Parul University
Faculty of Engineering and Technology
Department of Applied Science & Humanities
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Subject: Quant and Reasoning (303105311)
Branch: CSE/ IT

Unit-6 Permutation, Combinations and Probability

6.1 Permutations

Introduction

Permutations refer to different arrangements of things from a given lot taken one or more at a time whereas combinations refer to different sets or groups made out of a given lot, without repeating an element, taking one or more of them at a time. The distinction will be clear from the following illustration of combinations and permutations made out of a set of three elements {a,b,c}.

	Combinations	Permutations
(i) one at a time :	{a},{b},{c}	{a},{b},{c}
(ii) two at a time :	{a,b} {b,c} {a,c}	{a,b} {b,a} {b,c} {c,d}{a,c} {c,a}
(iii) Three at a time:	{a,b,c}	{a,b,c} {a,c,b} {b,c,a} {b,a,c} {c,a,b} {c,b,a}

Fundamental principle of Counting

Fundamental Counting Principle can be used to determine the number of possible outcomes when there are two or more characteristics.

(1) Principal of Multiplication: If an event can occur in m different ways, and following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is **m**×**n**.

For example: In a class of 10 Boys & 8 Girls, in how many ways, We can form a Couple (a Boy & a Girl)?
Solution: 10×8 = 80.

(2) Principal of addition: If an event can occur in m different ways, or another event can occur in n different ways, then the total number of occurrence of the events in the given order is **m + n.**

For example: In a class of 10 Boys & 8 Girls, in how many ways, we can form a Monitor (a Boy & a Girl)?
Solution: 10+8=18

REPETITION OF AN EVENT:
If one event with n outcomes occurs r times with repetition allowed then the number of ordered arrangements is n^r

Example:

What is the number of arrangements if a die is rolled? (a) 2 times ? (b) 3 times ? (c) r times ?

Solution: (a) $6 \times 6 = 6^2$ (b) $6 \times 6 \times 6 = 6^3$ (c) $6 \times 6 \times 6 \times \ldots$(r times) $= 6^r$

FACTORIAL REPRESENTATION
Factorial notation is denoted by !. The Product of the first n natural numbers 1,2,3,……,n is called n factorial and it is written as n!.
i.e. n! =1 ×2 ×3…….(n-2)(n-1)n
For example 5! = 5 × 4 × 3 × 2 × 1
Note 0! = 1
Example:
a) In how many ways can 6 people be arranged in a row?
Solution: 6 × 5 × 4 × 3 × 2 × 1 = 6!

SOLVED EXAMPLES

Example 1: Evaluate $\frac{8!}{3!\cdot 5!}$
Solution: $\frac{8!}{3!\cdot 5!} = \frac{8\cdot 7\cdot 6\cdot 5!}{3!\cdot 5!} = 8\cdot 7 = 56$

Example 2: show that $\frac{10!}{8!} = 90$
Solution: $\frac{10!}{8!} = \frac{10\times 9\times 8!}{8!} = 90$

Example 3: If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$
Solution:
$$\frac{9! + 8!}{8! \times 9!} = \frac{x}{10!}$$
$$\Rightarrow \frac{(9 \times 8!) + 8!}{8! \times 9!} = \frac{x}{10!}$$
$$\Rightarrow \frac{8!\,(9 + 1)}{8! \times 9!} = \frac{x}{10!}$$
$$\Rightarrow \frac{10}{9!} = \frac{x}{10 \times 9!}$$
$$\Rightarrow x = 100$$

Solved Examples

Example 1: How many telephone connections can be allowed with 5 and 6 digits from the natural numbers 1 to 9 inclusive?

Solution: As per the rules of counting, the total number of telephone connection can be

$9\times9\times9\times9\times9 = 9^5$

$9\times9\times9\times9\times9\times9 = 9^6$

Example 2: In how many ways a chairman and a vice-chairman of a board of 6 members can occupy their seats?

Solution: Whoever is chosen first, he would be seated in 6 ways and having seated, the other one can be seated in 5 ways because one person cannot hold both the seats.

Therefore, both chairman and the vice-chairman can be seated in $6\times5=30$ways.

Example 3: Three persons go into a railway carriage, where there are 8 seats. In how many ways can they seat themselves?

Solution: Since there are 8 vacant seats, the first men can choose any one of these 8 seats. There are thus 8 ways of filling the first seat, when that one is occupied 7 seats are left , therefore the second men can occupy any one of the 7 seats. The last men can now seat in one of the remaining 6 seats.

