Note on Attribution

This note follows the structure and flow of Prof. Athreya Shankar's lectures for the course *PH5620: Coherent and Quantum Optics, Jan-May 2025* at IIT Madras. The material has been reorganized and expanded where helpful, and written up in a form intended to complement the structure of our notes. Any mistakes are ours.

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1 Quantization of Light and Matter

Previous Notes

Why bother building 'quantum mechanics' as a framework?

In the first note, we looked at a series of experiments and phenomena to motivate why we would need a quantum mechanical framework and how classical mechanics left us without enough concepts to describe the world we live in.
For material on this section, please refer: Note 1 | Why QM

Quantum Mechanics

In the second note, we introduced some of the postulates of Quantum Mechanics using the Stern-Gerlach experiment as an example that shows inherently *quantum* behaviour.
For material on this section, please refer: Note 2 | QM Framework + Stern Gerlach Experiment

Supplementary Note 1: Mathematical Interlude: *Infinite Dimensional Vector Spaces*

After motivating quantum mechanics and introducing its postulates, we now develop the mathematical tools needed to work with the theory, focusing on the wavefunction and the uncertainty principle. For material on this section, please refer: Supplementary 1 | Wavefunction

The Harmonic Oscillator

The harmonic oscillator is a central model in physics that is exactly solvable classically, exhibits uniquely quantum features when quantized, and approximates many real systems such as molecular vibrations and light modes.

Key Takeaway: By promoting conjugate variables to operators that satisfy appropriate commutation relations, we can study the 'quantum' versions of classical Hamiltonians. The Quantum Harmonic Oscillator has discrete energy levels, which can be enumerated by an excitation number.

The Harmonic Oscillator

In the previous notes, we motivated the need for quantum mechanics through concrete physical phenomena and introduced its basic postulates using simple, idealized experiments. At this stage, the theory exists as a set of rules, but it is not yet clear how these rules play out in realistic physical systems or how familiar classical behaviour emerges from them. To move forward, we now focus on a single, solvable model that appears repeatedly in nature and experiment. The harmonic oscillator provides this starting point: it captures small oscillations, bound motion, and normal modes, all of which arise ubiquitously in both matter and light.

Introduction to Note 3

In the early days of quantum mechanics, many of its ideas were introduced through thought experiments and indirect observations. Concepts such as energy quantization, superposition, and measurement were inferred from spectral lines or theoretical arguments, rather than observed directly. Over the past few decades, this has changed in a striking way, largely due to the development of quantum optics. Some of the most famous modern experiments in physics, such as the double slit experiment performed with single photons, the trapping and cooling of individual atoms using laser light, and the controlled interaction of light and matter inside optical cavities, have made quantum mechanics directly visible in the laboratory. Light is no longer just a probe but a powerful tool for control, allowing physicists to prepare quantum states with high precision and follow their evolution in real time. As a result, quantum mechanics has moved from an abstract mathematical framework to a theory that describes systems we can actively engineer and observe.

In this modern setting, simple quantum models become surprisingly powerful. One of the most important of these is the quantum harmonic oscillator. Despite its mathematical simplicity, it appears in an enormous range of physical systems, from trapped atoms and ions, to vibrations in solids, to quantized modes of light in optical cavities. As we study the harmonic oscillator, we will see how quantization emerges naturally and how several fundamental ideas of quantum mechanics are built into this single model. The remaining postulates of quantum mechanics that were not introduced earlier will appear organically during this discussion, making the harmonic oscillator both a physical system of interest and a guide to understanding the structure of quantum theory itself.

Classical Harmonic Oscillator

You may already be familiar with the harmonic oscillator from a purely Newtonian treatment. To place it on a more systematic footing and to help us draw useful analogies later on, we will briefly revisit the harmonic oscillator using the Lagrangian and Hamiltonian formalisms. This will also serve as a short and self-contained introduction to the ideas of classical dynamics that will be important when we move on to the quantum description.

1.0.1 Lagrangian Formulation

The Lagrangian for the harmonic oscillator is given by the difference between kinetic energy (T) and potential energy (V):

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{1}$$

We define the momenta as follows:

- **Mechanical momentum:** $\Pi = m\dot{x}$
- **Canonical momentum:** $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

In this case, the canonical and mechanical momenta coincide. This happens because the potential energy is independent of the velocity. This equivalence is not generic; for example, a charged particle in a magnetic field has a velocity-dependent potential, leading to $p \neq m\dot{x}$.

