Some Discrete Probability Distributions

The Bernoulli Process

An experiment often consists of repeated trials, each with two possible outcomes that may be labeled success or failure.

Eg: Tossing a coin ten times and finding the probability of number of heads.

Head — Success

Tail — Failure.

The process is referred to as a Bernoulli process. Each trial is called a Bernoulli trial.

The Bernoulli ~~trial~~ process must possess the following properties:

1. The experiment consists of repeated trials.
2. Each trial results in an outcome that may be classified as a success or a failure.
3. The probability of success, denoted by p, remains constant from trial to trial.
4. The repeated trials are independent.

Note: In drawing cards from a deck, the probabilities for repeated trials change if the card is not replaced.

The probability of selecting a heart on first draw is $\frac{13}{52}$ and on second draw it is conditional probability, $\frac{13}{51}$ or $\frac{12}{51}$ depending on whether a heart appeared on first draw.

$\therefore$ This would not be considered as a bernoulli trial.

Binomial Distribution

The number X of successes in n Bernoulli trials is called a binomial random variable. The probability distribution of this discrete random variable is called the binomial distribution and its value is denoted by $b(x; n, p)$.

Def: A Bernoulli trial can result in a success with probability p and a failure with probability $q = 1-p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is

$$b(x; n, p) = {}^nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, ---, n.$$

$$\boxed{{}^nC_x = \binom{n}{x}}$$

Ex: Three items are selected at random from a manufacturing process, inspected and classified as defective or nondefective. Find the probability distribution for number of defectives assuming that 25% items are defective.

Sol. Let X be a random variable representing number of defectives. $P(S) = p = \frac{1}{4}, \; q = \frac{3}{4}$

S = { NNN, NDN, NND, DNN, NDD, DND, DDN, DDD }

X	0	1	2	3
$f(x)$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

$$f(0) = P(NNN) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$$

$$f(1) = 3 \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$$

$$f(2) = 3 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{9}{64}$$

$$f(3) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64}$$

$$b\left(x; 3, \frac{1}{4}\right) = {}^3C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x},$$

$n = 0, 1, 2, 3.$

Ex: ~~Three~~ Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.

Sol: p = Probability of getting a head $= \frac{1}{2}$

q = Probability of not getting a head $= \frac{1}{2}$

The probability of getting x heads in a random throw of 10 coins is

$$b(x; 10, \tfrac{1}{2}) = {}^{10}C_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{10-x}, \quad x = 0, 1, 2, \ldots, 10$$

Probability of getting at least 7 heads

$$= P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10).$$

$$= {}^{10}C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^3 + {}^{10}C_8 \left(\tfrac{1}{2}\right)^8 \left(\tfrac{1}{2}\right)^2 + {}^{10}C_9 \left(\tfrac{1}{2}\right)^9 \left(\tfrac{1}{2}\right)$$

$$+ {}^{10}C_{10} \left(\tfrac{1}{2}\right)^{10} \left(\tfrac{1}{2}\right)^0$$

$$= \left(\tfrac{1}{2}\right)^{10} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right]$$

$$= \frac{1}{1024} \left[\frac{10!}{7!\,3!} + \frac{10!}{8!\,2!} + \frac{10!}{9!\,1!} + \frac{10!}{0!\,10!} \right]$$

$$= \frac{1}{1024} \left[\frac{10 \cdot 9 \cdot 8}{3 \cdot 2} + \frac{10 \cdot 9}{2} + 10 + 1 \right]$$

$$= \frac{1}{1024} \left[120 + 45 + 11 \right]$$

$$= \frac{176}{1024}.$$

Ex The probability that a certain kind of component will survive a shock test is $\frac{3}{4}$. Find the probability that exactly 2 of the next 4 components tested survive.

Sol: $p = \frac{3}{4}, \; q = \frac{1}{4}$

$$b\left(2; 4, \frac{3}{4}\right) = {}^4C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^2 = 6 \cdot \frac{9}{16} \cdot \frac{1}{16} = \frac{27}{128}.$$

Ex The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that

(a) at least 10 survive

(b) from 3 to 8 survive

(c) exactly 5 survive?

Sol :- Let X be the number of people who survive.

(a) $P(X \geq 10) = 1 - P(X < 10) = 1 - \sum_{x=0}^{9} b(x; 15, 0.4)$

$= 1 - 0.9662$

$= 0.0338$

(b) $P(3 \leq X \leq 8) = \sum_{x=3}^{8} b(x;15,0.4) = \sum_{x=0}^{8} b(x;15,0.4) - \sum_{x=0}^{2} b(x;15,0.4)$

$= 0.9050 - 0.0271$

$= 0.8779$

(C) $P(X = 5) = b(5; 15, 0.4) = \sum_{x=0}^{5} b(x; 15, 0.4) - \sum_{x=0}^{4} b(x; 15, 0.4)$

$= 0.4032 - 0.2173$

$= 0.1859$

$$\boxed{\begin{aligned} & {}^{15}C_5 (0.4)^5 (0.6)^{10} \\ &= (3003)(0.01024)(0.006) \\ &= 0.185 \end{aligned}}$$

Areas of Application

1. Industrial Process
2. Medical
3. Military
4. Traffic light
5. Participating in a lucky draw
6. Participating in an Election
7. Supporting a particular sports team.

Mean and Variance of the binomial distribution $b(x; n, p)$

$\mu = np, \quad \sigma^2 = npq.$

In the previous ex,

$\mu = (15)(0.4) = 6$

$\sigma^2 = (15)(0.4)(0.6) = 3.6$

Multinomial Experiments and Multinomial Distribution

The binomial experiment becomes a multinomial experiment if we let each trial have more than two possible outcomes.

Eg:
1. The classification of a manufactured product as being light, heavy or acceptable.
2. The drawings of a card with replacement if the 4 suits are the outcomes of the interest.

