ecen 4632 problem set 1

Problem 1

Using Euler's formula we get

$$X(e^{j\omega}) = \frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega} + \frac{j}{2}e^{-2j\omega} - \frac{j}{2}e^{j2\omega}$$

and thus

$$x[n] = \begin{cases} \frac{1}{2}j & \text{if } n = -2 \\ \frac{1}{2} & \text{if } n = -1 \\ 0 & \text{if } n = 0 \\ \frac{1}{2} & \text{if } n = 1 \\ -\frac{1}{2}j & \text{if } n = 2 \end{cases}$$

Problem 2.

We use the properties of the Fourier transform.

a) $X(e^{j\omega})\Big|_{\omega=0} = \sum x[n] = 0$

c) $\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega = 2\pi\, x[0]$ (reconstruction formula)

$= 8\pi$

d) $X(e^{j\pi}) = \sum (-1)^n x[n] = 4$

e) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum |x[n]|^2$ (Parseval)

$= 56\pi$

b) Because $x[n]$ is real and even, $X(e^{j\omega})$ is real and even so

$\measuredangle X(e^{j\omega}) = 0$ (or π if $X(e^{j\omega}) < 0$)

f) $\frac{d}{d\omega} X(e^{j\omega})\Big|_{\omega=0} = -j \sum n\, x[n] = 0$

Problem 2

1) $y[n] = e^{j\pi n} x[n]$

So $Y(e^{j\omega}) = X(e^{j(\omega-\pi)})$

2) $\sum_{n=-\infty}^{\infty} (-1)^n x[n] = X(e^{j\omega})|_{\omega=\pi}$

3) The system is best described in the time domain, where

$$y[n] = x[n] + (-1)^n x[n].$$

or
$$\begin{cases} y[2n] = 2x[2n] \\ y[2n+1] = 0 \end{cases}$$

The system is linear, but time variant:

consider $x[n] = \delta[n]$, then $y[n] = 2\delta[n]$

also $x[n-1] = \delta[n-1]$ is mapped to 0 for all n, which is not $y[n-1] = 2\delta[n-1]$.

5) Obvious from the expression of y

2.47. $$x[n] = w[n]\cos(\omega_0 n)$$

A. Fourier transforming gives

$$\begin{aligned} X\left(e^{j\omega}\right) &= \frac{1}{2\pi} W\left(e^{j\omega}\right) * \left\{\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right\} \\ &= \frac{1}{2\pi}\left\{\pi W\left(e^{j(\omega-\omega_0)}\right) + \pi W\left(e^{j(\omega+\omega_0)}\right)\right\} \\ &= \tfrac{1}{2} W\left(e^{j(\omega-\omega_0)}\right) + \tfrac{1}{2} W\left(e^{j(\omega+\omega_0)}\right), \end{aligned}$$

for $-\pi < \omega \le \pi$.

B. We know from tables that if

$$y[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise,} \end{cases}$$

then the DTFT $Y\left(e^{j\omega}\right)$ is

$$Y\left(e^{j\omega}\right) = \frac{\sin\left(\omega(M+1)/2\right)}{\sin(\omega/2)} e^{-j\omega M/2}$$

Let $M = 2L$. Then we have

$$y[n] = \begin{cases} 1, & 0 \le n \le 2L \\ 0, & \text{otherwise,} \end{cases}$$

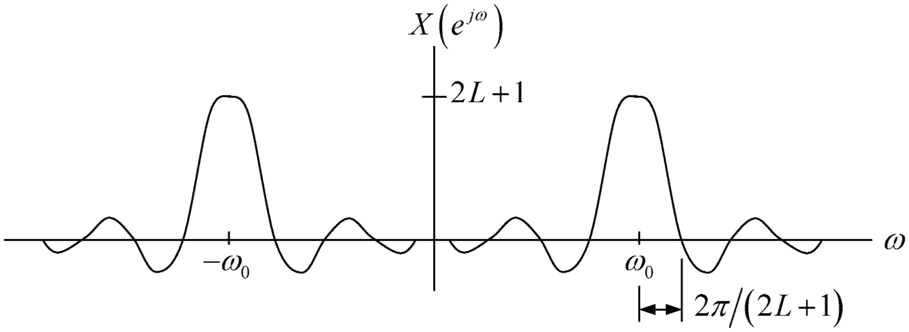
with DTFT

$$Y\left(e^{j\omega}\right) = \frac{\sin\left(\omega(2L+1)/2\right)}{\sin(\omega/2)} e^{-j\omega L}$$

Now $w[n] = y[n+L]$, which implies $W\left(e^{j\omega}\right) = Y\left(e^{j\omega}\right)e^{j\omega L}$. That is,

$$W\left(e^{j\omega}\right) = \frac{\sin\left(\omega(2L+1)/2\right)}{\sin(\omega/2)}$$

C. $X\left(e^{j\omega}\right) = \tfrac{1}{2} W\left(e^{j(\omega-\omega_0)}\right) + \tfrac{1}{2} W\left(e^{j(\omega+\omega_0)}\right)$



As ω_0 gets closer to $\omega = 0$, the two peaks merge into a single peak. We will have two distinct peaks if $\omega_0 \ge \dfrac{2\pi}{2L+1}$.

