Lecture 30: The Trapezium Rule and Fourier Coefficients

In this lecture we start to introduce the *discrete Fourier transform* (DFT). The *Fourier series* is an expansion

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f_k} e^{ik\theta}$$

where

$$\hat{f_k} \triangleq \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta$$

We will explore calculating these coefficients using the Trapezium rule. Note that the Trapezium rule for periodic functions satisfies (since $f(2\pi) = f(0)$)

$$\frac{2\pi}{n} \left[\frac{f(0)}{2} + \sum_{j=1}^{n-1} f(\theta_j) + \frac{f(2\pi)}{2} \right] = \frac{2\pi}{n} \sum_{j=1}^{n} f(\theta_j)$$

where $\theta_j = hj = \frac{2\pi j}{n}$. Thus define

$$Q_n[f] \triangleq \frac{2\pi}{n} \sum_{j=1}^{n} f(\theta_j)$$

$$f_k^n \triangleq \frac{1}{2\pi} Q_n[f(\theta) e^{-ik\theta}] = \frac{1}{n} \sum_{j=1}^{n} f(\theta_k) e^{-ik\theta_j}$$

Observations about the first coefficient

The key to the DFT is based on the following observation. Consider $f(\theta) = cos 8\theta$ and the first coefficient:

$$f_0^n = \frac{1}{2\pi} Q_n[f]$$

we want this to approximate

$$\hat{f_0} = \frac{1}{2\pi} \int_0^{2\pi} \cos 8\theta d\theta = \frac{\sin 8 * 2\pi - \sin 0}{8} = 0.$$

Firsst define our Trapezium rule:

In [1]:

```
function trap(f,a,b,n)
    h=(b-a)/n
    x=linspace(a,b,n+1)

    v=f(x)
    h/2*v[1]+sum(v[2:end-1]*h)+h/2*v[end]
end
```

Out[1]:

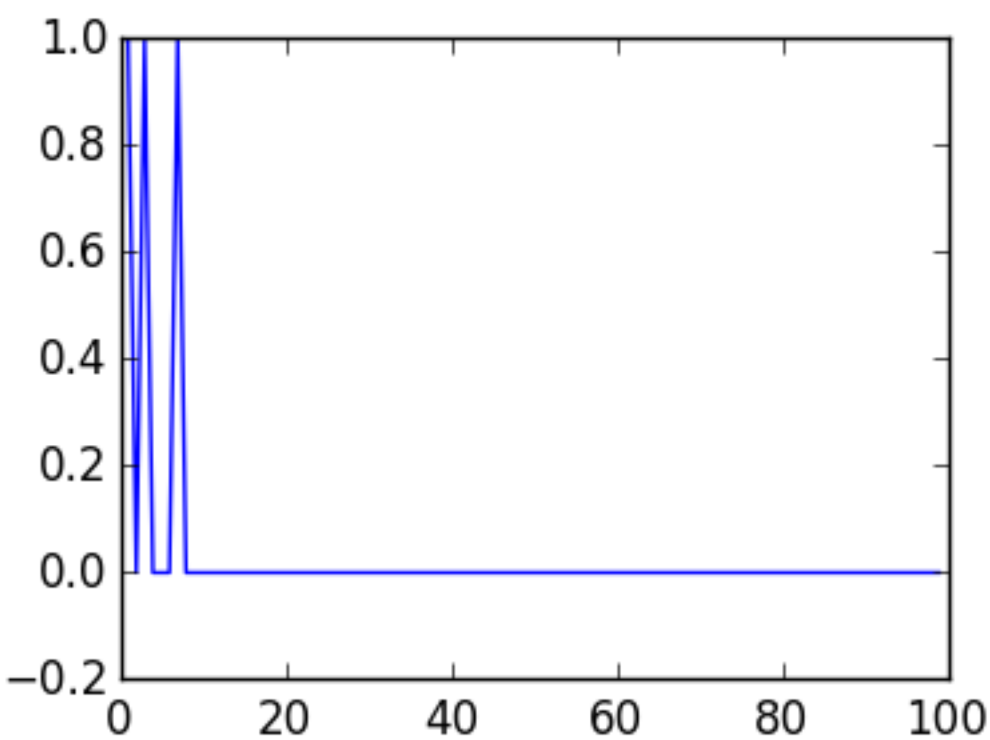
```
trap (generic function with 1 method)
```

Depending on the choice of n, we either get the exact result, or 1.

