11

Eigenvalue Decomposition and Diagonalization

Would it save you a lot of time if I just gave up and went mad now?

—Douglas Adams

We have completed the basics of Linear Algebra. We may have gone a bit beyond the basics in its algebraic and geometric aspects. Now it is time to switch gears and look at some topics that have enormous impact in computer science as well as more classical sciences. Eigenvalues are eigenvectors are the entry point to such topics. Although it appears in the "Advanced Topics" part of this book, eigenvalue decomposition and the associated discussion usually appear toward the end of all undergraduate-level courses on Linear Algebra.

The word "eigen" is German, and it means *own* or *characteristic*. We may, therefore, see some people calling the eigenvectors the characteristic vectors, although it is not common. What is much more common is to call the expression that gives us the eigenvalues of a matrix its *characteristic polynomial*.

11.1 Definition and Notation

We talked about $\boldsymbol{Ax} = \boldsymbol{b}$ as a transformation $\boldsymbol{A} \in \mathbb{R}^{m \times n} : \mathbb{R}^n \mapsto \mathbb{R}^m$. If we consider square matrices $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, then the mapping is from $\mathbb{R}^n$ to $\mathbb{R}^n$. We can then say that the matrix $\boldsymbol{A}$ is a mapping from a space to itself: It takes vectors in one space and transforms them to other vectors in the *same* space.

Are there vectors in the space that get transformed to a scalar multiple of itself? If there are, such vectors are called eigenvectors. Writing this statement in symbols, we come up with their definition.

Eigenvectors and eigenvalues

Definition: For $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $\boldsymbol{s} \in \mathbb{R}^n \neq \boldsymbol{0}$ is an eigenvector if $\boldsymbol{As} = \lambda \boldsymbol{s}$ with the eigenvalue λ.

A few points to note about eigenvalues and eigenvectors:

1. For $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, its eigenvalues do not have to be in $\mathbb{R}$. In other words, just because we have square matrix over the field of reals ($\boldsymbol{A} \in \mathbb{R}^{n \times n}$), we cannot assume that its eigenvalues are real; it may have complex or imaginary eigenvalues.

2. Similarly, not all real matrices ($\boldsymbol{A} \in \mathbb{R}^{n \times n}$) have real eigenvectors.

3. If $\boldsymbol{s}$ is an eigenvector of $\boldsymbol{A}$ with an eigenvalue λ, so is any scaled version of it ($r\boldsymbol{s}$), with the *same* eigenvalue λ. Proof:

$$\boldsymbol{A}(r\boldsymbol{s}) = r\boldsymbol{A}\boldsymbol{s} = r\lambda s = \lambda(r\boldsymbol{s})$$

4. As a special case, λ can be zero. When $\lambda = 0$, we have $\boldsymbol{As} = \boldsymbol{0} \implies \boldsymbol{s} \in \mathcal{N}(\boldsymbol{A})$. Note that $\boldsymbol{s} \neq \boldsymbol{0}$.

5. Although presented here in terms of matrices and vectors, the ideas behind the eigenvectors came from other fields, such as physics. For this reason, eigenvectors are defined as those vectors that do not change their "direction" when $\boldsymbol{A}$ applies on them.

   In this book, however, for our own overly pedantic reasons, we stay away from the notion of "direction" of vectors to the extent possible.

11.2 Examples of Eigenvalues and Eigenvectors

Before writing down a general method for finding eigenvalues and eigenvectors, let's look at a few examples.

11.2.1 Permutation Matrix

A permutation matrix is the one that shuffles the elements of a vector (or the rows of a matrix, as we saw earlier in §4.3.4, page 94, when dealing with elementary matrices). In $\mathbb{R}^2$, the only possible permutation is $r_1 \leftrightarrow r_2$ with the following $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \boldsymbol{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{A}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

As we can see above, we have two eigenvectors for this matrix, with eigenvalues 1 and -1.

11.2.2 Projection Matrix

We came across projection matrices $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, which take any vector $\boldsymbol{x} \in \mathbb{R}^n$ to its projection on to a subspace $\mathcal{S} \subset \mathbb{R}^n$. If the vector $\boldsymbol{x}$ that $\boldsymbol{P}$ is acting on is already in the subspace $\mathcal{S}$, we know that its projection is itself. Calling these vectors $\boldsymbol{x}_{\parallel}$, we can write:

$$\boldsymbol{P}\boldsymbol{x}_{\parallel} = \boldsymbol{x}_{\parallel} = 1 \times \boldsymbol{x}_{\parallel}$$

So we have a whole bunch of eigenvectors $\boldsymbol{x}_{\parallel}$ in $\mathcal{S}$. Furthermore, the eigenvalues for these eigenvectors would be one, $\lambda = 1$.

We also know that if $\boldsymbol{x}$ is orthogonal to the subspace $\mathcal{S}$, the projection will be the zero vector. Calling such vectors $\boldsymbol{x}_{\perp}$, we write:

$$\boldsymbol{P}\boldsymbol{x}_{\perp} = \boldsymbol{0} = 0 \times \boldsymbol{x}_{\perp}$$

So $\boldsymbol{x}_{\perp}$ is an eigenvector with $\lambda = 0$. Note that for any matrix $\boldsymbol{A}$, we can always write $\boldsymbol{A}\boldsymbol{0} = \boldsymbol{0}$, but $\boldsymbol{0}$ is *not* an eigenvector by our definition above.

11.2.3 Shear Matrix

A shear matrix transforms a square to a parallelogram[1]. Here is an example of a horizontal shear matrix, as shown in Figure 11.1.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \implies \boldsymbol{A}\boldsymbol{q_1} = \boldsymbol{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{A}\boldsymbol{q_2} = \boldsymbol{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

When any vector is transformed by $\boldsymbol{A}$, its first component is not affected. A horizontal shear leaves the x axis alone, and therefore $\boldsymbol{q_1}$ is an eigenvector with eigenvalue $\lambda = 1$.

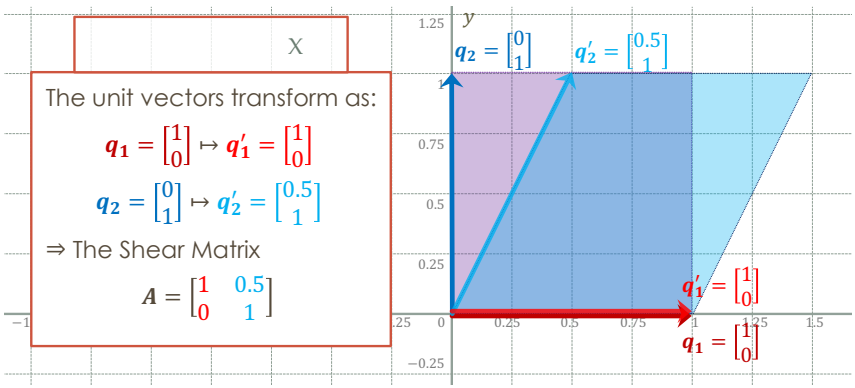

Fig. 11.1 An example shear matrix, showing a square being transformed into a parallelogram.

11.2.4 Rotation Matrix

We looked at rotation matrices earlier, when talking about orthogonal matrices. In $\mathbb{R}^2$, the rotation matrix is

$$\boldsymbol{Q_\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

[1] Although we state it like this, we should note that squares and parallelograms do not exist in a vector space. They live in coordinate spaces, and this statement is an example of the Notational Abuse, about which we complained in a box earlier. What we mean is that the two vectors forming the sides of a unit square get transformed such that they form sides of a parallelogram. We should perhaps eschew our adherence to this pedantic exactitude, now that we are in the advanced section.

For any non-trivial θ (which means $\theta \neq 2k\pi$ for integer k), we can see that $\boldsymbol{Q_\theta}$ changes every single vector in $\mathbb{R}^2$. We have no eigenvectors for this matrix in $\mathbb{R}^2$.

