Lecture 33: Linear ODEs

We now turn our attention to the numerical solution of ordinary differential equations. We will focus on *linear* ODEs to begin with. The reason we do this is it allows us to view the numerical solution as reducing infinite-dimensional linear equations (the ODE) to finite-dimensional linear systems (matrices).

To emphasize the difference between linear and nonlinear, consider the following two examples:

1. Example of a linear ODE:
$$y'(t) - a(t)y(t) = 0, y(0) = 1$$
Write this as $Ly = 0, y(0) = 1$ where L is a *linear operator*:
$$L = D - a(t)$$
where D is the *derivative operator*. This is called a linear operator because, for constant c_1 and c_2 , we have
$$L(c_1 y_1 + c_2 y_2) = c_1 L y_1 + c_2 L y_2 = c_1 (y_1' - a(t)y_1) + c_2 (y_2' - a(t)y_2)$$

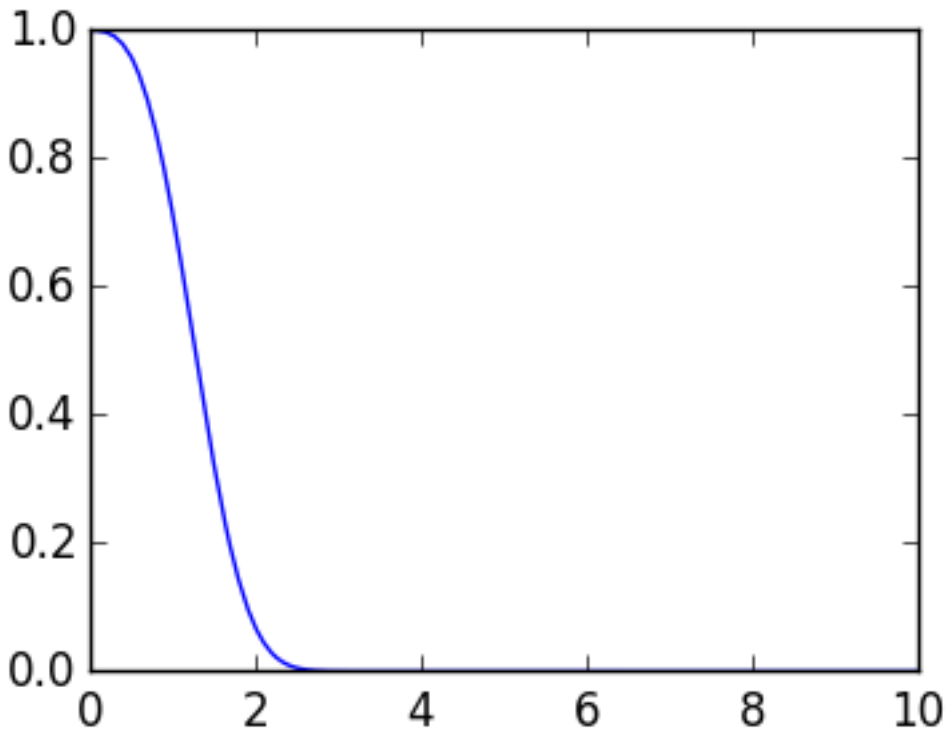
1. Example of a nonlinear ODE: the following logistic equation is a nonlinear ODE, since it depends quadratically on y:
$$y' - y(1 - y) = 0, y(0) = 0.5$$

Solving ode45

When the equation is an *initial value problem*, we can solve both linear and nonlinear ODEs using `ode45`. Here is an example of a linear ODE:

In [1]:

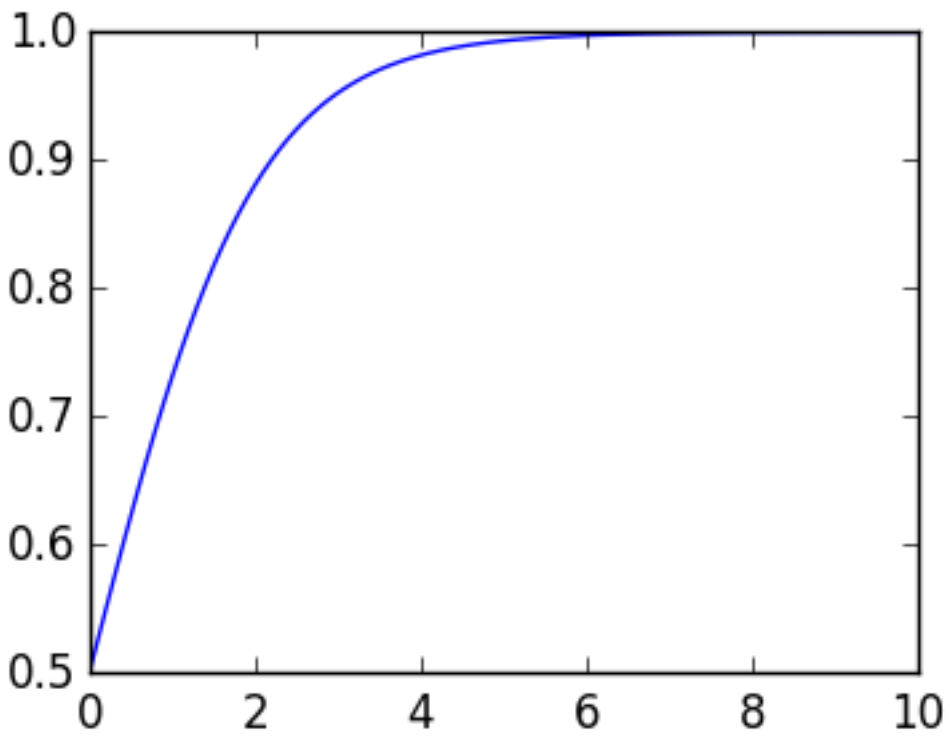
```
using ODE, PyPlot

a=t->-t^2
t,y=ode45((t,y)->a(t)*y,1.,0.:0.1:10.)
plot(t,y);
```



Similarly, we can solve a nonlinear ODE:

In [3]:

```
t,y=ode45((t,y)->y*(1-y),0.5,0.:0.1:10.)
plot(t,y);
```



Boundary value problems

The distinction between linear and nonlinear becomes greater when we consider other types of problems. The following is a linear intial value problem, since the constraints are specified at a single point, $t = 0$:

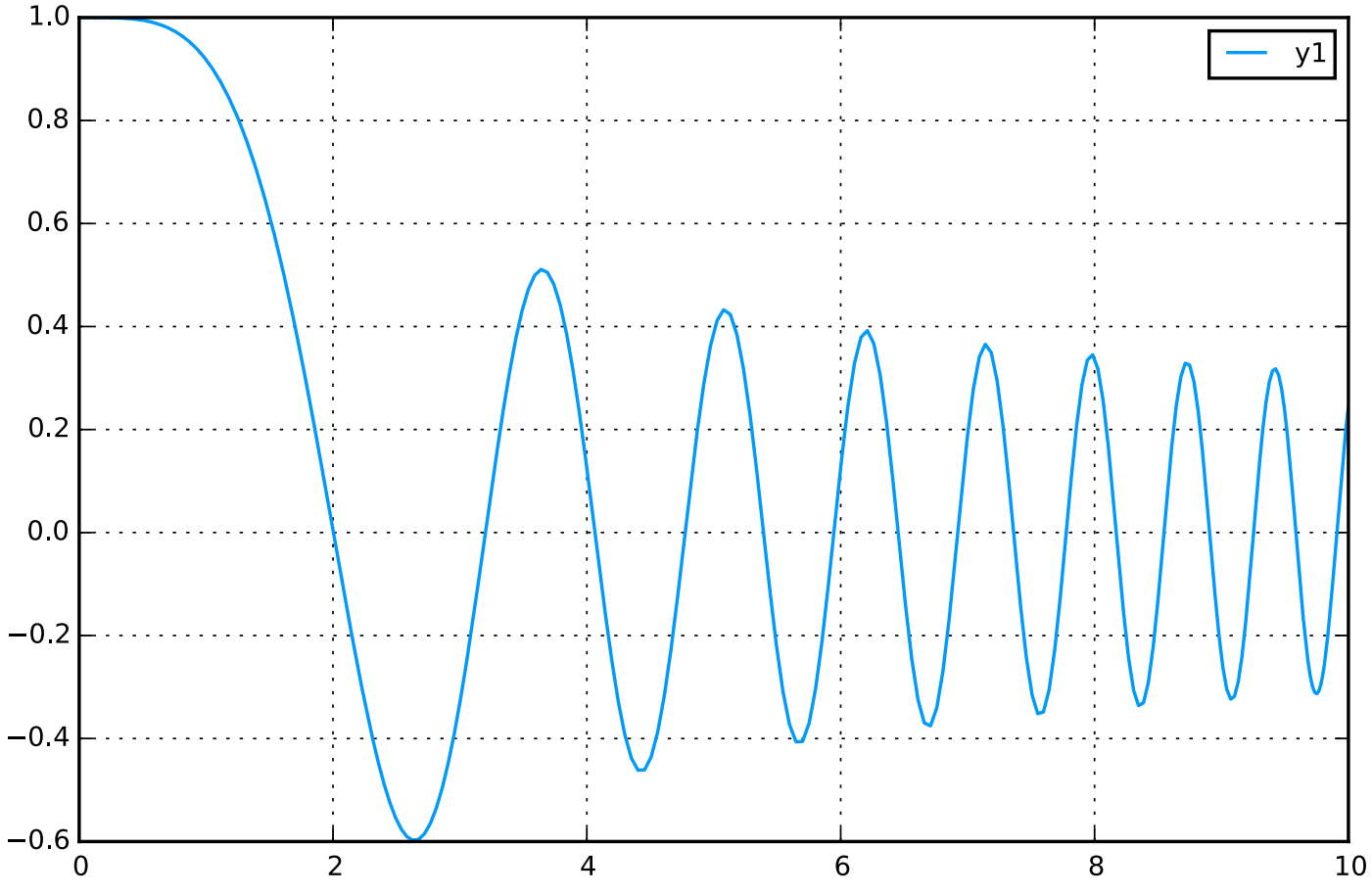
$$u'' - a(t)u = 0, u(0) = 1, u'(0) = 0$$

While we can solve this with `ode45` by rewriting as a system, we can also solve it in ApproxFun:

In [14]:

```
using ApproxFun

D=Derivative()   # the derivative operator
B=ivp()   # conditions corresponding to u(0),u(1)
a=Fun(t->-t^2,[0.,10.])
u=[B;D^2-a]\[1.,0.]
ApproxFun.plot(u)
```

Out[14]:



However, many problems are not initial value problems, but rather *boundary value problem*. One can imagine this coming up if, say, you observe a bullet at two different points, and know it solves an ODE, and want to recover the solution.