2.48. (a) Notice that $x_1[n] = x_2[n] + x_3[n+4]$, so if $T\{\cdot\}$ is linear,

$$\begin{aligned} T\{x_1[n]\} &= T\{x_2[n]\} + T\{x_3[n+4]\} \\ &= y_2[n] + y_3[n+4] \end{aligned}$$

From Fig P2.4, the above equality is not true. Hence, the system is NOT LINEAR.
(b) To find the impulse response of the system, we note that

$$\delta[n] = x_3[n+4]$$

Therefore,

$$\begin{aligned} T\{\delta[n]\} &= y_3[n+4] \\ &= 3\delta[n+6] + 2\delta[n+5] \end{aligned}$$

(c) Since the system is known to be time-invariant and not linear, we cannot use choices such as:

$$\delta[n] = x_1[n] - x_2[n]$$

and

$$\delta[n] = \frac{1}{2}x_2[n+1]$$

to determine the impulse response. With the given information, we can only use shifted inputs.

2.55. (a)

$$
\begin{aligned}
X(e^{j\omega})|_{\omega=0} &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}|_{\omega=0} \\
&= \sum_{n=-\infty}^{\infty} x[n] \\
&= 6
\end{aligned}
$$

(b)

$$
\begin{aligned}
X(e^{j\omega})|_{\omega=\pi} &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} \\
&= \sum_{n=-\infty}^{\infty} x[n](-1)^n \\
&= 2
\end{aligned}
$$

(c) Because $x[n]$ is symmetric about $n = 2$ this signal has linear phase.

$$X(e^{j\omega}) = A(\omega)e^{-j2\omega}$$

$A(\omega)$ is a zero phase (real) function of ω. Hence,

$$\angle X(e^{j\omega}) = -2\omega, \qquad -\pi \leq \omega \leq \pi$$

(d)

$$\int_{-\pi}^{\pi} X(e^{j\omega})e^{-j\omega n}d\omega = 2\pi x[n]$$

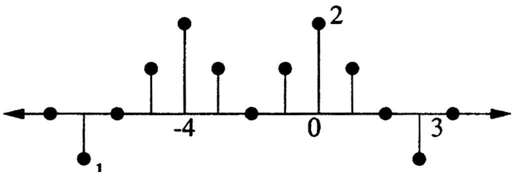
for $n = 0$:

$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0] = 4\pi$$

(e) Let $y[n]$ be the unknown sequence. Then

$$
\begin{aligned}
Y(e^{j\omega}) &= X(e^{-j\omega}) \\
&= \sum_n x[n]e^{j\omega n} \\
&= \sum_n x[-n]e^{-j\omega n} \\
&= \sum_n y[n]e^{-j\omega n}
\end{aligned}
$$

Hence $y[n] = x[-n]$.



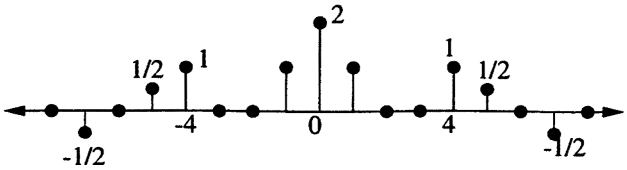
(f) We have determined that:

$$X(e^{j\omega}) = A(\omega)e^{-j2\omega}$$

$$\begin{aligned} X_R(e^{j\omega}) &= \mathcal{R}e\{X(e^{j\omega})\} \\ &= A(\omega)\cos(2\omega) \\ &= \frac{1}{2}A(\omega)\left(e^{j2\omega} + e^{-j2\omega}\right) \end{aligned}$$

Taking the inverse transform, we have

$$\frac{1}{2}a[n+2] + \frac{1}{2}a[n-2] = \frac{1}{2}x[n+4] + \frac{1}{2}x[n]$$



2.67. (a) Note that $x_2[n] = -\sum_{k=0}^{k=4} x[n-k]$. Since the system is LTI, we have:

$$y_2[n] = -\sum_{k=0}^{k=4} y[n-k].$$

(b) By carrying out the convolution, we get:

$$h[n] = \begin{cases} -1 & , \quad n = 0, n = 2 \\ -2 & , \quad n = 1 \\ 0 & , \quad \text{o.w.} \end{cases}$$

2.71. Let the input be $x[n] = \delta[n-1]$, if the system is causal then the output, $y[n]$, should be zero for $n < 1$. Let's evaluate $y[0]$:

$$
\begin{aligned}
y[0] &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} Y(e^{j\omega})\, d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-j\omega} e^{-j\omega/2}\, d\omega \\
&= -\frac{2}{3\pi} \\
&\neq 0.
\end{aligned}
$$

This proves that the system is not causal.