11.2.5 Differentiation

We can think of the set of all functions (of one variable, for instance) as a vector space. It satisfies all the requisite properties. The calculus operation of differentiation is a linear transformation in this space; it satisfies both the homogeneity and additivity properties of linearity.

$$\frac{d}{dx}\, e^{ax} = ae^{ax} \implies e^{ax} \text{ is an eigenvector with eigenvalue } a$$

$$\frac{d^2}{dx^2}\, \sin x = -\sin x \implies \sin x \text{ is an eigenvector with eigenvalue } -1$$

11.3 Computing Eigenvalues and Finding Eigenvectors

In order to find the eigenvalues and then eigenvectors, we start from their definitions.

$$\boldsymbol{A}\boldsymbol{s} = \lambda \boldsymbol{s} \implies (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = \boldsymbol{0}$$

Remembering that $\boldsymbol{s} \neq \boldsymbol{0}$, we can see that $\boldsymbol{A} - \lambda \boldsymbol{I}$ has a non-trivial null space (to which the eigenvector $\boldsymbol{s}$ belongs). Since $\mathcal{N}(\boldsymbol{A} - \lambda \boldsymbol{I})$ has non-zero vectors in it, $\boldsymbol{A} - \lambda \boldsymbol{I}$ is singular and its determinant has to be zero, which gives us an equation for eigenvalues.

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

Thus, we get rid of $\boldsymbol{s}$ and end up with a polynomial in λ (when we expand the determinant) equalling zero from which we can solve for the possible values of λ. This polynomial is called the *characteristic polynomial* of the matrix.

For a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, when we expand the determinant $|\boldsymbol{A} - \lambda \boldsymbol{I}|$ using the Laplace formula in Eqn (3.5), we get a polynomial of order n in λ, which should give us n roots, but not all of them may be real. Once we have the eigenvalues, we can put them back in $(\boldsymbol{A} - \lambda \boldsymbol{I}) = \boldsymbol{0}$ to find the eigenvectors $\boldsymbol{s}$, which is the same as finding the null space of $(\boldsymbol{A} - \lambda \boldsymbol{I})$.

Let's look at the examples from the previous section again to see how we get the eigenvalues and eigenvectors.

Permutation Matrix

Starting from the permutation matrix in $\mathbb{R}^2$, here are the steps.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad |\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \implies \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\text{Expanding the determinant, } \lambda^2 - 1 = 0 \implies \lambda = \pm 1$$

As we saw earlier, we have two eigenvalues, 1 and -1. To find the corresponding eigenvectors, we substitute the λ values in either $\boldsymbol{A}\boldsymbol{s} = \lambda \boldsymbol{s}$ and solve, or, equivalently, find the null space of $\boldsymbol{A} - \lambda \boldsymbol{I}$.

$$\text{With } \lambda = 1, \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \boldsymbol{s} = \boldsymbol{0} \implies \boldsymbol{s} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{With } \lambda = -1, \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \boldsymbol{s} = \boldsymbol{0} \implies \boldsymbol{s} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Here, to find $\boldsymbol{s}$, we are still using the column-picture of matrix multiplication and figuring out what linear combinations of the columns of $\boldsymbol{A} - \lambda \boldsymbol{I}$ give the zero vector. As we can see, we get two distinct eigenvectors. We can also see that the eigenvectors are actually orthogonal to each other. Note that any scaled versions of the eigenvectors are still eigenvectors with the same λ. For this reason, we typically normalize them.

The fact that we got real eigenvalues and distinct and orthogonal eigenvectors is not an accident. Real symmetric matrices (as our $\boldsymbol{A}$ was, in the case of this permutation matrix) always have real eigenvalues and a full set of orthogonal eigenvectors. They are the best matrices to work with.

Projection Matrix

As a concrete example, let's consider the projection matrix in $\mathbb{R}^3$ that projects vectors to the xy plane. Still calling it $\boldsymbol{A}$, what $\boldsymbol{A}$ needs to do is to leave the first two components alone and make the last component zero.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad |\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \implies \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\text{Expanding the determinant, } -\lambda(1-\lambda)^2 = 0 \implies \lambda = 0, 1, 1$$

Again, as we saw earlier, we have two eigenvalues, 0 and 1. But note that $\lambda = 1$ is repeated. We state this fact more fancifully, that the *algebraic multiplicity* of the eigenvalue ($\lambda = 1$) is two. Let's go ahead and try to find the corresponding eigenvectors.

$$\text{With } \lambda = 0, \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{s} = \boldsymbol{0} \implies \boldsymbol{s} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{With } \lambda = 1, \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \boldsymbol{s} = \boldsymbol{0} \implies \boldsymbol{s} = \begin{bmatrix} t_1 \\ t_2 \\ 0 \end{bmatrix}$$

Where t_1 and t_2 are any real numbers. We see that we have a small issue with the second eigenvalue with algebraic multiplicity two: The eigenvectors corresponding to it span a subspace, which is called the *eigenspace* associated with the second eigenvalue ($\lambda = 1$). The dimension of this eigenspace is two, which is called its *geometric multiplicity*. All eigenvalues have eigenspaces associated with them, with geometric multiplicity of at least one.

What we need to do when we have an eigenvalue with a geometric multiplicity greater than one is to select any full set of linearly independent vectors that span its eigenspace. In other words, we take its basis as the eigenvectors. In this particular case of projecting to the xy plane, the perfect basis would be the unit vectors along x and y directions. Putting it all together, here is the full solution:

$$\lambda_1 = 0, \boldsymbol{s_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \lambda_2 = 1, \boldsymbol{s_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{s_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Shear Matrix

Moving on to our next example,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \quad |\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \implies \begin{vmatrix} 1 - \lambda & 0.5 \\ 0 & 1 - \lambda \end{vmatrix} = 0$$

$$\text{Expanding the determinant, } (1 - \lambda)^2 = 0 \implies \lambda = 1$$

As we saw earlier, we have a single eigenvalue of 1, but with an algebraic multiplicity of two. The corresponding eigenvector is.

$$\text{With } \lambda = 1, \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} \boldsymbol{s} = \boldsymbol{0} \implies \boldsymbol{s} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The eigenvalue has an algebraic multiplicity of two, and a geometric multiplicity of one. We cannot find a full set of (real) eigenvectors, and we are in trouble because for $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, we would like to have n eigenvectors.

Rotation Matrix

We saw that there were no eigenvectors for a rotation matrix in $\mathbb{R}^2$. Let's consider a $\frac{\pi}{2}$-rotation, call the matrix $\boldsymbol{A}$ and attempt to find its eigenvalues and eigenvectors.

$$\boldsymbol{Q_\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \boldsymbol{A}$$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \implies \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\text{Expanding the determinant, } \lambda^2 + 1 = 0 \implies \lambda = \pm i$$

We have no real eigenvalues. For the sake of completeness, we can try to find the eigenvectors, although we do not expect to find any in $\mathbb{R}^2$.

$$\text{With } \lambda = i, \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \boldsymbol{s} = \boldsymbol{0} \implies \boldsymbol{s} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\text{With } \lambda = -i, \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \boldsymbol{s} = \boldsymbol{0} \implies \boldsymbol{s} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Thus, if we allow ourselves to step into the field of complex numbers, we can find eigenvectors of the rotation matrix $\boldsymbol{s} \in \mathbb{C}^2$. Physically, the vector that is conserved during rotation is perpendicular to the plane of rotation, which is why gyroscopes work the way they do. How that fact corresponds to the actual eigenvectors of the matrix we computed above, however, is a fairly tortured explanation.

11.4 Properties

The eigenvalues and eigenvectors provide deep insights into the structure of the matrix, and have properties related to the properties of the matrix itself. Here are some of them with proofs, where possible. It is worth our time to verify these properties on the examples we worked out above.

11.4.1 Eigenvalues

1. The sum of eigenvalues equals the trace of the matrix. For an $n \times n$ matrix $\boldsymbol{A} = \left[a_{ij}\right]$,

$$\sum_{i=1}^{n} \lambda_i = \text{trace}(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii}$$

Proof: Since the characteristic polynomial has roots λ_i, we can write:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

where we constructed the RHS to have roots λ_i and to match the coefficient of λ^n in the determinant on the LHS, which expands to give a polynomial in λ with the coefficient of $\lambda^n = (-1)^n$. On the LHS, the coefficient of λ^{n-1} is

$$(-1)^n \sum_{i=1}^{n} a_{ii}$$

On the RHS, the coefficient of λ^{n-1} is

$$(-1)^n \sum_{i=1}^{n} \lambda_i$$

Since the LHS and RHS coefficients have to match, we see that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \text{trace}(\boldsymbol{A})$$

which proves the property.