Aside: Euler–Lagrange Equation and Classical Motion

The equations of motion follow from the Euler–Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{2}$$

For the harmonic oscillator Lagrangian,

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{3}$$

$$\frac{\partial L}{\partial x} = -kx. \tag{4}$$

Substituting into the Euler–Lagrange equation gives

$$m\ddot{x} + kx = 0. \tag{5}$$

This is precisely the familiar equation of motion obtained from Newton's second law for a Hookean spring. Thus, the Lagrangian formalism reproduces classical mechanics while providing a framework that generalizes naturally to quantum theory.

1.0.2 Hamiltonian Formulation

The Hamiltonian is defined via a Legendre transform of the Lagrangian:

$$H(x, p) = p\dot{x} - L. \tag{6}$$

To express H purely in terms of x and p, we eliminate $\dot{x}$ using the definition of the canonical momentum:

$$p = m\dot{x} \quad \Rightarrow \quad \dot{x} = \frac{p}{m}. \tag{7}$$

Substituting into the Legendre transform,

$$H = p\left(\frac{p}{m}\right) - \left[\frac{1}{2}m\left(\frac{p}{m}\right)^2 - \frac{1}{2}kx^2\right] \tag{8}$$

$$= \frac{p^2}{2m} + \frac{1}{2}kx^2. \tag{9}$$

Aside: Canonical vs Mechanical Momentum

In the case of the Harmonic Oscillator, the canonical momentum coincides with the mechanical (kinetic) momentum, so we can directly invert the relation to obtain

$$\dot{x} = \frac{p}{m}.$$

In general, however, the canonical momentum need not equal $m\dot{x}$. A physically important example is a charged particle in an electromagnetic field, with Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 + q\dot{x}A(x) - q\phi(x),$$

which leads to

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA(x).$$

In such cases, the Hamiltonian contains additional structure beyond the familiar kinetic and potential energy terms.

The Hamiltonian represents the total energy of the system expressed in terms of the phase-space variables that generate time evolution; in classical mechanics it governs the motion via Hamilton's equations, and in quantum mechanics it becomes the operator that generates time evolution of the state, as we will see later in the note.

1.0.3 Classical Dynamics and Phase Space

Having constructed the Hamiltonian, we now examine how it generates the classical dynamics of the system.

The equations of motion are given by Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p}, \tag{10}$$

$$\dot{p} = -\frac{\partial H}{\partial x}. \tag{11}$$

For the harmonic oscillator Hamiltonian,

$$\dot{x} = \frac{p}{m}, \tag{12}$$

$$\dot{p} = -kx, \tag{13}$$

which reproduces Hooke's law.

The solutions are

$$x(t) = A\cos(\omega t + \phi), \tag{14}$$

$$p(t) = -mA\omega\sin(\omega t + \phi), \tag{15}$$

where $\omega = \sqrt{k/m}$.

Since the Hamiltonian has no explicit time dependence, the total energy is conserved:

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = E. \tag{16}$$

Aside: Conservation Laws in Hamiltonian Mechanics

In Hamiltonian mechanics, the time evolution of any phase-space function $F(x, p, t)$ is given by

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}.$$

In particular, for the Hamiltonian itself,

$$\frac{dH}{dt} = \{H, H\} + \frac{\partial H}{\partial t}.$$

Since the Poisson bracket of any function with itself vanishes, $\{H, H\} = 0$, it follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

Therefore, if the Hamiltonian has no explicit time dependence,

$$\frac{\partial H}{\partial t} = 0,$$

the Hamiltonian is conserved along the motion. In such cases, the Hamiltonian coincides with the total energy of the system.

Rewriting,

$$\frac{x^2}{\left(\frac{2E}{m\omega^2}\right)} + \frac{p^2}{2mE} = 1, \tag{17}$$

which describes an ellipse in the (x, p) phase space. Each trajectory corresponds to motion at fixed energy.

