Norms

We now consider the different norms discussed in lecture:

$$
\begin{aligned}
||\mathbf{v}||_1 &= \sum_{k=1}^{n} |v_k| \\
||\mathbf{v}||_2 &= \sqrt{\sum_{k=1}^{n} v_k^2} \\
||\mathbf{v}||_\infty &= \max |v_k|
\end{aligned}
$$

We can use the inbuilt `norm` function to calculate norms:

In [4]:

```
norm([1,2,3])==norm([1,2,3],2)==sqrt(1^2+2^2+3^2)
```

Out[4]:

```
true
```

In [5]:

```
norm([1,-2,3],1)==1+2+3
```

Out[5]:

```
true
```

In [6]:

```
norm([1,-2,3],Inf)==3
```

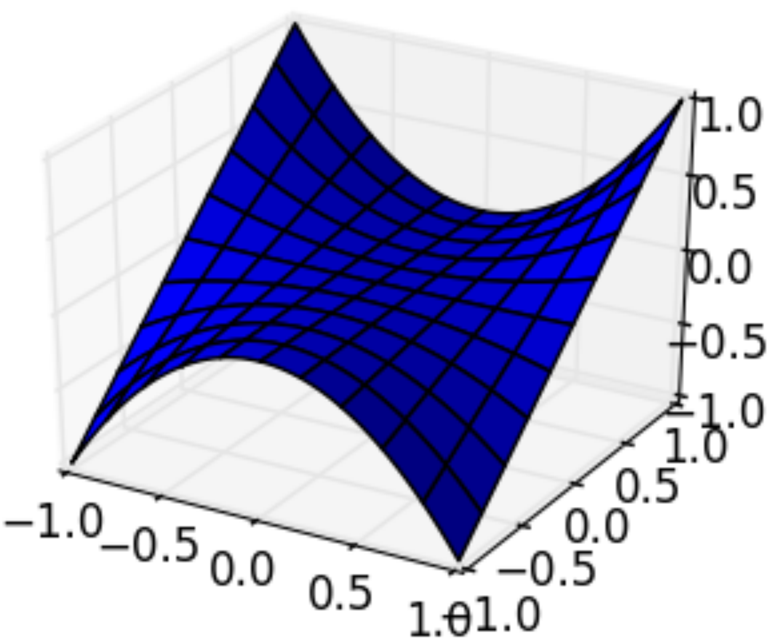
Out[6]:

```
true
```

We will investigate these norms by plotting are the level curves for different norms. First, we discuss how to do a surface plot. The following plots $y * x^2$:

In [3]:

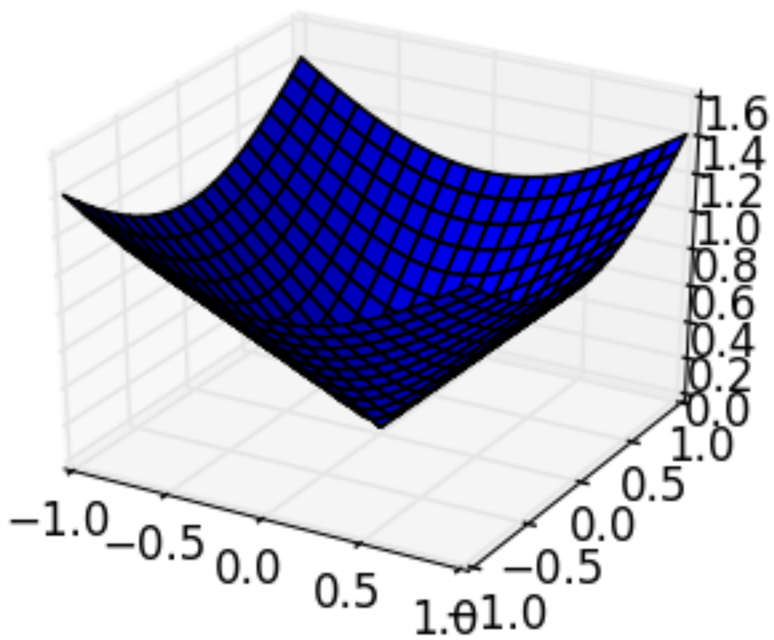
```
f(x,y)=y*x^2

# this is short hand for
function f(x,y)
    y*x^2
end

x=y=linspace(-1.,1.,100)


z=Float64[
    f(x[j],y[k])
    for k=1:length(y), j=1:length(x)]



using PyPlot

surf(x,y,z)  # 3D plot;
```



