) A will serve only if he is president.

We have two situations

(i) A is selected as president.

Choices $= (1)(49) = 49.$

(ii) president and treasurer are choosen from the remaining 49 people without A.

$\therefore$ No of choices $= {}^{49}P_2 = \frac{49!}{47!} = 49 \cdot 48 = 2352.$

$\therefore$ Total number of choices $= 49 + 2352 = 2401.$

(C) B and C will serve together or not at all.

(i) The number of selections when B and C serve together is 2.

(ii) The number of selections, when B and C are not choosen is

$$^{48}P_2 = \frac{48!}{46!} = 48 \cdot 47 = 2256.$$

$\therefore$ Total choices $= 2 + 2256 = 2258.$

(d) D and E will not serve together.

D,E will serve together in 2 ways only.

$\therefore$ D and E will not serve together in $2450 - 2$
$= 2448$ ways.

Result 1. The number of permutations of n objects arranged in a circle is $(n-1)!$

2. The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, --- n_k of kth kind is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Ex: In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

Sol:
$$\frac{10!}{1!\,2!\,4!\,3!} = \frac{10 \cdot 9 \cdot \cancel{8}^{4} \cdot \cancel{6} \cdot 5 \cdot \cancel{4}^{7}}{\cancel{2} \cdot \cancel{3 \cdot 2}}$$
$= 12,600.$

Result: The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second and so forth, is

$$\binom{n}{n_1, n_2 \cdots, n_r} = \frac{n!}{n_1! \, n_2! \cdots n_r!}$$

where $n_1 + n_2 + \cdots + n_r = n$.

Ex: In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel ~~d~~ rooms during a conference?

Sol:
$$\frac{7!}{3! \, 2! \, 2!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4}}{\cancel{2} \cdot \cancel{2}} = 210.$$

Combinations

In many problems, we are interested in the number of ways of selecting r objects from n without regard to order. These selections are called combinations.

The number of combinations of n distinct objects taken r at a time is

$$^{n}C_r = \frac{n!}{r!(n-r)!}$$

Ex: How many different letter arrangements can be made from the letters in the word STATISTICS?

Sol:
$$\frac{10!}{3! \, 3! \, 1! \, 2! \, 1!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4} \cdot 2}{\cancel{3 \cdot 2 \cdot 2}} = 50{,}400.$$

Ex:- In how many different ways can 3 of 20 labotatory assistants be choosen to assist with an experiment?

Sol:- $^{20}C_3 = \frac{20!}{3!\,17!} = \frac{20 \cdot 19 \cdot \cancel{18}^{3}}{\cancel{3 \cdot 2}} = 1140.$

Ex:- In how many ways a captain and vice captain be selected from a team of 11 players where every candidate is equally deserving for the post?

Sol:- $^{11}P_2 = \frac{11!}{9!} = 11 \cdot 10 = 110$

C	VC
A	B
B	A

] Two arrangements

Ex: In how many ways 2 players be selected from a team of 11 players?

AB ≡ BA

Sol:- $^{11}C_2 = \frac{11!}{2!\,9!} = \frac{11 \cdot 10}{2} = 55.$

Probability of an Event

Let set S represent the sample space and A be the set containing outcomes of an event, then

$$p(A) = \frac{n(A)}{n(S)}.$$

The probability of an event ranges between 0 to 1.

To every point in the sample space, we assign a probability such that the sum of all probabilities is 1.

$p(\text{impossible event}) = 0$

$p(\text{sure event}) = 1.$

Def : The probability of an event A is

$$p(A) = \frac{n(A)}{n(S)}$$

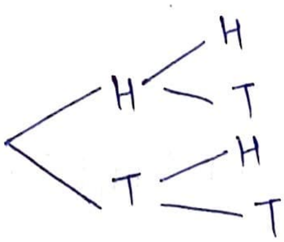
and $0 \leq P(A) \leq 1$, $P(\phi) = 0$, $P(S) = 1.$

If $A_1, A_2, \ldots, A_n, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Ex : A coin is tossed twice. What is the probability that atleast 1 head occurs?