As a simple example, we consider

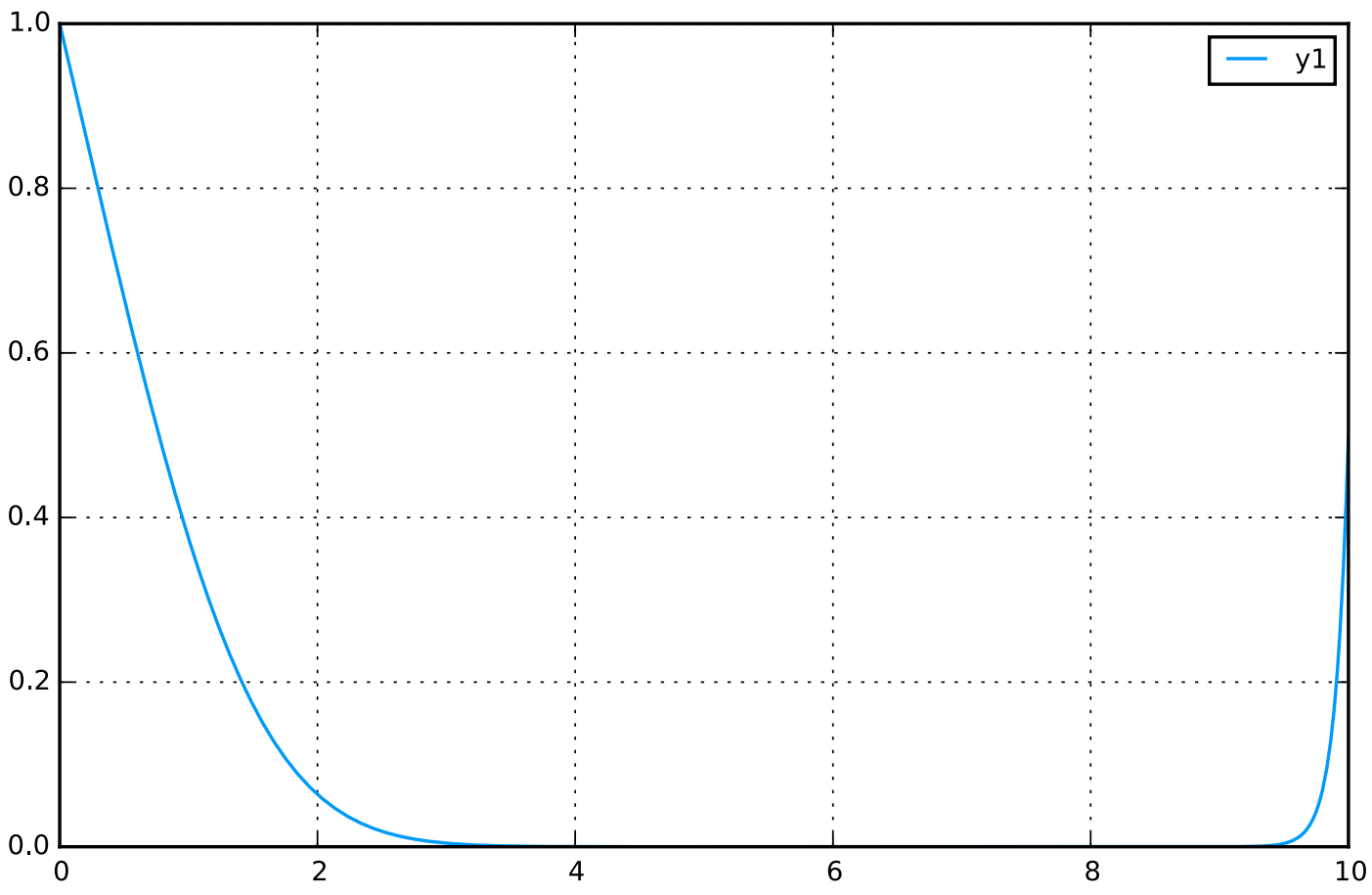
$$u'' - a(t)u = 0, u(0) = 1, u(10) = 0.5$$

Because it's linear, we can solve BVPs just as easily as initial value problems:

In [17]:

```
D=Derivative()
B=dirichlet()
a=Fun(t->t^2,[0.,10.])
u=[B;D^2-a]\[1.,0.5]
ApproxFun.plot(u)
```

Out[17]:



Euler's method

Our starting point is to construct *Euler's method* for the ODE

$$y' = f(t, y), y(0) = 1$$

The motivation is the observation that

$$y(h) = y(0) + hy'(0) + \frac{h^2}{2}y''(\chi) = y(0) + hy'(0) + O(h^2) = y(0) + hf(0, y(0)) + O(h^2)$$

where h is a small step. Thus we have $y(h) \approx y(0) + hf(0, y(0))$, and more generally, $y(t_{k+1}) \approx y(t_k) + hf(t_k, y(t_k))$ for $t_k \triangleq k * h$. We use this to construct an approximation $w_k \approx y(t_k)$ via:

$$w_{k+1} = w_k + hf(t_k, w_k)$$

For the linear ODE

$$y'(t) = a(t)y(t)$$

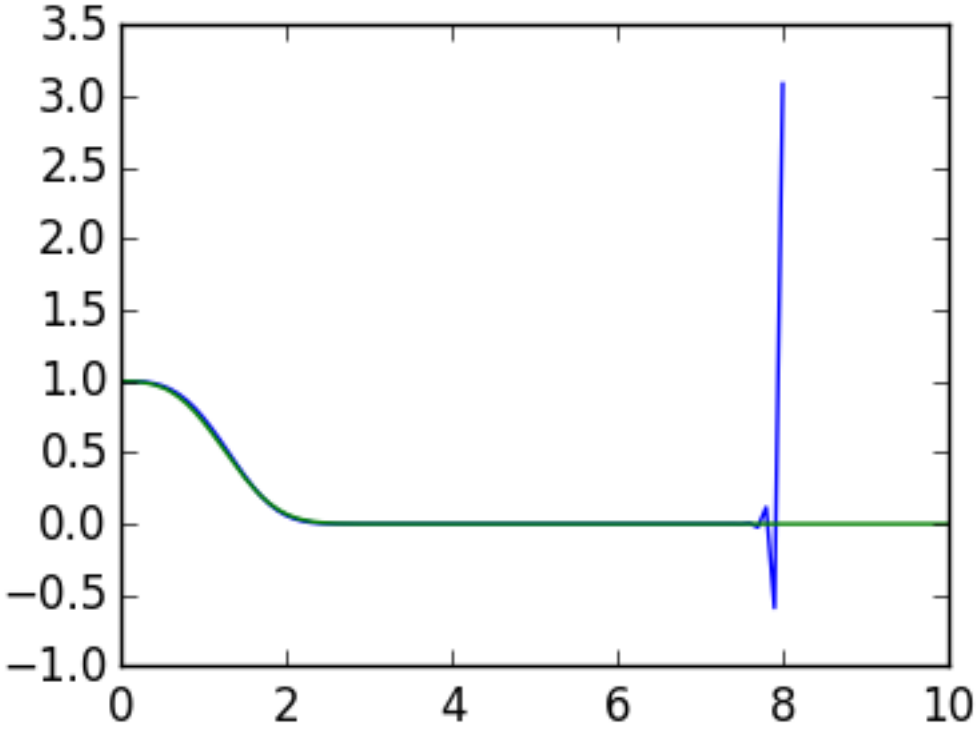
, this becomes

$$w_0 = y(0), w_{k+1} = (1 + ha(t_k))w_k$$

We observe that this does indeed approximate the solution, by comparing to `ode45`:

In [21]:

```
a=t->-t^2



# step 1: choose a time step
h=0.1
n=round(Int,8/h)
w=zeros(n+1)  # a vector that approximates y at each time step

t=0.:h:10.

w[1]=1.
M=length(w)-1
for k=1:M
    w[k+1]=(1 + h*a(t[k]))*w[k]
end


plot(t[1:M+1],w[1:M+1])


t,y=ode45((t,y)->a(t)*y,1.,t)
plot(t,y);
```



Taking a small step size and we get an accurate approximation, though it eventually blows up.

The key observation we make now is that we are in fact solving a lower triangular linear system with forward elimination:

$$\begin{pmatrix} 1 & & & & & \\ 1+ha(t_0) & -1 & & & & \\ & 1+ha(t_1) & -1 & & & \\ & & 1+ha(t_2) & -1 & & \\ & & & \ddots & \ddots & \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$