ECS 323: Control Systems

Planar VTOL System

Design Study

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Contents

1. Design Study Description

In this design study we need to design a control system for a planar VTOL system with the given parameters [1]:

- $M_c = 2$ kg
- $J_c = 0.009$ kg m^2
- $m_l = 0.3$ kg
- $m_r = 0.3$ kg
- $d = 0.28$ m
- $\mu = 0.21$ kg s^{-1}

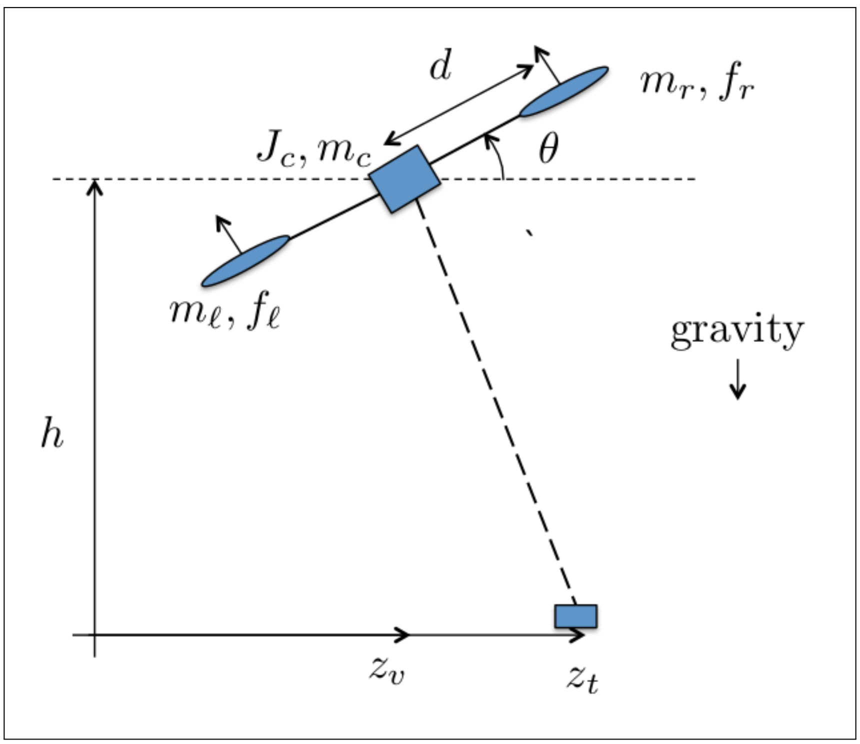

Figure 1: Planar VTOL System

2. Kinetic Energy

The postions of the various components of the VTOL are given by:

$$\begin{aligned}
\mathbf{p_c} &= (z_v,\ h) \\
\mathbf{p_l} &= (z_v - d\cos\theta,\ h - d\sin\theta) \\
\mathbf{p_r} &= (z_v + d\cos\theta,\ h + d\sin\theta)
\end{aligned}$$

So, the velocities can be written as:

$$\begin{aligned}
\mathbf{v_c} &= (\dot{z}_v,\ \dot{h}) \\
\mathbf{v_l} &= (\dot{z}_v + d\dot{\theta}\sin\theta,\ \dot{h} - d\dot{\theta}\cos\theta) \\
\mathbf{v_r} &= (\dot{z}_v - d\dot{\theta}\sin\theta,\ \dot{h} + d\dot{\theta}\cos\theta)
\end{aligned}$$

Kinetic energy of the centerpod is given by:

$$K_{pod} = \frac{1}{2}m_c\mathbf{v}_c^T\mathbf{v}_c + \frac{1}{2}\boldsymbol{\omega}_c^T J_c\boldsymbol{\omega}_c = \frac{1}{2}m_c(\dot{z}_v^2 + \dot{h}^2) + \frac{1}{2}J_c\dot{\theta}^2 \tag{1}$$

Kinetic energy of the left and right rotors is given by:

$$\begin{aligned}
K_{rotors} &= \frac{1}{2}m_l\mathbf{v}_l^T\mathbf{v}_l + \frac{1}{2}m_r\mathbf{v}_r^T\mathbf{v}_r \\
&= \frac{1}{2}m_l(\dot{z}_v + d\dot{\theta}\sin\theta)^2 + \frac{1}{2}m_l(\dot{h} - d\dot{\theta}\cos\theta)^2 \\
&\quad + \frac{1}{2}m_r(\dot{z}_v - d\dot{\theta}\sin\theta)^2 + \frac{1}{2}m_l(\dot{h} + d\dot{\theta}\cos\theta)^2 \\
&= \frac{1}{2}(m_l + m_r)(\dot{z}_v^2 + \dot{h}^2) + \frac{1}{2}(m_l + m_r)d^2\dot{\theta}^2 \\
&\quad + (m_l - m_r)(\dot{z}_v\sin\theta - \dot{h}\cos\theta)d\dot{\theta}
\end{aligned} \tag{2}$$

Now, the total kinetic energy of the VTOL will be given by the sum of (1) and (2):

$$\begin{aligned}
K_V &= K_{pod} + K_{rotors} \\
&= \frac{1}{2}(m_c + m_l + m_r)(\dot{z}_v^2 + \dot{h}^2) + \frac{1}{2}(m_l d^2 + m_r d^2 + J_c)\dot{\theta}^2 \\
&\quad + (m_l d - m_r d)(\dot{z}_v\sin\theta - \dot{h}\cos\theta)\dot{\theta}
\end{aligned} \tag{3}$$

As in the given parameters $m_l = m_r$, so the last term in the kinetic energy is zero and will be ignored in the rest of the report.

