Lecture 39: CFL Condition, nonlinear boundary value problems

This lecture tidies up some loose ends. We will discuss very briefly Runge–Kutta methods, CFL Conditions, the secant method, Newton iteration in higher dimensions and nonlinear boundary value problems

Runge–Kutta methods

We first mention that one should read up on Runge–Kutta methods (https://en.wikipedia.org/wiki/Runge–Kutta_methods), in particular RK4. This gives an explicit (like forward Euler) time-stepping scheme that converges like $O(h^4)$. That is, if we approximate

$$u' = f(t, u)$$

by w_k where w_{k+1} is found using RK4, with a step-size of h, then if we evolve to a time $T = nh$ where $h = T/n$, we have

$$w_n - u(T) = O(h^4).$$

This is a substantial improvement over forward Euler, which only achieved $O(h)$ convergence.

We can define RK4 as follows:

In [5]:

```
function RK4(L,y,h)
    k1=L(y)
    k2=L(y+.5h*k1)
    k3=L(y+.5h*k2)
    k4=L(y+h*k3)

    y+h*(k1+2k2+2k3+k4)/6.
end
```

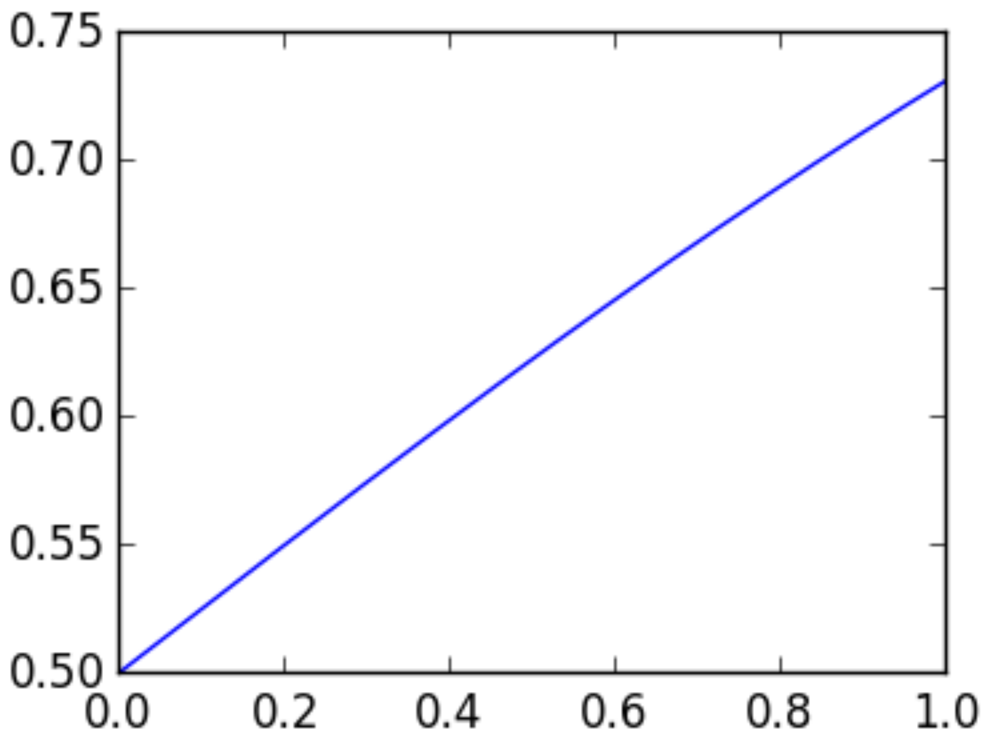
Out[5]:

```
RK4 (generic function with 1 method)
```

Here we solve the logistic equation $u' = u(1-u)$:

In [14]:

```
w=zeros(1001)
w[1]=0.5
h=0.001
for k=1:length(w)-1
    w[k+1]=RK4(u->u*(1-u),w[k],h)
end

using PyPlot
plot((0:h:1.),w);
```