Multinomial Distribution

If a given trial can result in the k outcomes $E_1, E_2, \dots, E_k$ with probabilities $p_1, p_2, \dots, p_k$, then the probability distribution of the random variables $X_1, X_2, \dots, X_k$, representing the number of occurrences for $E_1, E_2, \dots, E_k$ in n independent trials, is

$$f(x_1, x_2, \dots, x_k; p_1, p_2, \dots, p_k, n) = \binom{n}{x_1, x_2, \dots, x_k} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k},$$

with $\sum_{i=1}^{k} x_i = n, \quad \sum_{i=1}^{k} p_i = 1.$

Ex The complexity of arrivals and departures of planes at an airport is such that computer simulation is often used to model the "ideal" conditions. For a certain airport with three runways, it is known that in the ideal setting the following are the probabilities that the individual runways are accessed by a randomly arriving commericial jet:

Runway 1: $P_1 = \frac{2}{9}$,

Runway 2: $P_2 = \frac{1}{6}$,

Runway 3: $P_3 = \frac{11}{18}$.

What is the probability that 6 randomly arriving airplanes are distributed in the following fashion?

Runway 1: 2 airplanes

Runway 2: 1 airplanes

Runway 3: 3 airplanes.

Sol: $f\left(2, 1, 3; \frac{2}{9}, \frac{1}{6}, \frac{11}{18}, 6\right) = \frac{6!}{2!\,3!\,1!}\left(\frac{2}{9}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{11}{18}\right)^3$

$= \frac{6 \cdot 5 \cdot 4}{2} \cdot \frac{4}{81} \cdot \frac{1}{6} \cdot \frac{1331}{5832}$

$= 0.1127.$

Negative Binomial and Geometric Distributions

Negative Binomial Experiments

Consider an experiment where the properties are the same as those listed for a binomial experiment, with the exception that the trials will be replaced until a fixed number of successes occur.

Therefore, instead of the probability of x successes in n trials, where n is fixed, we are now interested in the probability that the kth success occurs on the xth trial. Experiments of this kind are called negative binomial experiments.

Negative Binomial Random Variable

The number X of trials required to produce k successes in a negative binomial experiment is called a negative binomial random variable and its probability distribution is called the negative binomial distribution.

Negative Binomial distribution

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1-p$, then the probability distribution of the random variable X, the number of the trial on which the kth success occurs, is

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}, \quad x = k, k+1, \text{-----}$$

1. Q:- Find the probability that a person flipping a coin gets

(a) the third head on the seventh flip

(b) the first head on the fourth flip.

Sol: (a) $b^*(7, 3, \frac{1}{2}) = {}^6C_2 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^4$

$= \frac{15}{128} = 0.1172$

~~$b^*(4, 1, \frac{1}{2})$~~ (b) $b^*(4, 1, \frac{1}{2}) = {}^3C_0 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = \frac{1}{16} = 0.0625$

Ex In an NBA (National Basketball Association) Championship series, the team that wins four games out of seven is the winner. Suppose that the teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

(a) What is the probability that team A will win the series in 6 games?

(b) What is the probability that team ~~B~~ A will win the series? ~~in 6 games?~~

(c) If teams A and B were facing each other in a regional playoff series, which is decided by winning three out of five games, what is the probability that team A would win the series?

Sol: (a) $b^*(6; 4, 0.55) = {}^5C_3 (0.55)^4 (0.45)^2$

$= 0.1853$

(b) P(A will win the series)

$= P(X \geq 4)$

$= b^*(4; 4, 0.55) + b^*(5; 4, 0.55) + b^*(6; 4, 0.55) + b^*(7; 4, 0.55)$

$= {}^3C_3 (0.55)^4 (0.45)^0 + {}^4C_3 (0.55)^4 (0.45)^1 + {}^5C_3 (0.55)^4 (0.45)^2 + {}^6C_3 (0.55)^4 (0.45)^3$

$= 0.0915 + 0.1647 + 0.1853 + 0.1668$

$= 0.6083.$

(c) P(team A wins the playoff)

$= P(X \geq 3)$

$= b^*(3; 3, 0.55) + b^*(4; 3, 0.55) + b^*(5; 3, 0.55)$

$= {}^2C_2 (0.55)^3 (0.45)^0 + {}^3C_2 (0.55)^3 (0.45)^1 + {}^4C_2 (0.55)^3 (0.45)^2$

$= 0.1664 + 0.2246 + 0.2021$

$= 0.5931$

If we consider the special case of the negative binomial distribution where $k=1$, we have a probability distribution for the number of trials required for a single success.

For eg: Tossing a coin until head occurs.

We might be interested in the probability that the first head occurs on the fourth toss.

The negative binomial reduces to

$$b^*(x; 1, p) = pq^{x-1}, \quad x = 1, 2, 3, ----$$

Geometric Distribution

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1-p$, then the probability distribution of the random variable X, the number of the trial on which the first success occurs, is

$$g(x; p) = pq^{x-1}, \quad x = 1, 2, 3, ----$$

1 a (b) $g(4; \frac{1}{2}) = (\frac{1}{2})(\frac{1}{2})^3 = \frac{1}{2^4} = \frac{1}{16} = 0.0625.$

Ex For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Sol: $g(5; 0.01) = (0.01)(0.99)^4$ $\quad\quad p = \frac{1}{100} = 0.01$

$= 0.0096$ $\quad\quad q = 0.99$

Ex: At a busy time, a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the number of attempts necessary in order to make a connection. Suppose that we let $p = 0.05$ be the probability of a connection during a busy time. We are interested in knowing the probability that 5 attempts are necessary for a successful call.

Sol: $g(5; 0.05) = (0.05)(0.95)^4 = 0.0407.$

Mean and Variance of a random variable following the Geometric Distribution:

$$\mu = \frac{1}{p}, \quad \sigma^2 = \frac{1-p}{p^2}.$$

Q: In the above example, find the expected number of calls necessary to make a connection?

Sol: $\mu = \frac{1}{0.05} = 20.$

Poisson Process and Poisson Distribution

Some experiments results in Counting the number of particular events occur in given time interval or in a specified region, known as Poisson Experiments.

The time interval may be of any length, such as a minute, a day, a week, a month or even a year.