3. Equations of Motion

(a) Now in order to determine the equations of motion of the VTOL, we first write its potential energy. The potential energy is due to the gravitational potential and can be written as the sum of potential energies of the individual components:

$$P_V = m_c gh + m_l gh + m_r gh = (m_c + m_l + m_r)gh \tag{4}$$

(b) Now as we are only considering the dynamics of the VTOL and not of the target so the generalized coordinates can be defined as:

$$\mathbf{q} = \begin{pmatrix} z_v \\ h \\ \theta \end{pmatrix}$$

Also as it is given in the project objective, the damping forces in the system are due to the momentum drag which is caused by the change in direction of the air when it flows through the rotors. This momentum drag can be modeled as $F_{drag} = -\mu \dot{z}_v$. So, we can write the dissipative (drag) forces as:

$$-B\dot{\mathbf{q}} = -\begin{pmatrix} \mu \dot{z}_v \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} \mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{z}_v \\ \dot{h} \\ \dot{\theta} \end{pmatrix}$$

(c) The total force on the COM of the VTOL is given by $\boxed{F = f_l + f_r}$. The torque due to the left rotor is $\tau_l = -f_l d$ (using right handed coordinates) and the torque due to the right rotor is $\tau_r = f_r d$. Hence, the total torque about the COM of the VTOL is $\boxed{\tau = (f_r - f_l)d}$. So, we can write the generalized forces as:

$$\mathbf{\Phi} = \begin{pmatrix} -F\sin\theta \\ F\cos\theta \\ \tau \end{pmatrix} = \begin{pmatrix} -(f_r + f_l)\sin\theta \\ (f_r + f_l)\cos\theta \\ (fr - f_l)d \end{pmatrix}$$

(d) Using the kinetic and the potential energies (3) and (4) we can write the Lagrangian as:

$$\mathcal{L} = K_V - P_V = \frac{1}{2}M_V(\dot{z}_v^2 + \dot{h}^2 - 2gh) + \frac{1}{2}J_V\dot{\theta}^2$$

Here, $M_V \equiv (m_c + m_l + m_r)$ and $J_V \equiv (m_l d^2 + m_r d^2 + J_c)$. Now, we can write:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \begin{pmatrix} M_V \dot{z}_v \\ M_V \dot{h} \\ J_V \dot{\theta} \end{pmatrix}$$

And,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ -M_V g \\ 0 \end{pmatrix}$$

Writing the Euler-Lagrange equations in matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{\Phi} - B\dot{\mathbf{q}}$$

$$\implies \begin{pmatrix} M_V \ddot{z}_v \\ M_V \ddot{h} \\ J_V \ddot{\theta} \end{pmatrix} - \begin{pmatrix} 0 \\ -M_V g \\ 0 \end{pmatrix} = \begin{pmatrix} -F\sin\theta \\ F\cos\theta \\ \tau \end{pmatrix} - \begin{pmatrix} \mu \dot{z}_v \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \boxed{\begin{pmatrix} M_V \ddot{z}_v \\ M_V \ddot{h} \\ J_V \ddot{\theta} \end{pmatrix} = \begin{pmatrix} -\mu \dot{z}_v - F\sin\theta \\ -M_V g + F\cos\theta \\ \tau \end{pmatrix}} \tag{5}$$

4. Linearize Equations of Motion

(a) First we determine the equilibrium points of the equations of motion (5) of the VTOL. At this point there is no motion in the system. This means:

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{z}_v \\ \dot{h} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

And consequently,

$$\ddot{\mathbf{q}} = 0 \implies \begin{pmatrix} M_V \ddot{z}_v \\ M_V \ddot{h} \\ J_V \ddot{\theta} \end{pmatrix} = 0$$

$$\begin{pmatrix} -\mu \dot{z}_v - F \sin\theta \\ M_V g + F\cos\theta \\ \tau \end{pmatrix} = 0 \implies \begin{pmatrix} -F\sin\theta \\ -M_V g + F\cos\theta \\ \tau \end{pmatrix} = 0$$

The solutions of the above equations are given by, $\theta_e = 0$ or $\theta_e = \pi$ and due to this $F_e = M_V g$ or $F_2 = -M_V g$ and $\tau_e = 0$. The values of z_e and h_e can be anything.

But taking the practical consideration that the rotors are able to rotate in only one direction the case where $\theta = \pi$ and $F_e = -M_V g$, the rotors can not generate a positive force in the opposite direction of the direction of lift. So, we will drop this case.

(b) In order to linearize the equations, we first write the equations seperately in the form:

$$\begin{aligned} \ddot{z}_v &= -\frac{\mu}{M_V}\dot{z}_v - \frac{F}{M_V}\sin\theta \\ \ddot{h} &= -g + \frac{F}{M_V}\cos\theta \\ \ddot{\theta} &= \frac{1}{J_c}\tau \end{aligned}$$

Now, we define:

$$\tilde{z} \equiv z_v - z_v^{(e)},\ \tilde{h} \equiv h - h_e,\ \tilde{\theta} \equiv \theta - \theta_e,\ \tilde{F} \equiv F - F_e,\ \tilde{\tau} \equiv \tau - \tau_e = \tau$$