Although we proved it by comparing the coefficient of λ^{n-1}, it is a lot easier to prove once we learn matrix similarity in the next chapter. Note that we have to include λ_i in the summation as many times as its algebraic multiplicity. Note also that for $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, this property means that if $\boldsymbol{A}$ has any complex eigenvalues, they should come in pairs of complex conjugates.

2. The product of eigenvalues equals the determinant of the matrix. For an $n \times n$ matrix $\boldsymbol{A}$,

$$\prod_{i=1}^{n} \lambda_i = |\boldsymbol{A}|$$

Proof: Again we start with the equality:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

Set $\lambda = 0$ to get

$$|\boldsymbol{A}| = (-1)^n(-1)^n\lambda_1\lambda_2\cdots\lambda_n = \lambda_1\lambda_2\cdots\lambda_n = \prod_{i=1}^{n}\lambda_i$$

A corollary of this property is that singular matrices have at least one zero eigenvalue.

3. The eigenvalues of a real, symmetric matrix are real.

$$\boldsymbol{A} \in \mathbb{R}^{n\times n}, \boldsymbol{A}^{\mathsf{T}} = \boldsymbol{A} \implies \lambda_i \in \mathbb{R}$$

Proof: To prove this property, we have to step into the scary field of complex numbers again. The strategy for proving something is real is to assume that it is complex, and then show that its conjugate (where we replace all i with $-i$) is the same as itself, which shows that it has to be real. Following this stragegy, let's say that $\lambda \in \mathbb{C}$ is a possibly complex eigenvalue of $\boldsymbol{A}$, and $\boldsymbol{s} \in \mathbb{C}^{n\times n}$ is the corresponding eigenvector. (Note that in step (2) below, we take the complex conjugate, which is defined as: $(a + ib)^* = a - ib$.)

(1) By the definition of eigenvalues, we have:	$\boldsymbol{A}\boldsymbol{s} = \lambda\boldsymbol{s}$
(2) Taking the complex conjugate, we get:	$\boldsymbol{A}^*\boldsymbol{s}^* = \lambda^*\boldsymbol{s}^*$
(3) Since $\boldsymbol{A}$ is real $\boldsymbol{A}^* = \boldsymbol{A}$. Therefore:	$\boldsymbol{A}\boldsymbol{s}^* = \lambda^*\boldsymbol{s}^*$
(4) Multiplying (1) on the left with $\boldsymbol{s}^{*\mathsf{T}}$:	$\boldsymbol{s}^{*\mathsf{T}}\boldsymbol{A}\boldsymbol{s} = \boldsymbol{s}^{*\mathsf{T}}\lambda\boldsymbol{s}$
(5) Flipping it around and reordering:	$\lambda\boldsymbol{s}^{*\mathsf{T}}\boldsymbol{s} = \boldsymbol{s}^{*\mathsf{T}}\boldsymbol{A}\boldsymbol{s}$
(6) Using the product rule of transposes:	$\lambda\boldsymbol{s}^{*\mathsf{T}}\boldsymbol{s} = \left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{s}^*\right)^{\mathsf{T}}\boldsymbol{s}$
(7) Since $\boldsymbol{A}$ is symmetric:	$= \left(\boldsymbol{A}\boldsymbol{s}^*\right)^{\mathsf{T}}\boldsymbol{s}$
(8) Using step (3) above:	$= \lambda^*\boldsymbol{s}^{*\mathsf{T}}\boldsymbol{s}$

Finally, from steps (6) and (8), we get:

$$\lambda\boldsymbol{s}^{*\mathsf{T}}\boldsymbol{s} = \lambda^*\boldsymbol{s}^{*\mathsf{T}}\boldsymbol{s} \implies (\lambda - \lambda^*)\boldsymbol{s}^{*\mathsf{T}}\boldsymbol{s} = 0 \implies \lambda = \lambda^*$$

since $\boldsymbol{s}^{*\mathsf{T}}\boldsymbol{s}$ is the square of the norm $\|\boldsymbol{s}\|^2$ (it is, for $\boldsymbol{s} \in \mathbb{C}^n$, as we shall see in the next chapter) of an eigenvector, which cannot be the zero vector $\boldsymbol{0}$. $\lambda = \lambda^*$ means $\lambda \in \mathbb{R}$.

4. The "opposite" of the previous property: The eigenvalues of a real, antisymmetric (AKA skew symmetric) matrix are imaginary.

$$\boldsymbol{A} \in \mathbb{R}^{n\times n}, \boldsymbol{A}^{\mathsf{T}} = -\boldsymbol{A} \implies \lambda_i = i\lambda_i', \text{ where } \lambda_i' \in \mathbb{R}$$

Proof: Identical to the previous proof, but with a negative sign in step (7), showing $\lambda = -\lambda^*$.

5. If we multiply a matrix ($\boldsymbol{A}$) by a scalar (α), then all its eigenvalues (λ_i) are multiplied by the same scalar.
Proof: $\boldsymbol{A}\boldsymbol{s} = \lambda\boldsymbol{s} \implies (\alpha\boldsymbol{A})\boldsymbol{s} = (\alpha\lambda)\boldsymbol{s}$. That is it.

6. The eigenvalues of $\boldsymbol{A} + \alpha\boldsymbol{I}$ are $\lambda_i + \alpha$, the eigenvalues of $\boldsymbol{A}$ "shifted" by α.
Proof: If $\boldsymbol{s}$ is an eigenvector of $\boldsymbol{A}$ with the eigenvalue λ, we have:

$$(\boldsymbol{A} + \alpha\boldsymbol{I})\boldsymbol{s} = \boldsymbol{A}\boldsymbol{s} + \alpha\boldsymbol{s} = \lambda\boldsymbol{s} + \alpha\boldsymbol{s} = (\lambda + \alpha)\boldsymbol{s}$$

which means $\boldsymbol{s}$ is an eigenvector of $\boldsymbol{A} + \alpha\boldsymbol{I}$ with the eigenvalue $\lambda + \alpha$, which goes for every eigenvector/eigenvalue pair.

7. The eigenvalues of real, symmetric matrices are related to the pivots in the row echelon form (REF, or $\boldsymbol{U}$ in $\boldsymbol{LU}$ decomposition): The number of positive eigenvalues is the same as the number of positive pivots. Same goes for negative ones too. We will leave this property without proof.

11.4.2 Eigenvectors

1. The eigenvectors of a matrix[2] corresponding to distinct eigenvalues are linearly independent.

$$\begin{gathered}\boldsymbol{A} \in \mathbb{R}^{n\times n} \text{ with } \boldsymbol{A}\boldsymbol{s_i} = \lambda_i\boldsymbol{s_i}, \boldsymbol{s_i} \in \mathbb{R}^n, \text{ for } 0 < i \leq n \\ \lambda_i \neq \lambda_j \text{ and } a_i\boldsymbol{s_i} + a_j\boldsymbol{s_j} = \boldsymbol{0} \\ \implies a_i = a_j = 0 \text{ for } 0 < i, j \leq n, i \neq j\end{gathered}$$

[2]We use $\mathbb{R}$ (the real field, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{s} \in \mathbb{R}^n$) in the mathematical statement and proof of this property for convenience and because of its relevance to computer science, but the property applies to $\mathbb{C}$ as well.

Proof: We need to prove that if $\boldsymbol{s_i}$ and $\boldsymbol{s_j}$ are eigenvectors of $\boldsymbol{A}$ with distinct eigenvalues λ_i and λ_j, we will not be able to find non-zero a_i and a_j such that $a_i\boldsymbol{s_i} + a_j\boldsymbol{s_j} = \boldsymbol{0}$.