In [4]:

```
using PyPlot
vals=Float64[trap(θ->cos(8θ),0.,2π,k)/(2π) for k=1:100]

plot(vals);
```



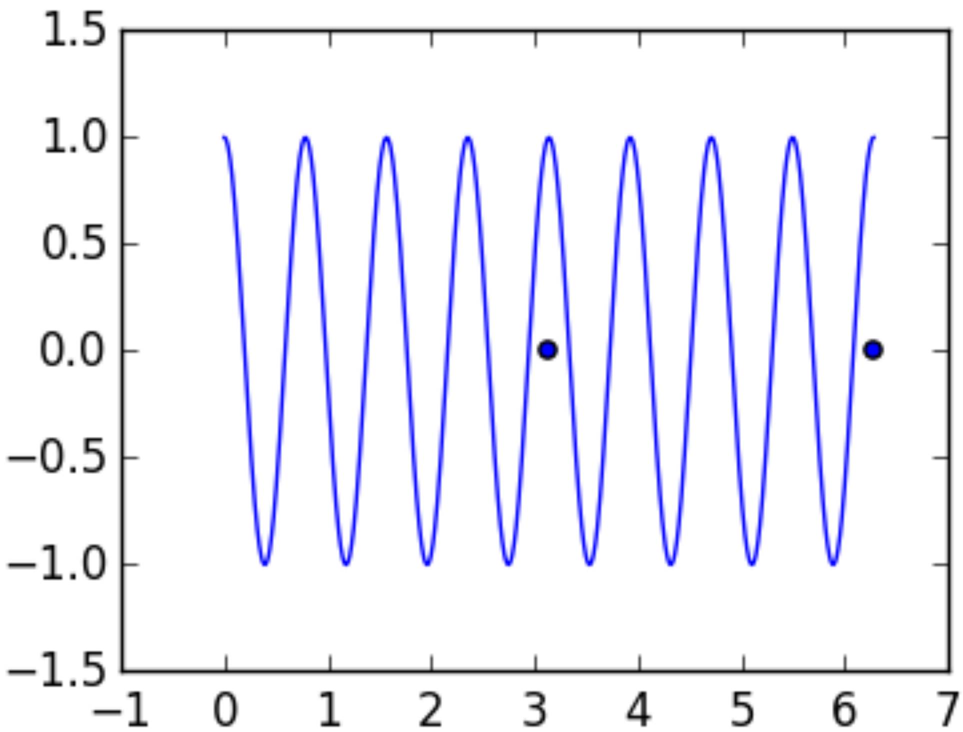
Before proving this, we do an experiment to get intuition. Suppose we take two points. Then we are sampling the function at precisely the peaks of $\cos 8\theta$:

In [8]:

```
g=linspace(0.,2π,1000)

# plot the function
plot(g,cos(8g))

# plot the grid
n=2
h=2π/n
scatter(linspace(h,2π,n),zeros(n));
```

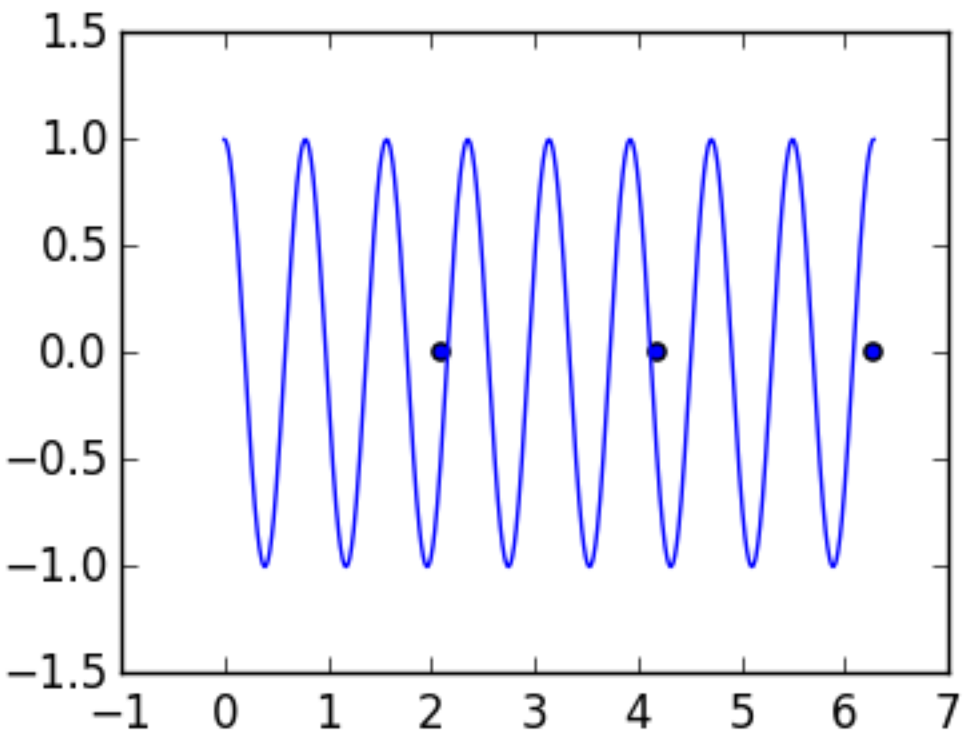


Thus $\cos 8\theta$ looks exactly like 1 at the sample points, and its returning 1. For $n = 3$, on the other hand, we cancel and get zero:

In [10]:

```
# plot the function
plot(g,cos(8g))

# plot the grid
n=3
h=2π/n
scatter(linspace(h,2π,n),zeros(n));
```