Sol : $S = \{HH, HT, TH, TT\}$



$P(\text{at least one head occurs})$

$$= \frac{3}{4}.$$

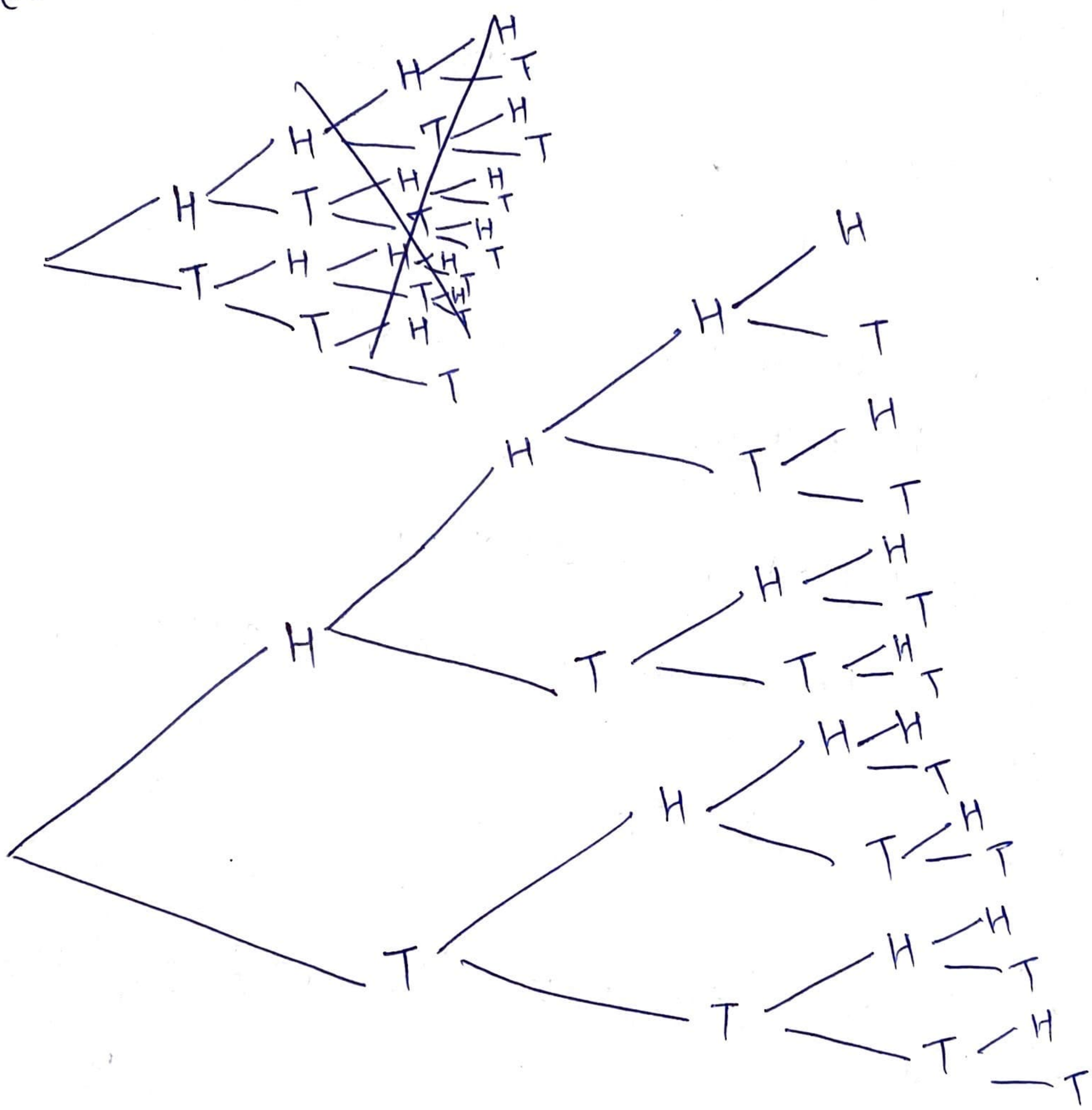
Favourable cases = HH, HT, TH

Ex: A fair coin is tossed 4 times. Define the sample space corresponding to this random experiment. Also give the subsets corresponding to the following events and find the respective probabilities.

(a) More heads than tails are obtained.

(b) Tails occur on the even numbered tosses.

Sol:



S = { HHHH, HHHT, HHTH, HHTT, HTHH, HTTT, HTHT, HTTH, THHH, THHT, THTH, THTT, TTHH TTHT, TTTH, TTTT }

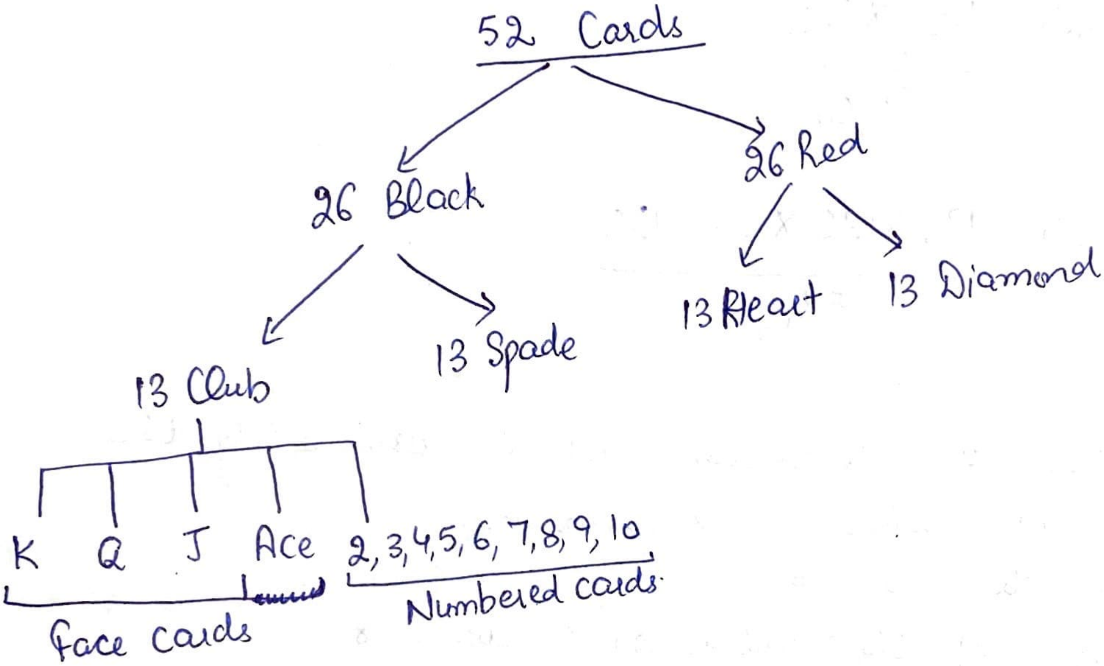
(a) A = { HHHH, HHHT, HHTH, HTHH, THHH }

$P(A) = \frac{5}{16}$, where A is the event that more heads occur than tails.

Let B be the event that tails occur on the even numbered toss.

$$B = \{HTTT, HTHT, TTHT, TTTT\}$$

$$P(B) = \frac{4}{16} = \frac{1}{4}.$$



face cards = 12

Numbered cards = 36.

<u>Ex</u> : (a) Four cards are drawn at random from a pack of 52 cards. Find the probability that

(i) They are a king, a queen, a jack and an ace.

(ii) Two are kings and two are queens.

(iii) Two are black and two are red.

(iv) There are two cards of hearts and two cards of diamonds.

(b) In shuffling a pack of cards, four are accidentally dropped, find the chance that the missing cards should be one from each suit ?.

<u>Sol</u> : (a) (i) Reqd. probability $= \dfrac{{}^4C_1\, {}^4C_1\, {}^4C_1\, {}^4C_1}{{}^{52}C_4} = \dfrac{256}{{}^{52}C_4}.$

(ii) Reqd prob $= \dfrac{{}^{4}C_2 \times {}^{4}C_2}{{}^{52}C_4}$

(iii) Reqd prob $= \dfrac{{}^{26}C_2 \times {}^{26}C_2}{{}^{52}C_4}$

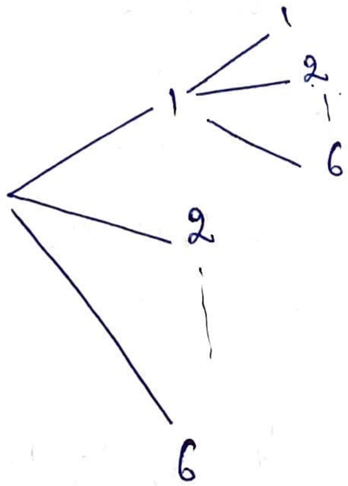
(iv) Reqd prob $= \dfrac{{}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_4}$.

