ECS 323: Control Systems

Planar VTOL System

Design Study

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1. Design Study Description

In this design study we need to design a control system for the given planar VTOL system with the parameters:

- $M_c = 2$ kg
- $J_c = 0.009$ kg m^2
- $m_l = 0.3$ kg
- $m_r = 0.3$ kg
- $d = 0.28$ m
- $\mu = 0.21$ kg s^{-1}

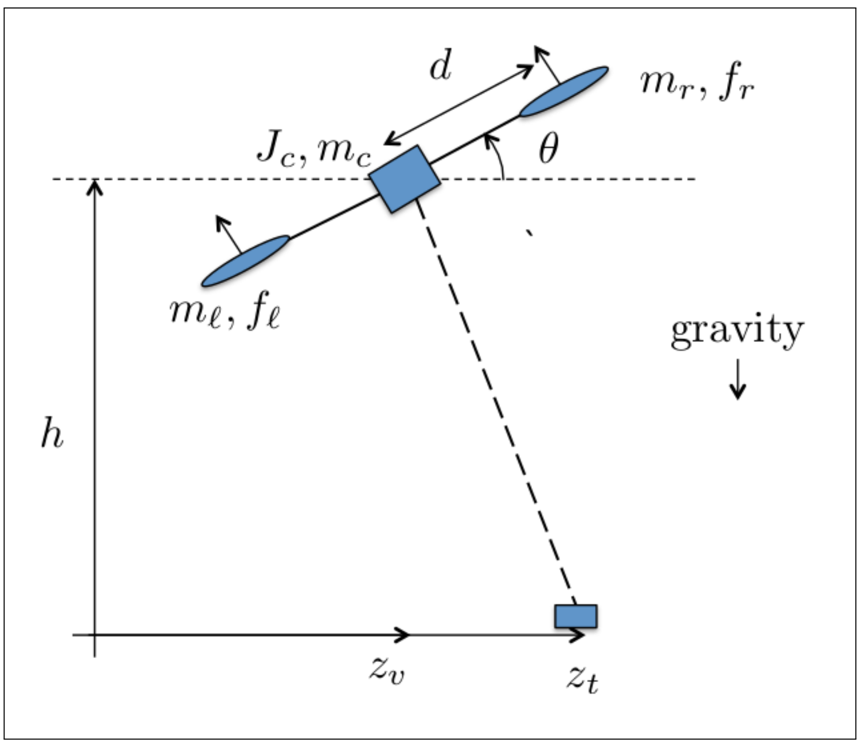

Figure 1: Planar VTOL System

2. Kinetic Energy

The postions of the various components of the VTOL are given by:

$$\begin{aligned}\mathbf{p_c} &= (z_v,\ h)\\ \mathbf{p_l} &= (z_v - d\cos\theta,\ h - d\sin\theta)\\ \mathbf{p_r} &= (z_v + d\cos\theta,\ h + d\sin\theta)\end{aligned}$$

So, the velocities can be written as:

$$\begin{aligned}\mathbf{v_c} &= (\dot{z}_v,\ \dot{h})\\ \mathbf{v_l} &= (\dot{z}_v + d\dot{\theta}\sin\theta,\ \dot{h} - d\dot{\theta}\cos\theta)\\ \mathbf{v_r} &= (\dot{z}_v - d\dot{\theta}\sin\theta,\ \dot{h} + d\dot{\theta}\cos\theta)\end{aligned}$$

Kinetic energy of the centerpod is given by:

$$K_{pod} = \frac{1}{2}m_c\mathbf{v}_c^T\mathbf{v}_c + \frac{1}{2}\boldsymbol{\omega}_c^T J_c\boldsymbol{\omega}_c = \frac{1}{2}m_c(\dot{z}_v^2 + \dot{h}^2) + \frac{1}{2}J_c\dot{\theta}^2 \tag{1}$$

Kinetic energy of the left and right rotors is given by:

$$\begin{aligned}K_{rotors} &= \frac{1}{2}m_l\mathbf{v}_l^T\mathbf{v}_l + \frac{1}{2}m_r\mathbf{v}_r^T\mathbf{v}_r\\ &= \frac{1}{2}m_l(\dot{z}_v + d\dot{\theta}\sin\theta)^2 + \frac{1}{2}m_l(\dot{h} - d\dot{\theta}\cos\theta)^2\\ &\quad + \frac{1}{2}m_r(\dot{z}_v - d\dot{\theta}\sin\theta)^2 + \frac{1}{2}m_l(\dot{h} + d\dot{\theta}\cos\theta)^2\\ &= \frac{1}{2}(m_l + m_r)(\dot{z}_v^2 + \dot{h}^2) + \frac{1}{2}(m_l + m_r)d^2\dot{\theta}^2\\ &\quad + (m_l - m_r)(\dot{z}_v\sin\theta - \dot{h}\cos\theta)d\dot{\theta}\end{aligned} \tag{2}$$