Eg:- 1. ~~tot~~ Number of telephone calls received per hour by an office.

2. How many vehicles pass through a traffic signal in a day.

3. How many people arrive at a railway station from 9 am to 11 am.

4. How many people enter in the door of a shopping mall in January.

The specified region can be a line segment, an area, a volume or perhaps a piece of material.

Eg:- 1. Number of field mice per acre.

2. Number of typing errors per page.

Poisson Process

Poisson Process represents observations | occurrences | happenings over time | area.

Properties of Poisson Process

1. The number of outcomes | occurrences during disjoint time intervals are independent.

Eg:- No. of earthquakes (accidents) recorded in 2021-22 is independent of the no. of earthquakes (accidents) recorded in 2001-02.

2. The probability of a single occurrence during a small time interval is proportional to the length of the interval.

$P_1(h) = P(X(h) = 1) = \lambda h$ (Rate of occurrence of an event)

3. The probability of more than one occurrence during a small time interval in negligible.

Eg.: If there a train accident at 9.00 am at a particular place, then it is higely ~~that~~ unlikely that there will be a train accident at 9.03 am.

Poisson Random Variable and Poisson Distribution

The number X of outcomes occurring during a Poisson experiment is called a Poisson Random Variable and its probability distribution is called the Poisson distribution.

Poisson Distribution

Def.: The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region denoted by t, is

$$p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where λ is the average number of outcomes per unit time, distance, area or volume and $e = 2.71828\ldots$

Ex During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 milisecond is 4. What is the probability that 6 particles enter the counter in a given milisecond?

Sol:- $\lambda t = 4$, $x = 6$

$$p(6;4) = \frac{e^{-4}(4)^6}{6!} = \frac{(0.0183)(4096)}{720} = \frac{74.9568}{720} = 0.1041$$

Mean and Variance of Poisson Distribution $p(x;\lambda t)$

$$\boxed{\mu = \lambda t, \ \sigma^2 = \lambda t}$$

$$\mu = E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda t}(\lambda t)^x}{x!} = \sum_{x=1}^{\infty} \frac{x e^{-\lambda t}(\lambda t)^x}{x!}$$

$$= e^{-\lambda t}(\lambda t) \sum_{x=1}^{\infty} \frac{(\lambda t)^{x-1}}{(x-1)!}$$

$$= e^{-\lambda t}(\lambda t)\left[1 + \frac{\lambda t}{1} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots\right]$$

$$= \lambda t \, e^{-\lambda t} e^{\lambda t}$$

$$\boxed{\mu = \lambda t}$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda t}(\lambda t)^x}{x!} - \lambda^2 t^2$$

$$= \sum_{x=1}^{\infty} \frac{(x^2 - x + x)e^{-\lambda t}(\lambda t)^x}{x!} - \lambda^2 t^2$$

$$= \sum_{x=2}^{\infty} \frac{\cancel{x(x-1)}\, e^{-\lambda t} (\lambda t)^x}{\cancel{x(x-1)}(x-2)!} + \lambda t - \lambda^2 t^2$$

$$= e^{-\lambda t} (\lambda t)^2 \sum_{x=2}^{\infty} \frac{(\lambda t)^{x-2}}{(x-2)!} + \lambda t - \lambda^2 t^2$$

$$= e^{-\lambda t} (\lambda t)^2 \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots \right] + \lambda t - \lambda^2 t^2$$

$$= e^{-\lambda t} (\lambda t)^2 e^{\lambda t} + \lambda t - \lambda^2 t^2$$

$$= \lambda^2 t^2 + \lambda t - \lambda^2 t^2$$

$$= \lambda t$$

$$\boxed{\sigma^2 = \lambda t}$$

Approximation of Binomial Distribution by a Poisson Distribution

Poisson distribution is a limiting case of the binomial distribution under the following conditions:

(i) n, the number of trials is indefinitely large, i.e, $n \to \infty$

(ii) p, the constant probability of success for each trial is indefinitely small, i.e., $p \to 0$.

(iii) $\cancel{np = \lambda}$, $np = \mu$, is finite.

Thm Let X be a binomial random variable with probability distribution $b(x; n, p)$. When $n \to \infty$, $p \to 0$ and $np \xrightarrow{n\to\infty} \mu$ remains constant,

$$b(x; n, p) \xrightarrow{n\to\infty} p(x; \mu).$$

$$p(x; \mu) = \frac{e^{-\mu} \mu^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Ex In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other.

Sol: (a) What is the probability that in any given period of 400 days there will be an accident on one day?

(b) What is the probability that there are at most three days with accidents?

Sol: Let X be a binomial random variable with $n=400$ and $p=0.005$.

Thus, $np = 400 \times 0.005 = 2$

Using Poisson Process,

(a) $P(X=1) = \frac{e^{-2} 2^1}{1!} = (0.1353)(2) = 0.2706$

(b) $P(X \leq 3) = P(X=1) + P(X=2) + P(X=3) + P(X=0)$

$$= 0.2706 + \frac{e^{-2} 2^2}{2!} + \frac{e^{-2} 2^3}{3!} + \frac{e^{-2} 2^0}{0!}$$

$$= 0.2706 + \frac{(0.1353)(4)}{2} + \frac{(0.1353)(8)}{6} + 0.1353$$

$$= 0.2706 + 0.2706 + 0.1804 + 0.1353$$

$$= 0.7216 + 0.1353$$

$$= 0.8569$$

Ex In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles. What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?

Sol: It is a binomial experiment with $n = 8000$ and $p = 0.001$. Since p is very close to 0 and n is quite large, we will use Poisson distribution.

$$\mu = 8000 \times 0.001 = 8.$$

Let X represent the number of bubbles.

$$P(X<7) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)$$

$$= e^{-8}\left[\frac{8^0}{0!}+\frac{8^1}{1!}+\frac{8^2}{2!}+\frac{8^3}{3!}+\frac{8^4}{4!}+\frac{8^5}{5!}+\frac{8^6}{6!}\right]$$

$$= e^{-8}\left[1+8+32+85.3333+170.6667+273.0667+364.0889\right]$$

$$= \cancel{0.00039}[934.1556] \times 0.0003355$$

$$= 0.3134$$

Ex A manufacturer, who produces medicine bottles, finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottles. Using Poisson distribution, find how many boxes will contain:

(i) no defective (ii) at least two defectives.