Phase-space trajectories for 2 A values such that $A_1 > A_2$

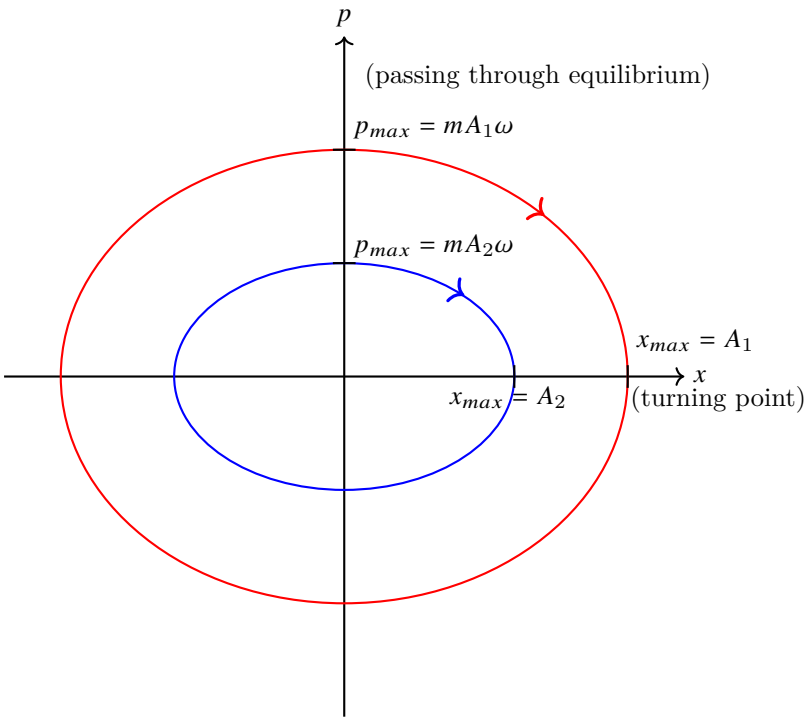

Figure 1: Phase-space diagram of a Classical Harmonic Oscillator

1.0.4 From Classical to Quantum: Quantization

The classical harmonic oscillator admits a complete description in terms of trajectories in phase space, with the Hamiltonian function generating time evolution through Hamilton's equations. Quantum mechanics does not modify this picture gradually; instead, it replaces the classical description at a fundamental level.

Quantization is introduced as a structural postulate. Classical phase space variables are promoted to operators acting on a space of states, and the classical Hamiltonian function is replaced by a Hamiltonian operator. The algebraic structure of classical mechanics, encoded in the Poisson bracket, is preserved in quantum mechanics through commutation relations:

$$\{x, p\} = 1 \quad \longrightarrow \quad [\hat{x}, \hat{p}] = i\hbar. \tag{18}$$

This replacement has immediate physical consequences. Because operators need not commute, the quantum theory forbids the simultaneous specification of position and momentum with arbitrary precision, and classical phase space trajectories cease to be meaningful. Instead, the state of the system is described by a single object that encodes all measurable information. This is dealt with in greater detail in the supplementary material (please refer: Sup 1 | Infinite Dimensional Vector spaces).

We now apply this quantization procedure to the harmonic oscillator.

1.1 Quantum Harmonic Oscillator

The quantum harmonic oscillator is defined by promoting the classical Hamiltonian (see Equation 9) to an operator,[1]

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \tag{19}$$

where the position and momentum operators satisfy the canonical commutation relation

$$[\hat{x}, \hat{p}] = i\hbar. \tag{20}$$

Our goal is to determine the eigenvalues and eigenstates of this Hamiltonian and to understand how the quantum behavior of the oscillator differs from its classical counterpart.

It is convenient to rewrite $\hat{H}$ in a suggestive form,

$$\hat{H} = \frac{1}{2m}[\hat{p}^2 + m^2\omega^2\hat{x}^2]. \tag{21}$$

If $\hat{x}$ and $\hat{p}$ were ordinary numbers rather than operators, we could attempt to factor this expression using identities of the form $(u + iv)(u - iv)$. However, since $\hat{x}$ and $\hat{p}$ do not commute, operator ordering matters and the factorization is not straightforward. Nevertheless, this observation motivates the introduction of the operators

$$\hat{a} = \sqrt{\frac{1}{2m\omega\hbar}}\,(i\hat{p} + m\omega\hat{x})\,, \tag{22}$$

$$\hat{a}^\dagger = \sqrt{\frac{1}{2m\omega\hbar}}\,(-i\hat{p} + m\omega\hat{x})\,. \tag{23}$$

Let us now compute their product. Using the canonical commutation relation, we find

$$\hat{a}^\dagger\hat{a} = \frac{1}{2m\omega\hbar}(-i\hat{p} + m\omega\hat{x})(i\hat{p} + m\omega\hat{x}) \tag{24}$$

$$= \frac{1}{2m\omega\hbar}\left(\hat{p}^2 + m^2\omega^2\hat{x}^2 + im\omega[\hat{x}, \hat{p}]\right) \tag{25}$$

$$= \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}. \tag{26}$$

Rearranging, the Hamiltonian can be written compactly as

$$\hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right), \tag{27}$$

where we have defined the number operator

$$\hat{n} = \hat{a}^\dagger\hat{a}. \tag{28}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ are called the annihilation and creation operators, respectively. They are also known as ladder operators, for reasons that will become clear shortly.