Therefore, number of ways in which three persons can occupy 8 seats is $8\times7\times6=336$

Exercise:

1. There are five routes for journey from station A to station B. In how many different ways can a man go from A to B and return, if for returning (i) any of the routes is taken, (ii) the same route is taken, (iii) the same route is not taken.

Ans: (i) 25 (ii) 5 (iii) 20

2. In how many different ways, 3 rings of a lock can combine when each ring has 10 digits 0 to 9? If the lock opens in only one combination of 3 digits how many unsuccessful events are possible?

Ans: 999

Permutations

Permutations: A permutation is an arrangement of objects in a definite order.

Notice: order matters!

To find the number of Permutations of n items, we can use the Fundamental Counting Principle or factorial notation.

Permutation of n different things, taken r at a time is denoted by the symbol P(n,r) or ${}_nP_r$ or ${}^nP_r$.

Theorem 1: To find the number of permutations of n items chosen r at a time, you can use the formula (**when repetition is not allowed**).

${}_{n}P_{r} = \frac{n!}{(n-r)!}$ $where\ 0 \le r \le n.$

Where n=number of objects and r= number of positions

For example, ${}_{5}P_{3} = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$

Solved Examples

Example-1: From a club of 24 members, a President, Vice President, Secretary, Treasurer and Historian are to be elected. In how many ways can the offices be filled?

Solution: There are 24 members. Therefore, n=24.and r=4.

Hence, The Office will be filled in following ways:

${}_{24}P_{5} = \frac{24!}{(24-5)!} = \frac{24!}{19!} = 24 \times 23 \times 22 \times 21 \times 20$

Example-2: Indicate how many 4 digit numbers greater than 7000 can be formed from the digits 3,5,7,8,9.

Solution: If the digits are to be greater than 7000 then the first digit can be any one of the 7,8,and 9.

Now, the first digit can be chosen in 3 ways, because ${}_{3}P_{1}$= 3 and the remaining three digits can be any of the four digits left, which can be chosen in ${}_{4}P_{3}$ ways.

Therefore, the total number of ways = 3 × ${}_{4}P_{3}$ = $3 \times \frac{4!}{1!} = 3 \times 4 \times 3 \times 2 = 72$

Example-3: Find how many four letter words can be formed out of the word LOGARITHMS. (The words may not have any meaning.)

Solution: There are 10 different letters, therefore, n is equal to 10 and since we have to find four letter words, r is 4.

Hence, the required numbers of words are ${}_{10}P_{4}$= $\frac{10!}{(10-4)!} = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7 = 5040.$

NOTE: The number of permutations of *n objects* taken all at a time, denoted by the symbol ${}_{n}P_{n}$ and is given by ${}_{n}P_{n}$ = n!

Theorem-2: When repetition of objects is allowed

The number of permutations of n things taken all at a time, when repetition of objects is allowed is n^n and the number of permutations of n objects, taken r at a time, when repetition of objects is allowed, is n^r .

Solved Examples

Example 1: How Many new Permutations of all letters of the word TUESDAY are possible?

Solution: The Number of arrangements is $_7P_7$= 7!

Now, 7!= 7×6×5×4×3×2×1= 5040

Hence, 5040 arrangements exist.

Example 2: How many 3 letter words with or without meaning can be formed by word NUTS when repetition is allowed?

Solution: Here: n= 4 (no of letters we can choose from)

r = 3 (no of letters in the required word)

Thus, by Theorem 2: $n^r = 4^3 = 64$

Thus, 64 words are possible.

Theorem 3: The number of permutations of n objects or things of which p things are of one kind, q things are of second kind, r things are of third kind and all the rest are different is given by $\frac{n!}{p!\times q!\times r!}$

Solved Examples

Example 1: Find number of permutations of word ALLAHABAD.

Solution: Here total number of word n = 9

Number of repeated A's = p = 4

Number of repeated L's = q = 2

Rest all letters are different.

Thus applying theorem 3,

we have: $\frac{n!}{p!\times q!} = \frac{9!}{4!\times 2!} = \frac{9\times 8\times 7\times 6\times 5\times 4!}{4!\times 2!} = 7560\ ways$

Example 2: How many numbers greater than a million can be formed with the digits 4,5,5,0,4,5,3?

