Lecture 34: Discretizing linear ODEs

Our approach to solving a linear ODE like

$$u' - a(t)u = f(t), u(0) = c$$

is to replace this infinite-dimensional equation by a finite-dimensional linear system. To do this systematic, we will replicate the procedure of the trapezium rule.

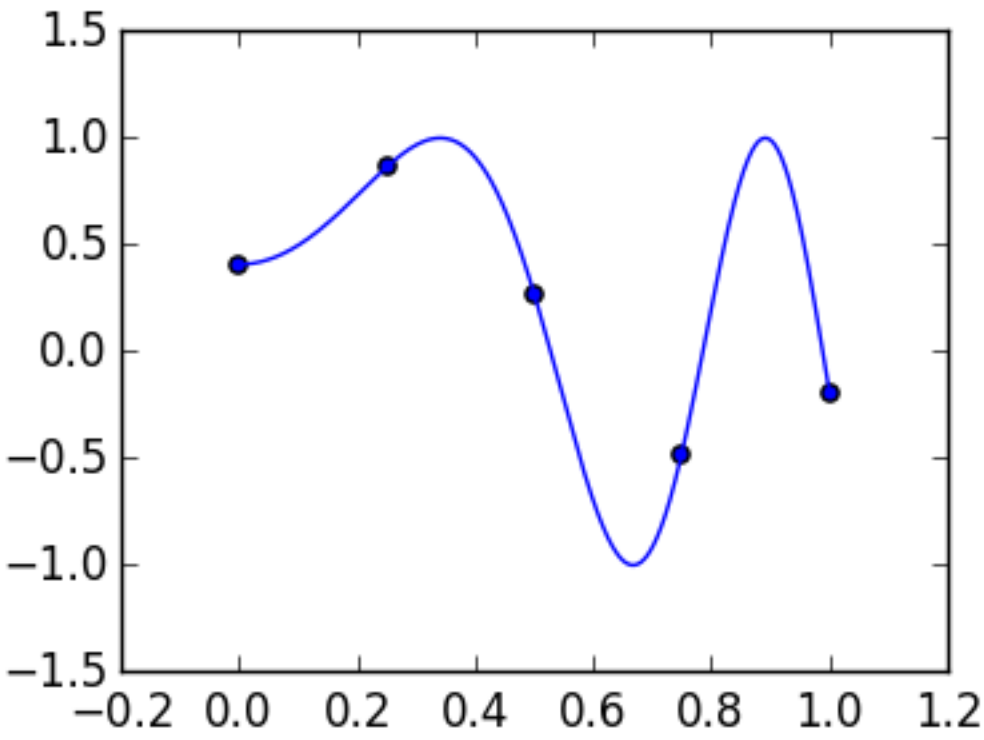
Trapezium rule revisited

The trapezium rule can be thought of as a system of transformations:

1) Discretize: $f(x) \mapsto [f(x_0), \ldots, f(x_n)]^\top$

In [30]:

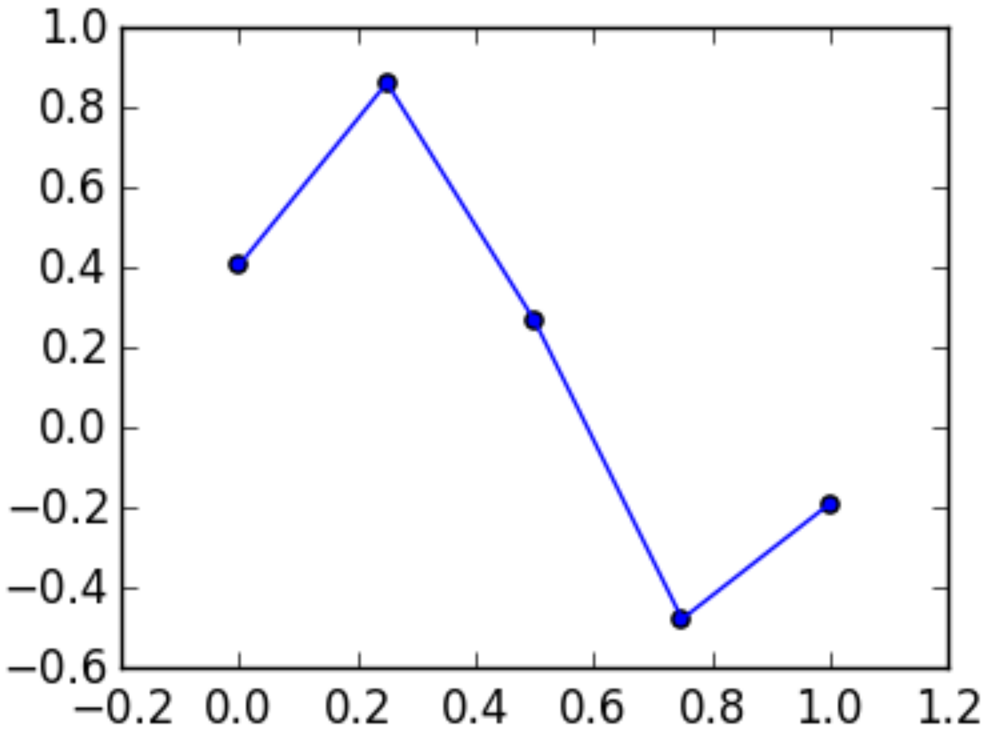
```
f=x->cos(20cos(x))
n=4
x=linspace(0.,1.,n+1)
vals=f(x)

g=linspace(0.,1.,1000)

plot(g,f(g))
scatter(x,vals);
```