Sol: $n = 500$, $p = 0.001$, $np = 0.5$

Let X be a random variable denote the number of defective bottles in a box of 500. The prob of x defective bottle in a box is

$$\therefore P(X=x) = \frac{e^{-0.5}(0.5)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$\therefore$ The number of boxes containing x defective bottles in a consignment of 100 boxes is

$$100 \times P(X=x) = 100 \times \frac{e^{-0.5} \times (0.5)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) Number of boxes containing no defective bottles is

$$100 \times P(X=0) = 100 \times \frac{e^{-0.5} \times (0.5)^0}{0!}$$

$$= 100 \times 0.6065$$

$$= 6065$$

$$\simeq 61.$$

(ii) Number of boxes containing at least two defective bottles is

$$100 \times P(X \geq 2) = 100 \left[1 - P(X<2)\right]$$

$$= 100\left[1 - P(X=0) - P(X=1)\right]$$

$$= 100\left[1 - 0.6065 \times 1 - \frac{0.6065.(0.5)}{1!}\right]$$

$$= 100\left[1 - 0.6065 - 0.3033\right]$$

$$= 100\left[0.0902\right]$$

$$= 9.02$$

$$\simeq 9.$$

Gamma and Exponential Distributions

Gamma Distribution is used to find the time until k events occur. When we are interested in the time until rth event occurs.

Ex: 1. The time until k customers arrive.

2. The time until you've been invited to k parties.

(Poisson: The prob that k customers will arrive in a fixed interval).

Gamma Function

The gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx, \text{ for } \alpha > 0.$$

Ex: $\int_0^\infty x^2 e^{-x} \, dx = \Gamma(3)$.

Properties:
1. $\Gamma n = (n-1)(n-2) \dots 1 \Gamma 1$, for positive integer n.
2. $\Gamma n = (n-1)!$ for a positive integer n.
3. $\Gamma 1 = 1$.
4. $\Gamma \frac{1}{2} = \sqrt{\pi}$.

Gamma Distribution

The continuous random variable X has a gamma distribution with parameters α and β, if its density function is given by

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma \alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$, $\beta > 0$.

The special gamma distribution for which $\alpha = 1$ is called the exponential distribution.

Exponential Distribution

The continuous random variable X has an exponential distribution, with parameter β, if its density function is given by

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\beta > 0$.

	μ	σ^2
Gamma distribution	$\alpha\beta$	$\alpha\beta^2$
Exponential distribution	β	β^2.

Gamma Distribution

- Model (represent) the time between independent events that occur at constant average rate.

Ex: Modeling the time until 3rd or 4th accident occurs.

- Model the elapsed time between various number of events.

$\beta \rightarrow$ Mean time between events.

If $\beta = 2$ and measuring the time between the vehicles passing through a traffic signal in minutes.

$\Rightarrow$ There are ~~4 vehicles~~ 2 minutes between vehicles passing on average.

Exponential Distribution :- Model the time till next event occurs.

Note: $\lambda = \frac{1}{\beta}$,

where λ represents the average number of events per unit time and β is mean time between events.

Ex Suppose that telephone calls arriving at a call center follow a poisson process with an average of 5 calls per minute. What is the probability that upto a minute will elapse by the time 2 calls have come in to the call center?

Sol: The poisson process applies, with time until 2 Poisson events occur following gamma distribution.

$\therefore \beta = \frac{1}{5}, \quad \alpha = 2.$

$$P(X \leq 1) = \int_0^1 25 \frac{1}{\Gamma 2} x e^{-5x} dx$$

$$= 25 \int_0^1 x e^{-5x} dx \qquad [\Gamma n = (n-1)!]$$

$$= 25 \left[x \frac{e^{-5x}}{-5} - 1 \cdot \frac{e^{-5x}}{25} \right]_0^1$$

$$= \cancel{\frac{25}{5}\left\{-e^{-5} + e^{-5} + 0 + 1\right\}}$$

$$= -\frac{25}{25} \left[5x e^{-5x} + e^{-5x} \right]_0^1$$

$$= -1 \left[5e^{-5} + e^{-5} - 1 \right]$$

$$= -6e^{-5} + 1 = 1 - 6(0.0067)$$

$$= 0.96.$$

Ex In a biomedical study with rats, a dose response investigation is used to determine the effect of the dose of a toxicant on their survival time. For a certain dose of the toxicant, the study determines that the survival time, in weeks, has a gamma distribution with $\alpha=5$ and $\beta=10$. What is the probability that a rat survives no longer than 60 weeks?

Sol: Let the random variable X be the survival time.

The reqd probability is

$$P(X\leq 60) = \frac{1}{\beta^5}\int_0^{60} \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma 5}\,dx$$

$\alpha=5$, $\beta=10$

$$\Rightarrow P(X\leq 60) = \frac{1}{4!\,10^5}\int_0^{60} x^4 e^{-x/10}\,dx$$

$$= \frac{1}{4!\,10^5}\left[x^4\frac{e^{-x/10}}{-1/10} - 4x^3\frac{e^{-x/10}}{1/10^2} + 12x^2\frac{e^{-x/10}}{-1/10^3} - 24x\frac{e^{-x/10}}{1/10^4} + \frac{24e^{-x/10}}{-1/10^5}\right]_0^{60}$$

$$= \frac{1}{4!\,10^5}\left[\left(-10x^4 - 400x^3 - 12000x^2 - 240000x - 2400000\right)e^{-x/10}\Big|_{x=60} + 2400000\right]$$

$$= \frac{1}{4!\,10^5}\left[0.0025\left[(-129600000) - 86400000 - 43200000 - 2400000\right) + 2400000\right]$$

$$= \frac{1}{4!\,10^5}\left[-654000 + 2400000\right] = \frac{1746000}{2400000} = 0.72.$$

Ex Suppose that a system contains a certain type of Component whose time, in years, to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta = 5$. If 5 of these Components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Sol: The probability that a given component is still functioning after 8 years is

$$P(T > 8) = \frac{1}{5}\int_8^\infty e^{-t/5}\, dt = \frac{1}{5}\left[\frac{e^{-t/5}}{-1/5}\right]_8^\infty$$

$$= -\left[0 - e^{-8/5}\right]$$

$$= e^{-8/5} = 0.2$$

Let X represent the number of components functioning after 8 years.

