Lecture 5: Rounding and arithmetic in IEEE Floating

Before we begin, we need to setup printing floating point numbers in colour.

In [2]:

```
printred(x)=print("\x1b[31m"*x*"\x1b[0m")
printgreen(x)=print("\x1b[32m"*x*"\x1b[0m")
printblue(x)=print("\x1b[34m"*x*"\x1b[0m")

function printbits(x::Float16)
       bts=bits(x)
       printred(bts[1:1])
       printgreen(bts[2:7])
       printblue(bts[8:end])
end

function printbits(x::Float32)
   bts=bits(x)
    printred(bts[1:1])
    printgreen(bts[2:2+8-1])
    printblue(bts[2+8:end])
end

function printbits(x::Float64)
   bts=bits(x)
    printred(bts[1:1])
    printgreen(bts[2:2+11-1])
    printblue(bts[2+11:end])
end
```

Out[2]:

```
printbits (generic function with 3 methods)
```

Let's start with the example from last lecture. We want to understand how numbers are rounded, by looking at rounding a `Float64` to a `Float32`. Note that putting `f0` at the end of a number forces it to be a `Float32`. Therefore, `1.3f0` is equivalent to calling `Float32(1.3)`.

In [4]:

```
printbits(1.3) # Float64
```

```
0011111111110100110011001100110011001100110011001100110011001101
```

In [5]:

```
printbits(1.3f0) #Float32
```

00111111101001100110011001100110

Note that the last two bits are different. This is because the number has been rounded to the nearest Float32. Because the exponents of both 1.3 and 1.3f0. Both 1.3 and 1.3f0 have the same exponent:

In [9]:

```
exponent(1.3),exponent(1.3f0)
```

Out[9]:

(0,0)

We can therefore focus on the significands. The following commands get the significands of the two numbers:

In [10]:

```
x=1.3
str=bits(1.3)
bts64=str[13:end]  # lets get the bits for the significand.  This uses the
                   # `end` keyword for getting all the characters of a strin
g
                   # up to the last one
```

Out[10]:

"0100110011001100110011001100110011001100110011001101"

In [11]:

```
x=1.3
str=bits(Float32(1.3))
bts32=str[10:end]  # lets get the bits for the significand.  This uses the
                   # `end` keyword for getting all the characters of a strin
g
                   # up to the last one
```

Out[11]:

"01001100110011001100110"

A brief aside: `parse` interprets a string as a number in base-2, *not* a sequence of bits. For example, a negative number is given using a minus sign:

In [15]:

```
parse(Int64,"-111111111101001100110011001100110011001100110011001100110011001101
",2)
```

Out[15]:

```
-4608533498688228557
```

Let's create a new Float64 by writing bits directly, and see how it is rounded when it becomes a `Float32`. We begin with the bits of `1.3`. The following parses the bts as an unsigned 64-bit integer, then reinterpret the bits as a `Float64`, creating a `Float64` called `x64` with the precise bits specified in the original string. We then round it to a `Float32` called `x32`. Checking the bits of `x32` shows the result of rounding.

In [93]:

```
# this is the 32 bit string, aligned so that the significands are in the sam
e column
       "00111111101001100110011001100110"
str="0011111111110100110011001100110011001100110011001100110011001101"

bts=parse(UInt64,str,2)
x64=reinterpret(Float64,bts)
x32=Float32(x)
bits(x32)
```

Out[93]:

```
"00111111101001100110011001100111"
```

If we change the first bit after where the significand is truncated to 1, it now rounds up to the nearest `Float32`:

In [95]:

```
# this is the old 32 bit string, aligned so that the significands are in the
 same column
       "00111111101001100110011001100110"
str="0011111111110100110011001100110011011100110011001100110011001101"

bts=parse(UInt64,str,2)
x=reinterpret(Float64,bts)
x32=Float32(x)
bits(x32)
```

Out[95]:

```
"00111111101001100110011001100111"
```

We are rounding to the nearest representable `Float32`. But if all the bits are zero, there is no longer a unique nearest `Float32`. The default behaviour is to round to the nearest `Float32` that has a zero in the last bit:

In [96]:

```
"00111111101001100110011001100110"
str="0011111111110100110011001100110011001101000000000000000000000000"

bts=parse(UInt64,str,2)
x=reinterpret(Float64,bts)
bits(Float32(x))
```

Out[96]:

```
"00111111101001100110011001100110"
```

The default rounding mode can be changed:

In [97]:

```
with_rounding(Float32,RoundUp) do
    bits(Float32(1.3) )
end
```

Out[97]:

```
"00111111101001100110011001100111"
```

In [98]:

```
with_rounding(Float32,RoundDown) do
    bits(Float32(1.3) )
end
```

Out[98]:

```
"00111111101001100110011001100110"
```

In [100]:

```
with_rounding(Float32,RoundNearest) do
    bits(Float32(1.3) )
end
```

Out[100]:

```
"00111111101001100110011001100110"
```

In [101]:

```
with_rounding(Float32,RoundToZero) do
    bits(Float32(1.3) )
end
```

Out[101]:

```
"00111111101001100110011001100110"
```

A real number can have an infinite number of digits to represent exactly. Define the operation that takes a real number to its `Float64` representation as `round`.

The Arithmetic operations '+', '-', '*', '/' are defined by the property that they are exact up to rounding. That is, if `x` and `y` are `Float64`, we have

$$x \oplus y = \text{round}(x + y)$$

where in this formula $\oplus$ denotes the floating point definition of + and + denotes the mathematical definition of +.

This has some bizarre effects. For example, 1.1+0.1 gives a different result than 1.2:

In [110]:

```
x=1.1
y=0.1
x+y-1.2
```

Out[110]:

```
2.220446049250313e-16
```

This is because $\text{round}(1.1) \neq 1 + 1/10$, but rather:

$$\text{round}(1.1) = 1 + 2^{-4} + 2^{-5} + 2^{-8} + 2^{-9} + \cdots + 2^{-48} + 2^{-49} + 2^{-51} = \frac{2476979795053773}{2251799813685248} =$$

In [111]:

```
printbits(1.1)
```

```
0011111111110001100110011001100110011001100110011001100110011010
```

In [112]:

```
printbits(x)
```

```
0011111111110001100110011001100110011001100110011001100110011010
```

In [113]:

```
printbits(y)
```

```
0011111110111001100110011001100110011001100110011001100110011010
```

In [114]:

```
printbits(x+y)
```

```
0011111111110011001100110011001100110011001100110011001100110100
```

In [115]:

```
printbits(1.2)
```

```
0011111111110011001100110011001100110011001100110011001100110011
```

Cancellation and calculating derivatives

How do I calculate $f'(0)$? The definition

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

tells us that

$$f'(0) \approx \frac{f(h) - f(0)}{h}$$

provided that h is sufficiently small.

Let's try sin(x)

In [118]:

```
h=0.000001
y=abs(sin(h)./h-1)
```

Out[118]:

```
1.66644475996236e-13
```

We can do a plot to see how fast the error goes down as we let h become small.

Here, we use the notation `x:st:y` to denote a range of numbers from `x` to `y` in steps of size `st`. So `0:-1:-10` is code for `0,-1,-2,…,-10`. Similarly, 1:2:4 is code for `1,3`.

We then use `10.0.^(0:-1:-10)` to create vector `h` whose entries are `1,.1,.01,…,1E-10`. The `.^` syntax is used instead of `^` because, mathematically, a number raised to a vector is not defined. The `.^` syntax means perform the operation entrywise.

