Note on Attribution

This note follows the structure and flow of Prof. Athreya Shankar's lectures for the course *PH5620: Coherent and Quantum Optics, Jan-May 2025* at IIT Madras. The material has been reorganized and expanded where helpful, and written up in a form intended to complement the structure of our notes. Any mistakes are ours.

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1 Quantization of Light and Matter

Previous Notes

Why bother building 'quantum mechanics' as a framework?

In the first note, we looked at a series of experiments and phenomena to motivate why we would need a quantum mechanical framework and how classical mechanics left us without enough concepts to describe the world we live in.
For material on this section, please refer: Note 1 | Why QM

Quantum Mechanics

In the second note, we introduced some of the postulates of Quantum Mechanics using the Stern-Gerlach experiment as an example that shows inherently *quantum* behaviour.
For material on this section, please refer: Note 2 | QM Framework + Stern Gerlach Experiment

Supplementary Note 1: Mathematical Interlude: ***Infinite Dimensional Vector Spaces***

After motivating quantum mechanics and introducing its postulates, we now develop the mathematical tools needed to work with the theory, focusing on the wavefunction and the uncertainty principle.
For material on this section, please refer: Supplementary 1 | Wavefunction

The Harmonic Oscillator

The harmonic oscillator is a central model in physics that is exactly solvable classically, exhibits uniquely quantum features when quantized, and approximates many real systems such as molecular vibrations and light modes.
For material on this section, please refer:Quantum Harmonic Oscillator

Time Evolution and Schrödinger Equation

We look at how the state ket evolves in time and derive the Schrodinger Wave Equation from the Time-Evolution postulate, which bridges the 'wave-mechanics' approach and 'matrix-mechanics' approach that authors use to teach QM.

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2 Time Evolution

So far, our discussion has only involved trying to understand the static solutions of the system, we now tackle the question of how the state of the system evolves with time.

2.1 The Time-Evolution Operator

The text below borrows heavily from: Littlejohn (2021)

Postulate 5: Time-Evolution of a Quantum State
The state of a system at a later time t is related to the state at an initial time t_0 by a linear transformation for a closed quantum system.[1]

Let the state at time t_0 be described by the ket $|\psi(t_0)\rangle$, and the state at time t be $|\psi(t)\rangle$. According to this postulate, these two states are related by a linear operator $\hat{U}(t, t_0)$:

$$|\psi(t)\rangle = \hat{U}(t, t_0)\,|\psi(t_0)\rangle \tag{1}$$

Since the total probability of finding a particle must remain unity (assuming particles are neither created nor destroyed):

$$\begin{aligned} \langle\psi(t)|\psi(t)\rangle &= \langle\psi(t_0)|\hat{U}(t, t_0)^\dagger \hat{U}(t, t_0)\psi(t_0)\rangle \\ &= 1 \\ \implies \hat{U}(t, t_0)^\dagger \hat{U}(t, t_0) &= \mathbb{1} \end{aligned}$$

That is, the linear operator introduced is unitary.

$$\hat{U}(t, t_0)^{-1} = \hat{U}(t, t_0)^\dagger \tag{2}$$

Additionally, the operator must satisfy the identity property at the initial time:

$$\hat{U}(t_0, t_0) = \mathbb{1} \tag{3}$$

Furthermore, the time evolution must be composable. If we evolve the system from t_0 to an intermediate time t_1, and then from t_1 to t_2, the result must be the same as evolving directly from t_0 to t_2:

$$\hat{U}(t_2, t_1)\hat{U}(t_1, t_0)|\psi(t_0)\rangle = \hat{U}(t_2, t_0)|\psi(t_0)\rangle \tag{4}$$

Thus,

$$\hat{U}(t_2, t_1)\hat{U}(t_1, t_0) = \hat{U}(t_2, t_0) \tag{5}$$

2.2 The Hamiltonian and the Schrödinger Equation

Aside: Functions of an Operator

In quantum mechanics, functions of operators are generally defined by their power series (Taylor series) expansions. For a scalar function $f(x) = \sum_{n=0}^{\infty} c_n x^n$, the corresponding operator function $f(\hat{A})$ is defined as:

$$f(\hat{A}) = \sum_{n=0}^{\infty} c_n \hat{A}^n = c_0\mathbb{I} + c_1\hat{A} + c_2\hat{A}^2 + \dots$$

Where c_i are the coefficients of the Taylor Expansion.

