Lecture 21: Images

In this lecture we consider manipulating images in Julia. Before that, we wrap up some loose ends on stability:

Back substition is backward stable

Using similar logic to the dot product case from last lecture, the following theorem can be proven, showing that back substitution is *backward stable*:

Theorem Approximating the problem $f(U) = U^{-1}\mathbf{b}$ by backsubstition `\tilde f(U) = {\rm backsubstitution(U,\mathbf b)` in floating point arithmetic satisfies

$$f(\tilde{U}) = f(U + \Delta U)$$

where the relative backward error satisfies

$$\frac{\|\Delta U\|_\infty}{\|U\|_\infty} \leq \frac{n\epsilon}{1 - n\epsilon}.$$

A trivial consequence is that the forward error is small provided that the ∞-condition number

$$\kappa_\infty(U) \triangleq \|U\|_\infty \|U^{-1}\|_\infty$$

is small:

Corollary

$$\frac{\|f(U) - f(\tilde{U})\|_\infty}{\|f(U)\|_\infty} \leq \kappa_\infty(U) \frac{n\epsilon}{1 - n\epsilon}$$

We omit the precise statement, but we also have that QR with Given's rotations is backward stable.

PLU is not stable

We finally come to a surprise: the PLU decomposition is not stable! We can demonstrate this on a very simple example:

$$A = \begin{pmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & 1 & & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}$$

This matrix is well-conditioned:

In [1]:

```
n=100

A=2eye(n)-tril(ones(n,n))
A[1:n-1,end]=ones(n-1)

cond(A)
```

Out[1]:

```
44.80225124630286
```

The QR Decomposition, because it is stable, preserves this condition number:

In [2]:

```
Q,R=qr(A)

cond(Q),cond(R)
```

Out[2]:

```
(1.0000000000000018,44.80225124630287)
```

The PLU Decomposition, on the other hand, has very badly conditioned components (in this case $P = I$:

In [3]:

```
L,U,p=lu(A)

cond(L),cond(U)
```

Out[3]:

```
(9.345713008686627e17,8.451004001521529e29)
```

This bad conditioning translates into very inaccurate solution, even for the inbuilt \ command. This compares unfavourably with the QR Decomposition, which is perfectly accurate:

In [7]:

```
b=rand(n)
x_backslash=A\b
x_QR=(R\(Q'*b))
x_LU=U\(L\b)
x_inv=inv(A)*b

norm(x_inv-x_QR),norm(x_LU-x_inv)
```

Out[7]:

```
(4.943922429291265e-15,9.685147727311062e6)
```

We can check the *error in residual*: that is, see how well the approximation satisfies $Ax = b$:

In [8]:

```
norm(A*x_inv-b),norm(A*x_LU-b)
```

Out[8]:

```
(9.27040380578984e-15,7.746521550447148e7)
```

Perturbing the matrix A by a small amount causes the PLU decomposition to become stable:

In [44]:

```
n=100

A=2eye(n)-tril(ones(n,n))
A[1:n-1,end]=ones(n-1)

A=A+0.0001*randn(n,n)

Q,R=qr(A)
L,U,p=lu(A)

cond(A),cond(L),cond(U)
```

Out[44]:

```
(44.80474788446134,210.96228170781936,68.99809165265951)
```

This is a big open problem: explaining why with high probability that PLU Decomposition is stable. This is the reason \ uses PLU: the chance of failure is small, and PLU is roughly 2x as fast as QR.

Images in Julia

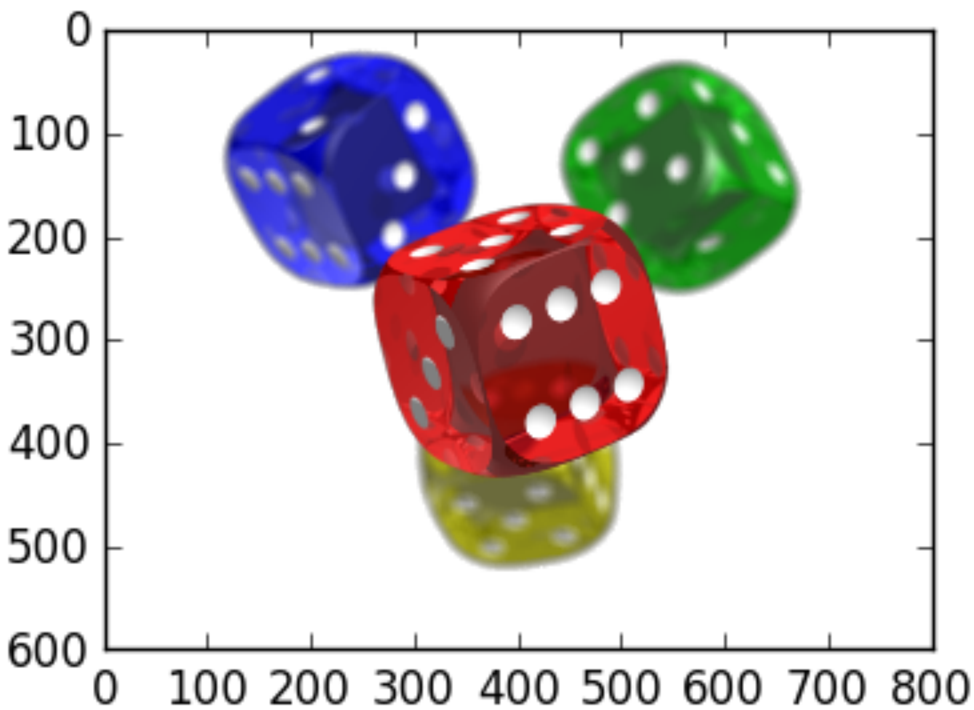
We now consider image analysis. We can load an image in Julia using PyPlot:

In [11]:

```
using PyPlot

img=imread("/Users/solver/Desktop/PNG_transparency_demonstration_1.png");

imshow(img);
```



This is a 600 x 800 pixel image. Each pixel has 4 channels: the red, green, blue and alpha component. This is stored by a 600 x 800 x 4 *tensor*:

In [14]:

```
size(img)
```

Out[14]:

```
(600,800,4)
```

Each value is between 0. and 1.:

In [23]:

```
maximum(img),minimum(img)
```

Out[23]:

```
(1.0f0,0.0f0)
```

A tensor is like a matrix just with one more index. For example:

In [15]:

```
A=rand(3)     # random vector of length 3
A=rand(3,3)   # random matrix of size 3 x 3
A=rand(3,3,3) # random matrix of size 3 x 3 x 3
```

Out[15]:

```
3x3x3 Array{Float64,3}:
[:, :, 1] =
 0.225026  0.679698   0.875816
 0.161986  0.0402596  0.269381
 0.515684  0.640174   0.654053

[:, :, 2] =
 0.175512  0.0302496  0.329362
 0.893919  0.262348   0.400141
 0.720663  0.82769    0.650222

[:, :, 3] =
 0.353372  0.81447    0.0399681
 0.079338  0.390212   0.735976
 0.668979  0.0419518  0.502528
```

The entries are accessed using three components:

In [16]:

```
A[1,3,2]
```

Out[16]:

```
0.3293624516851563
```

Just like matrices, tensors are actually stored as a single vector in memory:

In [18]:

```
vec(A)
```

Out[18]:

```
27-element Array{Float64,1}:
 0.225026
 0.161986
 0.515684
 0.679698
 0.0402596
 0.640174
 0.875816
 0.269381
 0.654053
 0.175512
 0.893919
 0.720663
 0.0302496
 ⋮
 0.329362
 0.400141
 0.650222
 0.353372
 0.079338
 0.668979
 0.81447
 0.390212
 0.0419518
 0.0399681
 0.735976
 0.502528
```

We can access the R,G,B and A components by creating 600 x 800 matrices follows:

In [20]:

```
R=img[:,:,1]
G=img[:,:,2]
B=img[:,:,3]
A=img[:,:,3]

size(R)
```

Out[20]:

```
(600,800)
```

We now create a grey-scale image, by removing the A component and setting the R,G and B components to the same values. We also swap white and black to make it more visible:

In [28]:

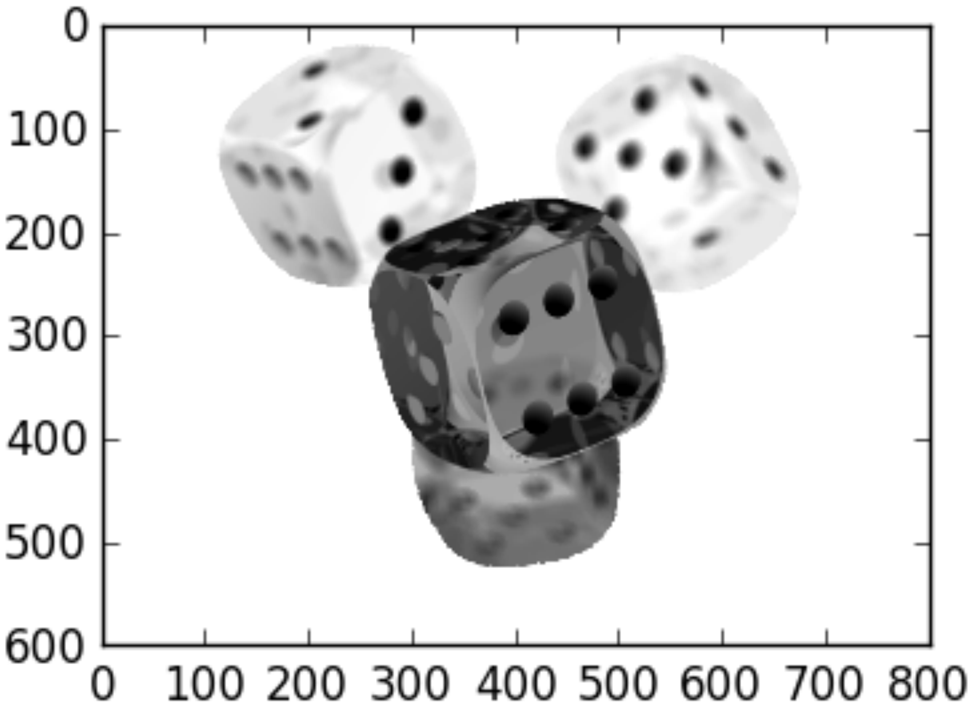
```
myimg=zeros(size(img,1),size(img,2),3)



myimg[:,:,1]=1-R
myimg[:,:,2]=1-R
myimg[:,:,3]=1-R

grey=myimg[:,:,1]

imshow(myimg)
```



Out[28]:

PyObject <matplotlib.image.AxesImage object at 0x3152d4e50>