Choosing `f` to be the 2-norm, we can see how the norm grows:

In [4]:

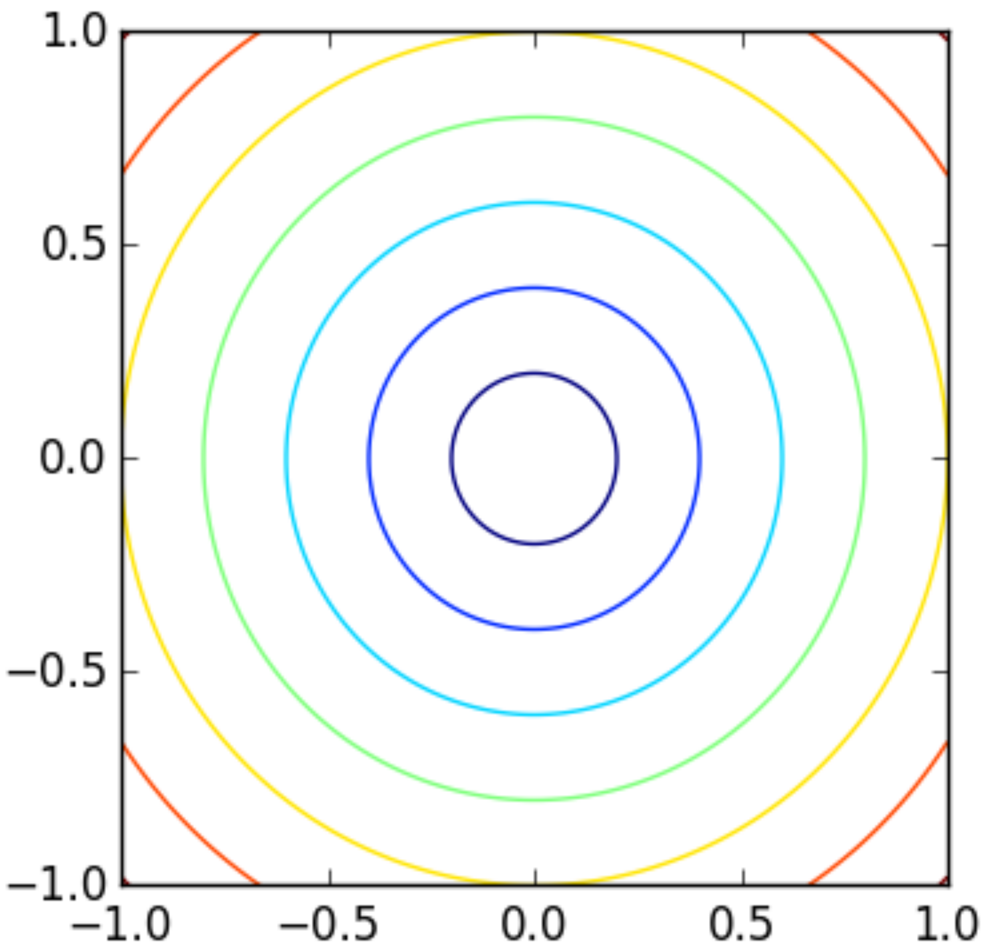
```
f(x,y)=norm([x,y],2)

x=y=linspace(-1.,1.,200)

z=Float64[
    f(x[j],y[k])
    for k=1:length(y), j=1:length(x)]


using PyPlot

surf(x,y,z)  # 3D plot;
```