(1) Starting from:	$a_i\boldsymbol{s_i} + a_j\boldsymbol{s_j} = \boldsymbol{0}$
(2) Multiplying on the left by $\boldsymbol{A}$:	$a_i\boldsymbol{A}\boldsymbol{s_i} + a_j\boldsymbol{A}\boldsymbol{s_j} = \boldsymbol{0}$
(3) Since $\boldsymbol{s_i}$ and $\boldsymbol{s_j}$ are eigenvectors:	$a_i\lambda_i\boldsymbol{s_i} + a_j\lambda_j\boldsymbol{s_j} = \boldsymbol{0}$
(4) Multiplying (1) with λ_i, we get:	$a_i\lambda_i\boldsymbol{s_i} + a_j\lambda_i\boldsymbol{s_j} = \boldsymbol{0}$
(5) Subtracting (3) from (4):	$a_j(\lambda_i - \lambda_j)\boldsymbol{s_j} = \boldsymbol{0}$
(6) Since $\lambda_i \neq \lambda_j$ and $\boldsymbol{s_j} \neq \boldsymbol{0}$:	$a_j = 0$
(7) Putting a_j in (1), we can show:	$a_i = 0$

The converse of this statement is *not* true: If the eigenvalues are *not* distinct, the eigenvectors may still be linearly independent, as in the case of the projection-matrix. Or the identity matrix $\boldsymbol{I}_n$, which has the eigenvalue one repeated n times, but is already diagonalized.

2. Real symmetric matrices have a full set of orthogonal eigenvectors.

$$\boldsymbol{A} \in \mathbb{R}^{n\times n} \text{ with } \boldsymbol{A}\boldsymbol{s_i} = \lambda_i\boldsymbol{s_i}, \boldsymbol{s_i} \in \mathbb{R}^n, \text{ for } 0 < i \leq n$$
$$\boldsymbol{A}^\mathsf{T} = \boldsymbol{A} \implies \boldsymbol{s_i} \perp \boldsymbol{s_j} \text{ for } 0 < i, j \leq n, i \neq j$$

Proof:

(1) By the definition of eigenvalues:	$\boldsymbol{A}\boldsymbol{s_i} = \lambda_i\boldsymbol{s_i}$
(2) Taking the dot product with $\boldsymbol{s_j}$:	$(\boldsymbol{A}\boldsymbol{s_i})^\mathsf{T}\boldsymbol{s_j} = \lambda_i\boldsymbol{s_i}^\mathsf{T}\boldsymbol{s_j}$
(3) Using the product rule of transposes:	$\boldsymbol{s_i}^\mathsf{T}\boldsymbol{A}^\mathsf{T}\boldsymbol{s_j} = \lambda_i\boldsymbol{s_i}^\mathsf{T}\boldsymbol{s_j}$
(4) Since $\boldsymbol{A}$ is symmetric:	$\boldsymbol{s_i}^\mathsf{T}\boldsymbol{A}\boldsymbol{s_j} = \lambda_i\boldsymbol{s_i}^\mathsf{T}\boldsymbol{s_j}$
(5) Since $\boldsymbol{s_j}$ is an eigenvector of $\boldsymbol{A}$:	$\boldsymbol{s_i}^\mathsf{T}\lambda_j\boldsymbol{s_j} = \lambda_i\boldsymbol{s_i}^\mathsf{T}\boldsymbol{s_j}$
(6) Reordering:	$\lambda_j\boldsymbol{s_i}^\mathsf{T}\boldsymbol{s_j} = \lambda_i\boldsymbol{s_i}^\mathsf{T}\boldsymbol{s_j}$
(7) Gathering terms:	$(\lambda_j - \lambda_i)\boldsymbol{s_i}^\mathsf{T}\boldsymbol{s_j} = 0$
(8) Since $\lambda_j \neq \lambda_i \implies$	$\boldsymbol{s_i}^\mathsf{T}\boldsymbol{s_j} = 0$
(9) $\boldsymbol{s_i}^\mathsf{T}\boldsymbol{s_j} = 0 \implies$	$\boldsymbol{s_i} \perp \boldsymbol{s_j}$

It may happen that the eigenvalues are repeated. For instance, the identity matrix $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ has n repeated eigenvalues of 1.

Every vector in $\mathbb{R}^{n\times n}$ is an eigenvector of $\boldsymbol{I}$. In this case also, we can *choose* an orthogonal set of vectors as the full set of eigenvectors for the matrix.

In some cases, the eigenvectors may span a subspace (called eigenspace, of course), as in the case of the projection matrix. Here again, we can choose an orthogonal eigenbasis for the eigenspace associated with the repeated eigenvalue.

3. The eigenvectors of $\boldsymbol{A}+\alpha\boldsymbol{I}$ are the same as those of $\boldsymbol{A}$.
Proof: As we already proved, if $\boldsymbol{s}$ is an eigenvector of $\boldsymbol{A}$ with the eigenvalue λ,

$$(\boldsymbol{A}+\alpha\boldsymbol{I})\boldsymbol{s}=\boldsymbol{A}\boldsymbol{s}+\alpha\boldsymbol{s}=\lambda\boldsymbol{s}+\alpha\boldsymbol{s}=(\lambda+\alpha)\boldsymbol{s}$$

which should be proof enough. If not, it means $\boldsymbol{s}$ is an eigenvector of $\boldsymbol{A}+\alpha\boldsymbol{I}$ with the eigenvalue $\lambda+\alpha$.

11.5 Unit Circles and Ellipses

One fair question we may have at this point is why we are doing all this. It is all an academic exercise in intellectual acrobatics? We may not be able to answer this question completely yet, but we can look at a linear transformation and see what the eigenvalue analysis tells us about it. In the last chapter, we will see how these insights are harnessed in statistical analyses.

Let's start with an example $\boldsymbol{A}\in\mathbb{R}^{2\times 2}$, find its eigenvalues and eigenvectors, and look at them in the coordinate space $\mathbb{R}^2$.

$$\boldsymbol{A}=\frac{1}{4}\begin{bmatrix}5 & -\sqrt{3}\\ -\sqrt{3} & 3\end{bmatrix}\qquad \lambda_1=\frac{3}{2};\ \boldsymbol{s_1}=\frac{1}{2}\begin{bmatrix}\sqrt{3}\\ -1\end{bmatrix}\qquad \lambda_2=\frac{1}{2};\ \boldsymbol{s_2}=\frac{1}{2}\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}$$

As we can see from Figure 11.2, $\boldsymbol{A}$ takes the first basis vector ($\boldsymbol{q_1}$, shown in red, dashed arrow) to its first column vector (a_1 shown bright red arrow): $\boldsymbol{q_1}\mapsto\boldsymbol{a_1}$. Similarly for the second one as well, $\boldsymbol{q_2}\mapsto\boldsymbol{a_2}$, shown in various shades of blue. What happens to the basis vectors happens to all vectors, and therefore, the unit circle in the figure gets mapped to the ellipse, as shown.

Although we know about this unit-circle-to-ellipse business, from the matrix $\boldsymbol{A}$ itself, we know very little else. Note that the vectors to which the unit vectors transform (in $\boldsymbol{q_i}\mapsto\boldsymbol{a_i}$) are nothing special;

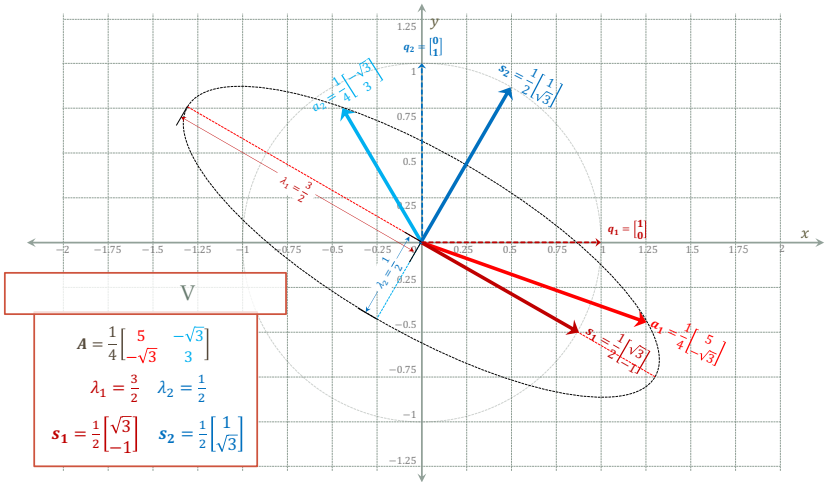

Fig. 11.2 Visualization of eigenvalues and eigenvectors: $\boldsymbol{A}$ transforms the unit circle into a rotate ellipse. The eigenvalues specify the lengths of its major and minor axes. And the eigenvectors specify the orientation of the axes.

they are on the ellipse somewhere. What we would like to know are the details of the ellipse, like its size and orientation, which is exactly what the eigenvalues and eigenvectors tell us. The eigenvalues λ_i are the lengths of the major and minor axes of the ellipse and the eigenvectors are the unit vectors along the axes. In Figure 11.2, the eigenvectors ($\boldsymbol{s_1}$ and $\boldsymbol{s_2}$) are shown in darker shades of red and blue, while the corresponding eigenvalues are marked as the lengths of the axes.