(b) Reqd prob $= \dfrac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4}$

Ex Two unbiased dice are thrown. Find the probability that
(i) both the dice show the same number.
(ii) the first die shows 6
(iii) the total of the numbers on the dice is 8.
(iv) the total of the numbers on the dice is greater than 8
(v) the total of the numbers on dice is 13 and
(vi) the total of the numbers on dice is any number from 2 to 12, both inclusive.

Sol :

Total number of cases = 36.



(i) Let A be the event that both dice show the same number.

$A = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$P(A) = \frac{6}{36} = \frac{1}{6}$.

(ii) Let B be the event that first die shows 6.

$B = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$.

$P(B) = \frac{1}{6}$.

(iii) Let C be the event that total of the numbers on the dice is 8.

$C = \{(2,6), (6,2), (3,5), (5,3), (4,4)\}$

$P(C) = \frac{5}{36}$.

(iv) Let D be the event that total of the numbers on dice is greater than 8.

$D = \{(3,6), (6,3), (4,6), (6,4), (5,6), (6,5), (5,5), (6,6), (4,5), (5,4)\}$.

$P(D) = \frac{10}{36} = \frac{5}{18}$.

(v) Let E be the event that total of the numbers on dice is 13. ⇒ E is impossible event.

$P(E) = \frac{0}{36} \Rightarrow P(E) = 0$.

(vi) Let F be the event that total of the numbers on dice is any number from 2 to 12, both inclusive ⇒ F is certain event.

~~$P(F) = \frac{}{36}$~~ $P(F) = 1$.

$F = \{(1,1), (1,2), (2,1), (2,2), (1,3), (3,1), \ldots (6,6)\}$

Ex :- (a) Among the digits 1, 2, 3, 4, 5 at first one is choosen and then second selection is made among the remaining four digits. Assuming that all twenty possible outcomes have equal probabilities, find the probability that an odd digit will be selected

(i) the first time (ii) second time (iii) both times.

(b) From 25 tickets, marked with first 25 numerals, one is drawn at random. Find the chance that (i) it is multiple of 5 or 7 and (ii) it is a multiple of 3 or 7.

Sol :- (a) Total number of cases = $5 \times 4 = 20$.

(i)

$\underset{\substack{3 \text{ choices} \\ 1,3,5}}{\underline{\ \ }}\ \underset{4 \text{ choices.}}{\underline{\ \ }}$

Favourable cases $= 3 \times 4 = 12$

Reqd prob $= \frac{12}{20} = \frac{3}{5}$.

[(1, 2), (1,3), (1, 4), (1, 5), (3, 1), (3, 2), (3, 4), (3, 5), (5, 1), (5, 2), (5, 3), (5, 4)]

(ii) Reqd prob $= \frac{3}{5}$.

(iii)

$\underset{3 \text{ choices}}{\underline{\ \ }}\ \underset{2 \text{ choices.}}{\underline{\ \ }}$

Reqd prob $= \frac{6}{20} = \frac{3}{10}$.

[(1, 3), (1, 5), (3, 1), (3, 5), (5, 1), (5, 3)]

(b) (i) Reqd prob $= \frac{8}{25}$

Favourables cases = 5, 10, 15, 20, 25, 7, 14, 21

(ii) Reqd prob $= \frac{10}{25} = \frac{2}{5}$.

Favourable cases = 3, 6, 9, 12, 15, 18, 21, 24, 7, 14.

Ex :- An urn contains 6 white, 4 red and 9 black balls. If 3 balls are drawn at random, find the probability that

(i) two of the balls drawn are white

(ii) one of each colour

(iii) none is red

(iv) at least one is white.

Sol :- Total cases $= {}^{19}C_3$ as there are $6+4+9=19$ balls in the urn.

(i) Reqd prob $= \dfrac{{}^{6}C_2 \times {}^{13}C_1}{{}^{19}C_3}$

(ii) Reqd prob $= \dfrac{{}^{6}C_1 \times {}^{4}C_1 \times {}^{9}C_1}{{}^{19}C_3}$

(iii) Reqd prob $= \dfrac{{}^{15}C_3}{{}^{19}C_3}$

(iv) Reqd prob $= 1 - P(\text{None of the three balls is white})$

$= 1 - \dfrac{{}^{13}C_3}{{}^{19}C_3}$

Ex :- (a) A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find P(E).

(b) Let A be the event that an even number turns up and B be the event that a number divisible by 3 occurs. Find $P(A \cup B)$ and $P(A \cap B)$.

Sol →

$S = \{1, 2, 3, 4, 5, 6\}$

Let $P(\text{odd number}) = k$

$P(\text{even number}) = 2k$.

Total probability $= 1$.

$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$

$k + 2k + k + 2k + k + 2k = 1$

$9k = 1$

$k = \frac{1}{9}$.

$E = \{1, 2, 3\}$

$P(E) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \cancel{\frac{6}{9}} = \cancel{\frac{2}{3}} = \frac{4}{9}$

(b) $A = \{2, 4, 6\}$, $B = \{3, 6\}$

$A \cup B = \{2, 3, 4, 6\}$, $A \cap B = \{6\}$

$P(A \cup B) = \frac{2}{9} + \frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{7}{9}$.

$P(A \cap B) = \frac{2}{9}$.

Ex A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are choosen from the lot at random. Find the probability that (i) both are good (ii) both have major defects (iii) at least 1 is good (iv) at most 1 is good (v) exactly 1 is good (vi) neither has major defects (v) neither is good.

Sol: Total articles $= 10 + 4 + 2 = 16$.

Total cases $= {}^{16}C_2$.

(i) P(both are good) $= \frac{{}^{10}C_2}{{}^{16}C_2} = \frac{3}{8}$.

(ii) P(both have major defects) $= \frac{{}^{2}C_2}{{}^{16}C_2} = \frac{1}{120}$.