2) Interpolate: $[f(x_0), \ldots, f(x_n)]^\top \mapsto p(x)$ where p is piecewise affine

In [31]:

```
scatter(x,vals)
plot(x,vals);
```



3) Integrate: Calculate $\int_0^1 p(x)dx$ exactly

Discretizing derivatives

We want to replicate this procedure to construct an approximation to the derivative of $f(x)$. Note that steps 1 and 2 are the same as before.

1) Discretize: $f(x) \mapsto [f(x_0), \ldots, f(x_n)]^\top$

2) Interpolate: $[f(x_0), \ldots, f(x_n)]^\top \mapsto p(x)$ where p is piecewise affine

3) Differentiate: $p(x) \mapsto p'(x)$ where $p'(x)$ is the *exact* derivative of $p(x)$

Note that in each panel we have

$$p(x) = f(x_k) + x\frac{f(x_{k+1}) - f(x_k)}{h} \qquad \text{for} \qquad x_k \leq x \leq x_{k+1}.$$

Therefore,

$$p'(x) = \frac{f(x_{k+1}) - f(x_k)}{h} \qquad \text{for} \qquad x_k \leq x \leq x_{k+1}.$$

Note that the `diff(v)` command is a convenient shortcut to construct

```
[v[2]-v[1],v[3]-v[2],…,v[end]-v[end-1]]
```

In [20]:

```
diff([1,3,7,4])
```

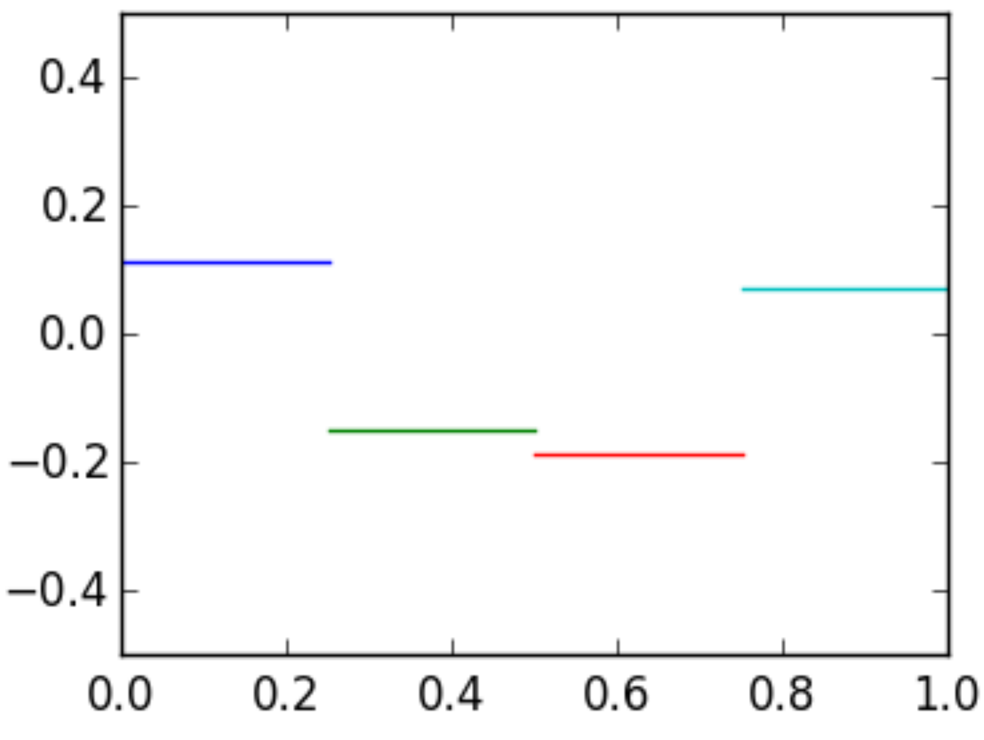
Out[20]:

```
3-element Array{Int64,1}:
  2
  4
 -3
```

Thus the following calculates the values of $p'(x)$ and plots it:

In [34]:

```
h=1/n

dvals=diff(vals)*h

for k=1:n
    plot([(k-1)*h,k*h],[dvals[k],dvals[k]])
end

axis([0,1,-0.5,0.5]);
```