It is helpful to plot a contour plot, to see where the curves of constant value are. Here, we see that the 2-norm forms circles:

In [7]:

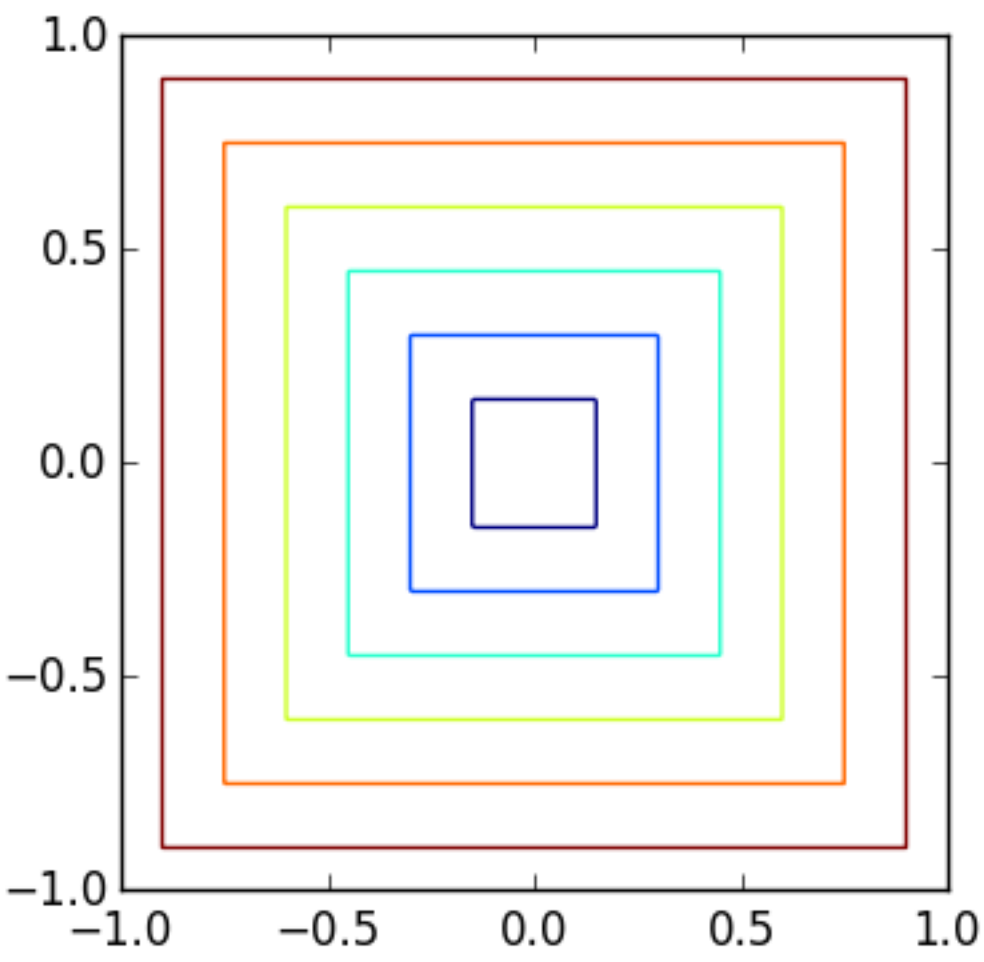
```
f(x,y)=norm([x,y],2)

x=y=linspace(-1.,1.,200)

z=Float64[
    f(x[j],y[k])
    for k=1:length(y), j=1:length(x)]


using PyPlot

contour(x,y,z)  # contour plot of f

gcf()[:set_size_inches](4,4)   # make the plot a square;
```



The ∞-norm has squares of constant norm:

In [8]:

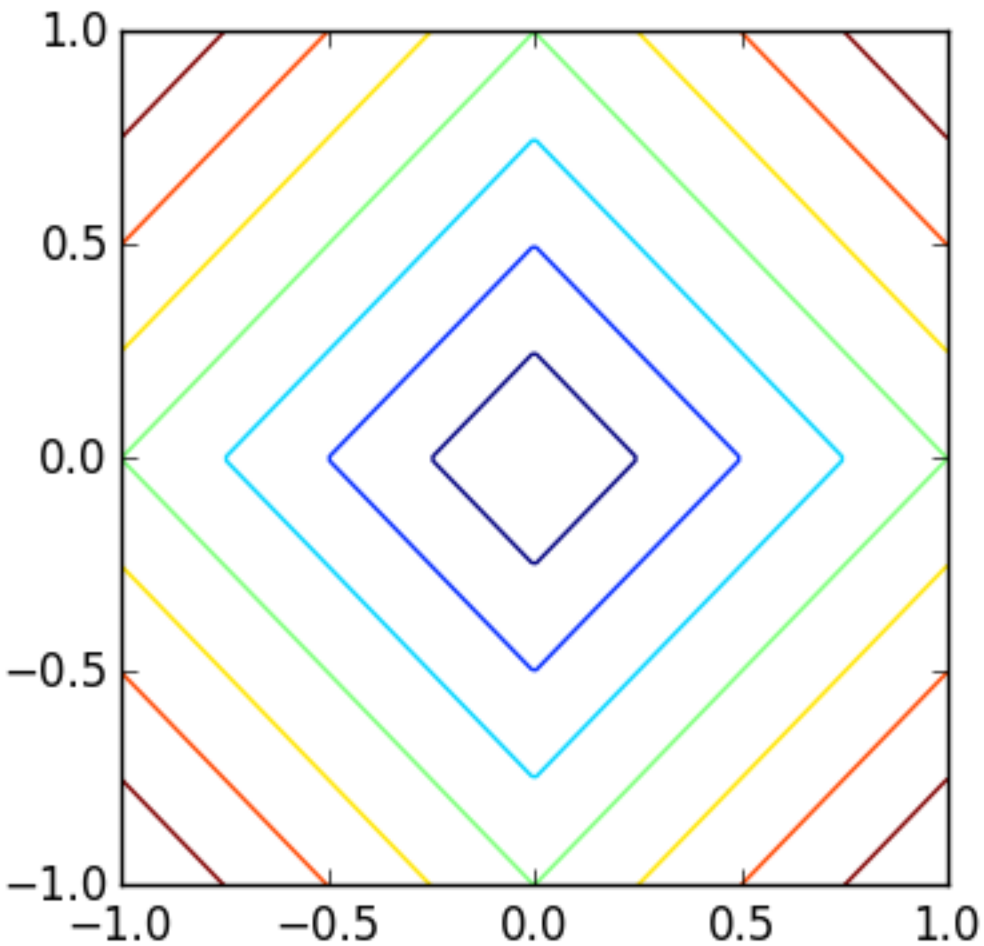
```
f(x,y)=norm([x,y],Inf)

x=y=linspace(-1.,1.,200)

z=Float64[
    f(x[j],y[k])
    for k=1:length(y), j=1:length(x)]


using PyPlot

contour(x,y,z)  # 3D plot;

gcf()[:set_size_inches](4,4)   # make the plot a square
```



The 1-norm has diamonds:

In [9]:

```
f(x,y)=norm([x,y],1)

x=y=linspace(-1.,1.,200)

z=Float64[
    f(x[j],y[k])
    for k=1:length(y), j=1:length(x)]


using PyPlot

contour(x,y,z)  # 3D plot;


gcf()[:set_size_inches](4,4)   # make the plot a square
```