When we move on to higher dimensions, ellipses become ellipsoids or hyper-ellipsoids, and the axes are their principal axes. The mathematics of eigen-analysis still stays the same: We get the directions and lengths of the principal axes. And, if the matrix on which we are performing the eigen-analysis happens to contain the covariance of the variables in a dataset, then what the eigen-analysis gives us are insights about the directions along which we can decompose the covariance. If we sort the directions by the eigenvalues, we can extract the direction for the highest variance, second highest variance and so on. We will revisit this idea in more detail in one of our last topics, the Principal Component Analysis, which is the mainstay of dimensionality reduction in data science.

11.6 Diagonalization

We learned quite a bit about eigenvalues and eigenvectors by now. We might still wonder why at this point. What is the point in learning all this trivia about them? We hinted at its significance in data analytics for dimensionality reduction. We have one more good reason; there is a method to this madness. Once we have the eigenvectors of a matrix, we can *diagonalize* it. And once we diagonalize, we can immediately see how it can help in computing the powers of the matrix. Why would we want to take powers of matrices? Because it is the basis of modeling time-varying systems mathematically.

11.6.1 S and Λ

Suppose $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ has its eigenvalues λ_i and the corresponding eigenvectors $\boldsymbol{s_i}$. Let's construct two matrices, arranging the eigenvectors as columns, and the eigenvalues as diagonal elements:

$$\boldsymbol{S} = \begin{bmatrix} | & | & \cdots & | \\ \boldsymbol{s_1} & \boldsymbol{s_2} & \cdots & \boldsymbol{s_n} \\ | & | & \cdots & | \end{bmatrix} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

For simplicity, we may write $\boldsymbol{S} = [\boldsymbol{s}]$ and $\boldsymbol{\Lambda} = [\lambda]$. With these new matrices, we arrive at the most important result from this chapter:

$$\boldsymbol{AS} = \boldsymbol{S\Lambda}$$

The LHS is a matrix multiplication, where the product matrix has columns that are product of $\boldsymbol{A}$ and the corresponding column in $\boldsymbol{S}$. In other words, $\boldsymbol{AS} = \boldsymbol{A}[\boldsymbol{s_i}] = [\boldsymbol{As_i}]$. We think of the RHS using the column picture of matrix multiplication again: The columns of $\boldsymbol{S\Lambda} = [\boldsymbol{s_i}]\boldsymbol{\Lambda}$ are the linear combinations of $\boldsymbol{s_i}$ taken with the scaling factors in the columns of $\boldsymbol{\Lambda}$. But the scaling factors are merely λ_i in the i^{th} place. Therefore, the i^{th} column in $\boldsymbol{AS} = \boldsymbol{S\Lambda}$ is the same as $\boldsymbol{As_i} = \boldsymbol{s_i}\lambda_i$, which is the now-familiar the definition of eigenvalues and eigenvectors.

11.6.2 The Decomposition

If we know that $\boldsymbol{S}$ is invertible, we can go one step further and write:

$$\text{If } \boldsymbol{S^{-1}} \text{ exixts, } \boldsymbol{AS} = \boldsymbol{S\Lambda} \implies \boldsymbol{A} = \boldsymbol{S\Lambda S^{-1}}$$

This is the famous eigenvalue decomposition of a real, square, *diagonalizable* matrix. Most matrices are diagonalizable over the field of complex numbers.

11.6.3 Powers of $\boldsymbol{A}$

As previously advertised, the reason for this decomposition is that it gives us a way to express the powers of the matrix. For a square, diagonalizable matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, we can write,

$$\begin{aligned}\boldsymbol{A} = \boldsymbol{S\Lambda S^{-1}} \implies \boldsymbol{A}^2 &= \boldsymbol{S\Lambda S^{-1}S\Lambda S^{-1}} = \boldsymbol{S\Lambda}(\boldsymbol{S^{-1}S})\boldsymbol{\Lambda S^{-1}} \\ &= \boldsymbol{S\Lambda I\Lambda S^{-1}} = \boldsymbol{S\Lambda\Lambda S^{-1}} \\ \boldsymbol{A}^2 &= \boldsymbol{S\Lambda}^2\boldsymbol{S^{-1}}\end{aligned}$$

Similarly, we can easily see that $\boldsymbol{A}^k = \boldsymbol{S\Lambda}^k\boldsymbol{S^{-1}}$. While this result may look fairly mundane, let's think about a medium-sized matrix, say 100×100, and $k = 60$. Imagine the number of operations required to compute $\boldsymbol{A}^{60}$, which is probably of the order of $60\times100^3 = 6\times10^7$ (for the 60 matrix multiplications). But if we have the decomposition, taking the 60^{th} power of $\boldsymbol{\Lambda}$ is trivial, merely 100 exponentiations. The two matrix multiplications cost about $2\times100^3 = 2\times10^6$ operations and the exponentiations of the 100 diagonal elements, a negligible amount. The overall saving in computational time, therefore, is roughly 30.

The best algorithms for matrix multiplications take about $n^{2.3}$ operations. If we are computing the k^{th} power of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, therefore, it would cost us $kn^{2.3}$ operations. But with diagonalization, it will cost us n exponentiations of the diagonal elements of $\boldsymbol{\Lambda}$ and two multiplications, or a total of $n + 2n^{2.3}$ operations, which implies a reduction of $\frac{kn^{2.3}}{n+2n^{2.3}} \approx \frac{k}{2}$ for large n.

Since we have any power of $\boldsymbol{A}$ being expressed as, essentially, powers of $\boldsymbol{\Lambda}$, we can make the same statement about polynomials of $\boldsymbol{A}$.

Invertibility vs. Diagonalizability

Not all invertible matrices can be diganolized. We saw an example earlier. The shear matrix has an inverse that is easily written down.:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{A^{-1}} = \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}$$

But $\boldsymbol{A}$ has only one eigenvector, and is not diagonalizable.

Not all diagonalizable matrices are invertible. We already know that the projection matrix is not invertible because it destroys information; it is a many-to-one (or injective) mapping. But it can be diagonalized.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{S^{-1}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \boldsymbol{S^{\mathsf{T}}}$$

$$\boldsymbol{A} = \boldsymbol{S\Lambda S^{-1}}$$

In fact, $\boldsymbol{A}$ is already a diagonal matrix: It has non-zero elements only along its diagonal.

We know the condition for $\boldsymbol{A}$ to be invertible, or for $\boldsymbol{A^{-1}}$ to exist. Let's state it several different ways, as a means to remind ourselves. $\boldsymbol{A}$ is invertible if:

- $|\boldsymbol{A}| \neq 0$. Otherwise, as Eqn (5.1) clearly shows, we cannot compute $\boldsymbol{A^{-1}}$ because $|\boldsymbol{A}|$ appears in the denominator.
- $\mathcal{N}(\boldsymbol{A}) = \boldsymbol{0}$, its null space contains only the zero vector. Otherwise, for some $\boldsymbol{x}$, we have $\boldsymbol{Ax} = \boldsymbol{0}$, and there is no way we can invert it to go from $\boldsymbol{0}$ to $\boldsymbol{x}$.
- $\lambda_i \neq 0$, all its eigenvalues are non-zero. Otherwise, $|\boldsymbol{A}|$, being the product of eigenvalues, would be zero.
- $\lambda_i \neq 0$, all its eigenvalues are non-zero. Another reason, otherwise, for the zero λ, we have $\boldsymbol{Ax} = \boldsymbol{0}$, which implies the existence of a non-trivial null space.