(iii) P(at least 1 is good) = P(exactly 1 is good or both are good)

$$= \frac{{}^{10}C_1 \times {}^{6}C_1 + {}^{10}C_2}{{}^{16}C_2} = \frac{7}{8}.$$

or 1 − P(None is good)

$$= 1 - \frac{{}^{6}C_2}{{}^{16}C_2} = \frac{7}{8}.$$

(iv) P(at most 1 is good) = P(none is good or 1 is good and 1 is bad)

$$= \frac{{}^{6}C_2 + {}^{10}C_1 \times {}^{6}C_1}{{}^{16}C_2} = \frac{5}{8}.$$

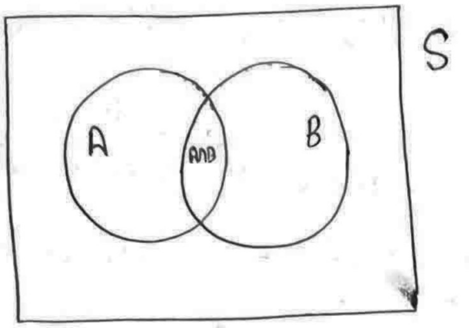
(v) P(exactly 1 is good) $= \frac{{}^{10}C_1 \times {}^{6}C_1}{{}^{16}C_2} = \frac{1}{2}$.

(vi) P(neither has major defects) $= \frac{{}^{14}C_2}{{}^{16}C_2} = \frac{91}{120}$

(vii) P(neither is good) $= \frac{{}^{6}C_2}{{}^{16}C_2} = \frac{1}{8}$.

Additive Rules

Rule 1 If A and B are two events,
then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.



* If A and B are mutually exclusive,
then $P(A \cup B) = P(A) + P(B)$. $[\because A \cap B = \phi]$

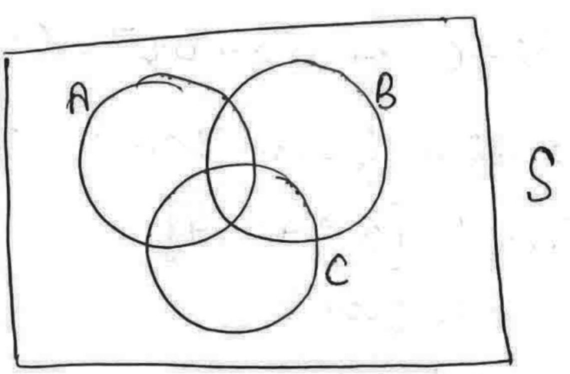
* If $A_1, A_2, \ldots, A_n$ are mutually exclusive events,
then $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$.

* A collection of events $\{A_1, A_2, \ldots, A_n\}$ of a sample space S is called a partition of S if $A_1, A_2, \ldots, A_n$ are mutually exclusive and $A_1 \cup A_2 \cup \ldots \cup A_n = S$.

* If $A_1, A_2, \ldots, A_n$ is a partition of S, then
$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) = P(S) = 1$.

Rule 2: For three events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$



Ex: What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

Sol: Let A be the event that the integer selected at random is divisible by 2, B be the event that it is divisible by 5.

Then, $A \cup B$ is the event that it is divisible by either 2 or 5.

Also, $A \cap B$ is the event that it is divisible by both 2 and 5.

$\therefore$ $|A| = 50$, $|B| = 20$, $|A \cap B| = 10$

$$P(A \cup B) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{60}{100} = \frac{6}{10} = \frac{3}{5}.$$

Result: $P(\bar{A}) = 1 - P(A)$, where $\bar{A}$ is the complement of event A.

or $P(A) + P(A') = 1$.

Ex: If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7 or 8 or more cars on any given workday are, resp, 0.12, 0.19, 0.28, 0.24, 0.10 and 0.07. What is the probability that he will service at least 5 cars on his next day at work?

Sol: Let A be the event that at least 5 cars are serviced. $P(A) = 0.28 + 0.24 + 0.10 + 0.07 = 0.69$.

or Thus, A' is the event that fewer than 5 cars are serviced.

$\therefore$ $P(A') = 1 - P(A) = 1 - [0.12 + 0.19]$

$= 1 - 0.31$

$= 0.69.$

$$P(A \cup B) = 1 - P(\overline{A} \cap \overline{B})$$
$$= 1 - 0.2$$
$$= 0.8$$

Ex: Three newspapers A, B and C are published in a certain city. It is estimated from a survey that of the adult Population: 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read both B and C, 2% read all three. Find what percentage read at least one of the papers.

Sol: Let E: The adult reads newspaper A
F: The adult reads newspaper B
G: The adult reads newspaper C.

$P(E) = \frac{20}{100}$, $P(F) = \frac{16}{100}$, $P(G) = \frac{14}{100}$, $P(E \cap F \cap G) = \frac{2}{100}$

$P(E \cap F) = \frac{8}{100}$, $P(E \cap G) = \frac{5}{100}$, $P(F \cap G) = \frac{4}{100}$,

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)$$
$$= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{5}{100} - \frac{4}{100} + \frac{2}{100}$$
$$= \frac{35}{100} = 0.35.$$

Hence, 35% of the adult population reads at least one of the paper.

What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?

Sol : Let A: 7 occurs $\Rightarrow A = \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$

B: 11 occurs $\Rightarrow B = \{(5,6),(6,5)\}$

$P(A) = \frac{6}{36} = \frac{1}{6}$, $P(B) = \frac{2}{36} = \frac{1}{18}$.

$P(A \cap B) = 0$

$P(A \cup B) = \frac{1}{6} + \frac{1}{18} = \frac{4}{18} = \frac{2}{9}$

Q. The probability that a student passes a Physics test is $\frac{2}{3}$ and the probability that he passes both a physics and an English test is $\frac{14}{45}$. The probability that he passes at least one test is $\frac{4}{5}$. What is the probability that he passes the English test?

Sol: Let A be the event that the student passes a Physics test, B be the event that the student passes an English test.

$$P(A) = \frac{2}{3}, \quad P(A \cap B) = \frac{14}{45}, \quad P(A \cup B) = \frac{4}{5}.$$

$$P(A) + P(B) - P(A \cap B) = P(A \cup B)$$

$$\Rightarrow \quad P(B) = P(A \cup B) - P(A) + P(A \cap B)$$

$$= \frac{4}{5} - \frac{2}{3} + \frac{14}{45}$$

$$= \frac{36 - 30 + 14}{45} = \frac{20}{45} = \frac{4}{9}.$$

Ex: An MBA applies for a job in two firms X and Y. The probability of his being selected in firm X is 0.7 and being rejected at Y is 0.5. The probability of at least one of his applications being rejected is 0.6. What is the probability that he will be selected in at least one of the firms?

