Given's rotations and the QR decomposition

In this lecture we introduce Given's rotations as a way to calculate the QR decomposition of a matrix.

We derived an algorithm for calculating the LU Decomposition by applying lower triangular operations on 2 rows at time: thus it came down to the matrix

$$L = \begin{pmatrix} 1 & \\ -\frac{b}{a} & 1 \end{pmatrix}$$

so that

$$L \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

In [57]:

```
x=rand(2)
a,b=x;
L=[1    0;
   -b/a 1]

L*x
```

Out[57]:

```
2-element Array{Float64,1}:
 0.398033
 1.11022e-16
```

This had the issue that it degenerated as a becomes small.

As an alternative, we will *rotate* the vector to introduce a zero. Recall that the angle of a point (a, b) with the origin is given by $\mathrm{atan}\frac{b}{a}$, thus we want to rotate by $\theta = -\mathrm{atan}\frac{b}{a}$:

In [59]:

```
θ=-atan2(b,a)   # equivalent to  θ=atan(b/a)
Q=[cos(θ) -sin(θ);
    sin(θ) cos(θ)]

Q*x
```

Out[59]:

```
2-element Array{Float64,1}:
 0.959959
 5.55112e-17
```

Unlike lower triangularizations, this works well even for points on the y-axis:

In [61]:

```
x=[0.,2.]
a,b=x

θ=-atan2(b,a)
Q=[cos(θ) -sin(θ);
    sin(θ) cos(θ)]

Q*x
```

Out[61]:

```
2-element Array{Float64,1}:
 2.0
 1.22465e-16
```

Effect of error

On a computer, we don't know exactly where a point is: every point can have a small ϵ of error. Thus to understand the robustness of an algorithm, we need to understand what happens to balls of radius ϵ around where we think of the point. This can be used to demonstrate why rotations are preferred to lower triangular operations.

In the following, we design a function that plots a circle around a point `[a,b]` of size `ε`, both before and after a matrix `L` is applied:

In [62]:

```
using PyPlot



function plotmat(a,b,ε,L)
    t=linspace(0.,2π,100)
    x,y=(a+ε*cos(t),b+ε*sin(t))
    plot(x,y)

    Lx=zeros(length(x))
    Ly=zeros(length(x))

    for k=1:length(x)
        Lx[k],Ly[k]=L*[x[k],y[k]]
    end
    plot(Lx,Ly)
    axis([-3,3,-3,3])
end
```

Out[62]:

```
plotmat (generic function with 1 method)
```

Here we demonstrate that the effect of `L` is to stretch the circles: our error can be amplified:

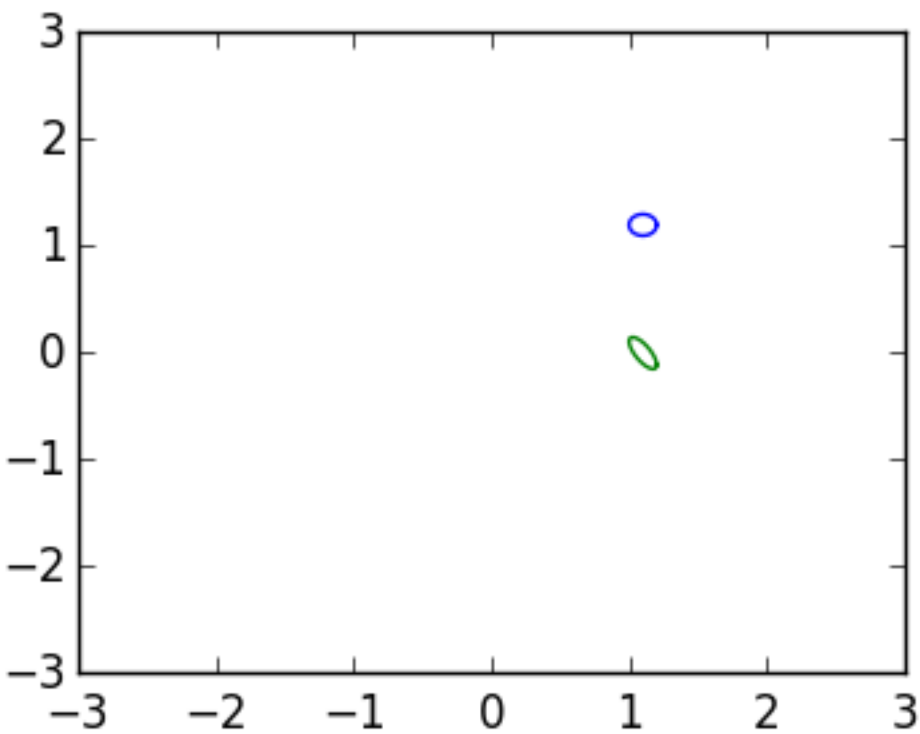
In [64]:

```
a,b=[1.1,1.2]

L=[1    0;
   -b/a 1]

ε=0.1

plotmat(a,b,ε,L)
```



Out[64]:

```
4-element Array{Int64,1}:
 -3
  3
 -3
  3
```

As a becomes small, this error amplification becomes greater: in the following, we go from knowing the true point with accuracy 0.1 to only knowing it with about accuracy 1:

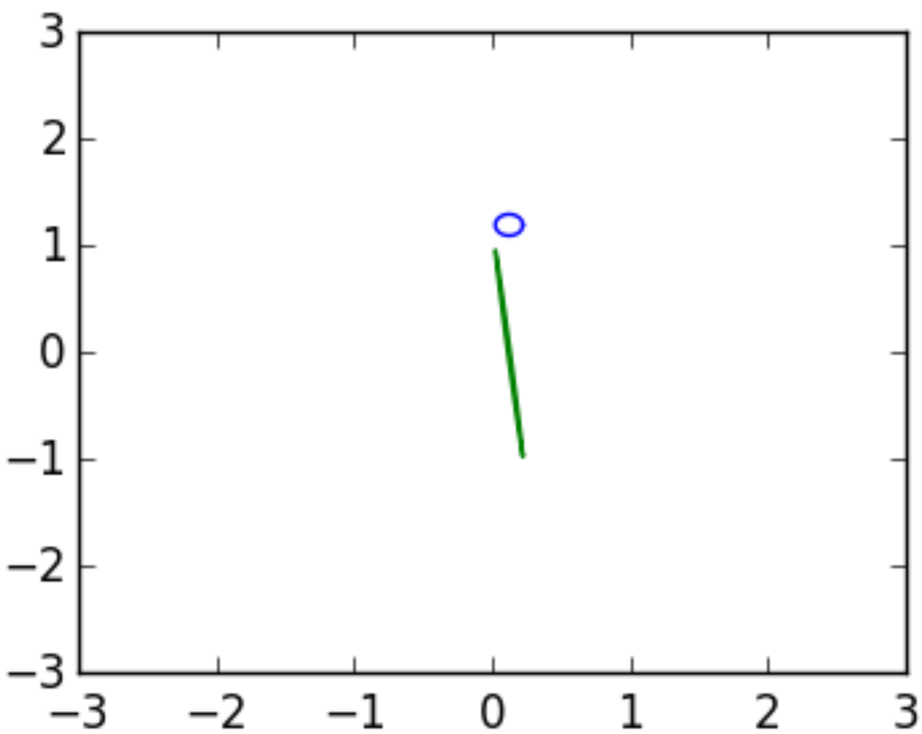
In [66]:

```
a,b=[0.125,1.2]

L=[1    0;
   -b/a 1]

ε=0.1

plotmat(a,b,ε,L)
```



Out[66]:

```
4-element Array{Int64,1}:
 -3
  3
 -3
  3
```

Rotations perform much better: the circles are only rotated, and are not magnified at all:

In [67]:

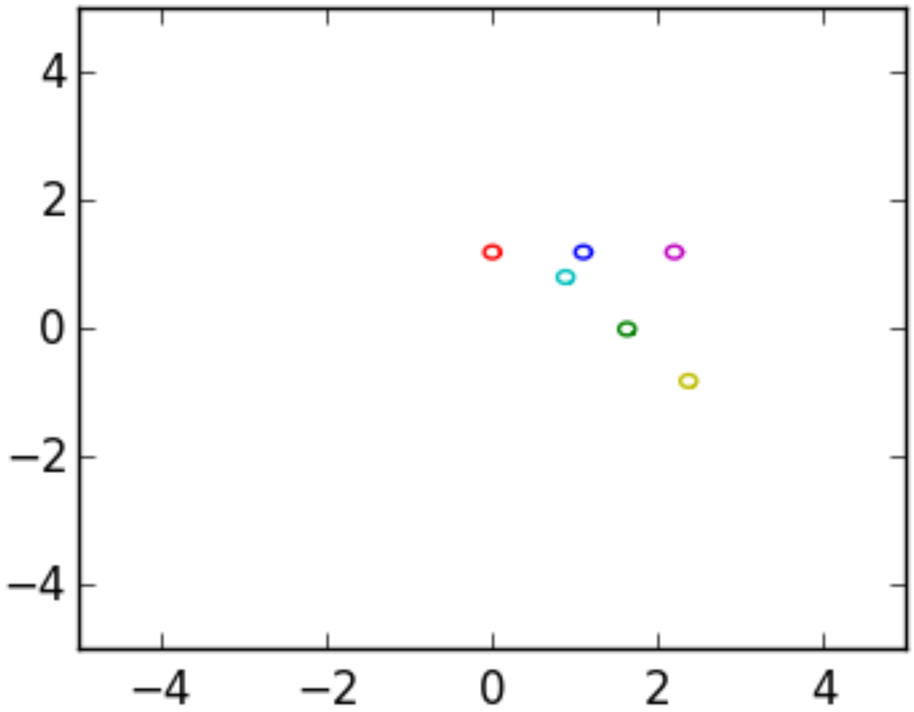
```
a,b=[1.1,1.2]

θ=-atan2(b,a)
Q=[cos(θ) -sin(θ);
    sin(θ) cos(θ)]

ε=0.1

plotmat(a,b,ε,Q)
plotmat(0,b,ε,Q)

plotmat(2a,b,ε,Q)
axis([-5,5,-5,5])
```



Out[67]:

```
4-element Array{Int64,1}:
 -5
  5
 -5
  5
```

Simpler definition

We can construct a simpler expression for `Q` as

$$\frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

In [52]:

```
Q=[cos(θ) -sin(θ);
    sin(θ) cos(θ)]
```

Out[52]:

```
2x2 Array{Float64,2}:
  0.675725  0.737154
 -0.737154  0.675725
```

In [54]:

```
[a b; -b a]/sqrt(a^2+b^2)
```

Out[54]:

```
2x2 Array{Float64,2}:
  0.675725  0.737154
 -0.737154  0.675725
```