The diagonalizability of $\boldsymbol{A}$ is tested using the invertibility of its eigenvector matrix $\boldsymbol{S}$. Although this point is probably not critical for our view of Linear Algebra as it applies to computer science, we might as well state it here. For a matrix to be non-diagonalizable, the algebraic multiplicity of one of its eigenvalues (the number of times it is repeated) has to be greater than its geometric multiplicity (the number of associated eigenvectors), which means the characteristic polynomial needs to have repeated roots to begin with. The roots are repeated if the discriminant of the polynomial (similar to $b^2 - 4ac$ in the quadratic case) is zero. The discriminant being a continuous function of the coefficients of the polynomial, which are the elements of the matrix, it being zero happens with a frequency of the order of zero. But the roots being complex happens half the time because the discriminant is less than zero half the time.

11.6.4 Inverse of $\boldsymbol{A}$

Since we have a product for $\boldsymbol{A}$, we can take its inverse using the product rule of inverses.

$$\boldsymbol{A^{-1}} = \left(\boldsymbol{S\Lambda S^{-1}}\right)^{-1} = \left(\boldsymbol{S^{-1}}\right)^{-1} \boldsymbol{\Lambda^{-1} S^{-1}} = \boldsymbol{S\Lambda^{-1} S^{-1}}$$

Since $\boldsymbol{\Lambda}$ is a diagonal matrix with λ_i along the diagonal, its inverse is another diagonal matrix with the elements equal to the reciproval of λ_1. Therefore, $\boldsymbol{A^{-1}}$ has the same eigenvectors $\boldsymbol{s_i}$ as $\boldsymbol{A}$, with eigenvalues equal to the reciprocals of the eigenvalues of $\boldsymbol{A}$: $\frac{1}{\lambda_i}$. It can be seen even more directly as follows:

$$\boldsymbol{A s_i} = \lambda_i \boldsymbol{s_i} \implies \boldsymbol{s_i} = \lambda_i \boldsymbol{A^{-1} s_i} \implies \boldsymbol{A^{-1} s_i} = \frac{1}{\lambda_i} \boldsymbol{s_i}$$

Enough said.

Since $\boldsymbol{A^{-1}} = \boldsymbol{S \Lambda^{-1} S^{-1}}$, we can see that $\boldsymbol{A}^k = \boldsymbol{S \Lambda}^k \boldsymbol{S^{-1}}$ holds for $k < 0$ as well. Extrapolating even further, through the Taylor series expansion, we can compute entities like $e^{\boldsymbol{A}}$ (matrix exponentiation), which are essential in solving differential equations—a topic we consider beyond the scope of this book.

11.6.5 Difference Equations

In building mathematical models of systems that evolve in time, we may come across the situation where the state of the system at any time step depends on the state at the previous step. If all the parameters specifying the state can be written as a vector, we may be able to write the time evolution as $\boldsymbol{x_{k+1}} = \boldsymbol{A x_k}$.

If we know the initial conditions at time step zero, we can write:

$$\boldsymbol{x_k} = \boldsymbol{A x_{k-1}} = \boldsymbol{A^2 x_{k-2}} = \cdots = \boldsymbol{A^k x_0}$$

And, if we have the eigenvalue decomposition of $\boldsymbol{A} = \boldsymbol{S \Lambda S^{-1}}$, we know that we can compute the matrix raised to the power k without worrying too much about the computational cost.

11.6.6 Eigenbasis

If we have a full set of real eigenvectors for $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, we can use them as a basis for $\mathbb{R}^n$, which we will call the eigenbasis. We can then express any vector $\boldsymbol{x_0} \in \mathbb{R}^n$ as a linear combination of the eigenbasis vectors, remembering what we learned about changing the bases of vectors earlier in §7.2 (page 154).

$$\boldsymbol{x_0} = \boldsymbol{s_1} c_1 + \boldsymbol{s_2} c_2 + \cdots + \boldsymbol{s_n} c_n = \sum_{i=1}^{n} \boldsymbol{s_i} c_i = \boldsymbol{S c}$$

where c_i are the coordinates of $\boldsymbol{x}$ in the eigenbasis. (We wrote the scalar after the vector in the summation so that the matrix product is easier to spot.)

Once we have the vector $\boldsymbol{x_0}$ in the eigenbasis of $\boldsymbol{A}$, we can do simplify the multiplications of the powers of $\boldsymbol{A}$ with $\boldsymbol{x}$ as in the following:

$$\boldsymbol{A}\boldsymbol{x_0} = \boldsymbol{A}\sum_{i=1}^{n} \boldsymbol{s_i} c_i = \sum_{i=1}^{n} \boldsymbol{A}\boldsymbol{s_i} c_i = \sum_{i=1}^{n} \lambda_i \boldsymbol{s_i} c_i = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{c}$$

$$\boldsymbol{A}^k\boldsymbol{x_0} = \sum_{i=1}^{n} \lambda_i^k \boldsymbol{s_i} c_i = \boldsymbol{S}\boldsymbol{\Lambda}^{\boldsymbol{k}}\boldsymbol{c}$$

Why does this matter? Why use the eigenbasis? Let's think of $\boldsymbol{A}$ as the transformation encoding the time evolution of a system with $\boldsymbol{x_k}$ its state at a given step (or iteration, or a point in time). Given the state of the system at one step, we evolve it to the next step by multiplying with $\boldsymbol{A}$ to get $\boldsymbol{x_{k+1}} = \boldsymbol{A}\boldsymbol{x_k}$.

Knowing the initial state $\boldsymbol{x_0}$ and, more importantly, the transition matrix $\boldsymbol{A}$, what can we say about the stability of the system? We can say the following:

$$\lim_{k\to\infty} \boldsymbol{x_k} = \lim_{k\to\infty} \boldsymbol{A}^k\boldsymbol{x_0} = \sum_{i=1,|\lambda_i|>1}^{n} \lambda_i^k \boldsymbol{s_i} c_i$$

In other words, in the sum that makes up $\boldsymbol{x_k}$, only those eigenvalues (whether they are real or complex) whose absolute value is greater than 1 matter; they are the only ones that will survive when we take the limit $k \to \infty$.

11.7 Fibonacci Numbers

As an example of how this idea of the powers of a matrix applies to a real-world problem, let's look at the Fibonacci numbers. This problem may be academic in nature, but it does show the kind of thinking that goes into transforming a problem to bring it into the domain of eigenvalues.

The famous Fibonacci sequence appears in nature in unexpected ways, and is heavily used in mathematics and, closer to home, in

computer science. The Wikipedia entry on it has a comprehensive listing of its properties and interesting facts.

To see how we connect the eigenvalue computation with Fibonacci numbers, let's start by writing them down. The sequence of numbers is: $0, 1, 1, 2, 3, 5, 8, 13, \cdots$: Each number in the sequence (after the first two) is the sum of the previous two. Calling them $f_0, f_1, f_2, \cdots, f_k, \cdots$, we can say that $f_{k+2} = f_{k+1} + f_k$. This is what we might call a second-order difference equation because each number depends on the previous two. We do not see any vector or matrix here, do we? In order to reveal them, let's make a vector out of two Fibonacci numbers:

$$\boldsymbol{x_k} = \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix} \implies \boldsymbol{x_{k+1}} = \begin{bmatrix} f_{k+2} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} f_{k+1} + f_k \\ f_{k+1} \end{bmatrix}$$

which gives us a chance to write it as matrix equation, and connect $\boldsymbol{x_k}$ to $\boldsymbol{x_0}$:

$$\boldsymbol{x_{k+1}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{x_k} = \boldsymbol{A}\boldsymbol{x_k} = \boldsymbol{A}^{k+1}\boldsymbol{x_0}$$

From the Fibonacci sequence, we can see that the numbers are growing, and we may want to find out how fast they are growing. Or maybe we want to have an approximation for the k^{th} Fibonacci number. The first question about the growth rate is directly answered by the eigenvalues of $\boldsymbol{A}$, and the second one by the eigenbasis representation of $\boldsymbol{x_k}$.