Also, we can note that as $\dot{\mathbf{q}}_e = 0$ and $\ddot{\mathbf{q}}_e = 0$, so we can write:

$$\dot{\tilde{z}} = \dot{z}_v,\ \ddot{\tilde{z}} = \ddot{z}_v$$

$$\dot{\tilde{h}} = \dot{h},\ \ddot{\tilde{h}} = \ddot{h}$$

$$\dot{\tilde{\theta}} = \dot{\theta},\ \ddot{\tilde{\theta}} = \ddot{\theta}$$

Now we linearize the non-linear terms in these equations as follows:

$$
\begin{aligned}
\frac{\mu}{M_V}\dot{z}_v &\approx \frac{\mu}{M_V}\dot{z}_v^{(e)} + \frac{\partial}{\partial \dot{z}_v}\left(\frac{\mu}{M_V}\dot{z}_v\right)\bigg|_{eq.}(\dot{z}_v - \dot{z}_v^{(e)}) \\
&= \frac{\mu}{M_V}\dot{\tilde{z}} \\
\frac{F}{M_V}\sin\theta &\approx \frac{F}{M_V}\sin\theta_e + \frac{\partial}{\partial\theta}\left(\frac{F}{M_V}\sin\theta\right)\bigg|_{eq.}(\theta - \theta_e) \\
&= \frac{F_e}{M_V}\tilde{\theta}\cos\theta_e \\
&= g\tilde{\theta} \\
\frac{F}{M_V}\cos\theta &\approx \frac{F}{M_V}\cos\theta_e + \frac{\partial}{\partial\theta}\left(\frac{F}{M_V}\cos\theta\right)\bigg|_{eq.}(\theta - \theta_e) \\
&= \frac{F}{M_V} + \frac{F_e}{M_V}\tilde{\theta}\sin\theta_e \\
&= \frac{F}{M_V} \\
&= \frac{\tilde{F}}{M_V} + g
\end{aligned}
$$

So, the linearized equations of motion are:

$$
\begin{aligned}
\ddot{\tilde{z}} &= -\frac{\mu}{M_V}\dot{\tilde{z}} - g\tilde{\theta} \\
\ddot{\tilde{h}} &= \frac{\tilde{F}}{M_V} \\
\ddot{\tilde{\theta}} &= \frac{1}{J_c}\tilde{\tau}
\end{aligned}
\tag{6}
$$

(c) The equations of motion can only be feedback linearized if we can write the equations in the from given below where $y(t)$ is the output and $u(t)$ is the input with the function $g(y, \dot{y})$ giving the non-linear terms.

$$a\ddot{y} + b\dot{y} + cy = g(y, \dot{y}) + u(t)$$

But in the case of the equations of motion of the planar VTOL given by (5), the non-linerity is present in the input term itself (F is the input), and hence we can't seperately define $g(y, \dot{y})$. Hence, the system can not be feedback linearized.

5. Transfer Function Model

(a) We have the linearized equations of motion given in (6). If we take the laplace transform of those equations we get the following:

$$\begin{aligned} s^2\tilde{Z}(s) &= -\frac{\mu}{M_V}s\tilde{Z}(s) - g\tilde{\Theta}(s) \\ s^2\tilde{H}(s) &= \frac{1}{M_V}\tilde{F}(s) \\ s^2\tilde{\Theta}(s) &= \frac{1}{J_c}\tilde{\tau}(s) \end{aligned}$$

(b) For longitudinal dynamics we have the following equation:

$$\begin{aligned} s^2\tilde{H}(s) &= \frac{1}{M_V}\tilde{F}(s) \\ \implies &\boxed{\tilde{H}(s) = \frac{1}{s^2M_V}\tilde{F}(s)} \end{aligned}$$

Hence, in the longitudinal dynamics we have $\tilde{F}(s)$ as the input variable and $\tilde{H}(s)$ as the output with transfer function $\frac{1}{s^2M_V}$.

(c) For lateral dynamics first we have the equation for $\tilde{\Theta}(s)$:

$$\begin{aligned} s^2\tilde{\Theta}(s) &= \frac{1}{J_c}\tilde{\tau}(s) \\ \implies &\boxed{\tilde{\Theta}(s) = \frac{1}{s^2J_c}\tilde{\tau}(s)} \end{aligned}$$

And, also we have the equation for $\tilde{Z}(s)$ given by:

$$\begin{aligned} s^2\tilde{Z}(s) &= -\frac{\mu}{M_V}s\tilde{Z}(s) - g\tilde{\Theta}(s) \\ \implies \tilde{Z}(s) + \frac{\mu}{sM_V}\tilde{Z}(s) &= -\frac{g}{s^4J_c}\tilde{\tau}(s) \\ \implies \tilde{Z}(s) &= -\frac{\frac{g}{s^4J_c}\tilde{\tau}(s)}{1+\frac{\mu}{sM_V}} \\ \implies &\boxed{\tilde{Z}(s) = -\frac{M_Vg}{s^3J_c(sM_V+\mu)}\tilde{\tau}(s)} \end{aligned}$$

Hence, in the lateral dynamics we have $\tilde{\tau}(s)$ is the input variable, $\tilde{\Theta}(s)$ is the intermediate variable with transfer function $\frac{1}{s^2J_c}$ and $\tilde{Z}(s)$ is the output variable with transfer function $-\frac{M_Vg}{s^3J_c(sM_V+\mu)}$.

(d) The block diagrams of open loop longitudinal and lateral systems are:

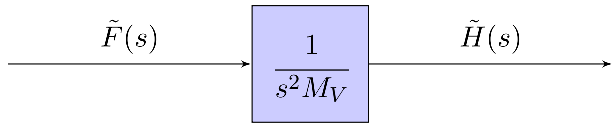

Figure 2: Longitudinal open loop system

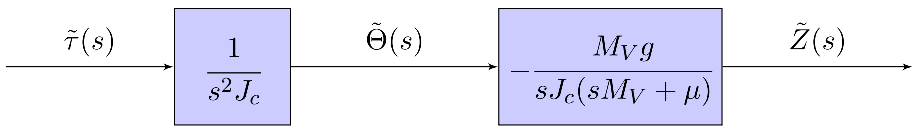

Figure 3: Lateral open loop system

6. State Space Model

(a) For the longitudinal dynamics we define the state of the system as $\tilde{x}_{lon} = \begin{pmatrix} \tilde{h} & \dot{\tilde{h}} \end{pmatrix}^{\mathrm{T}}$, the input as $\tilde{u}_{lon} = \tilde{F}$ and the output as $\tilde{y}_{lon} = \tilde{h}$. Now we can write:

$$\begin{aligned}
\dot{\tilde{x}}_{lon} &= \begin{pmatrix} \dot{\tilde{h}} \\ \ddot{\tilde{h}} \end{pmatrix} = \begin{pmatrix} \dot{\tilde{h}} \\ \frac{\tilde{F}}{M_V} \end{pmatrix} \\
\implies \dot{\tilde{x}}_{lon} &= \begin{pmatrix} \dot{\tilde{h}} \\ \frac{\tilde{u}_{lon}}{M_V} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{h} \\ \dot{\tilde{h}} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M_V} \end{pmatrix} \tilde{u}_{lon} \\
\implies \dot{\tilde{x}}_{lon} &= A\tilde{x}_{lon} + B\tilde{u}_{lon}
\end{aligned}$$

Similary for the output we have:

$$\begin{aligned}
\tilde{y}_{lon} &= \tilde{h} \\
\implies \tilde{y}_{lon} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{h} \\ \dot{\tilde{h}} \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} \tilde{u}_{lon} \\
\implies \tilde{y}_{lon} &= C\tilde{x}_{lon} + D\tilde{u}_{lon}
\end{aligned}$$

(b) For the lateral dynamics we define the state as $\tilde{x}_{lat} = \begin{pmatrix} \tilde{z} & \tilde{\theta} & \dot{\tilde{z}} & \dot{\tilde{\theta}} \end{pmatrix}^{\mathrm{T}}$ with the input $\tilde{u}_{lat} = \tilde{\tau}$ and output $\tilde{y}_{lat} = \begin{pmatrix} \tilde{z} & \tilde{\theta} \end{pmatrix}^{\mathrm{T}}$. So for this we have:

$$\begin{aligned}
\dot{\tilde{x}}_{lat} &= \begin{pmatrix} \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \\ \ddot{\tilde{z}} \\ \ddot{\tilde{\theta}} \end{pmatrix} = \begin{pmatrix} \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \\ -\frac{\mu}{M_V}\dot{\tilde{z}} - g\tilde{\theta} \\ \frac{1}{J_c}\tilde{\tau} \end{pmatrix} \\
\implies \dot{\tilde{x}}_{lat} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -g & -\frac{\mu}{M_V} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \\ \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_c} \end{pmatrix} \tilde{\tau} \\
\implies \dot{\tilde{x}}_{lat} &= A\tilde{x}_{lat} + B\tilde{u}_{lat}
\end{aligned}$$

Similary, for the output we have:

$$\begin{aligned}
\tilde{y}_{lat} &= \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \\ \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} \tilde{\tau} \\
\implies \tilde{y}_{lat} &= C\tilde{x}_{lat} + D\tilde{u}_{lat}
\end{aligned}$$

7. Pole Placement using PD

(a) The open loop transfer function from $\tilde{F}(s)$ to $\tilde{H}(s)$ is given by:

$$\tilde{H}(s) = \frac{1}{s^2 M_V}\tilde{F}(s)$$

So, the open loop poles can be written as the roots of the equation:

$$\Delta_{ol} = s^2 M_V$$

That is $p_{ol} = 0$

(b) Using the figure referred to in the problem we can write the closed loop transfer function equation from $\tilde{h}_r$ to $\tilde{h}$ as:

$$\tilde{H}(s) = \left(\frac{1}{s^2 M_V}\right)(k_p(\tilde{H}_r(s) - \tilde{H}(s)) - k_D s\tilde{H}(s))$$

$$\implies \tilde{H}(s) = \frac{\frac{k_p}{M_V}}{s^2 + \frac{k_D}{M_V}s + \frac{k_P}{M_V}}\tilde{H}_r(s)$$

Hence, the closed loop poles are given by the roots of the equation:

$$\Delta_{cl} = s^2 + \frac{k_D}{M_V}s + \frac{k_P}{M_V}$$

That is:

$$p_{cl} = -\frac{k_D}{2M_V} \pm \sqrt{\left(\frac{k_D}{2M_V}\right)^2 - \frac{k_P}{M_V}}$$

(c) The desired closed loop poles are at $p = -0.2$ and $p = -0.3$. So, the characteristic polynomial of the desired poles will be:

$$\Delta_{cl}^d = (s + 0.2)(s + 0.3)$$

Equating with the closed loop characteristic equation we have:

$$s^2 + 0.5s + 0.06 = s^2 + \frac{k_D}{M_V}s + \frac{k_P}{M_V}$$

$$\implies \frac{k_D}{M_V} = 0.5 \text{ and } \frac{k_P}{M_V} = 0.06$$

$$\implies k_D = 0.5M_V \text{ and } k_P = 0.06M_V$$

(d) The response for longitudinal dynamics is given below:

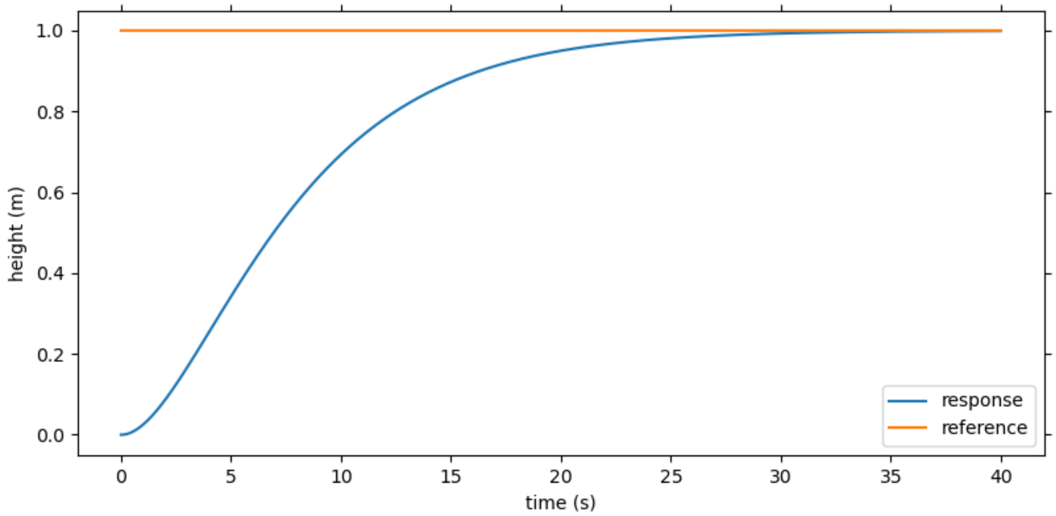

Figure 4: Step Response for Longitudinal Controller

8. Successive Loop Closure

(a) Given that the rise time is given by $t_r = 8$ seconds. The the natural frequency of is:

$$\omega_n = \frac{\pi}{2t_r\sqrt{1-\zeta^2}} \approx \frac{2.2}{t_r} \qquad \text{(For } \zeta = 0.707\text{)}$$

$$\implies \omega_n = 0.275$$

So, the desired closed loop characteristic polynomial is:

$$\Delta_{cl}^d = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 0.38885s + 0.075625$$

Equating with the closed loop characteristic polynomial we get:

$$k_{D_h} = 0.38885 M_V \text{ and } k_{P_h} = 0.075625 M_V$$

(b) For the inner loop in which we use $\tilde{\tau}$ to regulate $\tilde{\theta}$ we have the following transfer function equation using the transfer function of $\tilde{\tau}(s)$ to $\tilde{\theta}(s)$:

$$\tilde{\Theta}(s) = \frac{1}{s^2 J_c}(k_p(\tilde{\Theta}_r(s) - \tilde{\Theta}(s)) - sk_D\tilde{\Theta}(s))$$

$$\implies \tilde{\Theta}(s) = \frac{\frac{k_P}{J_c}}{s^2 + \frac{k_D}{J_c}s + \frac{k_P}{J_c}}\tilde{\Theta}_r(s)$$

So, the closed loop characteristic polynomial for roll angle is:

$$\Delta_{cl,\theta} = s^2 + \frac{k_D}{J_c}s + \frac{k_P}{J_c}$$

Now, given the rise time of $t_{r\theta} = 0.8$ seconds and damping of $\zeta_z = 0.707$ we have that the natural frequency is $\omega_{n\theta} = \frac{2.2}{t_{r\theta}} = 2.75$.

Then the desired characteristic polynomial for roll angle is:

$$\Delta^d_{cl,\theta} = s^2 + 2\zeta\omega_{n\theta}s + \omega^2_{n\theta} = s^2 + 3.8885s + 7.5625$$

Comparing this equation with the closed loop characteristic polynomial we have:

$$k_{D\theta} = 3.8885J_c \text{ and } k_{P\theta} = 7.5625J_c$$

(c) The DC gain of the inner loop is given by:

$$k_{DC_\theta} = \lim_{s\to 0} \frac{\frac{k_P}{J_c}}{s^2 + \frac{k_D}{J_c}s + \frac{k_P}{J_c}} = 1$$

(d) As the DC gain of the inner loop system is just 1, so the transfer function equation of the outer loop system will be given by:

$$\tilde{Z}(s) = \frac{-M_V g}{s^2 M_V + s\mu}(k_P(\tilde{Z}_r(s) - \tilde{Z}(s)) - sk_D\tilde{Z}(s))$$

Rearranging this equation we get:

$$\tilde{Z}(s) = \frac{-gk_P}{s^2 + (\frac{\mu}{M_V} - gk_D)s + -gk_P}\tilde{Z}_r(s)$$

Hence, the close loop characteristic polynomial of $\tilde{z}$ will be:

$$\Delta_{cl,z} = s^2 + (\frac{\mu}{M_V} - gk_D)s - gk_P$$

Now, given the rise time of $t_{rz} = 10t_{r\theta} = 8$ seconds and damping of $\zeta_z = 0.707$ we have that the natural frequency is $\omega_{nz} = \frac{2.2}{t_{rz}} = 0.275\ \text{s}^{-1}$.

Then the desired characteristic polynomial for translation is:

$$\Delta^d_{cl,z} = s^2 + 2\zeta_z\omega_{nz}s + \omega^2_{nz} = s^2 + 0.38885s + 0.075625$$

Comparing this equation with the closed loop characteristic polynomial we have:

$$k_{D_z} = \frac{1}{g}\left(\frac{\mu}{M_V} - 0.38885\right) \text{ and } k_{P_z} = \frac{-0.075625}{g}$$

(e) See code for the implementation. The respose is given in the following plot:

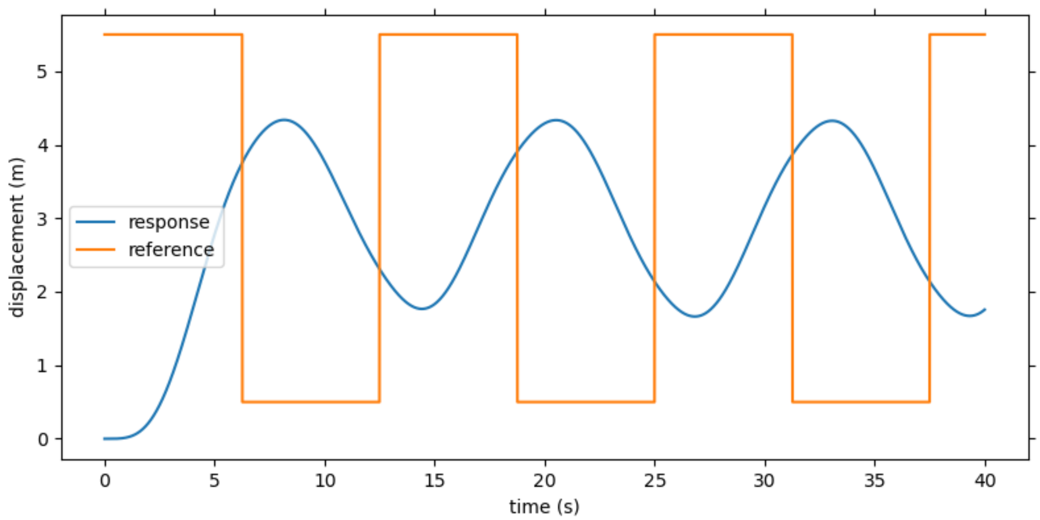

Figure 5: Square Response for Lateral Controller

(f) We are given the saturation limits: $0 \leq f_l \leq 10$N and $0 \leq f_r \leq 10$N. So, as $\tau = d * (f_r - f_l)$ and $F = f_r + f_l$. So, $F_{max} = 20$ N and $\tau_{max} = 10d$ Nm.

9. Integrators and System Type

(a) For the longitudinal dynamics, we have that the error term $e_h(t)$ with the reference input $h^d(t)$ in the transfer model can be written as:

$$E_h(s) = \frac{1}{1 + \frac{sk_{D_h} + k_{P_h}}{s^2 M_V}} H^d(s)$$

Now, for step input, we have $H^d(s) = \frac{1}{s}$. So, the steady state error is given by:

$$\begin{aligned}
\lim_{t\to\infty} e_h(t) &= \lim_{s\to 0} sE_h(s) \\
&= \lim_{s\to 0} s \frac{1}{1 + \frac{sk_{D_h} + k_{P_h}}{s^2 M_V}} \frac{1}{s} \\
&= \frac{1}{1 + \lim_{s\to 0} \frac{sk_{D_h} + k_{P_h}}{s^2 M_V}} \\
&= 0
\end{aligned}$$

For ramp input, we have $H^d(s) = \frac{1}{s^2}$. So, the steady state error is given by:

$$\begin{aligned}
\lim_{t\to\infty} e_h(t) &= \lim_{s\to 0} sE_h(s) \\
&= \lim_{s\to 0} s\frac{1}{1+\frac{sk_{D_h}+k_{P_h}}{s^2M_V}}\frac{1}{s^2} \\
&= \lim_{s\to 0} \frac{1}{s+\frac{s^2k_{D_h}+sk_{P_h}}{s^2M_V}} \\
&= \frac{1}{\lim_{s\to 0}\frac{s^2k_{D_h}+sk_{P_h}}{s^2M_V}} \\
&= 0
\end{aligned}$$

For parabolic response, we have $H^d(s) = \frac{1}{s^3}$. So, the steady state error is given by:

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\frac{1}{1+\frac{sk_{D_h}+k_{P_h}}{s^2M_V}}\frac{1}{s^3} \\
&= \lim_{s\to 0} \frac{1}{s^2+\frac{s^3k_{D_h}+s^2k_{P_h}}{s^2M_V}} \\
&= \frac{1}{\lim_{s\to 0}\frac{s^3k_{D_h}+s^2k_{P_h}}{s^2M_V}} \\
&= \frac{M_V}{k_{P_h}}
\end{aligned}$$

As the steady state error in the parabolic input of type $H^d(s) = \frac{1}{s^3}$ is finite so, we have that the system is a *type 2 system*.

(b) For the inner loop lateral dynamics, we have that the error term $e_\theta(t)$ with the reference input $\theta^d(t)$ in the transfer model can be written as:

$$E_\theta(s) = \frac{1}{1+\frac{sk_{D_\theta}+k_{P_\theta}}{s^2J_c}}\Theta^d(s)$$

Now, for step response, we have $\Theta^d(s) = \frac{1}{s}$. So, the steady state error is given by:

$$\begin{aligned}
\lim_{t\to\infty} e_\theta(t) &= \lim_{s\to 0} sE_\theta(s) \\
&= \lim_{s\to 0} s\frac{1}{1+\frac{sk_{D_\theta}+k_{P_\theta}}{s^2J_c}}\frac{1}{s} \\
&= \frac{1}{1+\lim_{s\to 0}\frac{sk_{D_\theta}+k_{P_\theta}}{s^2J_c}} \\
&= 0
\end{aligned}$$

For ramp response, we have $\Theta^d(s) = \frac{1}{s^2}$. So, the steady state error is given by:

$$\begin{aligned}
\lim_{t\to\infty} e_\theta(t) &= \lim_{s\to 0} sE_\theta(s) \\
&= \lim_{s\to 0} s\frac{1}{1+\frac{sk_{D_\theta}+k_{P_\theta}}{s^2 J_c}}\frac{1}{s^2} \\
&= \lim_{s\to 0} \frac{1}{s+\frac{s^2k_{D_\theta}+sk_{P_\theta}}{s^2 J_c}} \\
&= \frac{1}{\lim_{s\to 0}\frac{s^2k_{D_\theta}+sk_{P_\theta}}{s^2 J_c}} \\
&= 0
\end{aligned}$$

For parabolic response, we have $\Theta^d(s) = \frac{1}{s^3}$. So, the steady state error is given by:

$$\begin{aligned}
\lim_{t\to\infty} e_\theta(t) &= \lim_{s\to 0} sE_\theta(s) \\
&= \lim_{s\to 0} s\frac{1}{1+\frac{sk_{D_\theta}+k_{P_\theta}}{s^2 J_c}}\frac{1}{s^3} \\
&= \lim_{s\to 0} \frac{1}{s^2+\frac{s^3k_{D_\theta}+s^2k_{P_\theta}}{s^2 J_c}} \\
&= \frac{1}{\lim_{s\to 0}\frac{s^3k_{D_\theta}+s^2k_{P_\theta}}{s^2 J_c}} \\
&= \frac{J_c}{k_{P_\theta}}
\end{aligned}$$

As the steady state error in the parabolic input of type $\Theta^d(s) = \frac{1}{s^3}$ is finite so, we have that the system is a *type 2 system*.