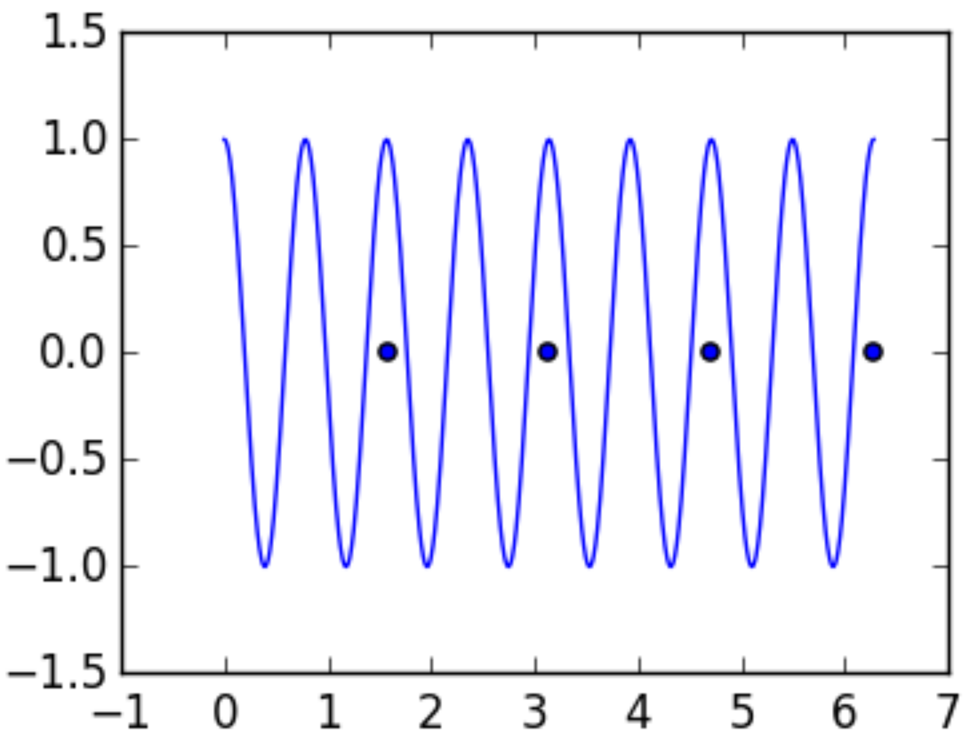


But $n = 4$ hits the peaks just perfectly, and again we get 1:

In [11]:

```
# plot the function
plot(g,cos(8g))

# plot the grid
n=4
h=2π/n
scatter(linspace(h,2π,n),zeros(n));
```



Proving that we get 1 for this example when we hit the frequencies just right is straightforward and intuitive. Explaining why otherwise we get exactly zero requires some effort.

A simple sum of exponentials

Consider

$$\frac{1}{2\pi} Q_n[e^{ik\theta}] = \sum_{j=1}^{n} e^{ik\theta_j}$$

for $\theta_j = \frac{2\pi}{n} j$. We'll show the following result that will explain the phenomenon observed above:

Theorem

$$\sum_{j=1}^{n} e^{ik\theta_j} = \begin{cases} n & k = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Proof

Consider $\omega \triangleq e^{i\theta_1} = e^{\frac{2\pi i}{n}}$. This is an nth root of unity: $\omega^n = 1$. Note that $e^{i\theta_j} = e^{\frac{2\pi i j}{n}} = \omega^j$.

(Case 1) Suppose k is a multiple of n, that is, $k = Mn$ for an integer M. Then we have

$$\sum_{j=1}^{n} e^{ik\theta_j} = \sum_{j=1}^{n} \omega^{kj} = \sum_{j=1}^{n} (\omega^{Mn})^j = \sum_{j=1}^{n} 1 = n$$

(Case 2) Recall that

$$\sum_{j=0}^{n-1} z^j = \frac{z^n - 1}{z - 1}.$$

Then we have

$$\sum_{j=1}^{n} e^{ik\theta_j} = \sum_{j=1}^{n} (\omega^k)^j = \frac{\omega^{kn} - 1}{\omega^k - 1} = 0.$$

∎

Corollary

$$\frac{1}{2\pi} Q_n[e^{ik\theta}] = \begin{cases} 1 & k = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \text{otherwise} \end{cases}$$

We can thus explain the previous observation: for $f(\theta) = \cos 8\theta$, we have

$$f^{n}_{0} = \frac{1}{2\pi} Q_n[cos 8\theta] = \frac{1}{2\pi} Q_n[\frac{e^{i8\theta} + e^{-i8\theta}}{2}] = \frac{1}{4\pi} Q_n[e^{i8\theta}] + \frac{1}{4\pi} Q_n[e^{-i}$$

$$= \begin{cases} 1 & 8 \text{ is a multiple of } n \\ 0 & \text{otherwise} \end{cases}$$

Next lecture we will use this approach to deduce a very simple formula for f^{n}_{k} in terms of the true Fourier coefficients.