Lecture 38: Nonlinear problems

Suppose we want to solve a nonlinear ODE

$$u'(t) = f(t, u), u(0) = c$$

We saw already the forward Euler method:

$$w_{k+1} = w_k + hf(t_k, w_k)$$

where $w_0 = c$ and $t_k = kh$. This is an *explicit method*, because we get the value at w_{k+1} explicitely from the previous value w_k.

If we try to implement backward Euler method, we get

$$w_{k+1} - hf(t_{k+1}, w_{k+1}) = w_k$$

This is an *implicit method*, since the next value is implicitely defined. To solve this, we need to do root finding, that is, solve $g(w_{k+1}) = 0$ for

$$g(x) = x - hf(t_{k+1}, x) - w_k$$

Newton iteration

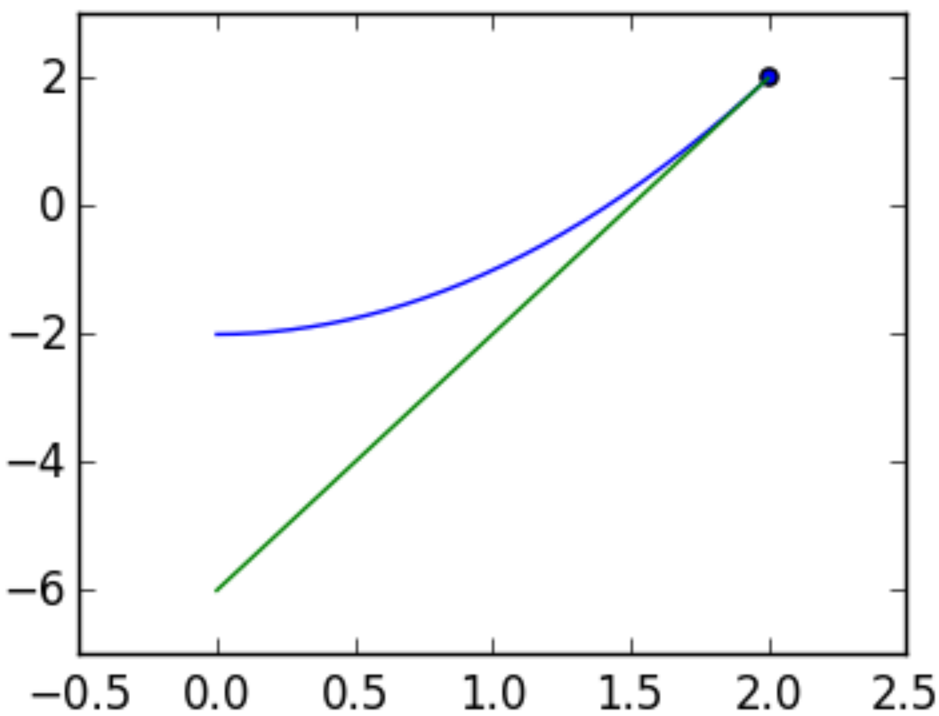
Suppose we want to solve

$$g(r) = 0$$

for $g(x) = x^2 - 2$. We can do so with *Newton iteration*: start at an guess $x_0 = 2$, approximating $g(x)$ by the tangent line at the point $g(x_0)$:

In [2]:

```
using PyPlot

g=x->x.^2-2
gp=x->2x

p=linspace(0.,2.,1000)

x=zeros(1000)
x[1]=2.


# g'(x[1])*(x-x[1])+g(x[1])

# g'(x[1])*(x[2]-x[1])+g(x[1])   ==0
# x[2]   ==x[1] - g(x[1])/g'(x[1])

N=1

for k=1:N-1
    x[k+1]=x[k]-g(x[k])/gp(x[k])
end

scatter(x[1:N],g(x[1:N]))

plot(p,g(p))
for k=1:N
    plot(p,gp(x[k])*(p-x[k])+g(x[k]))
end

x[N]-sqrt(2.)
```



Out[2]:

0.5857864376269049

Finding the root of this line gives us the second guess

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

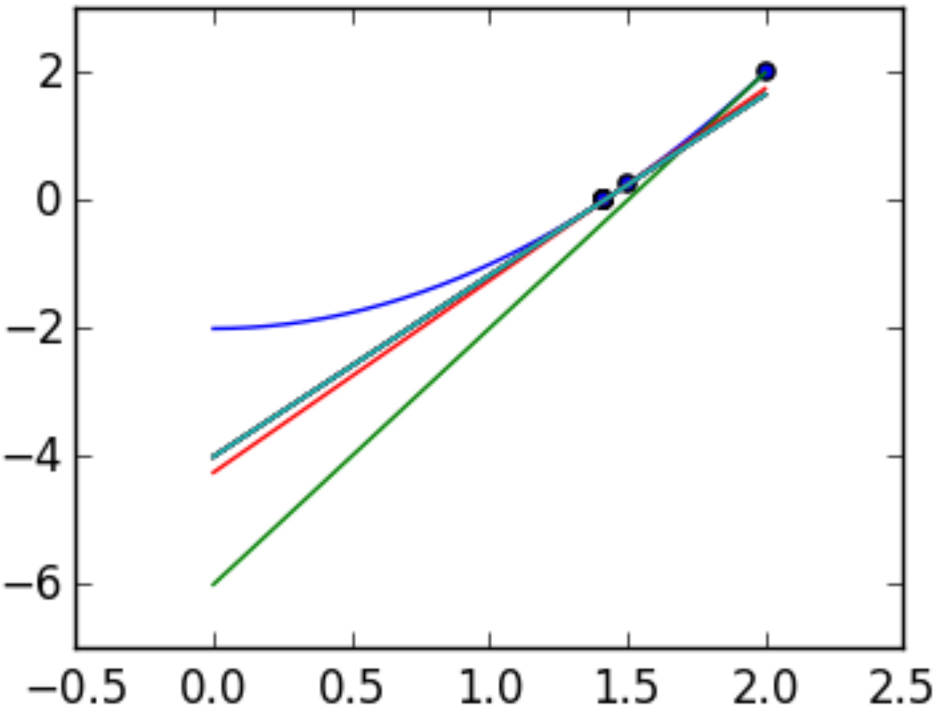
We can then repeat the process: approximate through the tangent line at x_k, then find the root

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

This converges remarkably quickly:

In [3]:

```
using PyPlot

g=x->x.^2-2
gp=x->2x

p=linspace(0.,2.,1000)

x=zeros(1000)
x[1]=2.


# g'(x[1])*(x-x[1])+g(x[1])

# g'(x[1])*(x[2]-x[1])+g(x[1])   ==0
# x[2]   ==x[1] - g(x[1])/g'(x[1])

N=10
for k=1:N-1
    x[k+1]=x[k]-g(x[k])/gp(x[k])
end

scatter(x[1:N],g(x[1:N]))

plot(p,g(p))
for k=1:N
    plot(p,gp(x[k])*(p-x[k])+g(x[k]))
end

x[N]-sqrt(2.)
```



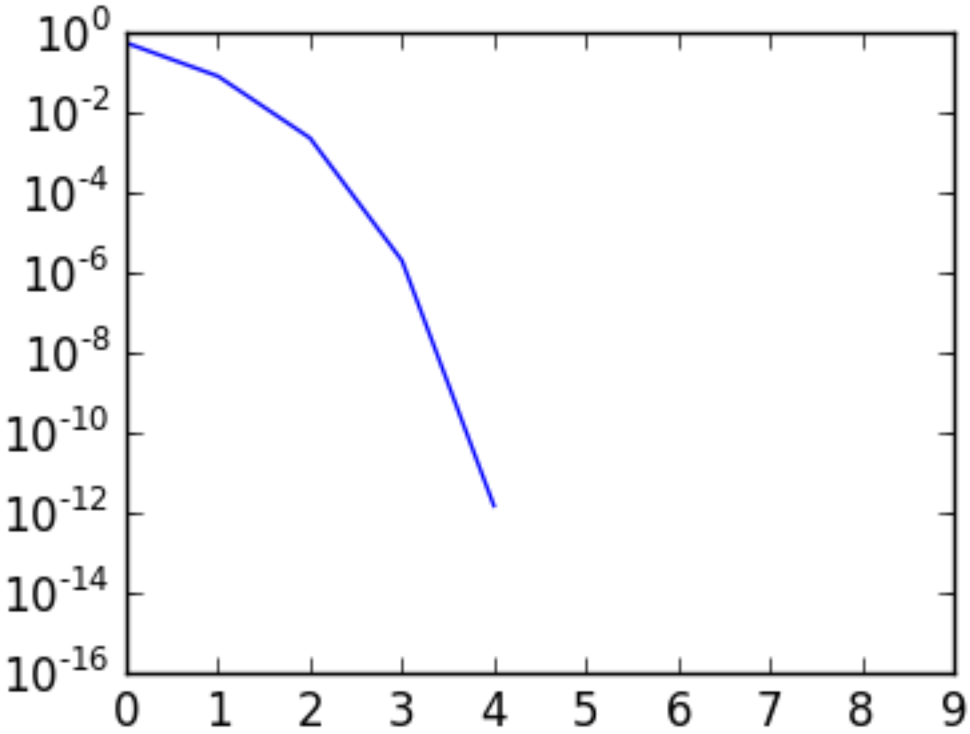
Out[3]:

```
0.0
```

The following plot emonstrates that it converges faster than exponentially:

In [7]:

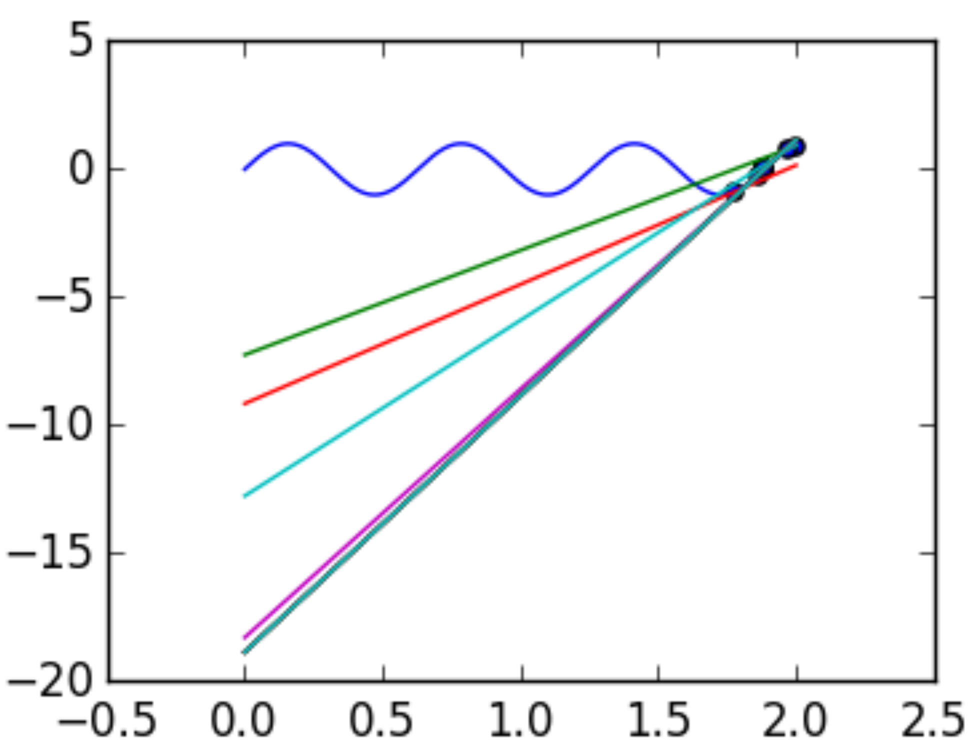
```
err=abs(x[1:N]-sqrt(2.))
semilogy(err);
```



For non-convex functions, the root it converges to (if any) is not predictible:

In [8]:

```
g=x->sin(10x)
gp=x->10cos(10x)

p=linspace(0.,2.,1000)

x=zeros(1000)
x[1]=2.



# g'(x[1])*(x-x[1])+g(x[1])

# g'(x[1])*(x[2]-x[1])+g(x[1])  ==0
# x[2]  ==x[1] - g(x[1])/g'(x[1])

N=10
for k=1:N-1
    x[k+1]=x[k]-g(x[k])/gp(x[k])
end

scatter(x[1:N],g(x[1:N]))

plot(p,g(p))
for k=1:N
    plot(p,gp(x[k])*(p-x[k])+g(x[k]))
end

sin(10x[N])
```



Out[8]:

-7.347880794884119e-16

Proof of quadratic convergence

We can prove this result using *Taylor series*:

$$
\begin{aligned}
0 = f(r) &= f(x_k) + f'(x_k)(x_k - r) + \frac{f''(\zeta_k)}{2}(x_k - r)^2 \\
&= f(x_k) + f'(x_k)\epsilon_k + \frac{f''(\zeta_k)}{2}\epsilon_k^2
\end{aligned}
$$

But we have

$$\epsilon_k = x_k - r = x_k - x_{k+1} + x_{k+1} - r = x_k - x_{k+1} + \epsilon_{k+1}$$

giving us

$$0 = f(x_k) + f'(x_k) (x*k - x*{k+1}) + f'(x*k)\epsilon*{k+1} + {f''(\zeta_k) \over 2} \epsilon_k^2

But be definition

$$0 = f(x_k) + f'(x_k)(x_k - x_{k+1})$$

Thus we have

$$\epsilon_{k+1} = -\frac{f''(\zeta_k)}{2f'(x_k)}\epsilon_k^2$$

This is called *quadratic convergence*, as the error is roughly the previous error squared. Bounding derivatives can be used to get rigorous bounds on the convergence, and guaranteed convergence.