Let's first compute the eigenvalues.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \implies -(1-\lambda)\lambda - 1 = 0 \text{ or } \lambda^2 - \lambda - 1 = 0$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$$

Since $|\lambda_1| > 1$ and $|\lambda_2| < 1$, it is the first eigenvalue that will dominate large Fibonacci numbers. In particular, f_{k+1} is going to be about 1.618 times bigger than f_k as $k \to \infty$.

Let's now try to find an equation, a closed-form formula, for f_k. We will start with the eigenvectors, express $\boldsymbol{x_0}$ in the eigenbasis and evolve it to $\boldsymbol{x_k}$. The eigenvectors of $\boldsymbol{A}$ are the solutions to $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = 0$.

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{s} = 0 \implies \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \boldsymbol{s} = 0$$

$$\implies \boldsymbol{s} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \text{ for } \lambda = \lambda_1, \lambda_2$$

$\boldsymbol{x_0}$ is a linear combination of $\boldsymbol{s_1}$ and $\boldsymbol{s_2}$.

$$\boldsymbol{x_0} = c_1\boldsymbol{s_1} + c_2\boldsymbol{s_2} \implies \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

After a bit of algebra, we will get:

$$c_1 = \frac{1}{\sqrt{5}}, c_1 = -\frac{1}{\sqrt{5}}$$

We know how the evolution of $\boldsymbol{x}$:

$$\boldsymbol{x_k} = \boldsymbol{A}^k\boldsymbol{x_0} = \boldsymbol{A}^k(c_1\boldsymbol{s_1} + c_2\boldsymbol{s_2}) = c_1\lambda_1^k\boldsymbol{s_1} + c_2\lambda_2^k\boldsymbol{s_2} = \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix}$$

We can now read the second element in $\boldsymbol{x_k}$ as f_k:

$$\begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix} = c_1\lambda_1^k \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2\lambda_2^k \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \implies f_k = c_1\lambda_1^k + c_2\lambda_2^k$$

Knowing that the second term vanishes for large k (because $|\lambda_2| = 0.618 < 1$) , we finally get an expression for f_k:

$$f_k \approx c_1\lambda_1^k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^k = f_k^{(\text{approx})}$$

How good this approximation is is shown in Table 11.1, which shows that the approximation is stunningly accurate. By the time we reach $k = 11$, the error is about 25 parts in a million.

11.8 Applications of Eigenvalues and Eigenvectors

The ideas behind eigenvalue decomposition has a multitude of applications, especially in physics and other physical sciences. In our domain of computer science, the Google Page Rank algorithm, described in its own box in the next chapter, is a brilliant success story of this line of thinking. Since this chapter is in the advanced part of this book, we do not list the applications here, but rather note that a good starting point to explore would be the Wikipedia page.

Table 11.1 Fibonacci numbers (f_k) vs. its approximation ($f_k^{(\text{approx})}$)

k	f_k	$f_k^{(\text{approx})}$	k	f_k	$f_k^{(\text{approx})}$	k	f_k	$f_k^{(\text{approx})}$
0	0	0.45	4	3	3.07	8	21	21.01
1	1	0.72	5	5	4.96	9	34	33.99
2	1	1.17	6	8	8.02	10	55	55.00
3	2	1.89	7	13	12.98	11	89	89.00

What We Learned

- In the transformation of $\boldsymbol{A} : \mathbb{R}^n \mapsto \mathbb{R}^n$ (same space and square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$), the vectors that transform to scaled versions of themselves are eigenvectors. The corresponding scaling factors are eigenvalues.
- $\boldsymbol{A}\boldsymbol{s}_i = \lambda_i \boldsymbol{s}_i \implies \boldsymbol{s}_i$ an eigenvector with the eigenvalue λ_i.
- Arrange the eigenvectors $\boldsymbol{s}_i$ as the columns in a matrix $\boldsymbol{S}$ and the eigenvalues as the diagonal elements in $\boldsymbol{\Lambda}$. Both $\boldsymbol{S}$ and $\boldsymbol{\Lambda}$ are of size $n \times n$.
- $\boldsymbol{AS} = \boldsymbol{S\Lambda}$: Always true by the definition of eigenvalues and vectors.
- If $\boldsymbol{S}$ is invertible, $\boldsymbol{A} = \boldsymbol{S\Lambda S^{-1}}$, and we say $\boldsymbol{A}$ is diagonalized.
- Diagonalization
  - The process of writing $\boldsymbol{A} = \boldsymbol{S\Lambda S^{-1}}$ is diagonalization.
  - We need to have a full set of linearly independent eigenvectors for $\boldsymbol{S}$ to be non-singular.
  - A real, symmetric matrix has full set of orthogonal eigenvectors and real eigenvalues.
  - Diagonalization makes it easy to take powers of a matrix and analyze difference equations.
  - The growth of the system depends on the dominant (meaning largest in absolute value) eigenvalue λ:
    - If $|\lambda| < 1$, the system collapses to zero over time (or as the number of steps tends to infinity).
    - If $|\lambda| > 1$, the system explodes.
    - If $|\lambda| = 1$, the system reaches a steady state.
- Some properties:
  - The eigenvectors corresponding to distinct eigenvalues are linearly independent.
  - The sum of the eigenvalues of a matrix is its trace.
  - The product of the eigenvalues of a matrix is its determinant.
  - Sylvester's Rule connects the signs of pivots and the eigenvalues for real symmetric matrices.

- Eigen-analysis gives us insights about the ellipsoid into which $\boldsymbol{A}$ transforms the unit sphere.
  - The axes of the ellipsoid are along the eigenvectors.
  - Their sizes are the eigenvalues.
  - If $\boldsymbol{A}$ is the covariance matrix of a data set, the eigenvalues and vectors specify the principal components and the variances along them.
  - For a (zero-centered) "tall" data matrix $\boldsymbol{A}$, $\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}$ is the (unnormalized) covariance matrix. The principal component analysis is the eigen-analysis of this so-called Gram matrix.
  - Spectral Theorem: For a real symmetric matrix, $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^{\mathsf{T}}$.

Exercises

A. Review Questions

If unsure of these review questions, go through the chapter again.

Multiple Choice Questions (Only one right answer)

1. What is an eigenvector of a matrix?

   (*a*) A vector that does not change when multiplied by the matrix

   (*b*) A vector that is definitely not in the column space of the matrix

   (*c*) A vector that gets scaled by the multiplication of the matrix

   (*d*) A vector that transforms orthogonal to itself

2. **True or False**: For any two square, invertible matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{AB}$ and $\boldsymbol{BA}$ have the same eigenvalues

   (*a*) True (*b*) False

3. If we take the ratio between the 101^{st} Fibonacci number to the 100^{th} one, what do we get?

   (*a*) The Golden Ratio = 0.618 (*b*) The Golden Ratio = 1.618

   (*c*) Golden Ratio to the power 100 (*d*) Golden Ratio to the power 101

4. If $\boldsymbol{\Lambda}$ is the eigenvalue matrix of $\boldsymbol{A}$, with eigenvectors in the columns of $\boldsymbol{S}$, which one is always true?

   (*a*) $\boldsymbol{\Lambda} = \boldsymbol{S}^{-1}\boldsymbol{AS}$ (*b*) $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$

   (*c*) $\boldsymbol{AS} = \boldsymbol{S}\boldsymbol{\Lambda}$ (*d*) $\boldsymbol{AS} = \boldsymbol{S}^{\mathsf{T}}\boldsymbol{\Lambda}$

5. If we have a real, symmetric matrix, what can we say about its eigenvalues?

(*a*) The eigenvalues are all real
(*b*) The eigenvalues are all positive
(*c*) The eigenvalues are all distinct
(*d*) All of the above

6. If we have a full-rank matrix, what can we say about its eigenvalues?

(*a*) All eigenvalues are real
(*b*) All eigenvalues are positive
(*c*) All eigenvalues are non-zero
(*d*) The eigenvalues are all distinct

7. If we have the largest eigenvalue $|\lambda| > 1$, the matrix represents an exponentially growing difference equation. What does it represent the largest eigenvalue $|\lambda| = 1$?