Solution: Each number must consist of 7 or more digits. There are 7 digits in all of which there are 2 four , 3 fives and the rest are different.

Therefore, the total numbers are $\frac{7!}{3!\times 2!} = 420$

Of these numbers, some begin with zero and are less than one million which must be rejected.

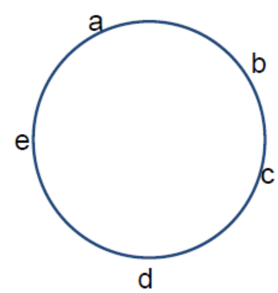
The numbers beginning with zero are $\frac{6!}{3!\times 2!} = 60$
Therefore, the required numbers are 420−60=360

Example 3: Find the number of arrangements that can be made out of the letters of the word "ASSASSINATION".
Solution: There are 13 letters in the word of which A occurs thrice, S occurs four times, I occurs twice and N occurs twice and rest are all different. Hence, the required number of arrangements is

$$\frac{13!}{3! \times 4! \times 2! \times 2!}$$

4.5 Circular Permutations

Circular permutations are related with arrangment in which objects are arranged in a **circle**. Consider arranging 5 objects (a, b, c, d, e) around a circular table. The arrangements abcde, bcdea, cdeab, deabc, eabcd are different in a line, but are identical around a circle.



To calculate the number of ways in which n objects can be arranged in a circle, we arbitrarily fix the position of one object, so the remaining (n-1) objects can be arranged as if they were on a straight line in (n-1)! ways.
i.e. the number of arrangements = (n – 1) ! in a circle.

<u>**Solved Examples**</u>

Example-1: In how many ways 5 boys and 5 girls be seated around a table so that 2 boys are adjacent.

Solution: Let the girls seated first. They can sit in 4! Ways.

Now since the place for the boys in between girls are fixed. The option is therefore the boys to occupy the remaining 5 places.

There are 5! Ways for the boys to occupy the remaining 5 places.

Thus, the total number of ways in which both girls and boys can seated such that no 2 boys are adjacent are 4!×5!=2880ways.

Example -2: At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if: a) there are no restrictions, b) men and women alternate, C) Ted and Carol must sit together?
Solution: (a) (12 – 1)! = 11! (b) (6 – 1)! × 6! = 5! × 6! (c) (TC) × other 10 = 2! × 10!

REMARK: Number of arrangements of n beads for forming a necklace is $\frac{(n-1)!}{2}$
(In case of necklace, anticlockwise and clockwise arrangements are same.)

Example 3: In how many ways can 8 differently colored beads be threaded on a string?
Solution: As necklace can be turned over, clockwise and anti-clockwise arrangements are the same.

$$\frac{(n-1)!}{2} = \frac{(8-1)!}{2} = \frac{7!}{2} = 2520$$

6.2 Combinations

A combination is an arrangement of items in which order does not matter.
NOTICE: order does not matter!
Since the order does not matter in combinations, there are fewer combinations than Permutations. The combinations are a “subset” of the permutations.

To find the number of combinations of n items chosen r at a time, you can use the formula:

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

An alternative (and more common) way to denote an r-combination is $\binom{n}{r}$and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Solved Examples

Example-1: Find the value of $\binom{6}{4}$.

Solution: $\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!\times 2!} = \frac{6\times 5}{2} = 15$

Example 2: A student must answer 3 out of 5 questions on a test. In how many different ways can student select the questions?

Solution: The required number of ways

$$\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5!}{3! \times 2!} = \frac{5 \times 4}{2} = 10$$

Example 3: In How many ways 4 white and 3 black balls be selected from a box containing 20 white and 15 black balls.

Solution:

(i) 4 out of 20 white balls can be selected in

$$\binom{20}{4} = \frac{20!}{4!\,(20-4)!} = \frac{20!}{4! \times 16!} = \frac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2} = 4845\,ways$$

(ii) 3 out of 15 white balls can be selected in

$$\binom{15}{3} = \frac{15!}{3!\,(15-3)!} = \frac{15!}{3! \times 12!} = \frac{15 \times 14 \times 13}{3 \times 2} = 455\,ways$$

Therefore, the two processes can be carried out together so in 4845×455 =22,04,475 ways.

Example 4: A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Mathematics Part II, unless Mathematics Part I is also borrowed. In how many ways can he choose the three books to be borrowed?