This definition is crucial when deriving the finite time-evolution operator from the infinitesimal form. We use this as the justification for a ‘first-order’ expansion for the Time-Evolution Operator to study how the state changes from time t to time $t + \epsilon$.

[1] For the sake of being pedantic. You could just consider the whole Universe as your system, and then your system would be closed.

Aside: The Hamitonian being the *generator* of time translations

In classical mechanics, the classical Hamiltonian H is considered to be the generator of time translations, i..e, for small time increments ϵ, we can calculate the change in a classical observable using the Poisson bracket with H.
For a small time increment ϵ, the change in a classical observable $f(x, p, t)$ is:

$$\begin{aligned} f(t+\epsilon) &= f(t) + \epsilon \frac{df}{dt} \\ &= f(t) + \epsilon(\dot{x}\frac{\partial f}{\partial x} + \dot{p}\frac{\partial f}{\partial p}) \\ &= f(t) + \epsilon(\frac{\partial H}{\partial p}\frac{\partial f}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial p}) \\ &= f(t) - \epsilon\{H, f\} \end{aligned}$$

Which can be re-written as:

$$\frac{df}{dt} = \{f, H\} \tag{6}$$

Keep this in mind, we will see the Quantum Mechanics analog to this very soon!

Consider an infinitesimal time evolution from t to $t + \epsilon$. We expand the operator to first order in ϵ:

$$\hat{U}(t+\epsilon, t) = \mathbb{1} - i\epsilon\hat{\Omega}(t) + \dots \tag{7}$$

where $\hat{\Omega}(t)$ is a Hermitian operator required to maintain the unitarity of U. We have, in effect, defined $\hat{\Omega}(t) = i\frac{\partial}{\partial t'}\hat{U}(t', t)\Big|_{t'=t}$

Exercise

Show that $\hat{\Omega}(t)$ being Herimitian implies that $\hat{U}(t+\epsilon, t) = \mathbb{1} - i\epsilon\hat{\Omega}(t)$ is unitary. Neglect higher order terms since ϵ is assumed to be infinitesimally small.
Hint: Find $U(t \hat{+} \epsilon, t)$ and $\hat{U}^\dagger(t+\epsilon, t)$ and verify whether their product gives Identity.

$\Omega(t)$ operator acts as the *generator* of time translations [2]. We identify $\Omega(t)$ with the Hamiltonian $\hat{H}$ (scaled by $\hbar$ for dimensional consistency since $\hat{\Omega}(t)$ has dimensions of $[\mathtt{T}]^{-1}$):

$$\hat{H}(t) = \hbar\Omega(t) \quad \Rightarrow \quad \hat{U}(t+\epsilon, t) = \mathbb{1} - \frac{i\epsilon}{\hbar}\hat{H}(t) \tag{8}$$

Assuming $\hat{H}$ is time-independent[3], we can compose this infinitesimal evolution N times over a finite duration $t - t_0$. Taking the limit as $N \to \infty$:

$$\hat{U}(t, t_0) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i(t-t_0)/N}{\hbar}\hat{H}\right)^N = \exp\left[-\frac{i\hat{H}(t-t_0)}{\hbar}\right] \tag{9}$$

[2] Time translations form a continuous symmetry, and in quantum mechanics such symmetries are generated by Hermitian operators. The operator that generates infinitesimal time translations therefore defines the Hamiltonian. From a physical viewpoint, invariance under time translations implies conservation of energy (Noether's theorem), which is why the generator is identified with the energy operator. The factor of $\hbar$ ensures the correct physical dimensions.

The reader is referred to: Physics StackExcange | Noether's theorem and Hamiltonian

[3] This result is true for time-dependent Hamiltonians provided that the Hamiltonian at any later time $\hat{H}(t)$ commutes with the the Hamiltonian at some time t_0, $\hat{H}(t_0)$.