In [121]:

```
h=10.0.^(0:-1:-10)
```

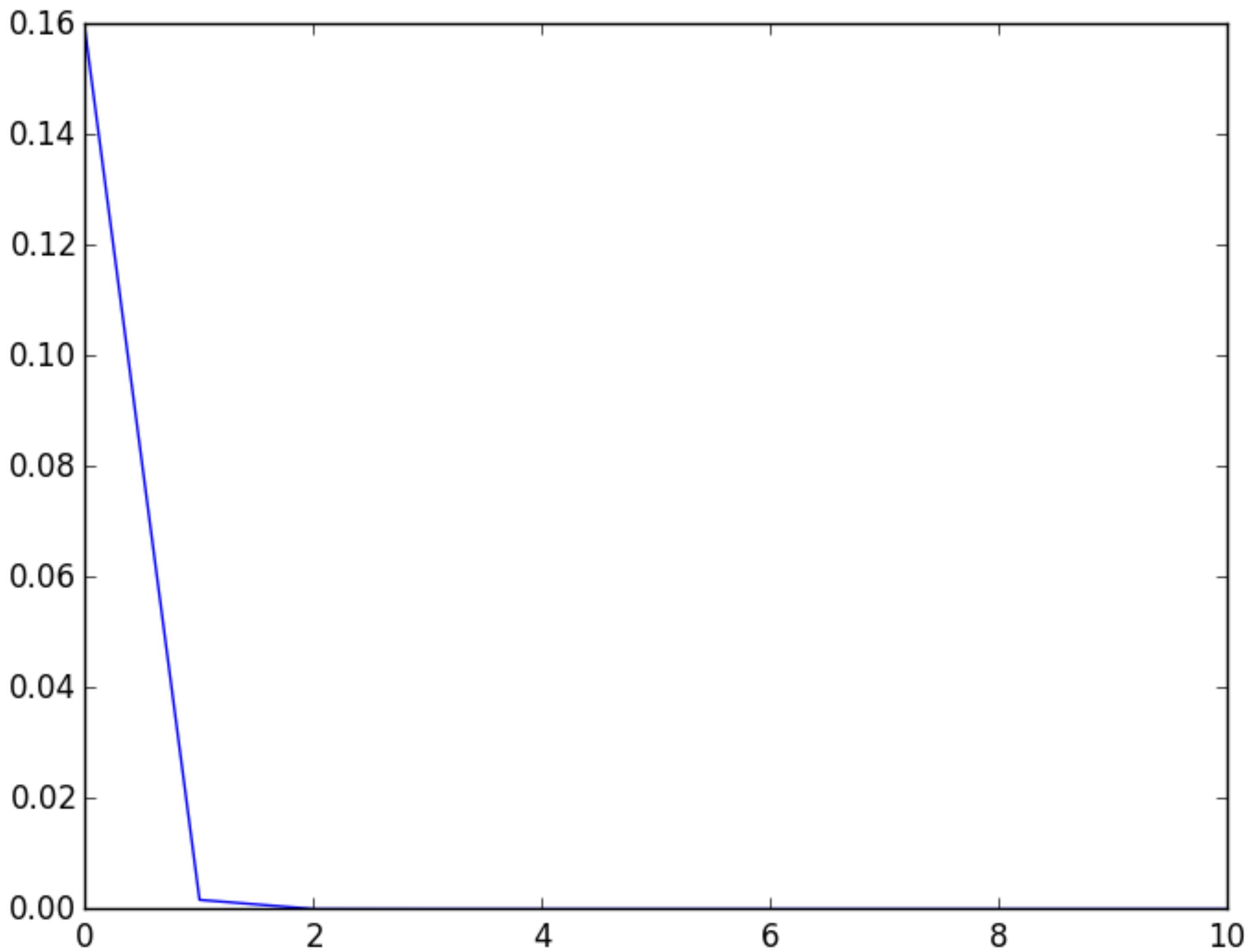
Out[121]:

```
11-element Array{Float64,1}:
 1.0
 0.1
 0.01
 0.001
 0.0001
 1.0e-5
 1.0e-6
 1.0e-7
 1.0e-8
 1.0e-9
 1.0e-10
```

We now use `PyPlot` to plot the result.