1.1.1 Commutation relations of ladder operators

Using the definitions of $\hat{a}$ and $\hat{a}^\dagger$, together with $[\hat{x}, \hat{p}] = i\hbar$, we find

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{29}$$

which is known as the bosonic commutation relation.

[1] A bit magical, but the reader is requested to bare with us. We will see that such a change makes 'sense' and that results have some sort of mathematical analogy with classical mechanics in various places as we continue developing the formalism.

Exercise

Using the definition of the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$, and the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, show that

$$[\hat{n}, \hat{a}] = -\hat{a}, \qquad [\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger. \tag{30}$$

1.1.2 Eigenstates of $\hat{n}$ and $\hat{H}$

Since $\hat{H}$ differs from $\hat{n}$ only by a constant, the eigenstates of $\hat{H}$ are the same as the eigenstates of $\hat{n}$. The number operator is Hermitian, so its eigenvalues are real and its eigenstates are orthogonal. We denote the eigenstates by $|n\rangle$, with

$$\hat{n}|n\rangle = n|n\rangle. \tag{31}$$

These states are called the Fock states. The eigenvalues n must be non-negative integers.

To see why $n \geq 0$, consider a normalized eigenstate $|n\rangle$ such that $\langle n|n\rangle = 1$. Then

$$\langle n|\hat{n}|n\rangle = n. \tag{32}$$

Define $|u\rangle = \hat{a}|n\rangle$. We have

$$\langle u|u\rangle = \langle n|\hat{a}^\dagger \hat{a}|n\rangle = n. \tag{33}$$

Since $\langle u|u\rangle \geq 0$ for any state $|u\rangle$, it follows that $n \geq 0$.

Next, consider the action of $\hat{a}$ on $|n\rangle$:

$$\hat{n}(\hat{a}|n\rangle) = (\hat{a}\hat{n} - [\hat{n}, \hat{a}])|n\rangle \tag{34}$$

$$= \hat{a}(n-1)|n\rangle. \tag{35}$$

Thus, $\hat{a}|n\rangle$ is an eigenstate of $\hat{n}$ with eigenvalue $n-1$, and we may write

$$\hat{a}|n\rangle = c\,|n-1\rangle. \tag{36}$$

To determine the constant c, note that

$$\langle n|\hat{n}|n\rangle = \langle n|\hat{a}^\dagger \hat{a}|n\rangle = |c|^2. \tag{37}$$

Choosing c to be real and positive, we obtain

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle. \tag{38}$$

Similarly, one finds

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle. \tag{39}$$

Repeated application of $\hat{a}$ and $\hat{a}^\dagger$ therefore moves us down and up the ladder of states, respectively. Since the spectrum is bounded below and $\hat{a}|0\rangle = 0$, the ladder terminates at the ground state $|0\rangle$. This requires the eigenvalues n to be non-negative integers.

Using $\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2})$, we find

$$\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle. \tag{40}$$

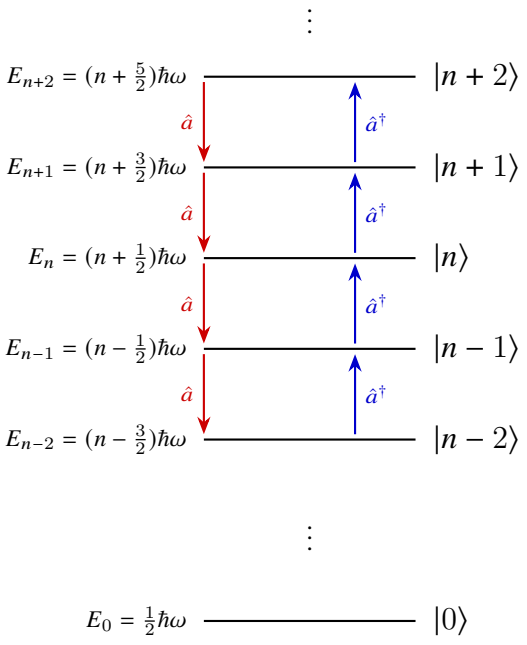

Figure 2: Repeated application of the creation operator ($\hat{a}^\dagger$) and annihilation operator ($\hat{a}$) moves us up and down the ladder of states. The energy levels differ by a constant amount $\hbar\omega$. The lowest possible energy level is $\frac{\hbar\omega}{2}$ with corresponding eigenstate $|0\rangle$