Therefore, the state at a later time t is given by:

$$|\psi(t)\rangle = exp(\frac{-i\hat{H}(t-t_0)}{\hbar})\,|\psi(t_0)\rangle \tag{10}$$

Deriving the Schrödinger Equation

To derive the differential equation for time evolution, we take the derivative of the defining equation for $U(t, t_0)$. Using the infinitesimal form:

$$\frac{\partial \hat{U}(t,t_0)}{\partial t} = \lim_{\epsilon\to 0}\frac{\hat{U}(t+\epsilon,t_0)-\hat{U}(t,t_0)}{\epsilon} = \left[\lim_{\epsilon\to 0}\frac{\hat{U}(t+\epsilon,t)-1}{\epsilon}\right]\hat{U}(t,t_0) \tag{11}$$

Substituting the expansion with the Hamiltonian (From Equation 8):

$$i\hbar\frac{\partial \hat{U}(t,t_0)}{\partial t} = \hat{H}(t)U(t,t_0) \tag{12}$$

Multiplying both sides by the initial state ket $|\psi(t_0)\rangle$, we obtain the Time-Dependent Schrödinger Equation (TDSE)!

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \tag{13}$$

Isn't it beautiful to see it arise directly from the formalism?
If the Hamiltonian is time-independent ($\frac{\partial \hat{H}}{\partial t} = 0$), we can look for solutions where the time dependence is separable from the spatial (ket) dependence. We propose the ansatz:

$$|\psi(t)\rangle = e^{-iEt/\hbar}|\phi\rangle \tag{14}$$

where $|\phi\rangle$ is a time-independent ket and E is a scalar energy. Substituting this into the left-hand side of the time-dependent equation:

$$i\hbar\frac{\partial}{\partial t}\left(e^{-iEt/\hbar}|\phi\rangle\right) = i\hbar\left(-\frac{iE}{\hbar}\right)e^{-iEt/\hbar}|\phi\rangle \tag{15}$$

$$= (-i^2)Ee^{-iEt/\hbar}|\phi\rangle \tag{16}$$

$$i\hbar\frac{\partial}{\partial t}\left(e^{-iEt/\hbar}|\phi\rangle\right) = Ee^{-iEt/\hbar}|\phi\rangle \tag{17}$$

Now, substituting the ansatz into the right-hand side (and noting that $\hat{H}$ commutes with the scalar exponential):

$$\hat{H}\left(e^{-iEt/\hbar}|\phi\rangle\right) = e^{-iEt/\hbar}\hat{H}|\phi\rangle \tag{18}$$

Equating the results from the left and right sides:

$$Ee^{-iEt/\hbar}|\phi\rangle = e^{-iEt/\hbar}\hat{H}|\phi\rangle \tag{19}$$

Since the exponential term $e^{-iEt/\hbar}$ is never zero, we can divide it out, leaving us with the Time-Independent Schrödinger Equation:

$$\hat{H}|\phi\rangle = E|\phi\rangle \tag{20}$$

Notice that this is exactly the form of the Hamiltonian we got when we quantized the classical Hamiltonian for the Harmonic Oscillator:

$$\hat{H}\,|n\rangle = \hbar\omega(n+\frac{1}{2})\,|n\rangle$$

Summary: Time Evolution and the Schrödinger Equation

- **Postulate 5:** The evolution of a quantum state is governed by a linear, unitary operator $\hat{U}(t, t_0)$, which ensures that probabilities remain conserved (sum to unity) over time.
- **Composition Property:** Time evolution is cumulative. Evolving from t_0 to t_2 is equivalent to evolving $t_0 \to t_1$ and then $t_1 \to t_2$.
- **The Generator:** The Hamiltonian $\hat{H}$ acts as the generator of time translations. For an infinitesimal time step ϵ, the evolution operator is $1 - (i\epsilon/\hbar)\hat{H}$.
- **TDSE:** The differential form of time evolution is the Time-Dependent Schrödinger Equation: $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$.
- **TISE:** If $\hat{H}$ is time-independent, the solution separates into spatial and temporal parts, leading to the eigenvalue equation $\hat{H}|\phi\rangle = E|\phi\rangle$, where E is the energy.