∴ The probability that x components will work after 8 years is

$$b(x; 5, 0.2) = {}^5C_x (0.2)^x (0.8)^{5-x}, \quad x = 0, 1, 2, \dots, 5.$$

Reqd probability is

$$P(X \geq 2) = 1 - P(X < 2) = 1 - \left[P(X=0) + P(X=1)\right]$$

$$= 1 - \left[(0.8)^5 + 5(0.2)(0.8)^4\right]$$

$$= 1 - [0.3277 + 0.4096]$$

$$= 0.2627.$$

Ex Based an extensive testing, it is determined that the time Y in years before a major repair is required for a certain washing machine is characterized by the density function

$$f(y) = \begin{cases} \frac{1}{4} e^{-y/4}, & y \geq 0, \\ 0, & \text{elsewhere}. \end{cases}$$

Note that Y is an exponential random variable with $\mu = 4$ years. The machine is considered a ~~good~~ bargain if it is unlikely to require a major repair before the sixth year. What is the probability $P(Y > 6)$?

What is the probability that a major repair is required in the first year?

Sol:

$$F(y) = \frac{1}{\beta} \int_0^y e^{-t/\beta}\, dt = \frac{1}{\beta} \left[\frac{-e^{-t/\beta}}{1/\beta}\right]_0^y$$

$$= \frac{-\beta}{\beta} \left[e^{-y/\beta} - 1\right]$$

$$= 1 - e^{-y/\beta}.$$

$\therefore P(Y \geq 6) = 1 - P(Y \leq 6)$

$= 1 - F(6)$

$= 1 - 1 + e^{-6/4}$

$= 0.2231.$

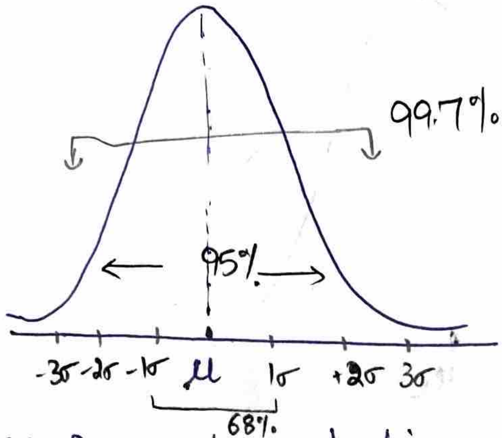
The probability that a major repair is required in the first year is

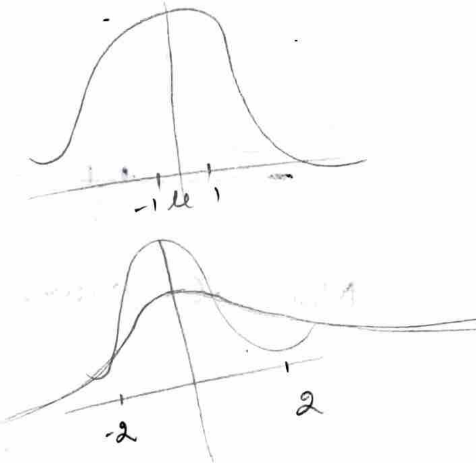
$P(Y<1) = 1 - e^{-1/4} = 1 - 0.7788 = 0.2212.$

In this case, machine is not a good bargain.

Normal Distribution

Normal Distribution is also known as Gaussian distribution or Bell-shaped Curve distribution. Its graph, called the normal curve, is the bell shaped curve.





Eg:- 1. Weight of newborn babies
2. Height of 15 year old girls.
3. ~~Shoe size~~ Temparature
4. Blood pressure
5. ~~Students marks~~ Rainfall.

A continuous random variable X having the bell-shaped distribution is called a <u>normal random variable</u>.

Normal Distribution

The density function of the normal random variable X with mean μ and variance σ^2, is

$$n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty,$$

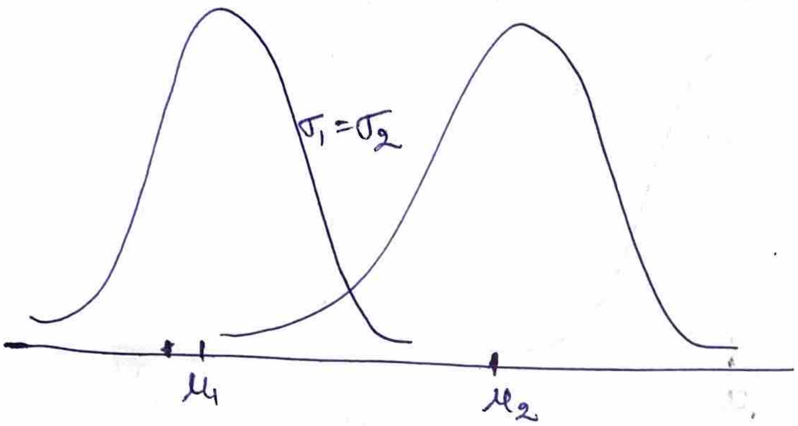
where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$