(c) For the outer lateral dynamics, we have that the error term $e_z(t)$ with the reference input $z^d(t)$ in the transfer model can be written as:

$$E_z(s) = \frac{1}{1-(sk_{D_z}+k_{P_z})\frac{M_V g}{s^3 J_c(sM_V+\mu)}}Z^d(s)$$

Now, for step response, we have $Z^d(s) = \frac{1}{s}$. So, the steady state error is given by:

$$\begin{aligned}
\lim_{t\to\infty} e_z(t) &= \lim_{s\to 0} sE_z(s) \\
&= \lim_{s\to 0} s\frac{1}{1-(sk_{D_z}+k_{P_z})\frac{M_V g}{s^3 J_c(sM_V+\mu)}}\frac{1}{s} \\
&= \lim_{s\to 0} \frac{1}{1-(sk_{D_z}+k_{P_z})\frac{M_V g}{s^3 J_c(sM_V+\mu)}} \\
&= 0
\end{aligned}$$

For ramp response, we have $Z^d(s) = \frac{1}{s^2}$. So, the steady state error is given by:

$$\begin{aligned}
\lim_{t\to\infty} e_z(t) &= \lim_{s\to 0} sE_z(s) \\
&= \lim_{s\to 0} s\frac{1}{1-(sk_{D_z}+k_{P_z})\frac{M_V g}{s^3 J_c(sM_V+\mu)}}\frac{1}{s^2} \\
&= \lim_{s\to 0} \frac{1}{s-(s^2k_{D_z}+sk_{P_z})\frac{M_V g}{s^3 J_c(sM_V+\mu)}} \\
&= \frac{-1}{\lim_{s\to 0}(s^2k_{D_z}+sk_{P_z})\frac{M_V g}{s^3 J_c(sM_V+\mu)}} \\
&= 0
\end{aligned}$$

For parabolic response, we have $Z^d(s) = \frac{1}{s^3}$. So, the steady state error is given by:

$$\begin{aligned}
\lim_{t\to\infty} e_z(t) &= \lim_{s\to 0} sE_z(s) \\
&= \lim_{s\to 0} s\frac{1}{1-(sk_{D_z}+k_{P_z})\frac{M_V g}{s^3 J_c(sM_V+\mu)}}\frac{1}{s^3} \\
&= \lim_{s\to 0} \frac{1}{s^2-(s^3k_{D_z}+s^2k_{P_z})\frac{M_V g}{s^3 J_c(sM_V+\mu)}} \\
&= \frac{-1}{\lim_{s\to 0}(s^3k_{D_z}+s^2k_{P_z})\frac{M_V g}{s^3 J_c(sM_V+\mu)}} \\
&= 0
\end{aligned}$$

Now we note that the steady state error in the input of type $Z^d(s) = \frac{1}{s^4}$ will be finite so, we have that the system is a *type 3 system*.

10. PID Control

The PID controller is implemented in the code files.

1. Some random variations are added in the system parameters. (file: `/src/parameters.py`)

2. The PID controller is implemented and the Integrator for lateral dynamics is tuned to $k_{I_z} = 0.000007$. (file: `/src/controller.py`). The steady state error for step response is given in the following plot.

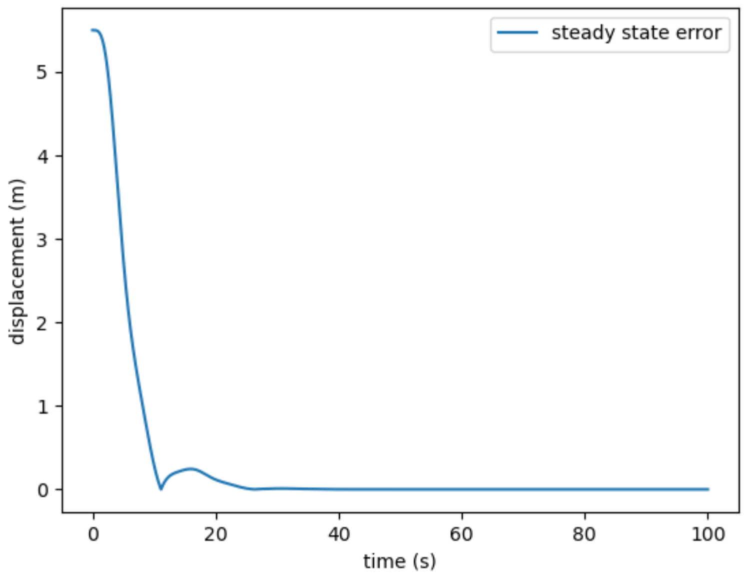

Figure 6: Steady state error for step response

11. Full State Feedback

The state space equations for the longitudinal dynamics of the VTOL are given by (after replacing numerical values):

$$\dot{\tilde{x}}_{lon} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{h} \\ \dot{\tilde{h}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0.384615 \end{pmatrix} \tilde{u}_{lon}$$

$$\tilde{y}_{lon} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{h} \\ \dot{\tilde{h}} \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} \tilde{u}_{lon}$$