2.3 Schrödinger Picture

The dynamics are governed by the Time-Dependent Schrödinger Equation (TDSE). For a time-independent Hamiltonian, the solution is:

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle \tag{21}$$

As verified in the previous section. Stationary states (eigenstates $|n\rangle$) only acquire a global phase factor. However, superpositions evolve non-trivially. For a superposition $|\psi(0)\rangle = c_n|n\rangle + c_{n+1}|n+1\rangle$:

$$|\psi(t)\rangle = e^{-i\omega t(n+1/2)}\left(c_n|n\rangle + c_{n+1}e^{-i\omega t}|n+1\rangle\right) \tag{22}$$

The relative phase change leads to oscillating fidelity $\mathcal{F}(t) = |\langle\psi(0)|\psi(t)\rangle|^2$.

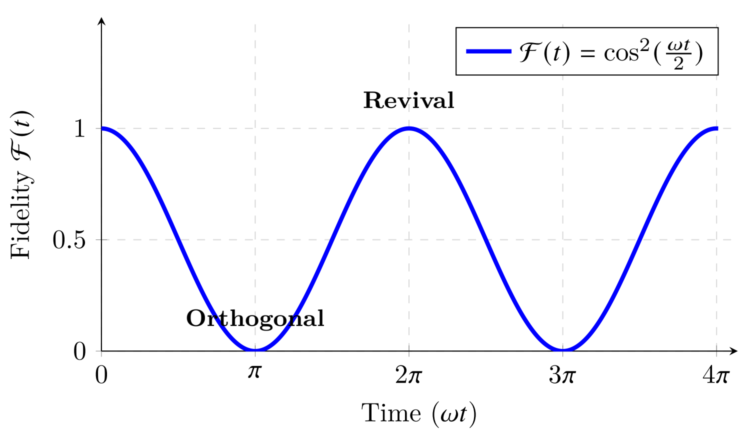
Exercise

Explicitly compute the Fidelity. Show that you get the following:

$$\mathcal{F}(t) = 1 - 4|c_n|^2|c_{n+1}|^2\sin^2\left(\frac{\omega t}{2}\right) \tag{23}$$

Consider the special case where $c_n = c_{n+1} = \frac{1}{\sqrt{2}}$, then:

$$\begin{aligned}\mathcal{F}(t) &= 1 - sin^2(\frac{\omega t}{2})\\ &= cos^2(\frac{\omega t}{2})\end{aligned}$$



Notice that the state at a later time can be orthogonal to the initial state!

2.4 Heisenberg Picture

Recall when you were first introduced to the rotation of a vector. There are two equivalent pictures to visualize this: either the basis remains fixed and the vector rotates, or the vector remains fixed, and the basis rotates.

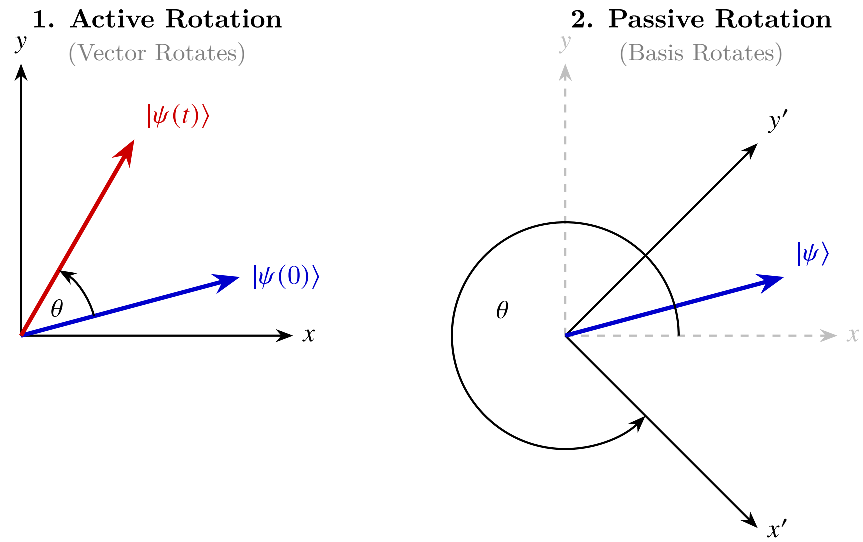


Figure 1: Two equivalent pictures of time evolution. **Left (Active/Schrödinger):** The coordinate system (observables) remains fixed, while the state vector $|\psi\rangle$ rotates by angle θ. **Right (Passive/Heisenberg):** The state vector remains fixed, while the coordinate system (observables) rotates by $-\theta$.