Properties of Normal Distribution

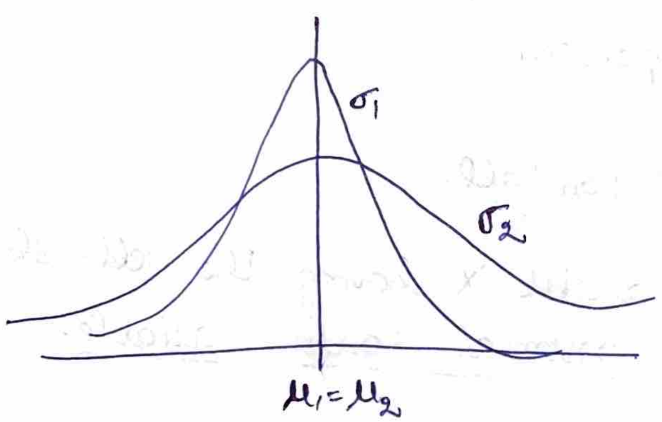
1. The mean, median and mode are all equal.
2. The curve is symmetric about the mean.
3. The total area under the curve and above the horizontal

axis is equal to 1. $\left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$

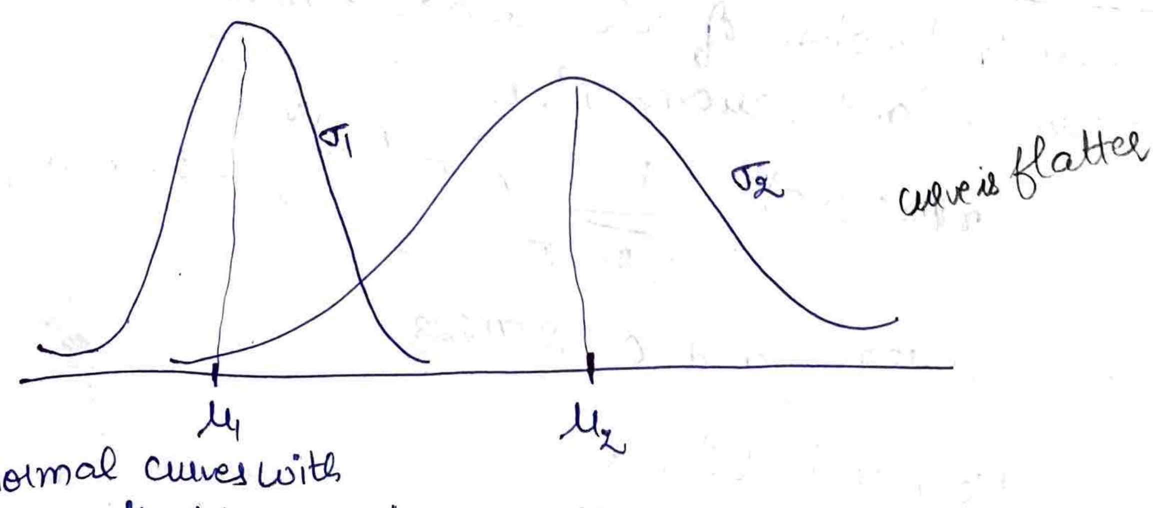
4. Area to the left and area to the right about the mean are same, i.e., 0.5.



Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$.

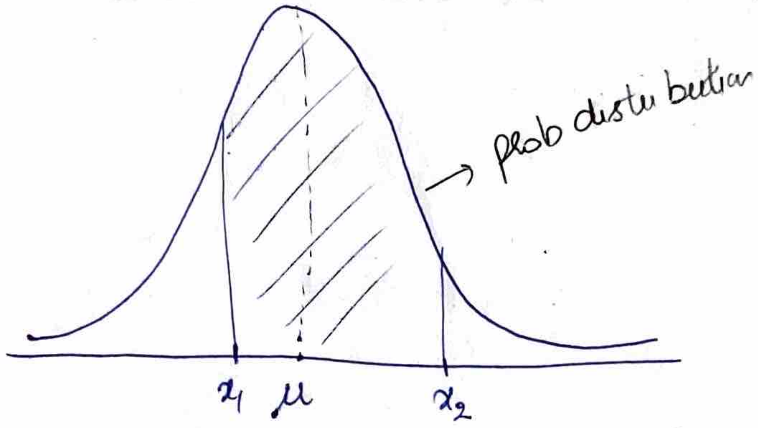


Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$.



Normal curves with
$\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$.

Areas under the Normal Curve



$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x; \mu, \sigma)\, dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}\, dx$$

For normal distributed variable X, the variable

$$Z = \frac{X - \mu}{\sigma}$$

is called standard normal variate.

$$\therefore P(x_1 < X < x_2) = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{\frac{-1}{2}z^2}\, dz$$

$$= \int_{z_1}^{z_2} n(z; 0, 1)\, dz$$

$$= P(z_1 < Z < z_2),$$

where Z is seen to be a normal random variable with mean 0 and variance 1.

Def : The distribution of a normal random variable with mean 0 and variance 1 is called a standard normal distribution.

Given a standard normal distribution, find the area under the curve that lies ③

(a) to the right of $z = 1.84$ and

(b) between $z = -1.97$ and $z = 0.86$.

$= 0.5 - P(0<z<1.84)$ 0 $z=1.84$

$P(0<z<z_1)$

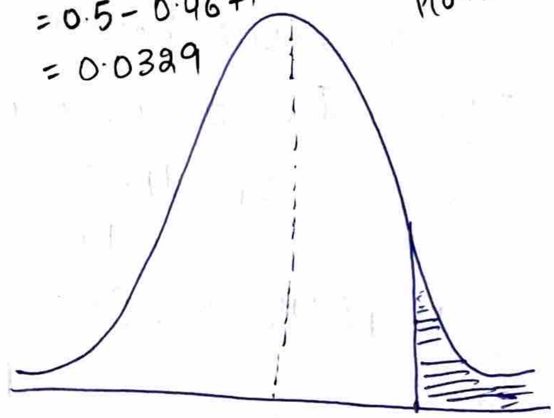
$= 0.5 - 0.4671$

$= 0.0329$

Sol: (a) $P(z > 1.84)$

$= 1 - 0.9671$

$= 0.0329$.