The CFL Condition

Here we explain roughly why we need to take $\Delta t \leq C \Delta x^2$ for some constant C when using explicit methods for time stepping. Consider the periodic Heat equation

$$u_t = u_{\theta\theta}$$

If we solve it with forward Euler and finite differences in space, we get

$$\mathbf{w}_{k+1} = \mathbf{w}_k + (\Delta t) D_n^2 \mathbf{w}_k$$

Consider the eigenvalue decomposition

$$D_n^2 = Q \Lambda Q^\top$$

Then define

$$\mathbf{z}_k \triangleq Q^\top \mathbf{w}_k$$

so that

$$\mathbf{z}_{k+1} = \mathbf{z}_k + (\Delta t) \Lambda \mathbf{z}_k$$

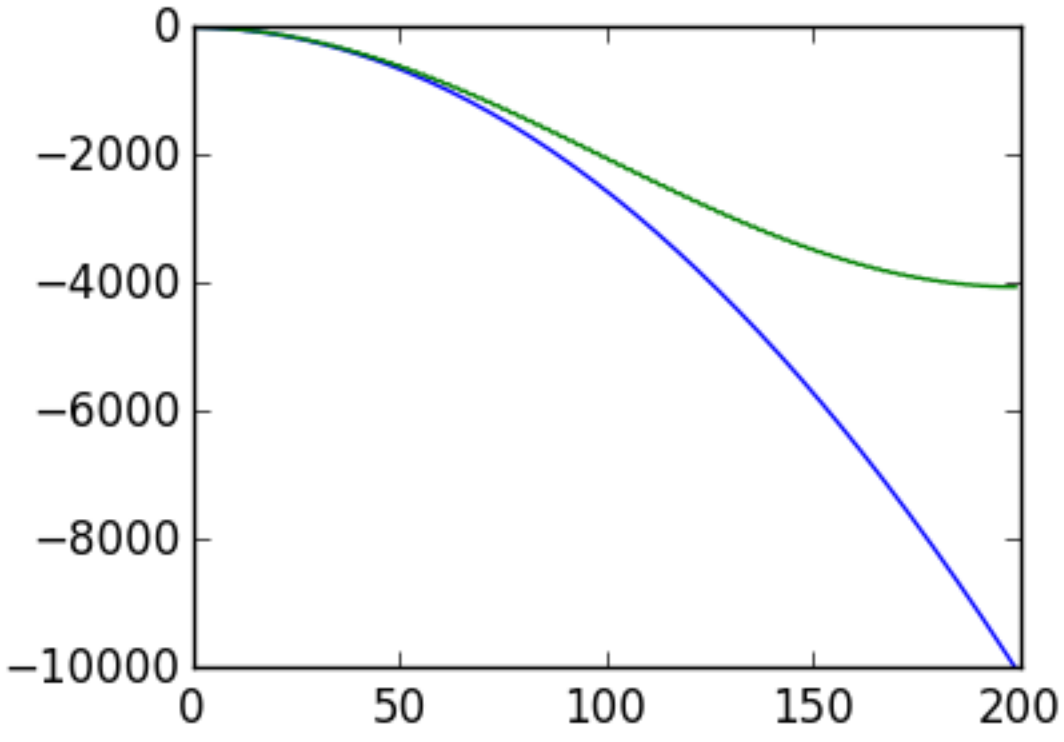
Each entry is thus evolved independentally:

$$\mathbf{e}_j^\top \mathbf{z}_{k+1} = \mathbf{e}_j^\top \mathbf{z}_k + (\Delta t) \lambda_j \mathbf{e}_j^\top \mathbf{z}_k$$

The eigenvalues grow roughly like $\lambda_j \sim -\tilde{C} j^2$:

In [23]:

```
function DL(n)
    h=2π/n
    ret=zeros(n,n)
    for k=1:n-1
        ret[k,k]=-1
        ret[k,k+1]=1
    end
    ret[n,1]=1
    ret[n,n]=-1
    ret/h
end


function DR(n)
    h=2π/n
    ret=zeros(n,n)
    for k=2:n
        ret[k,k]=1
        ret[k,k-1]=-1
    end
    ret[1,n]=-1
    ret[1,1]=1
    ret/h
end
n=200


Δθ=(2π)/n

D2=DL(n)*DR(n)


plot(-(1:n).^2/4)
plot(reverse(eigvals(D2)));
```



For forward Euler to be stable evolving

$$u' = au$$

,

we observed that we needed a step size satisfying $h \leq \frac{C_1}{|a|}$ for some constant C_1.