Now, the total kinetic energy of the VTOL will be given by the sum of 1 and 2:

$$\begin{aligned}K_V &= K_{pod} + K_{rotors}\\ &= \frac{1}{2}(m_c + m_l + m_r)(\dot{z}_v^2 + \dot{h}^2) + \frac{1}{2}(m_l d^2 + m_r d^2 + J_c)\dot{\theta}^2\\ &\quad + (m_l d - m_r d)(\dot{z}_v\sin\theta - \dot{h}\cos\theta)\dot{\theta}\end{aligned} \tag{3}$$

As in the given parameters $m_l = m_r$, so the last term in the kinetic energy is zero and will be ignored in the rest of the report.

3. Equations of Motion

(a) Now in order to determine the equations of motion of the VTOL, we first write its potential energy. The potential energy is due to the gravitational potential and can be written as the sum of potential energies of the individual components:

$$P_V = m_c gh + m_l gh + m_r gh = (m_c + m_l + m_r)gh \tag{4}$$

(b) Now as we are only considering the dynamics of the VTOL and not of the target so the generalized coordinates can be defined as:

$$\mathbf{q} = \begin{pmatrix} z_v \\ h \\ \theta \end{pmatrix}$$

Also as it is given in the project objective, the damping forces in the system are due to the momentum drag which is caused by the change in direction of the air when it flows through the rotors. This momentum drag can be modeled as $F_{drag} = -\mu \dot{z}_v$. So, we can write the dissipative (drag) forces as:

$$-B\dot{\mathbf{q}} = -\begin{pmatrix} \mu \dot{z}_v \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} \mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{z}_v \\ \dot{h} \\ \dot{\theta} \end{pmatrix}$$

(c) The total force on the COM of the VTOL is given by $\boxed{F = f_l + f_r}$. The torque due to the left rotor is $\tau_l = -f_l d$ (using right handed coordinates) and the torque due to the right rotor is $\tau_r = f_r d$. Hence, the total torque about the COM of the VTOL is $\boxed{\tau = (f_r - f_l)d}$. So, we can write the generalized forces as:

$$\mathbf{\Phi} = \begin{pmatrix} -F\sin\theta \\ F\cos\theta \\ \tau \end{pmatrix} = \begin{pmatrix} -(f_r + f_l)\sin\theta \\ (f_r + f_l)\cos\theta \\ (fr - f_l)d \end{pmatrix}$$

(d) Using the kinetic and the potential energies (eq. 3 and eq. 4) we can write the Lagrangian as:

$$\mathcal{L} = K_V - P_V = \frac{1}{2}M_V(\dot{z}_v^2 + \dot{h}^2 - 2gh) + \frac{1}{2}J_V\dot{\theta}^2$$

Here, $M_V \equiv (m_c + m_l + m_r)$ and $J_V \equiv (m_l d^2 + m_r d^2 + J_c)$. Now, we can write:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \begin{pmatrix} M_V \dot{z}_v \\ M_V \dot{h} \\ J_V \dot{\theta} \end{pmatrix}$$

And,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ -M_V g \\ 0 \end{pmatrix}$$

Writing the Euler-Lagrange equations in matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{\Phi} - B\dot{\mathbf{q}}$$

$$\implies \begin{pmatrix} M_V \ddot{z}_v \\ M_V \ddot{h} \\ J_V \ddot{\theta} \end{pmatrix} - \begin{pmatrix} 0 \\ -M_V g \\ 0 \end{pmatrix} = \begin{pmatrix} -F\sin\theta \\ F\cos\theta \\ \tau \end{pmatrix} - \begin{pmatrix} \mu \dot{z}_v \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \boxed{\begin{pmatrix} M_V \ddot{z}_v \\ M_V \ddot{h} \\ J_V \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \mu \dot{z}_v - F\sin\theta \\ M_V g + F\cos\theta \\ \tau \end{pmatrix}} \tag{5}$$