Solution: Let us make the following cases:
Case (i) Boy borrows Mathematics Part II, then he borrows Mathematics part I also. So the number of possible choices is ${}_6C_1$ = 6.
Case (ii) Boy does not borrow Mathematics Part II, and then the number of possible choices is ${}_7C_3$= 35.

Hence, the total number of possible choices is ${}_6C_1$ +${}_7C_3$ = 35 + 6 = 41.

Example 5: From 6 boys and 4 girls, 5 are to be selected for admissions for a particular course. In how many ways can't this be done if there must be exactly 2 girls?

Solution: There has to be exactly 2 girls and there should be 3 boys , the possible combination would be ${}_4C_2$ ×${}_6C_3$=120 ways.

Example 6: A party of 3 ladies and 4 gentlemen is to be formed from 8 ladies and 7 gentlemen. In how many different ways can the party be formed if Mrs.X and Mr. Y refuse to join the same party?
Solution: 3 ladies can be selected out of 8 ladies in 8C3 ways and 4 gentlemen can be selected out of 7 gentlemen in ${}_7C_4$ ways.

The number of ways of choosing the committee = ${}_8C_3 \times {}_7C_4$ = 1960 ways .
If both Mrs. X and Mr.Y are members, there remain to be selected 2 ladies from 7 ladies and 3 gentlemen from 6 gentlemen.
This can be done in = ${}_7C_2 \times {}_6C_3$ = 420 ways
The number of ways of forming the party in which Mrs.X and Mr.Y refuse to join is
= 1960-420
= 1540

EXERCISE:

1. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

2. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

3. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.

4. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

5. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

6. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

6.3 Probability

Introduction:
The 'Probability' or 'chance' is very commonly used in day-to-day life and generally people have a vague idea about its meaning.
For example statements like
(1) probably it may rain tomorrow.
(2) The chances of teams A and B winning a certain match is equal. And so on.

Basic Concept in Probability:

Experiment: The term experiment refers to describe an act which can be repeated under some given conditions.

Random Experiment or Trial: If in an experiment all possible outcomes are known in advance and none of the outcomes can be predicted with certainty, then such an experiment is called random experiment.

E.g.: Given any triangle, without knowing the three angles, we can definitely say that the sum of measure of angles is 180°.

An experiment is called random experiment if it satisfies the following two conditions: (i) It has more than one possible outcome. (ii) It is not possible to predict the outcome in advance with certainity.

Sample space: The set consisting of all outcomes of a random experiment is called sample space.
E.g.:
Sample space of tossing a coin once is {H, T},
Sample space of throwing a dice is {1, 2, 3, 4, 5, 6}.

Types of events

Certain event: An event whose occurrence is inevitable (or certain) is called a certain event.

Impossible event: An event whose occurrence is impossible is called impossible event.

To understand these let us consider the experiment of rolling a die. The associated sample space is S = {1, 2, 3, 4, 5, 6}

Let E be the event "the number appears on the die is a multiple of 7". Can you write the subset associated with the event E?
Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensures the occurrence of the event E. Thus, we say that the empty set only correspond to the event E.
In other words we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, **the event E = ∅ is an impossible event.**

Now let us take up another event F "the number turns up is odd or even". Clearly
F = {1, 2, 3, 4, 5, 6,} = S, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, **the event F = S is a sure event.**

Mutually exclusive events: Two events are said to be mutually exclusive or incompatible when both cannot happen simultaneously in a single trial.

Independent events: Two or more events are said to be independent when the outcome of one does not affect and is not affected by the other.

Dependent events: Dependent events are those events in which the occurrence or nonoccurrence of one event in any one trial affects the probability of other events in other trials.

Equally likely events: Events are said to be equally likely when one does not occur more often then the others.

Exhaustive events: Events are said to be exhaustive when their totality includes all the possible outcomes of a random experiment.