In the Schrödinger picture, we looked at how we could get the state vector at a later time $|\psi(t)\rangle$ by 'rotating' $|\psi(t_0)\rangle$ using a unitary $U(t, t_0)$.

Alternatively, we can let the operators evolve while the state remains fixed.

Irrespective of how we think about time evolution, the observations we see must be the same. For instance, if we compute the expectation of an operator $\hat{O}$ at time t for state $|\psi(t)\rangle$:

$$\begin{aligned}\langle\hat{O}\rangle &= \langle\psi(t)|\hat{O}|\psi(t)\rangle \\ &= \langle\psi(t_0)\hat{U}(t,t_0)^\dagger\hat{O}\hat{U}(t,t_0)|\psi(t_0)\rangle\end{aligned}$$

The Heisenberg operator $\hat{O}_H(t)$ is defined as:

$$\hat{O}_H(t) = \hat{U}(t,t_0)^\dagger\hat{O}_S\hat{U}(t,t_0) \tag{24}$$
$$= e^{i\hat{H}t/\hbar}\hat{O}_s e^{-i\hat{H}t/\hbar} \tag{25}$$
$$\tag{26}$$

Differentiating on both sides:

$$\frac{d\hat{O_H}(t)}{dt} = \frac{d}{dt}e^{i\hat{H}t/\hbar}\hat{O}_s e^{-i\hat{H}t/\hbar} \tag{27}$$
$$= \frac{i}{\hbar}[\hat{H}, \hat{O_H}(t)] \tag{28}$$

This is called the Heisenberg Equation of Motion:

$$\frac{d\hat{O_H}(t)}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{O_H}(t)] \tag{29}$$

Aside: Time-Evolution in Classical Mechanics and Quantum Mechanics

Remember Equation 6:

$$\frac{df}{dt} = \{f, H\}$$

We can re-write Equation 29 as:

$$\frac{d\hat{O}_H(t)}{dt} = \frac{1}{i\hbar}[\hat{O}_H(t), \hat{H}]$$

The 'Heisenberg Equation of Motion' was first written by P. A. M. Dirac, and this gives us an ansatz to derive Classical Mechanics from Quantum Mechanics by replacing the commutator with the Poisson bracket:

$$\frac{1}{i\hbar}[,] \longrightarrow \{,\}_{classical}$$

However, there are quantities like Spin that have no classical analogues. To read further, refer Section 2.2 from Sakurai & Napolitano (2011)

Exercise

Verify the form of the Heisenberg Equation of Motion by continuing the steps following Equation 29.

Exercise

Verify that the annihilation operator in the Heisenberg picture is given by:

$$\hat{a}_H(t) = \hat{a}_S e^{-i\omega t} \tag{30}$$

Hint: Use the Heisenberg Equation of Motion

Exercise

Compute $\langle \hat{x}(t) \rangle$, $\langle \hat{x}^2(t) \rangle$, $\langle \hat{p}(t) \rangle$ and $\langle \hat{p}^2(t) \rangle$

Aside: The Baker-Hausdorff lemma and Baker-Campbell-Hausdorff formula

1. The Baker-Hausdorff Lemma
There is a direct way to evaluate the Heisenberg picture operator using the Baker-Hausdorff lemma. For two operators $\hat{A}$ and $\hat{B}$, the lemma states that:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots \tag{31}$$

Defining the n-th nested commutator as $[\hat{A}, \hat{B}]_n = [\hat{A}, [\hat{A}, \dots [\hat{A}, \hat{B}] \dots]]$, we can write this compactly as:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{n=0}^{\infty} \frac{[\hat{A}, \hat{B}]_n}{n!} \tag{32}$$