1.1.3 Zero-point energy

The ground state $|0\rangle$ satisfies

$$\hat{H}|0\rangle = \frac{\hbar\omega}{2}|0\rangle. \tag{41}$$

Thus, even in its lowest energy state, the quantum harmonic oscillator has a non-zero energy $\hbar\omega/2$, known as the zero-point energy.

In contrast, the classical harmonic oscillator has a ground state at $x = 0$ and $p = 0$, corresponding to zero total energy. In quantum mechanics, however, the non-commutativity of $\hat{x}$ and $\hat{p}$ implies unavoidable quantum fluctuations, which give rise to the zero-point energy.

Review: Function Spaces

The reader is referred to the previous note: Supplementary 1 | Function Spaces for further reading.

1. **Completeness and orthogonality:**
   In function spaces, we have:

$$\int_{-\infty}^{\infty} |x\rangle \langle x| \, dx = \mathbb{1}, \; \langle x'|x\rangle = \delta(x - x')$$

   You can expand the state ket in terms of an infinite basis:

$$|\psi\rangle = \int_{-\infty}^{\infty} |x\rangle \langle x|\psi\rangle \, dx$$

   In infinite continuous dimensions (like position), that "column vector" becomes a smooth function: the **Wavefunction** $\psi(x)$.

2. **Probability Density (The Born Rule):**
   $\psi(x)$ is a *probability amplitude* (complex-valued). The physical probability is found by taking the absolute square:
   $$P(x) = |\psi(x)|^2 \tag{42}$$
   The probability of finding the particle in a tiny region of width dx is $|\psi(x)|^2 dx$.

3. **Normalization:**
   Just as discrete probabilities sum to 1, the total probability of finding the particle *somewhere* must be 1. We integrate over all space:
   $$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \tag{43}$$

1.1.4 Wavefunction

Apart from the energy, we may also be interested in trying to identify the probability with which a particle may be found around some interval in space, to do this, we would need to find the *position-space wavefunction*.
Ground State Wavefunction

We start by attempting to identify the probability of finding the particle when it is in the lowest energy state.

Remember the relation, $\hat{a}\,|0\rangle = 0$. Taking the inner product with $|x\rangle$ on both sides:

$$\begin{aligned}
\langle x|\,\hat{a}\,|0\rangle &= 0 \\
&= \langle x| \sqrt{\frac{1}{2m\hbar\omega}} (\hat{p} + im\omega\hat{x})\,|0\rangle \\
&= \sqrt{\frac{1}{2m\hbar\omega}} \left(\langle x|\,\hat{p}\,|0\rangle + im\omega\,\langle x|\,\hat{x}\,|0\rangle\right) \\
&= \sqrt{\frac{1}{2m\hbar\omega}} \left(-i\hbar\frac{\partial}{\partial x}\,\langle x|0\rangle + im\omega x\,\langle x|0\rangle\right) \\
&= \sqrt{\frac{1}{2m\hbar\omega}}\, i \left(-\hbar\frac{\partial}{\partial x} + m\omega x\right)\psi_0(x) \\
&\implies \left(\hbar\frac{\partial}{\partial x} + m\omega x\right)\psi_0(x) = 0 \\
&\implies \frac{\partial}{\partial x}\psi_0(x) = -\frac{x}{x_0^2}\psi_0(x)
\end{aligned} \tag{44}$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ is the characteristic length scale. The normalized solution is a Gaussian:

$$\psi_0(x) = \frac{1}{\pi^{1/4}\sqrt{x_0}} exp(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2) \tag{45}$$

Exercise: Solve the differential equation in Equation 44 and verify that the solution is a gaussian.