Sol: Let A: The person is selected in firm X.
B: The person is selected in firm Y.

$$P(A) = 0.7 \Rightarrow P(\bar{A}) = 1 - 0.7 = 0.3$$

$$P(\bar{B}) = 0.5 \Rightarrow P(B) = 1 - 0.5 = 0.5$$

$P(A) = 0.7$

$P(\bar{B}) = 0.5 \Rightarrow P(B) = 0.5.$

$P(\bar{A} \cup \bar{B}) = 0.6$

$\Rightarrow 1 - P(A \cap B) = 0.6$

$\Rightarrow P(A \cap B) = 1 - 0.6 = 0.4$

$P(A \cup B) = 0.7 + 0.5 - 0.4$

$= 0.8$

Conditional Probability & Multiplicative Rules

Conditional Probability

Ex: A couple has two children. Find the probability that both children are boys.

Sol: $S = \{bb, bg, gb, gg\}$

$P(\text{both children are boys}) = \frac{1}{4}$.

Ex: A couple has two children. Find the probability that both children are boys if it is known that

(i) atleast one of the children is a boy.

(ii) the older child is a boy.

Sol: (i) $S = \{bb, bg, gb\}$

~~P(at least~~ $P(\text{both children are boys}) = \frac{1}{3}$

(ii) $S = \{bb, bg\}$

$P(\text{both children are boys}) = \frac{1}{2}$.

Def: The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) > 0.$$

Remark : (i) $P(B|A) = \dfrac{P(B\cap A)}{P(A)}$ and $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

Thus, Conditional probabilities $P(B|A)$ and $P(A|B)$ are defined iff $P(A) \neq 0$ and $P(B) \neq 0$, resp.

(ii) If $P(B) = 0$, then $P(A|B)$ is not defined.

(iii) $P(B|B) = 1$.

Independent Events

If happening of an event is not dependent on any other event, the events are called independent.

Def : Two events A and B are independent iff

$$P(B|A) = P(B) \text{ or } P(A|B) = P(A).$$

Otherwise, A and B are dependent.

* $P(A\cap B) = P(A).P(B)$ iff Two events A and B are independent.

Ex : A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Sol : Let A: one of tubes drawn is good

B: the other tube is also good.

$P(A\cap B) = P(\text{both tubes drawn are good})$

$$= \frac{{}^6C_2}{{}^{10}C_2} = \frac{\frac{\cancel{3}\cancel{6}.5}{\cancel{2}}}{\frac{\cancel{2}\cancel{10}.\cancel{9}3}{\cancel{2}}} = \frac{1}{3}.$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{1/3}{6/10} = \frac{5}{9}$$

Ex: The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$, the probability that it arrives on time is $P(A) = 0.82$, and the probability that it arrives and departs on time is $P(D\cap A) = 0.78$. Find the probability that a plane

(a) arrives on time, given that it departed on time.

(b) departed on time, given that it has arrived on time.

Sol:

$P(D) = 0.83$

$P(A) = 0.82$

$P(D\cap A) = 0.78$

(a) $$P(A|D) = \frac{P(A\cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94$$

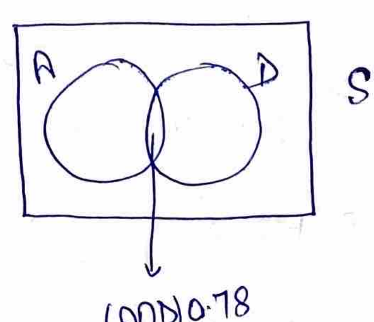
(b) $$P(D|A) = \frac{P(D\cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95$$

(c) Find the probability that a plane arrives on time, given that it did not depart on time.

Sol:

$$P(A|D') = \frac{P(A\cap D')}{P(D')}$$

$P(D') = 1 - 0.83 = 0.17$



$$P(A|D') = \frac{0.82 - 0.78}{0.17}$$

$$= \frac{0.04}{0.17} = 0.24.$$

$\frac{1}{2}$

Ex: Strips of a particular type of cloth are being produced in ~~an industrial~~ a textile industry. These strips can be defective in two ways, length and nature of texture. It is known that 10% of the strips fail the length test, 5% fail the texture test and only 0.8% fail both tests. If a strip is selected at random and a quick measurement identifies that it fails the length test, What is the probability that it is texture defective?

Sol: Let T: length defective
L: texture defective

$$P(T|L) = \frac{P(T\cap L)}{P(L)} = \frac{0.8/100}{10/100} = \frac{0.8}{10} = 0.08.$$

Ex: From a city population, the probability of selecting
(i) a male or a smoker is $\frac{7}{10}$.
(ii) a male smoker is $\frac{2}{5}$
(iii) a male, if a smoker is already selected is $\frac{2}{3}$.
Find the probability of selecting
(a) a non-smoker
(b) a male
(c) a smoker, if a male is first selected.

Sol: Let A: a male is selected.
B: a smoker is selected.

$$P(A\cup B) = \frac{7}{10}, \quad P(A\cap B) = \frac{2}{5}, \quad P(A|B) = \frac{2}{3}.$$

(a) $P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow P(B) = \frac{P(A\cap B)}{P(A|B)} = \frac{2}{5}\times\frac{3}{2} = \frac{3}{5}$

$$P(B') = 1-\frac{3}{5} = \frac{2}{5}.$$

(b) $P(A) + P(B) - P(A \cap B) = P(A \cup B)$

$\Rightarrow$ $P(A) = P(A \cup B) + P(A \cap B) - P(B)$

$= \frac{7}{10} + \frac{2}{5} - \frac{3}{5}$

$= \frac{7+4-6}{10} = \frac{5}{10} = \frac{1}{2}$

(C) $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{2/5}{1/2} = \frac{2}{5} \times \frac{2}{1} = \frac{4}{5}$.

Ex: A bag contains 10 gold and 8 silver coins. Two successive drawings of 4 coins are made such that

(i) Coins are replaced before the second trial

(ii) the coins are not replaced before the second trial.