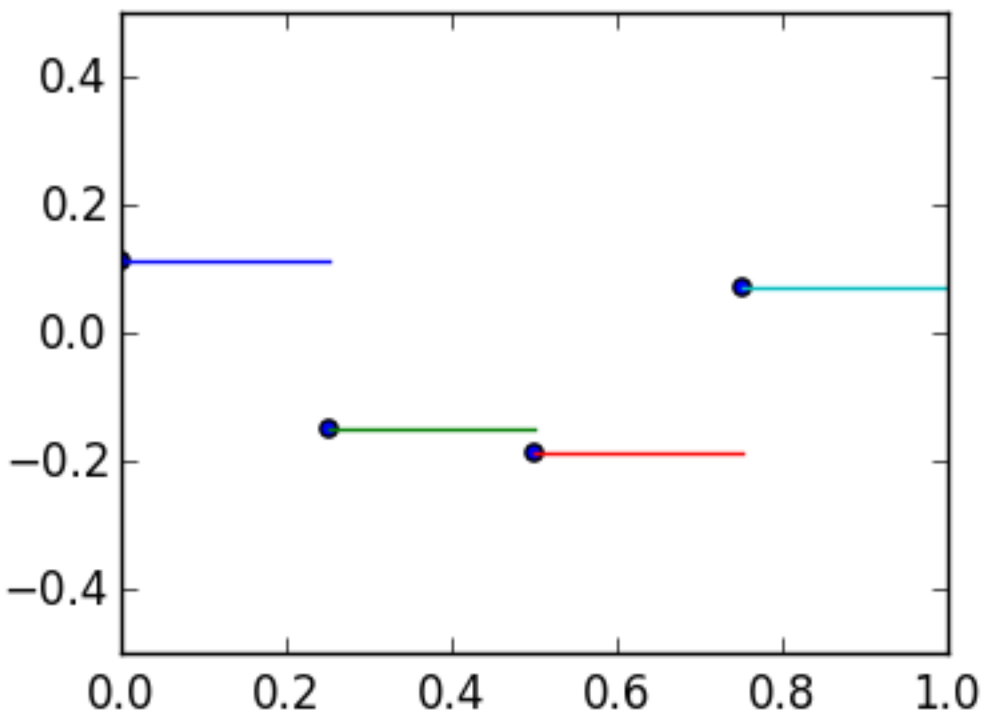
We now add a fourth step: discretize again, but now at $n-1$ points. Because p' jumps at x_k, we use the limits from the right, which we denote $x_k + 0$.

4) Discretize: $p'(x) \mapsto [p'(x_0 + 0), p'(x_1 + 0), \ldots, p'(x_{n-1} + 0)]$

This is precisely the `dvals` above:

In [37]:

```
dvals=diff(vals)*h


for k=1:n
    plot([(k-1)*h,k*h],[dvals[k],dvals[k]])

end

scatter(x[1:n],dvals)

axis([0,1,-0.5,0.5]);
```



Discrete derivative as a matrix

We can view this as a matrix:

$$D_n \begin{pmatrix} f(x_0) \\ \vdots \\ f(x_n) \end{pmatrix} = \begin{pmatrix} p'(x_0+0) \\ \vdots \\ f(x_{n-1}+0) \end{pmatrix} \approx \begin{pmatrix} f'(x_0) \\ \vdots \\ f'(x_{n-1}) \end{pmatrix}$$

for the $n-1 \times n$ matrix

$$D_n \triangleq \frac{1}{h} \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}$$

Discrete multiplication

We also have to construct a multiplication operator corresponding to multiplication by $a(x)$. Here, the input is a function evaluated at $x_0, \ldots, x_n$ while the output is a function evaluated at $x_0, \ldots, x_{n-1}$, to be consistent with D_n. This leads to

$$A_n \begin{pmatrix} f(x_0) \\ \vdots \\ f(x_n) \end{pmatrix} = \begin{pmatrix} a(x_0)f(x_0) \\ \vdots \\ a(x_n)f(x_{n-1}) \end{pmatrix}$$

for

$$A_n \triangleq \begin{pmatrix} a_0 & & & & \\ & a_1 & & & \\ & & \ddots & & \\ & & & a_{n-1} & 0 \end{pmatrix}$$

where $a_k \triangleq a(x_k)$.

Approximating the ODE

We thus approximate the ODE

$$u' - a(x)u = f(x)$$

to the discrete equation

$$(D_n + A_n) \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} f_0 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

where $f_k \triangleq f(x_k)$.

Expanding out the matrix, this gives us

$$\begin{pmatrix} -\frac{1}{h} - a_0 & \frac{1}{h} & & & \\ & -\frac{1}{h} - a_1 & \frac{1}{h} & & \\ & & \ddots & \ddots & \\ & & & -\frac{1}{h} - a_{n-1} & \frac{1}{h} \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} f_0 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

This is rectangular, so we need to incorporate the initial condition $u(0) = c$, in other words, we solve the lower triangular system

$$\begin{pmatrix} 1 & & & & \\ -\frac{1}{h} - a_0 & \frac{1}{h} & & & \\ & -\frac{1}{h} - a_1 & \frac{1}{h} & & \\ & & \ddots & \ddots & \\ & & & -\frac{1}{h} - a_{n-1} & \frac{1}{h} \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} c \\ f_0 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

When $f = 0$ and we multply through by h, we see that it is the Euler method.

Next lecture, we will see that replacing the choice of discretization points in step 4 can have dramatic effects.