Excited States

Excited states are generated by applying the creation operator: $|n\rangle \propto (\hat{a}^\dagger)^n\,|0\rangle$. The wavefunctions involve Hermite polynomials $H_n(\xi)$:

$$\psi_n(\xi) = \frac{1}{\sqrt{2^n n!}} \frac{1}{\pi^{1/4}\sqrt{x_0}} H_n(\xi) e^{-\xi^2/2}, \quad \text{where } \xi = \frac{x}{x_0} \tag{46}$$

Exercise: Recall the relation

$$\begin{aligned} \hat{a}^\dagger |n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \implies \hat{a}^\dagger |0\rangle &= |1\rangle \\ \implies \langle x|\hat{a}^\dagger|0\rangle &= \psi_1(x) \end{aligned}$$

Use it to find the analytical form of $\psi_1(x)$. Verify your answer by setting $n = 1$ in Equation 46.

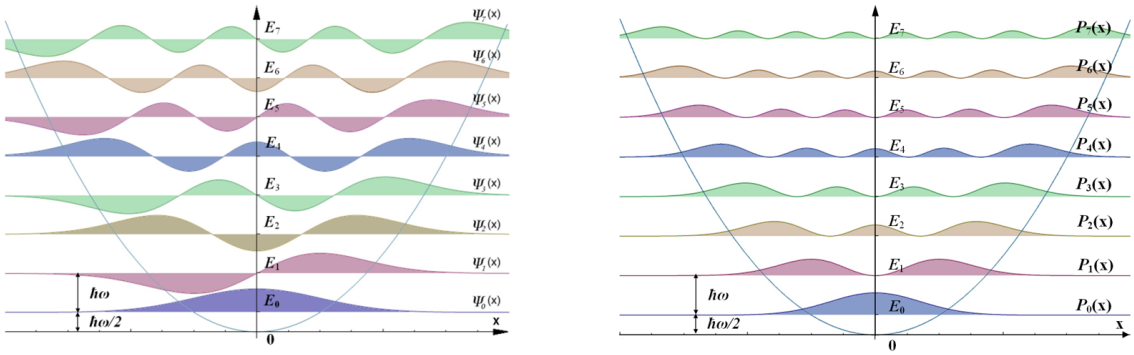

Figure 3: Image Source: Wikipedia | Quantum Harmonic Oscillator.
(a): The wavefunctions for the first eight eigenstates of the HO Hamiltonian (b): The corresponding probability densities ($|\psi(x)|^2$)

Summary: Quantum Harmonic Oscillator

1. Hamiltonians & Energy Spectrum

- **Hamiltonian Operator:**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

- **Energy Eigenvalues:** $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ for $n = 0, 1, 2, \ldots$
- **Zero-Point Energy:** $E_0 = \frac{\hbar\omega}{2}$ (Ground state energy $\neq 0$).

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2. Operator Algebra (Ladder Operators)

Definitions:

$$\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}}\left(i\hat{p} + m\omega\hat{x}\right), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}\left(-i\hat{p} + m\omega\hat{x}\right)$$

Commutation Relations:

$$[\hat{x}, \hat{p}] = i\hbar, \qquad [\hat{a}, \hat{a}^\dagger] = 1, \qquad [\hat{n}, \hat{a}] = -\hat{a}, \qquad [\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger$$

Action on States (Fock States $|n\rangle$):

$$\hat{n}\,|n\rangle = n\,|n\rangle \qquad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

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3. Position-Space Wavefunctions

- **Characteristic Length:** $x_0 = \sqrt{\frac{\hbar}{m\omega}}$

- **General State (n):** Gaussian distribution modulated by Hermite Polynomials H_n.

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{1}{\pi x_0^2}\right)^{1/4} H_n\left(\frac{x}{x_0}\right) \exp\left(-\frac{x^2}{2x_0^2}\right)$$

4. Comparison: Classical vs. Quantum

Classical Oscillator	**Quantum Oscillator**
Variables commute ($\{x, p\} = 1$)	Operators do not commute ($[\hat{x}, \hat{p}] = i\hbar$)
Spectrum is continuous ($E \geq 0$)	Spectrum is discrete ($E_n = \hbar\omega(n + 1/2)$)
Lowest energy is $E = 0$ (at rest)	Zero-Point Energy $E_0 = \hbar\omega/2$
Determinate trajectory $x(t)$	Probabilistic wavefunction $\psi(x)$
Particle confined to turning points	Wavefunction tunnels into classically forbidden regions

The time evolution and dynamics in the quantum picture will be explored in the next note.