Similary, for the lateral dynamics we have:

$$\dot{\tilde{x}}_{lat} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -9.81 & -0.0808 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \\ \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 111.111 \end{pmatrix} \tilde{\tau}$$

$$\tilde{y}_{lat} = \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \\ \dot{\tilde{z}} \\ \dot{\tilde{\theta}} \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} \tilde{\tau}$$

(a) For longitudinal dynamics, the open loop characteristic polynomial is given by:

$$\Delta_{ol,lon} = det(sI - A_{lon}) = s^2$$

So, we have:

$$\mathbf{a}_{A_{lon}} = (0, 0)$$
$$\mathcal{A}_{A_{lon}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, for $\zeta_h = 0.707$ and $\omega_{n_h} = 0.275$ the desired closed loop characteristic polynomial is given by:

$$\Delta^d_{cl,lon} = s^2 + 2\zeta_h\omega_{n_h}s + \omega^2_{n_h} = s^2 + 0.38885s + 0.075625$$

This implies that $\alpha_{lon} = (0.38885, 0.075625)$.

For lateral dynamics, the open loop characteristic polynomial is given by:

$$\begin{aligned}\Delta_{ol,lat} &= det(sI - A_{lat}) \\ &= s^4 - 0.0808s^3\end{aligned}$$

So, we have:

$$\mathbf{a}_{A_{lat}} = (-0.0808, 0, 0, 0)$$
$$\mathcal{A}_{A_{lat}} = \begin{pmatrix} 1 & -0.0808 & 0 & 0 \\ 0 & 1 & -0.0808 & 0 \\ 0 & 0 & 1 & -0.0808 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, for $\zeta_z = \zeta_\theta = 0.707$, $\omega_{n_z} = 0.275$ and $\omega_{n_\theta} = 2.75$ the desired closed loop characteristic polynomial is given by:

$$\begin{aligned}\Delta^d_{cl,lat} &= (s^2 + 2\zeta_z\omega_{n_z}s + \omega^2_{n_z})(s^2 + 2\zeta_\theta\omega_{n_\theta}s + \omega^2_{n_\theta}) \\ &= (s^2 + 0.38885s + 0.075625)(s^2 + 3.8885s + 7.5625) \\ &= s^4 + 4.27735s^3 + 9.15017s^2 + 3.23475s + 0.571914\end{aligned}$$

This implies that $\alpha_{lat} = (4.27735, 9.15017, 3.23475, 0.571914)$.

(b) The matrices are added in the file `/src/parameters.py`.

(c) The controllability matrix for lateral dynamics is given by,

$$\mathcal{C}_{lat} = \begin{pmatrix} B_{lat} & A_{lat}B_{lat} & A^2_{lat}B_{lat} & A^3_{lat}B_{lat} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 111.111 \\ 0. & 111.111 & 0. & 0. \\ 0. & 0. & -1090. & 0. \\ -1090. & 0. & 88.0719 & 0. \end{pmatrix}$$

Now, $det(\mathcal{C}_{lat}) = -1.46678\text{e}10 \neq 0$. Hence, the system is controllable.

The controllability matrix for longitudinal dynamics is given by,

$$\mathcal{C}_{lon} = \begin{pmatrix} B_{lon} & A_{lon}B_{lon} \end{pmatrix} = \begin{pmatrix} 0 & 0.384615 \\ 0.384615 & 0. \end{pmatrix}$$

Clearly, $det(\mathcal{C}_{lon}) \neq 0$, so the system is controllable.

(d) The gains for longitudinal dynamics are given by:

$$K_{lon} = (\alpha_{lon} - \mathbf{a}_{A_{lon}})\mathcal{A}_{A_{lon}}^{-1} C_{lon}^{-1} = (0.196625, 1.01101)$$

For $C_{r,lon} = (1, 0)$ the feedforward reference gain for the longitudinal dynamics is given by:

$$k_{r,lon} = \frac{-1}{C_{r,lon}(A_{lon} - B_{lon}K_{lon})^{-1}B_{lon}} = 1.65428$$

The gains for lateral dynamics are given by:

$$K_{lat} = (\alpha_{lat} - \mathbf{a}_{A_{lat}})\mathcal{A}_{A_{lat}}^{-1} C_{lat}^{-1} = (0.00805788, 0.0855209, -0.00399512, -0.00399831)$$

For $C_{r,lat} = (1, 0, 0, 0)$ the feedforward reference gain for the lateral dynamics is given by:

$$k_{r,lat} = \frac{-1}{C_{r,lat}(A_{lat} - B_{lat}K_{lat})^{-1}B_{lat}} = -0.0021735$$

(e) TODO

12. Full State With Integrator

(a) TODO

(b) TODO

(c) TODO

13. Observer Based Control

(a) *Completed in process.*

(b) For longitudinal dynamics, the observability matrix is given by:

$$\mathcal{O}_{lon} = \begin{pmatrix} C_{lon} \\ C_{lon}A_{lon} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Here, $det(\mathcal{O}_{lon}) \neq 0$ so the system is obeservable.

For lateral dynamics, the observability matrix is given by:

$$\mathcal{O}_{lat} = \begin{pmatrix} C_{lat} \\ C_{lat}A_{lat} \\ C_{lat}A_{lat}^2 \\ C_{lat}A_{lat}^3 \end{pmatrix}$$

Here, it was verified that $det(\mathcal{O}_{lat}) \neq 0$.

(c) TODO

(d) TODO

(e) TODO

14. Disturbance Observer

1. TODO
2. TODO

References

[1] Randal W Beard, Timothy W. McLain, Cammy Peterson, Marc Killpack (2021), "Introduction to Feedback Control using Design Studies", http://controlbook.byu.edu/doku.php