0 $z = 1.84$

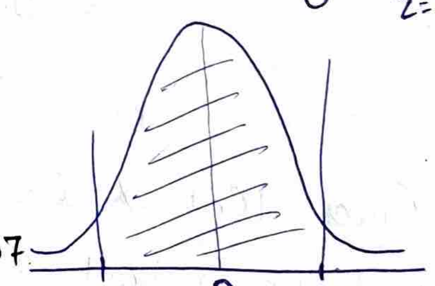
(b) $P(-1.97 < z < 0.86)$

$= P(z < 0.86) - P(z < -1.97)$

$= 0.8051 - 0.0244$

$= 0.7807$.

$$\left.\begin{aligned} P(0<z<0.86) &= 0.3051 \\ P(0<z<1.97) &= 0.4756 \end{aligned}\right\} + \to 0.7807$$



-1.97 0 0.86

Q: Given a standard normal distribution, find the value of K such that

(a) $P(Z > K) = 0.3015$ and

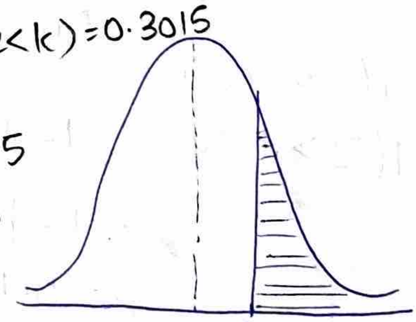
(b) $P(K < Z < -0.18) = 0.4197$.

$0.5 - P(0<z<k) = 0.3015$

$\Rightarrow P(0<z<k)$

$= 0.1985$

$k = 0.52$



k

Sol: (a) $P(z>k) = 1 - P(z<k) = 0.3015$

$\Rightarrow P(z<k) = 1 - 0.3015 = 0.6985$

$\therefore k = 0.52$.

(b) ~~P~~ $P(k < z < -0.18) = 0.4197$

$\Rightarrow P(z<-0.18) - P(z<k) = 0.4197$

$\Rightarrow 0.4286 - P(z<k) = 0.4197$

$\Rightarrow P(z<k) = 0.4286 - 0.4197 = 0.0089$

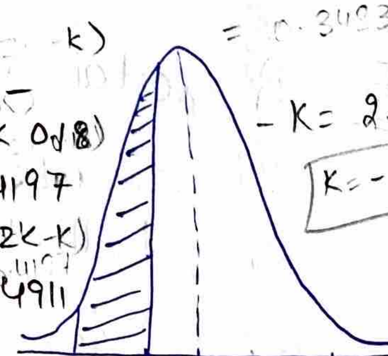
$P(0<z<-k) - P(0<z<0.18) = 0.4197$

$\Rightarrow P(0<z<-k) = 0.4911$

$\Rightarrow P(0<z<-k) = 0.3493$

$-k = 2.37$

$\boxed{k = -2.37}$



K -0.18 $-k$

$\Rightarrow P(Z<K) = 0.0089$

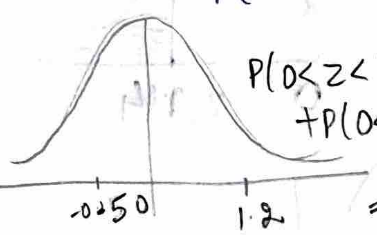
$K = -2.37.$

Ex: Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

Sol: The z value corresponding to $x_1 = 45$ and $x_2 = 62$ are

$$Z_1 = \frac{45-50}{10} = -0.5 \text{ and } Z_2 = \frac{62-50}{10} = 1.2.$$

$\therefore P(45 < X < 62) = P(-0.5 < Z < 1.2)$

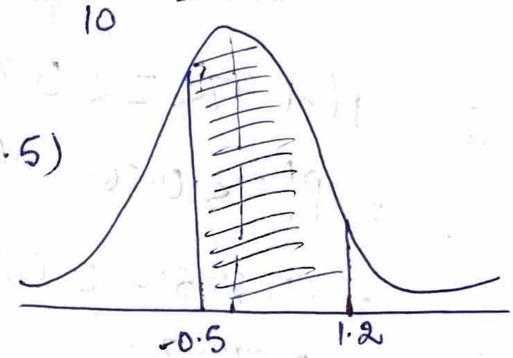


$= P(Z < 1.2) - P(Z < -0.5)$

$= 0.8849 - 0.3085$

$= 0.5764.$

$P(0<Z<1.2) + P(0<Z<0.5) = 0.3489 + 0.1915 =$



Ex: Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362.

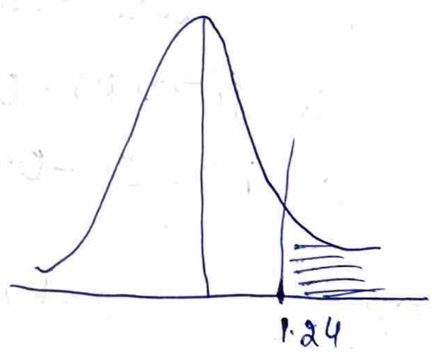
Sol: $Z = \frac{362-300}{50} = 1.24.$ $\left[Z = \frac{X-\mu}{\sigma} \right]$