(*a*) Expanding system
(*b*) Contracting system
(*c*) Has a steady state
(*d*) Cannot have such an eigenvalue

8. If $\boldsymbol{s}$ is an eigenvector of $\boldsymbol{A}$, so is any scalar multiple of $\boldsymbol{s}$. True or false?

(*a*) True
(*b*) False

9. A real, symmetric, full-rank matrix has:

(*a*) All positive eigenvalues
(*b*) All non-negative eigenvalues
(*c*) Full set of orthogonal eigenvectors
(*d*) Eigenvectors that can be chosen to be columns of the identity matrix

10. **True or False**: If $\boldsymbol{A}$ is full-row-rank matrix, so is $\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}$

(*a*) True
(*b*) False

11. **True or False**: For any matrix of any rank $\boldsymbol{A}$, either $\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}$ or $\boldsymbol{A}\boldsymbol{A}^{\mathsf{T}}$ is full rank.

(*a*) True
(*b*) False

Multi-Select Questions (About half the choices are right answers)

12. The eigenvalues of $\boldsymbol{A}$ come from the equation $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{s} = \boldsymbol{0}$. This equation implies:

(*a*) $\boldsymbol{A} - \lambda\boldsymbol{I}$ is a singular matrix
(*b*) $\boldsymbol{A}$ is a singular matrix
(*c*) $\boldsymbol{A}$ can be inverted
(*d*) $\boldsymbol{A} - \lambda\boldsymbol{I}$ has a null space with non-zero vectors
(*e*) The determinant $|\boldsymbol{A} - \lambda\boldsymbol{I}|$ is always zero regardless of what value we choose for the scalar λ
(*f*) The solutions λ_i to the eigenvalue equation are the eigenvalues of $\boldsymbol{A}$

13. The following statements are about a matrix $\boldsymbol{A} \in \mathbb{R}^{4\times 4}$ with $\text{rank}(\boldsymbol{A}) = 2$. Select the ones that are true.

    (a) If the column space of $\boldsymbol{A}$ is the same as its row space, then $\boldsymbol{A}$ is symmetric

    (b) $\boldsymbol{A}$ has two zero eigenvalues

    (c) $\boldsymbol{A}$ has two real, distinct eigenvalues

    (d) $\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}$ is full rank

    (e) If we run the Gram-Schmidt process on $\boldsymbol{A}$, we will get exactly two orthonormal vectors

    (f) A good basis for the null space of $\boldsymbol{A}$ would be the columns of the 2×2 identity matrix

B: Think, Explore & Self-learn: Harder Problems

Explore the web, other books or resources and attempt these questions.

14. In physics, a counterclockwise rotation in the xy-plane results in an angular momentum along the positive z direction. And the angular momentum is conserved. Therefore, it would be nice if the (complex) eigenvector of the rotation matrix somehow represented the (non-existent) z axis. Interpret the complex eigenvectors of the rotation matrix in $\mathbb{R}^2$.

Resources

Review the topic and get alternative viewpoints using these curated video links and links to problem sets.

- Lecture Videos by the Author: LA4CS, Chapter 11: Eigenvalue Decomposition and Diagonalization
- 3Blue1Brown: "Essence of Linear Algebra."
  - Video 14: Eigenvectors and eigenvalues | Chapter 14, Essence of linear algebra
  - Video 15: A quick trick for computing eigenvalues | Chapter 15, Essence of linear algebra
- Imperial College: "Mathematics for Machine Learning - Linear Algebra."
  - Watch Module 5 on eigenvectors and eigenvalues: From Video 28 to 35.
- MIT: Linear Algebra
  - Video 45: 21. Eigenvalues and Eigenvectors
  - Video 46: Eigenvalues and Eigenvectors
  - Video 47: 22. Diagonalization and Powers of A
  - Video 48: Powers of a Matrix

Glossary

The symbols, terms and abbreviations most commonly used in this book are listed and described below for easy reference.

We can also refer to an Online Glossary of Linear Algebra terms and definitions, courtesy of Robert Campbell of UMBC.

Scalar Scalars are written in lowercase letters, *e.g.*, s, similar to the notation for the elements of vectors and matrices.

Vectors Bold lowercase letters represent vectors, *e.g.*, $\boldsymbol{x}$, with elements x_i. Vectors are always column matrices, and are written with square brackets when needed. Note, however, that **SageMath** writes vectors as $(x_1, x_2, \cdots, x_m)$ (using parentheses with commas between the elements), and we may use that notation as well, albeit rarely.

Matrices Bold capital letters represent matrices, *e.g.*, $\boldsymbol{A}$. For the elements of $\boldsymbol{A}$, the corresponding lowercase letter, a_{ij} is used. When explicitly writing out the elements of the matrix, square brackets are used.

Elements of Matrices: The elements of matrices, being scalars, are written using lowercase letters. When we write $\boldsymbol{A} = \left[a_{ij}\right]$, we mean that $\boldsymbol{A}$ is a matrix with a general element a_{ij} in the i^{th} row and j^{th} column. We also write $\boldsymbol{A} = [\boldsymbol{a_j}]$ to indicate that $\boldsymbol{A}$ is composed of column vectors, with $\boldsymbol{a_i}$ as the j^{th} column. Although used only a couple of times in the book, we also write

$\boldsymbol{A} = [\boldsymbol{a}_i^\mathsf{T}]$ to denote the matrix $\boldsymbol{A}$ consisting of row vectors, $\boldsymbol{a}_i^\mathsf{T}$ as the i^{th} row.

Determinant: Our favorite symbol for determinant is a vertical line: The determinant of $\boldsymbol{A}$ is $|\boldsymbol{A}|$.

Transpose: The symbol T represents transpose. $\boldsymbol{A} \in \mathbb{R}^{m\times n} \implies \boldsymbol{A}^\mathsf{T} \in \mathbb{R}^{n\times m}$.

Hermitian Transpose: The complex conjugate transpose of $\boldsymbol{A}$ is indicated as $\boldsymbol{A}^\dagger$, although it is used very sparingly in this book

Fields: Our matrices and vectors are almost always over the field of reals, which is represented as $\mathbb{R}$. We will write $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{x} \in \mathbb{R}^m$. Although we may not use it, it is possible to have matrices and vectors over other fields and rings, such as integers ($\mathbb{Z}$), rationals ($\mathbb{Q}$) or complex ($\mathbb{C}$).

Spaces: We use calligraphic symbols such as $\mathcal{S}$ for spaces. In particular, the column space of $\boldsymbol{A}$ is $\mathcal{C}(\boldsymbol{A})$, row space $\mathcal{C}(\boldsymbol{A}^\mathsf{T})$, null space $\mathcal{N}(\boldsymbol{A})$ and its left null space is $\mathcal{N}(\boldsymbol{A}^\mathsf{T})$.

Math Symbols: In definitions and equations, we will use common mathematical symbols such as:

- $\forall$: For all or for any. $\forall\, \boldsymbol{x}$ means for any vector $\boldsymbol{x}$.
- $\in$: Is a member of the set. $s \in \mathbb{R}$ says that the scalar s is a member of the set of reals.
- $\implies$: Implies. $i, j \in \mathbb{Z} \implies \frac{i}{j} \in \mathbb{Q}$
- $\subset, \subseteq$: Subset of. $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Non-standard Notation: We use the symbol $\oplus$ in a manner not seen elsewhere: We write $\boldsymbol{A} = [\boldsymbol{I} \oplus \boldsymbol{F}]$, for instance, to indicate $\boldsymbol{A}$ is a matrix composed of the columns of the identity matrix and the matrix $\boldsymbol{F}$, but the columns are not necessarily in the order in which they appear in the constituent matrices. They may be "shuffled."

Norm: Double vertical lines indicate the norm (usually the Euclidean norm, usually of a vector). $\|\boldsymbol{x}\|$ is the norm of the vector $\boldsymbol{x}$.