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2. The Baker-Campbell-Hausdorff (BCH) Formula
For completeness, the BCH formula handles the exponential of a sum. For two operators $\hat{A}, \hat{B}$, such that both commute with their commutator $[\hat{A}, \hat{B}]$ (i.e., $[\hat{A}, [\hat{A}, \hat{B}]] = 0$ and $[\hat{B}, [\hat{A}, \hat{B}]] = 0$), we have:

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A}, \hat{B}]} \tag{33}$$

Solved Example
Determine the form of the annihilation operator in the Heisenberg picture using the BH Lemma. Use the Harmonic Oscillator Hamiltonian, i.e.,
Evaluate

$$\hat{O}_H(t) = e^{i\hat{H}t/\hbar}\,\hat{O}_S\,e^{-i\hat{H}t/\hbar}$$

where $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

Solution:

The BH Lemma states:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$$

Identifying $\hat{A} = i\hat{H}t/\hbar$ and $\hat{B} = \hat{a}_s$

$$\hat{a}_H(t) = e^{i\hat{H}t/\hbar}\,\hat{a}_s\,e^{-i\hat{H}t/\hbar}$$
$$\hat{H} = \hbar\omega\left(\hat{a}_s^\dagger\hat{a}_s + \frac{1}{2}\right)$$

Calculating the commutators:

$$\begin{aligned}[\hat{A},\hat{B}] &= \left[\frac{i\hat{H}t}{\hbar}, \hat{a}_s\right] \\ &= i\omega t[\hat{n},\hat{a}_s] + \left(\left[\frac{1}{2},\hat{a}_s\right]\right) \\ &= i\omega t(-\hat{a}_s)\end{aligned}$$

$$\begin{aligned}[\hat{A},[\hat{A},\hat{B}]] &= \left[\frac{i\hat{H}t}{\hbar}, -i\omega t\hat{a}_s\right] \\ &= (-1)(i\omega t)^2[\hat{n},\hat{a}_s] \\ &= (i\omega t)^2\hat{a}_s \\ &= -(\omega t)^2\hat{a}_s\end{aligned}$$

Induction hypothesis:

$$\begin{aligned}[\hat{A}, \underbrace{[\hat{A},\dots[\hat{A},\hat{B}]]]}_{k \text{ times}} &= [\hat{A},\hat{B}]_k \\ &= (-i\omega t)^k\hat{a}_s\end{aligned}$$

$$\begin{aligned}[\hat{A},\hat{B}]_{k+1} &= \left[\frac{i\hat{H}t}{\hbar}, (-i\omega t)^k\hat{a}_s\right] \\ &= (i\omega t)(-i\omega t)^k(-\hat{a}_s) \\ &= (-i\omega t)^{k+1}\hat{a}_s\end{aligned}$$

$\therefore$ By the induction principle,

$$\left[\frac{i\hat{H}t}{\hbar}, \hat{a}_s\right]_n = (-i\omega t)^n\hat{a}_s \quad \forall n \geq 1$$

Therefore:

$$\begin{aligned} e^{i\hat{H}t/\hbar}\hat{a}e^{-i\hat{H}t/\hbar} &= \hat{a}_s + (-i\omega t)\hat{a}_s + \frac{1}{2!}(-i\omega t)^2\hat{a}_s + \dots \\ &= \hat{a}_s \left(e^{-i\omega t}\right) \end{aligned}$$

Exercise: Introducing the Interaction Picture

Earlier in this note, we derived the Time-Dependent Schödinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H\hat{(}t)|\psi(t)\rangle \tag{34}$$

Subpart a: Change of frame

Sometimes, it is useful to study the time evolution of a system from a different frame. This is particularly useful, say if we have a system with two different characteristic frequencies (e.g., light with frequency ω_1 being incident on a harmonic oscillator whose spacing differs by $\hbar\omega_2$).
Define $|\phi(t)\rangle = \hat{U}(t)|\psi(t)\rangle$ where $\hat{U}(t)$ is an arbitrary Unitary operator, not necessarily related to the Hamiltonian in Equation 34
Re-write Equation 34 in terms of $|\phi(t)\rangle$. Verify that this gives:

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = [i\hbar\dot{\hat{U}}(t)\hat{U}(t)^\dagger + \hat{U}(t)\hat{H}(t)\hat{U}(t)^\dagger]|\phi(t)\rangle \tag{35}$$

Subpart b: The driven oscillator and Interaction Picture

Let the Hamiltonian in Equation 34 consist of a time-independent "simple" Hamiltonian $\hat{H}_0$ and a time-dependent interaction Hamiltonian $\hat{V}(t)$ giving

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t) \tag{36}$$

Let

$$\hat{U}(t) = exp(\frac{i\hat{H}_0(t)t}{\hbar}) \tag{37}$$

Replace Equation 37 and Equation 36 in Equation 35. Verify that you obtain:

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = [\hat{U}(t)\hat{V}(t)\hat{U}(t)^\dagger]|\phi(t)\rangle \tag{38}$$

Equation 38 is referred to as the "Interaction Picture".
Subpart c: Alternate form

$$\hat{H}_I(t) = \hat{U}(t)\hat{V}(t)\hat{U}(t)^\dagger \tag{39}$$

Derive that Equation 38 can be written as:

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = \hat{H}_I(t)|\phi_I(t)\rangle \tag{40}$$

Subpart d: The Rotating Frame

In Equation 35 let $\hat{U}(t) = exp(\frac{i\hat{H}_D t}{\hbar})$ such that $[\hat{H}_D, \hat{H}_0] = 0$
Verify that you obtain:

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = [\hat{H}_0 - \hat{H}_D + \hat{U}(t)\hat{V}(t)\hat{U}(t)^\dagger]|\phi(t)\rangle \tag{41}$$

Now, set

$$\hat{V}(t) = -\hbar g cos(w_d t)(\hat{a} + \hat{a}^\dagger) \tag{42}$$
$$\hat{H}_0 = \frac{\hbar\omega}{2}(\hat{a}^\dagger \hat{a} + \frac{1}{2}) \tag{43}$$

Evaluate $\hat{V}_I(t) = exp(\frac{i\hat{H}_0 t}{\hbar})\hat{V}(t)exp(\frac{-i\hat{H}_0 t}{\hbar})$

Verify that you obtain:

$$\hat{V}_I(t) = -\hbar g cos(w_d t)(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) \tag{44}$$

Expand the cosine in terms of $e^{iw_d t}$ and $e^{-iw_d t}$ and express frequencies in Equation 44 in terms of $\Delta = \omega - \omega_d$ and $\omega + \omega_d$

Finally, if $|\Delta| \ll \omega$, the terms involving $\omega + \omega_d$ oscillate rapidly and their contribution washes out in timescales governed by Δ. This is called the rotating wave approximation.
Verify that this leaves us with:

$$\hat{V}_I(t) = -\hbar g(\hat{a}e^{-i\Delta t} + \hat{a}^\dagger e^{i\Delta t}) \tag{45}$$

3 Analogy to Classical Oscillator

The classical position is $x(t) = A\cos(\omega t + \phi)$. The quantum position operator in the Heisenberg picture is:

$$\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_S e^{-i\omega t} + \hat{a}_S^\dagger e^{i\omega t}) \tag{46}$$

Here, $\hat{a}_S$ plays the role of the complex amplitude (phasor) in classical mechanics. We can define Quadrature Operators $\hat{X}_1$ and $\hat{X}_2$ (analogous to Real and Imaginary parts):

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \tag{47}$$

In-Chapter Visualization

We illustrate the examples used in this note with simulations on Python. The attached notebook also provides an introduction to using Numpy and QuTiP for simulating Quantum systems.
As the notes progress, these notebooks will expand your skill-set in simulations as well.
Run the interactive notebook here: Harmonic Oscillation + Time Evolution

References

Robert G. Littlejohn. Time evolution in quantum mechanics. Physics 221A Course Notes, University of California, Berkeley, 2021. URL `https://bohr.physics.berkeley.edu/classes/221/notes/tevolut.pdf`.

J. J. Sakurai and Jim Napolitano. *Modern Quantum Mechanics*. Addison-Wesley, San Francisco, CA, 2nd edition, 2011. ISBN 978-0-8053-8291-4.