$P(X > 362)$

$= P(Z > 1.24) = 1 - P(Z < 1.24)$

$= 1 - [0.8925]$

$= 0.1075.$

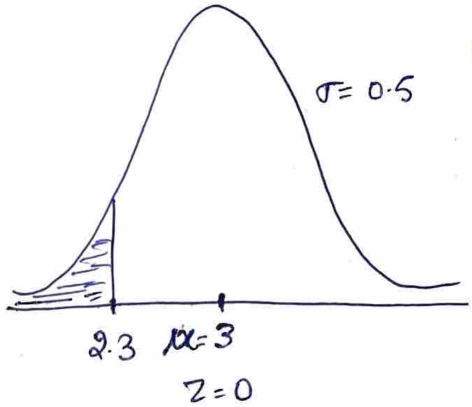


Ex: A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

$$Z = \frac{x-\mu}{\sigma} = \frac{2.3-3}{0.5} = -1.4.$$

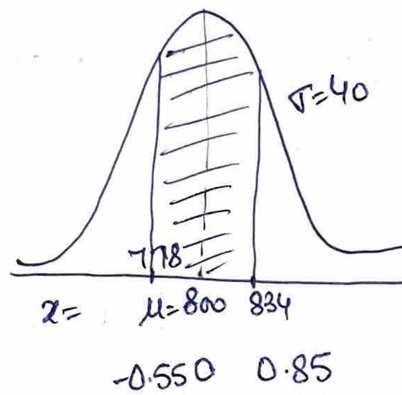
$P(\cancel{x} < 2.3) = P(Z < -1.4)$

$= 0.0808.$



Ex An elictrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

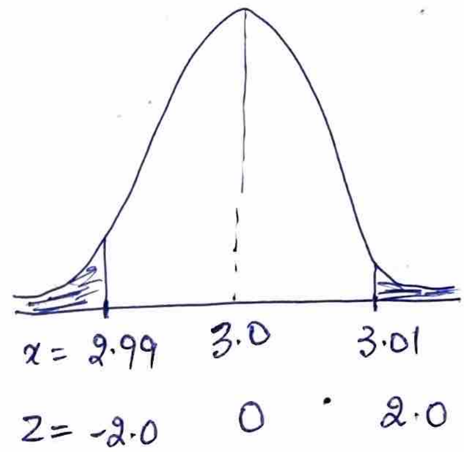
Sol →
$$Z = \frac{X-\mu}{\sigma} \Rightarrow Z_1 = \frac{X_1-\mu}{\sigma}$$
$$= \frac{778-800}{40}$$
$$= -0.55$$



$$Z_2 = \frac{x_2-\mu}{\sigma} = \frac{834-800}{40} = 0.85$$

$$P(778 < X < 834) = P(-0.55 < Z < 0.85)$$
$$= P(Z < 0.85) - P(Z < -0.55)$$
$$= 0.8023 - 0.2912$$
$$= 0.5111.$$

Ex In an industrial process, the diameter of a ball bearing is an important measure-~~of~~ment. The buyer sets specifications for the diameter to be 3.0 ± 0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that in the process the diameter of a ball bearing has a normal distribution with mean $\mu = 3.0$ and standard deviation $\sigma = 0.005$. On average, how many manufactured ball bearings will be scrapped?



Sol :- $Z_1 = \dfrac{2.99 - 3}{0.005} = -2.0$

$Z_2 = \dfrac{3.01 - 3}{0.005} = 2.0$

$\therefore \cancel{P(x > 2}$

$\therefore P(X < 2.99) = P(Z < -2)$

$= 0.0228$

and $P(X > 3.01) = P(Z > 2.0) = 1 - P(Z < 2)$

$= 1 - 0.9772$

$= 0.0228$

$\therefore$ The ball bearing will be scrapped if

$P(X < 2.99) + P(X > 3.01) = P(Z < -2) + P(Z > 2)$

$= 2(0.0228)$

$= 0.0456.$

$\therefore$ On average, ~~45.6%~~ 4.56% of the manufactured ball bearings will be scrapped.

Or by symmetry, $P(Z < -2) + P(Z > 2) = 2[P(Z < -2)]$

$= 2 \times 0.228$

$= 0.0456.$

$2[0.5 - P(Z < 2)] = [0.5 - 0.4772]2$

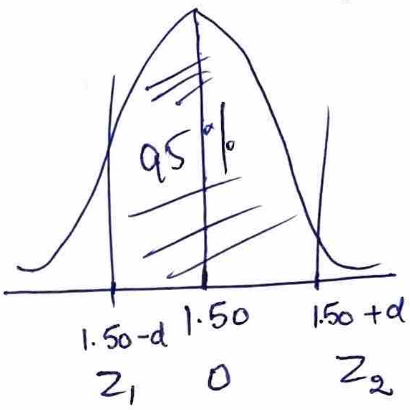
$= 0.0228 \times 2$

$= 0.0456$

Ex Gauges (Scales/Indicators) are used to reject all components for which a certain dimension is not within the specification $1.50 \pm d$. It is known that this measurement is normally

...istributed with mean 1.50 and standard deviation 0.2. Determine the value d such that the specifications cover 95% of the measurements.

Sol:



$$P(z_1 < z < z_2) = 0.95$$

$$2[0.5 - P(z < z_1)] = 0.95 \qquad 2 \times P(0 < z < z_1) = 0.95$$

$$\Rightarrow 1 - 2P(z < z_1) = 0.95 \qquad \Rightarrow P(0 < z < z_1) = 0.475$$

$$\Rightarrow 2P(z < z_1) = 0.05$$

$$\Rightarrow P(z < z_1) = 0.025 \qquad z_1 = 1.96$$

$$z_1 = -1.96$$

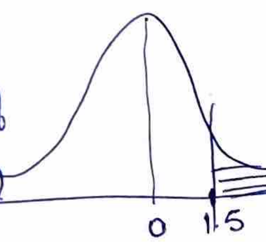
$$z_2 = 1.96$$

$$z_1 = \frac{x_1 - \mu}{\sigma} \Rightarrow -1.96 = \frac{1.50 - d - 1.50}{0.2}$$

$$\Rightarrow -1.96 = \frac{-d}{0.2}$$

$$\Rightarrow d = 0.392$$

Ex A certain machine makes electrical resistors having a mean resistance of 40 ohms (measurement of resistence) and a standard deviation of 2 ohms. Assuming that the resistance follows a normal distribution and can be measured to any degree of accuracy, what % of resistors will have a resistence exceeding 43 ohms?



Sol: $\mu = 40, \sigma = 2$

$\cancel{X = Z - \mu}$ $\quad Z = \frac{X - \mu}{\sigma} = \frac{43 - 40}{2} = 1.5$

$$P(X > 43) = P(Z > 1.5) = 1 - P(Z < 1.5) = 1 - 0.9332$$

$$= 0.0668.$$

Hence, 6.68% of the resistors will have a resistance exceeding 43 ohms.