In the previous case, its equivalent to evolving $u' = \lambda_j u$, hence we need

$$\Delta t \leq \frac{C_1}{\max |\lambda_j|} \leq \frac{C_2}{n^2} = C_3 \Delta x^2 .$$

Secant method

An issue with the Newton step

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

is that it requires knowing the derivative of f. We can get around this by approximating

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

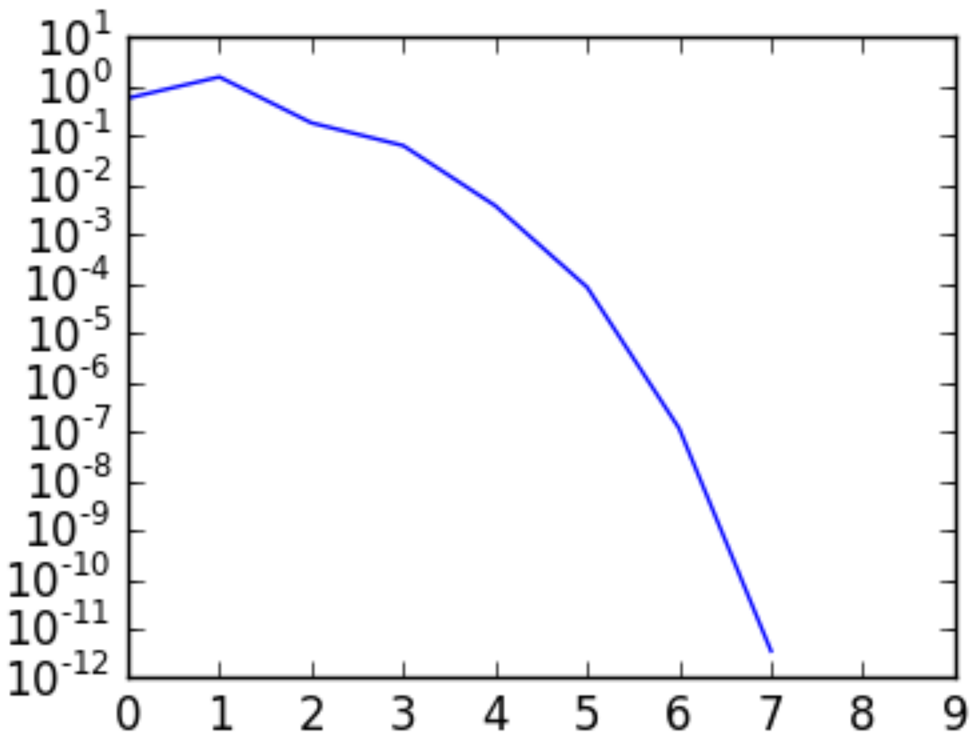
Leading to the *Secant method*

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

Note we need to specify both x_0 and x_1. We see that it still converges quite quickly:

In [28]:

```
f=x->x^2-2.

x=zeros(10)
x[1]=2.
x[2]=3.
for k=2:9
    x[k+1]=x[k]-f(x[k])*(x[k]-x[k-1])/(f(x[k])-f(x[k-1]))
end

semilogy(x-sqrt(2.))
```



Out[28]:

```
1-element Array{Any,1}:
 PyObject <matplotlib.lines.Line2D object at 0x3144fab10>
```

Higher-dimensional Newton iteration

Suppose we have $f : \mathbb{R}^n \to \mathbb{R}^n$ and we want to find $\mathbf{r}$ satisfying $f(\mathbf{r}) = 0$. We can do so using Newton iteration, where the derivative is replaced by the Jacobian.

Let's consider the $n = 2$ case, where we want to solve

$$f_1(x, y) = 0, f_2(x, y) = 0$$

The Jacobian is

$$J(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

Then Newton iteration is

$$\mathbf{x}_{k+1} = \mathbf{x}_k - J(\mathbf{x}_k)^{-1} f(\mathbf{x}_k)$$

We can try it out:

In [39]:

```
f=(x,y)->[x^2-y^2-2; x*y+3]
J=(x,y)->[2x -2y; y x]

x0=[1.,2.]
x=x0

for k=1:10
    x=x-J(x...)\f(x...)
end

x,f(x...)
```

Out[39]:

```
([-2.040166086417569,1.470468517231287],[-8.881784197001252e-16,0
.0])
```

Newton iteration for boundary value problems

Consider now a nonlinear boundary value problem:

$$u'' + xu + u^2 = 0, u(0) = a, u(1) = b$$

We can think of this as a root-finding problem $N(u) = 0$ where

$$N(u) = \begin{pmatrix} u(0) - a \\ D^2 u + xu + u^2 \\ u(1) - b \end{pmatrix}$$