Find the probability that the first drawing will give 4 gold and second 4 silver coins.

Sol: Let A: Getting 4 golds in first draw

B: Getting 4 silver coins in second draw.

We need $P(A \cap B)$.

(i) Since, coins are replaced, therefore, A and B are independent events.

$P(A \cap B) = P(A) \cdot P(B)$

$= \frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{18}C_4}$

(ii) Since, Coins are not replaced.

$\Rightarrow$ A and B are dependent events.

$P(A \cap B) = P(A)\, P(B|A)$

$= \frac{{}^{10}C_4}{{}^{18}C_4} \cdot \frac{{}^{8}C_4}{{}^{14}C_4}$

Ex: Plant I of XYZ manufacturing organisation employs 5 ~~pro~~ production and 3 maintenance engineers, another plant II of same organisation employs 4 production and 5 maintenance engineers. From any one of these plants, a single selection of two engineers is made. Find the probability that one of them would be production engineer and the other maintenance engineer.

Sol: Let A_1: Plant I is selected.

A_2: Plant II is selected.

B: In a selection of 2 persons, one is production engineer and the other is maintenance engineer.

Reqd prob $= P(A_1 \cap B) + P(A_2 \cap B)$

$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$

$P(A_1) = P(A_2) = \frac{1}{2}$.

$P(B|A_1) = \frac{5C_1 \times 3C_1}{8C_2} = \frac{15}{28}$.

$P(B|A_2) = \frac{4C_1 \times 5C_1}{9C_2} = \frac{5}{9}$

$\therefore$ Reqd prob $= \frac{1}{2}\cdot\frac{15}{28} + \frac{1}{2}\cdot\frac{5}{9} = \frac{135+140}{504} = \frac{275}{504}$.

$= \frac{\cancel{14+28}}{\cancel{504}}$

Product Rule or Multiplicative Rule

If in an experiment the events A and B can both occur, then $P(A \cap B) = P(A)P(B/A)$, provided $P(A) > 0$.

Remark : Since $A \cap B$ and $B \cap A$ are equivalent

$\therefore \quad P(A \cap B) = P(B \cap A) = P(B)\, P(A/B)$.

Ex Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Sol : Let A: The first fuse is defective

B: The second fuse is defective

Reqd prob

$$= P(A \cap B) = P(A)\, P(B/A) = \frac{5}{20} \cdot \frac{4}{19}$$

$$= \frac{1}{4} \cdot \frac{4}{19} = \frac{1}{19}.$$

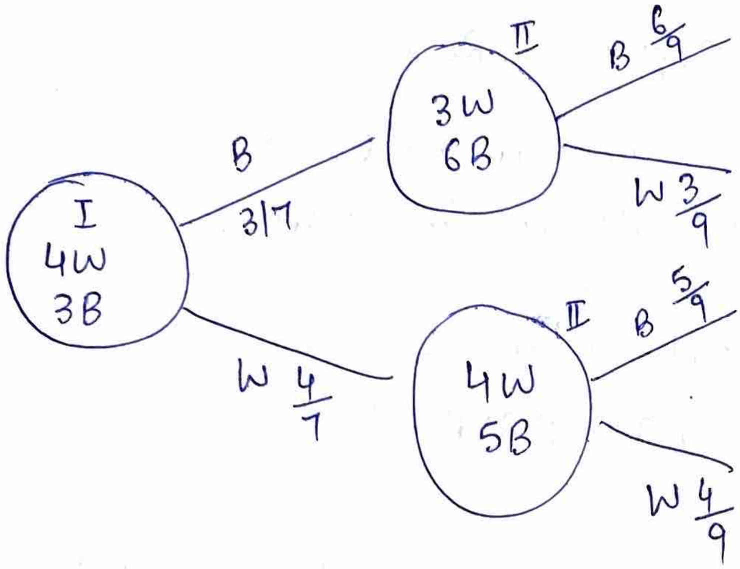
Ex : One bag contains 4 white balls and 3 black balls and second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in second bag. What is the probability that a ball now drawn from the second bag is black?

Sol : Let B_1 : Black ball drawn from bag I

B_2 : Black ball drawn from bag II

W_1 : White ball drawn from bag I

W_2 : White ball drawn from bag II.



$P(B_1 \cap B_2) = \frac{3}{7} \cdot \frac{6}{9}$

$P(B_1 \cap W_2) = \frac{3}{7} \cdot \frac{3}{9}$

$P(W_1 \cap B_2) = \frac{4}{7} \cdot \frac{5}{9}$

$P(W_1 \cap W_2) = \frac{4}{7} \cdot \frac{4}{9}$

$$\begin{aligned}
P\left[(B_1 \cap B_2) \text{ or } P(W_1 \cap B_2)\right] &= P[B_1 \cap B_2] + P(W_1 \cap B_2) \\
&= P(B_1)\, P(B_2 | B_1) + P(W_1)\, P(B_2 | W_1) \\
&= \frac{3}{7} \cdot \frac{6}{9} + \frac{4}{7} \cdot \frac{5}{9} \\
&= \frac{38}{63}.
\end{aligned}$$

<u>Result</u> :- If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then $P(A_1 \cap A_2 \cap \ldots \cap A_k)$

$$= P(A_1)\, P(A_2 | A_1)\, P(A_3 | A_1 \cap A_2) \ldots P(A_k | A_1 \cap A_2 \cap \ldots \cap A_{k-1})$$

If $A_1, A_2, \ldots, A_k$ are independent events,

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)\, P(A_2) \ldots P(A_k).$$

Ex: Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that second card is a 10 or a jack and A_3 is the event that third card is greater than 3 but less than 7.

Sol: $P(A_1) = \frac{2}{52}$, $\cancel{P(A_2) =}$ $P(A_2 | A_1) = \frac{8}{51}$, $P(A_3 | A_1 \cap A_2) = \frac{12}{50}$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2 | A_1)\, P(A_3 | A_1 \cap A_2)$$

$$= \frac{2}{52} \cdot \frac{8}{51} \cdot \frac{12}{50}$$

$$= \frac{8}{5525}.$$

Ex: Two defective tubes get mixed up with 2 good ones. The tubes are tested, one by one, until both defectives are found. What is the probability that the last defective tube is obtained on (i) the second test (ii) the third test (iii) the fourth test.