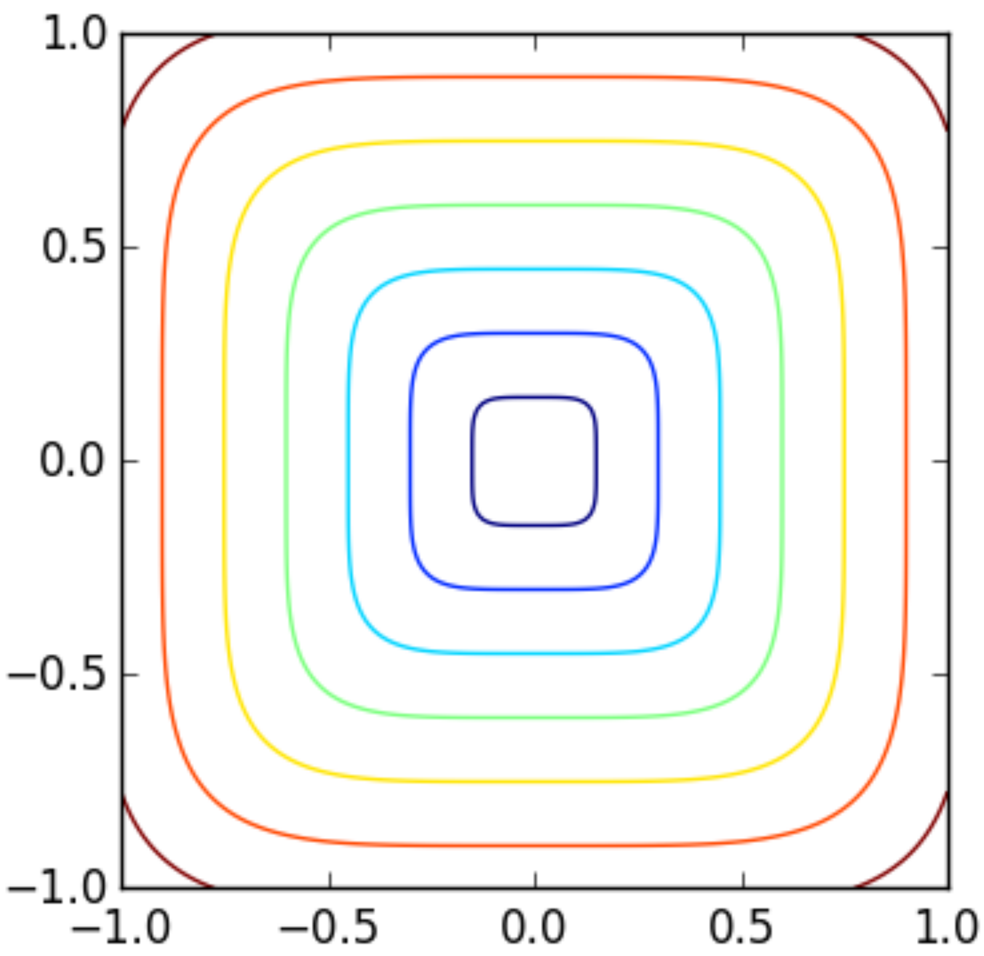
More generally, the p-norm

$$||\mathbf{v}||_p = \left(\sum_{k=1}^{n} |v_k|^p \right)^{1/p}$$

Is between a circle and a square:

In [10]:

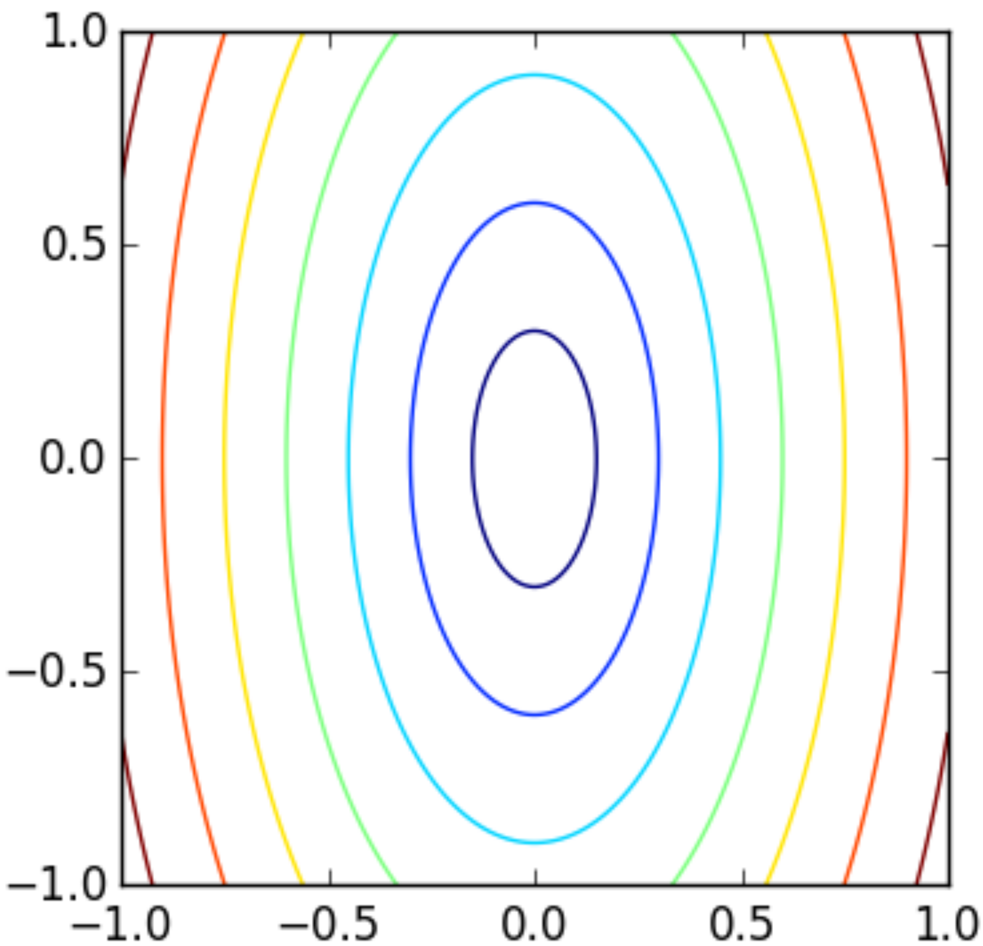
```
p=5

f(x,y)=norm([x,y],p)

x=y=linspace(-1.,1.,200)

z=Float64[
    f(x[j],y[k])
    for k=1:length(y), j=1:length(x)]

contour(x,y,z)  # 3D plot;
gcf()[:set_size_inches](4,4)   # make the plot a square
```



We can also weight the norm using a diagonal matrix. For the 2-norm, this changes circles to ellipses:

In [11]:

```
f(x,y)=norm([2 0; 0 1]*[x,y],2)

x=y=linspace(-1.,1.,200)

z=Float64[
    f(x[j],y[k])
    for k=1:length(y), j=1:length(x)]

contour(x,y,z)  # 3D plot;
gcf()[:set_size_inches](4,4)   # make the plot a square
```



\ and least squares

`x=A\b`

will find `x` so that `norm(A*x-b)` is minimized: that is, we are finding the vector that finds the best approximation to the equation in the 2-norm. This is called the least squares solution.

In [13]:

```
A=rand(10,5)
b=rand(10)

x=A\b
```

Out[13]:

```
5-element Array{Float64,1}:
 0.00826066
 0.151079
 0.881139
 0.144133
 0.350968
```

We thus know `norm(A*x-b)` should be the smallest possible value achievable by any vector `x`.

In [14]:

```
minnorm=norm(A*x-b)
```

Out[14]:

```
0.4485301788323935
```

We can check that this is true by sampling *many* random vectors `r`, and double checking that `norm(A*r-b)` is greater than `minnorm`. This sort of experiment is known as *Monte–Carlo simulation*.

In [15]:

```
randomminnorm=Inf   # start the smallest sampled norm at Inf

for k=1:1000000   # we will do a million trials
    r=rand(5)    # create a random vector of size 5
    newnrm=norm(A*r-b)     # Check ||A*r-b|| for  the random vector r r
    randomminnorm = min(newnrm,randomminnorm)   # the minimal sample norm is
 the min
end
randomminnorm  # this is greater than minnorm
```

Out[15]:

```
0.45086413589047813
```