Formally, we can differentiate this with respect to u to get an *operator*:

$$N'(u) = \begin{pmatrix} B_0 \\ D^2 + x + 2u \\ B_1 \end{pmatrix}$$

where $B_x u \triangleq u(x)$. We obtain a Newton iteration

$$u_{k+1} = u_k - N'(u_k)^{-1} N(u_k)$$

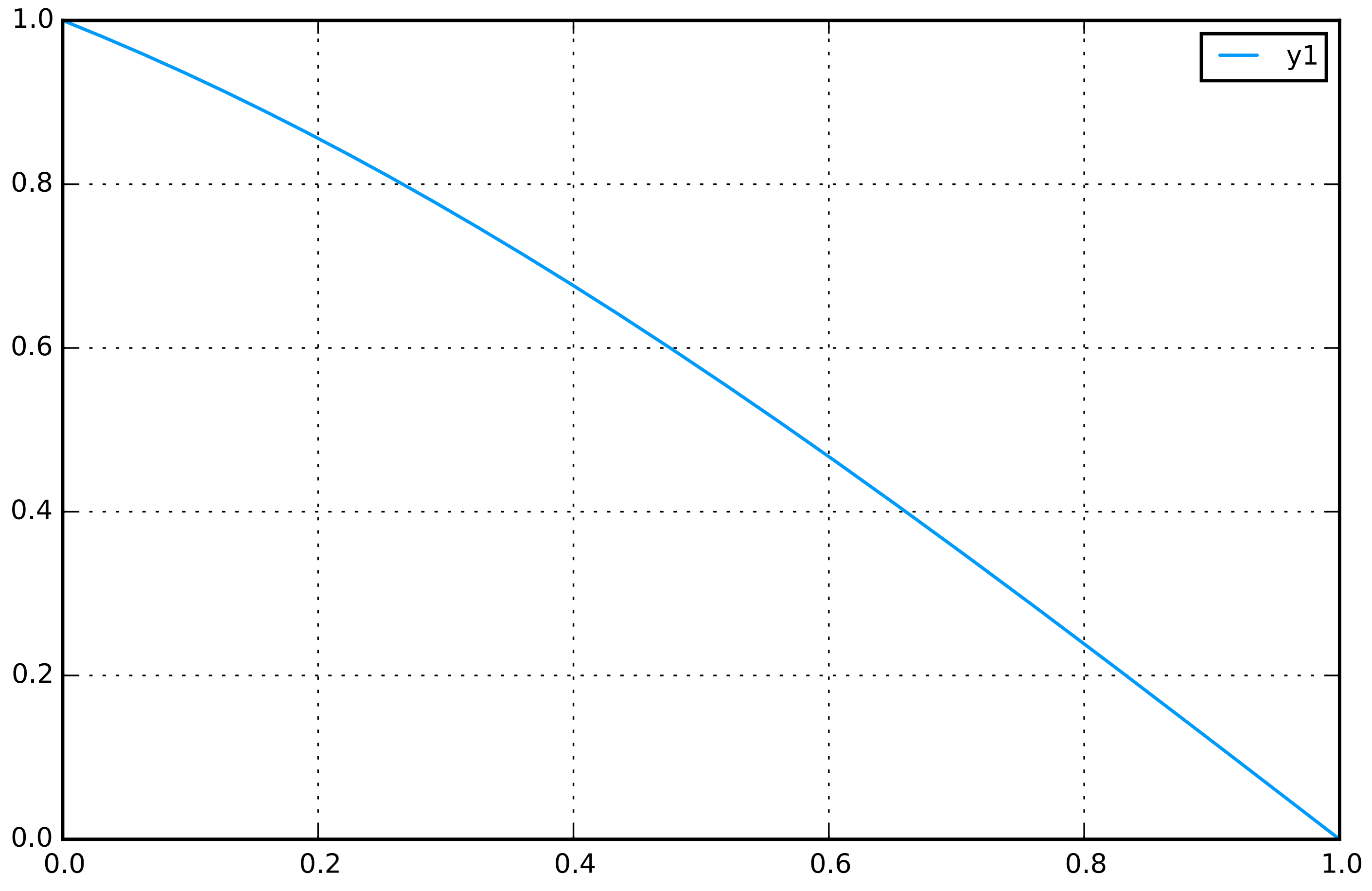
where $N'(u_k)^{-1}$ is interpreted as solving a linear boundary value problem.

We can implement this in practice:

In [42]:

```
using ApproxFun

u0=Fun(0.,[0.,1.])
D=Derivative(space(u0))
x=Fun(space(u0))

u=u0

a=1.;b=0.

for k=1:10
    u = u-  [dirichlet();D^2+x+2u]\[u(0.)-a;u(1.)-b; u''+x*u+u^2]
end


ApproxFun.plot(u)
```

Out[42]:



In [43]:

```
norm( u''+x*u+u^2)
```

Out[43]:

```
1.2389381059027248e-16
```

Exercise Re-implement this using finite section.