Sol: (i) $p = P(D_1 \cap D_2) = P(D_1)\, P(D_2) = \frac{2}{4} \cdot \frac{1}{3} = \frac{2}{12} = \frac{1}{6}.$

(ii) $p = P(D_1 \cap N_2 \cap D_3) + P(N_1 \cap D_2 \cap D_3) = \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} + \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2}$

$$= \frac{1}{3}.$$

(iii) $p = P(D_1 \cap N_2 \cap N_3 \cap D_4) + P(N_1 \cap D_2 \cap N_3 \cap D_4) + P(N_1 \cap N_2 \cap D_3 \cap D_4)$

$$= \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 1 + \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 1 + \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{2}{2} \cdot 1 = \frac{1}{2}.$$

Ex → In a shooting test, the probability of hitting the target is $\frac{1}{2}$ for A, $\frac{2}{3}$ for B and $\frac{3}{4}$ for C. If all of them fire at the target, find the probability that

(i) None of them hits the target

(ii) at least one of them hits the target.

Sol: Let A: A hits the target

B: B hits the target

C: C hits the target.

$$P(A)=\frac{1}{2},\ P(B)=\frac{2}{3},\ P(C)=\frac{3}{4}.$$

(i) $P(\bar{A}\cap\bar{B}\cap\bar{C}) = P(\bar{A})\,P(\bar{B})\,P(\bar{C})$

$= \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}$

$= \frac{1}{24}.$

(ii) P(at least one of them hits the target)

= 1 − P(none hits the target)

$= 1-\frac{1}{24} = \frac{23}{24}$

~~$P(A)P(\bar{B})P(\bar{C}) + P(\bar{A})P(B)P(\bar{C})$~~

~~or~~

Bayes' Rule

Partition

Let $A = \{A_1, A_2, ---, A_n\}$ be a finite collection of events. Then, A is a partition of sample space S if following three conditions holds

(i) $P(A_i) \geq 0$

(ii) Events A_i are pairwise disjoint or mutually exclusive i.e. $A_i \cap A_j = \phi$ for $i \neq j$.

(iii) Union of the events A_i equal to sample space S, i.e. $\bigcup_i A_i = S$.

Ex Consider an experiment of throwing a fair die. The sample space S is

$$S = \{1, 2, 3, 4, 5, 6\}$$

(i) $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, $A_3 = \{5, 6\}$

$\{A_1, A_2, A_3\}$ is a partition of S.

(ii) $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 6\}$ do not form a partition.

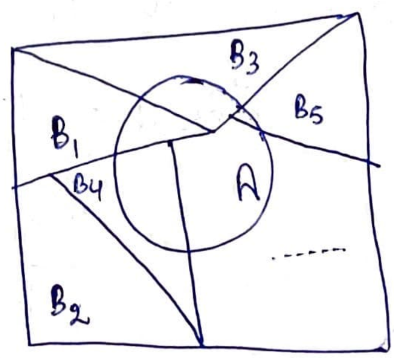
(iii) $A_1 = \{1, 2, 3\}$, $A_2 = \{3, 4, 5, 6\}$ do not form a partition.

Theorem of total probability (Rule of elimination)

If the events $B_1, B_2, ---, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, --, k$, then for any event A of S,

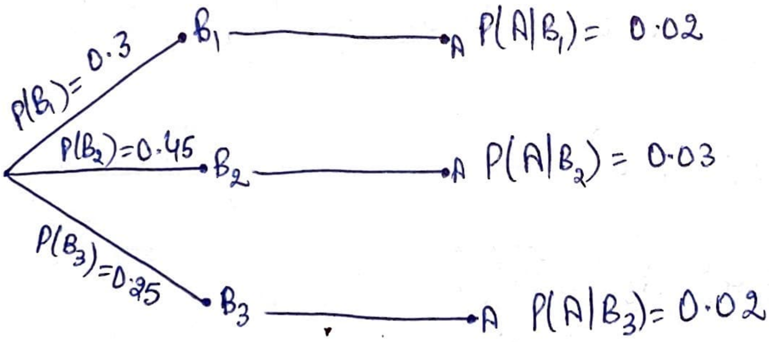
$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + --- + P(B_k)P(A|B_k)$$

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$



Ex. 1 In a certain assembly plant, three machines B_1, B_2 and B_3 make 30%, 45% and 25%, resp, of the products. It is known from past experience that 2%, 3% and 2% of the products made by each machine, resp, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Sol. Let A: the product is defective
B_1: the product is made by machine B_1
B_2: the product is made by machine B_2
B_3: the product is made by machine B_3.



$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$$
$$= (0.3)(0.02) + (0.45)(0.03) + (0.25)(0.02)$$
$$= 0.006 + 0.0135 + 0.005$$
$$= 0.0245.$$

Bayes' Rule

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)\,P(A|B_r)}{\sum_{i=1}^{k} P(B_i)\,P(A|B_i)} \quad \text{for } r = 1, 2, \ldots, k.$$

Ex: In the previous example, if a product was choosen randomly and found to be defective, what is the probability that it was made by machine B_3?

Sol:

$$P(B_3|A) = \frac{P(B_3)\,P(A|B_3)}{P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2) + P(B_3)\,P(A|B_3)}$$

$$= \frac{(0.25)\,(0.02)}{0.0245} = \frac{0.005}{0.0245} = 0.204$$

Ex A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20% and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$P(D|P_1) = 0.01$, $P(D|P_2) = 0.03$, $P(D|P_3) = 0.02$,

where $P(D|P_j)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

Sol: $P(P_1) = 0.30$, $P(P_2) = 0.20$, $P(P_3) = 0.50$

$$P(P_1|D) = \frac{P(P_1)\,P(D|P_1)}{P(P_1)\,P(D|P_1) + P(P_2)\,P(D|P_2) + P(P_3)\,P(D|P_3)}$$

$$= \frac{(0.30)(0.01)}{(0.30)(0.01) + (0.20)(0.03) + (0.50)(0.02)}$$

$$= \frac{0.003}{0.003 + 0.006 + 0.01} = \frac{0.003}{0.019} = 0.158$$

$$P(P_2|D) = \frac{P(P_2)\,P(D|P_2)}{P(P_1)\,P(D|P_1) + P(P_2)\,P(D|P_2) + P(P_3)\,P(D|P_3)}$$

$$= \frac{(0.20)(0.03)}{0.019} = \frac{0.006}{0.019} = 0.316$$

$$P(P_3|D) = \frac{P(P_3)\,P(D|P_3)}{P(P_1)\,P(D|P_1) + P(P_2)\,P(D|P_2) + P(P_3)\,P(D|P_3)}$$

$$= \frac{(0.50)(0.02)}{0.019} = \frac{0.01}{0.019} = 0.526.$$

The conditional probability of a defect given plan 3 is the largest of the three, thus a defective for a random product is most likely the result of the use of plan 3.