Mathematical or Classical approach:

If an experiment results in ‘n’ equally likely ways, and out of which ‘m’ are favorable to the happening of the event A, then the probability of happening of event A is defined as the ration of m/n.

$$P(A) = \frac{Number of favorable cases}{Total number of equally likely cases} = \frac{m}{n}$$

$$0 \le m \le n$$

$$0 \le \frac{m}{n} \le 1$$

$$0 \le P(A) \le 1$$

Thus the probability of an event is always between 0 and 1.
The probability of a certain event is always 1 and that of impossible event is 0.
If ‘m’ cases are favorable to the happening of event A, then ‘n-m’ are the cases not favorable to the happening of A i.e. favorable to the happening of complement of event A.

$$P(A') = 1 - \frac{m}{n}$$

$$P(A') = 1 - P(A)$$

$$P(A) + P(A') = 1$$

Modern or Axiomatic definition of Probability:

If A is any event from the sample space S, then (▯) is called the probability of A if it satisfies the following axioms:

i. $0 \le P(A) \le 1$
ii. $P(S) = 1$

iii. If A and B are mutually exclusive events then the probability of occurrence of either A or B is denoted by (▯ ∪ ▯)
▯(▯ ∪ ▯) =▯(▯) + ▯(▯)

Example-1:
A card is drawn at random from a well shuffled pack of 52 cards. Find the probability of getting
(i) a jack or a queen or a king,
(ii) a two of heart or diamond.
Solution:
(i) In a pack of 52 cards, we have: 4 jacks, 4 queens and 4 kings.

Now, clearly a jack and a queen and a king are mutually exclusive events.

$$\text{Also, } P(\text{a jack}) = \frac{{}^{4}C_1}{{}^{52}C_1} = \frac{4}{52} = \frac{1}{13}$$

$$P(\text{a queen}) = \frac{{}^{4}C_1}{{}^{52}C_1} = \frac{4}{52} = \frac{1}{13}$$

$$P(\text{a king}) = \frac{{}^{4}C_1}{{}^{52}C_1} = \frac{4}{52} = \frac{1}{13}.$$

By the addition theorem of Probability,

P(a jack or a queen or a king) = *P*(a jack) + *P*(a queen) + *P*(a king)

$$= \frac{1}{13} + \frac{1}{13} + \frac{1}{13} = \frac{3}{13}.$$

(ii) *P*(two of heart or two of diamond)

= *P*(two of heart) + *P*(two of diamond)

$$= \frac{1}{52} + \frac{1}{52} = \frac{2}{52} = \frac{1}{26}.$$

Example-2:

Find the probability of getting a sum of 7 or 11 in a simultaneous throw of two dice.

Solution:

When two dice are thrown we have observed that there are 36 possible outcomes.

Now, we can have a sum of 7 as

the six favourable cases are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1).

$$P(\text{a sum of } 7) = \frac{6}{36} = \frac{1}{6}.$$

Again, the favourable cases of getting a sum of 11 are (5, 6), (6, 5).

$$P(\text{a sum of } 11) = \frac{2}{36} = \frac{1}{18}$$

Since the events of getting ‘a sum of 7’ or ‘a sum of 11’ are mutually exclusive:

∴ *P*(a sum of 7 or 11) = *P*(a sum of 7) + *P*(a sum of 11)

$$= \frac{1}{6} + \frac{1}{18} = \frac{4}{18} = \frac{2}{9}.$$

Exercise

1. In a simultaneous toss of 2 coins, then find the probability of 2 tails.

(a) 1/2

(b) **1/4**

(c) 3/4

(d) 1/3

2. A coin is tossed 3 times. Find the chance that head and tail show alternately.

(a) 3/8
(b) **1/4**
(c) 1/8
(d) None of these

3. 2 dice are thrown. Find the probability of getting an odd number on 1 and a multiple of 3 on the other.
(a) 5/6
(b) 25/36
(c) 11/36
(d) 1/9

4. A card is drawn from a pack of 100 cards numbered 1 to 100. Find the probability of drawing a number which is a square.
(a) 1/10
(b) 9/10
(c) 1/5
(d) 2/5

5. What is the probability that 1 card drawn at random from the pack of playing cards may be either a queen or an ace?
(a) 1/13
(b) 2/13
(c) 3/13
(d) None of these

6. A box contains 36 tickets numbered 1 to 36, 1 ticket drawn at random. Find the probability that the number on the ticket is either divisible by 3 or is a perfect square.
(a) 2/9
(b) 4/9
(c) 5/9
(d) 1/3

7. A bag contains 3 red balls, 5 yellow balls and 7 pink banks. If one ball is drawn at random from the bag, what is the probability that it is either pink or red?
(a) 1/7
(b)2/3
(c) 4/9
(d) 5/7

8. Out of 5 girls and 3 boys, 4 children are to be randomly selected for a quiz contest. What is the probability that all the selected children are girls?
(a) 1/14
(b) 1/7
(c) 5/17
(d) 2/17