In [123]:

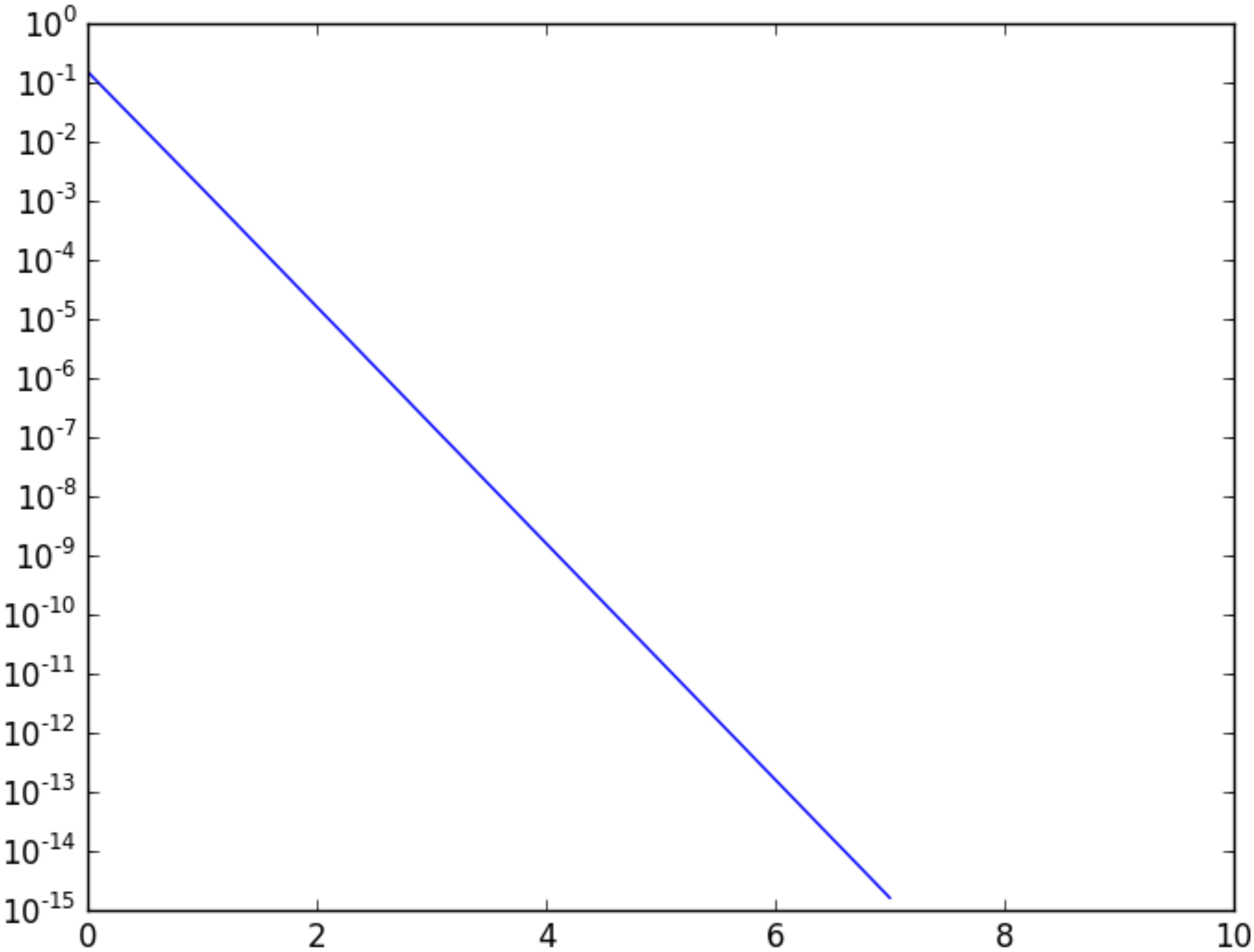
```
using PyPlot

h=10.0.^(0:-1:-10)
y=abs(sin(h)./h-1)
plot(y);
```



The error decays too fast to interpret the plot. Instead, we use `semilogy`, which creates a plot whose y-axis is scalled logarithmically.