P(B)

Ex In 2002, there will be three candidates for the position of principal - Mr Chatterji, Mr Ayangar and Dr Singh - whose chances of getting the appointment are in the proportion 4 : 2 : 3 respectively. The probability that Mr. Chatterji if selected introduce co-education in the college is 0.3. The probability of Mr. Ayangar and Dr. Singh doing the same are resp 0.5 and 0.8.

(i) What is probability that there will be co-education in the college in 2003?

(ii) If there is coeducation in the college in 2003, what is the probability that Dr. Singh is the principal?

Sol: Let us consider the events:

A: Introduction of co-education,

E_1: Mr. Chatterji is selected as principal

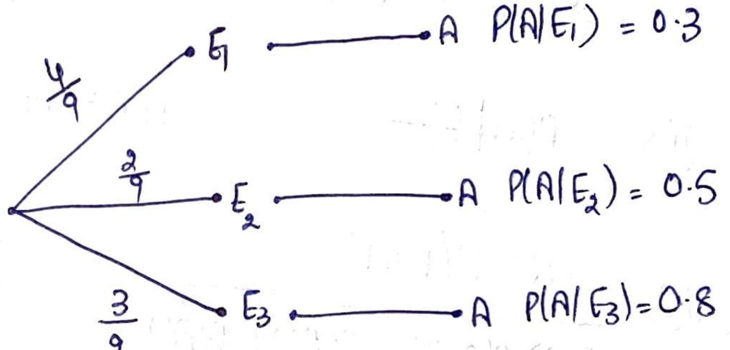
E_2: Mr. Ayangar is selected as principal

E_3: Dr. Singh is selected as principal

Then, $P(E_1) = \frac{4}{9}$, $P(E_2) = \frac{2}{9}$, $P(E_3) = \frac{3}{9}$

$P(A|E_1) = 0.3$, $P(A|E_2) = 0.5$, $P(A|E_3) = 0.8$

(i) $P(A) = P(E_1)P(A|E_1) +$
$P(E_2)P(A|E_2) +$
$P(E_3)P(A|E_3)$



$$= \frac{4}{9}(0.3) + \frac{2}{9}(0.5) + \frac{3}{9}(0.8)$$

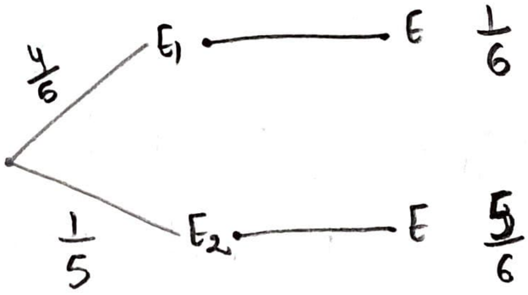
$$= \frac{4}{9}\cdot\frac{3}{10} + \frac{2}{9}\cdot\frac{5}{10} + \frac{3}{9}\cdot\frac{8}{10}$$

$$= \frac{23}{45}.$$

(ii) $$P(E_3|A) = \frac{P(E_3)P(A|E_3)}{P(A)} = \frac{\left(\frac{3}{9}\right)\left(\frac{8}{10}\right)}{\frac{23}{45}}$$

$$= \frac{12}{23}.$$

Ex A speaks truth 4 out of 5 times. A die is tossed. He reports that there is six. What is the chance that actually there was six?



Sol: Consider E : A reports a six

E_1: A speaks truth

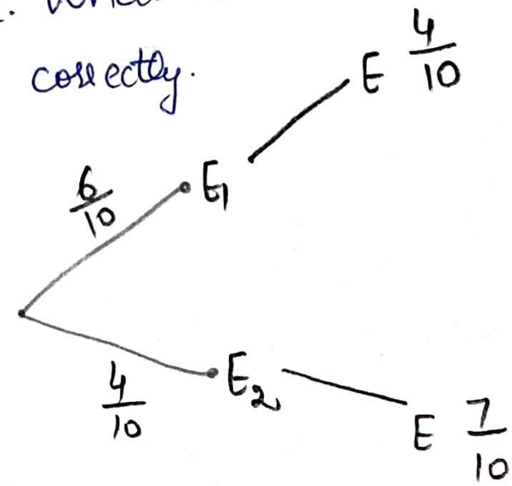
E_2: A tells a lie

$P(E_1) = \frac{4}{5}$, $P(E_2) = \frac{1}{5}$, $P(E|E_1) = \frac{1}{6}$, $P(E|E_2) = \frac{5}{6}$

Thus,
$$P(E_1|E) = \frac{P(E_1)P(E|E_1)}{P(E_1)P(E|E_1) + P(E_2)P(E|E_2)} = \frac{\frac{4}{5}\cdot\frac{1}{6}}{\frac{4}{5}\cdot\frac{1}{6} + \frac{1}{5}\cdot\frac{5}{6}}$$

$$= \frac{4}{9}.$$

Ex The chances that doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease X, died. What is the chance that his disease was diagnosed correctly.



Sol: E_1: The disease is diagnosed correctly by doctor A.

E_2: The disease is not diagnosed correctly by Dr. A.

E: A patient, who had disease X, died.

$P(E_1) = \frac{6}{10}, \quad P(E_2) = \frac{4}{10}$

$P(E|E_1) = \frac{4}{10}, \quad P(E|E_2) = \frac{7}{10}$

$$P(E_1|E) = \frac{P(E_1)\,P(E|E_1)}{P(E_1)\,P(E|E_1) + P(E_2)\,P(E|E_2)}$$

$$= \frac{\frac{6}{10} \cdot \frac{4}{10}}{\frac{6}{10} \cdot \frac{4}{10} + \frac{4}{10} \cdot \frac{7}{10}} = \frac{24}{52} = \frac{6}{13}$$