In [125]:

```
h=10.0.^(0:-1:-10)
y=abs(sin(h)./h-1)
semilogy(y);
```



So far so good, but now let's try $\exp(x)$:

In [126]:

```
h=0.0000000000001
(exp(h)-1)/h
```

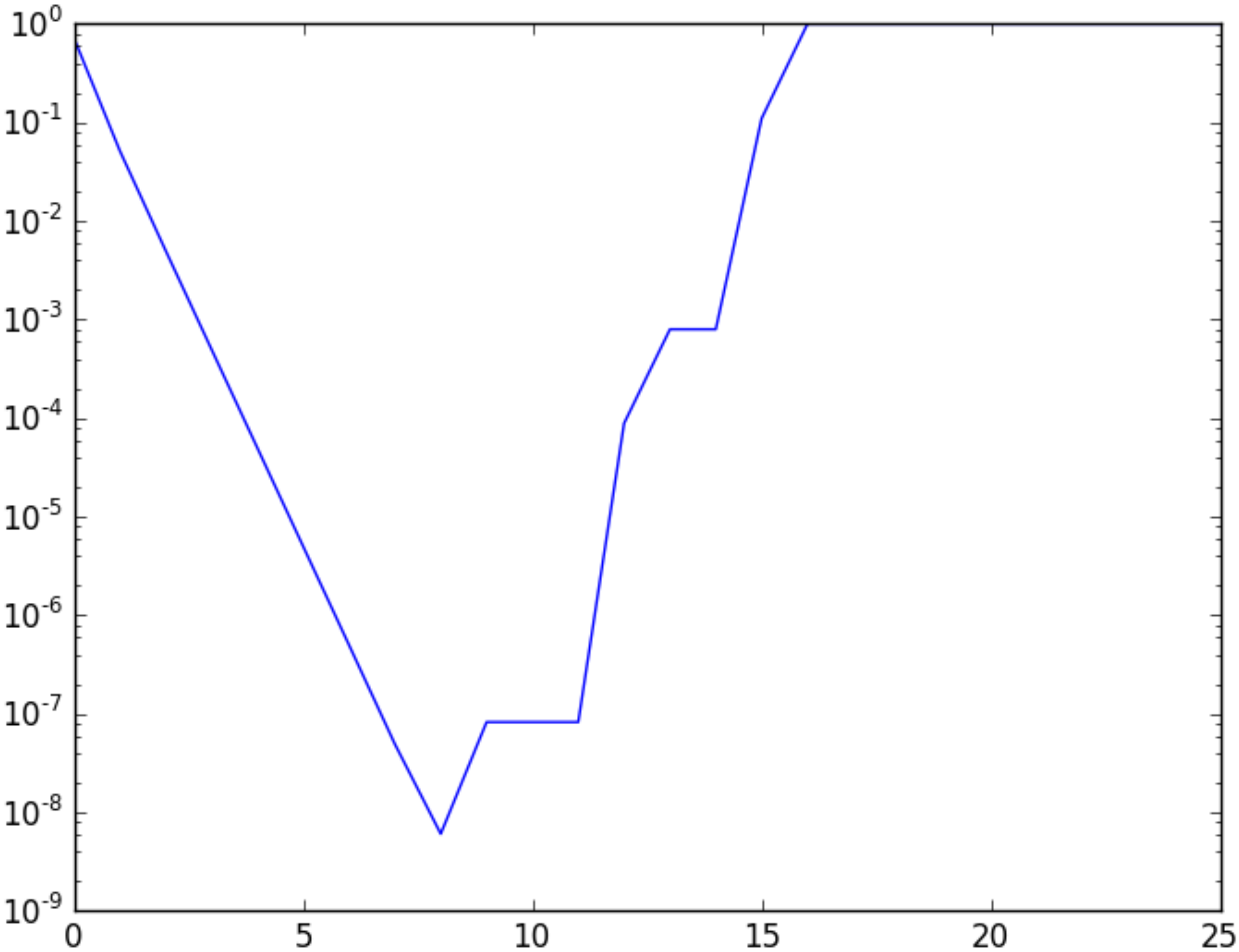
Out[126]:

```
0.9992007221626409
```

The error is much higher than expected!

In [127]:

```
h=10.0.^(0:-1:-25)
y=abs((exp(h)-1)./h-1)
semilogy(y);
```



This phenomenon is caused by the effect of rounding when subtracting off two numbers that are close in magnitude, and will be investigated in a future lab.