Problem 1

We consider a band-limited signal $x(t)$,

$$X(j\Omega) = 0 \quad \text{if } |\Omega| \geq |\Omega_1.$$

We consider the impulse response of a filter, $h(t)$, which is also band-limited,

$$H(j\Omega) = 0 \quad \text{if } |\Omega| \geq |\Omega_2.$$

The continuous signal $x(t)$ is convolved with the filter $h(t)$,

$$y(t) = h * x(t).$$

The signal $y(t)$ is then sampled with a sampling period T,

$$y[n] = y(nT).$$

Finally, we also sample the impulse response $h(t)$ with the sampling period T,

$$h[n] = h(nT).$$

We recall the following identity,

$$X(e^{j\omega}) = X_s(j\omega/T),$$

where X_s is the Fourier transform of the analog sampled signal, $x_s(t)$, obtained by sampling $x(t)$ with the period T. You can apply this identity to any signal sampled with the period T.

1. Express the Fourier transform of $y[n]$, $Y(e^{j\omega})$, as a function of $X(j\Omega)$ and $H(j\Omega)$.
2. We wish to reverse the order of the sampling and filtering. We therefore sample $x(t)$ with the same sampling period, T,

   $$x[n] = x(nT).$$

   Express the Fourier transform of $x[n]$, $X(e^{j\omega})$, as a function of $X(j\Omega)$
3. We then filter the digital signal $x[n]$ using the digital filter,

   $$g[n] = Th[n].$$

   Express the Fourier transform of $g[n]$, $G(e^{j\omega})$, as a function of $H(j\Omega)$.
4. Let $w[n]$ be the output of the convolution of $x[n]$ and $g[n]$,

   $$w[n] = g * x[n].$$

   Express the Fourier transform of $w[n]$, $W(e^{j\omega})$, as a function of $X(j\Omega)$ and $H(j\Omega)$.
5. What is the condition on T for the output of the two systems to be equal,

   $$\forall n \in \mathbb{Z}, w[n] = y[n].$$

   Hint: this is equivalent to proving that $W(e^{j\omega}) = Y(e^{j\omega})$.

Problem 2

Consider the multirate system shown in Fig. 1 with input $x[n]$. The Fourier transform of $x[n]$ and the frequency response of H_0 and H_1 are shown in Fig. 2. Let $Y_0(e^{j\omega})$ and $Y_1(e^{j\omega})$ be the Fourier transforms of the outputs $y_0[n]$ and $y_1[n]$, respectively.

1. Express $Y_0(e^{j\omega})$ and $Y_1(e^{j\omega})$ in terms of $X(e^{j\omega})$.
2. Sketch $Y_0(e^{j\omega})$ and $Y_1(e^{j\omega})$.

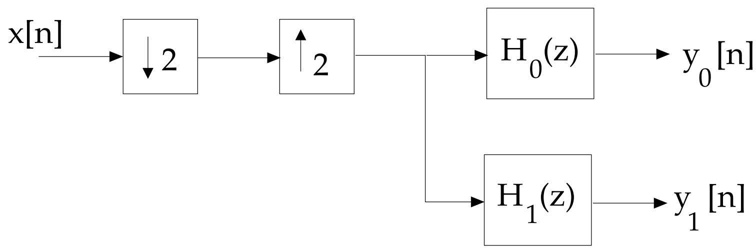

Figure 1:

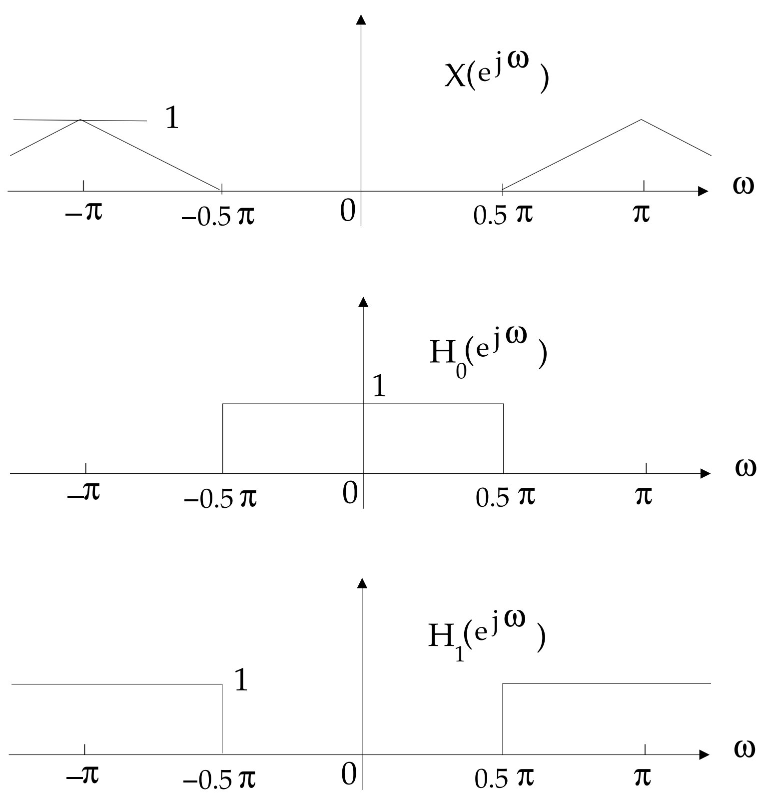

Figure 2:

Problem 3

The continuous time signal $x_c(t)$ is band limited, and has the Fourier transform

$$X_c(j\Omega) = \begin{cases} \frac{\pi}{\Omega_0}\left[1+\cos(\frac{\pi\Omega}{\Omega_0})\right] & \text{if } |\Omega| \le \Omega_0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The signal $x_c(t)$ is sampled with a period $T = 2\pi/\Omega_0$.

1. Determine the expression of $x[n] = x_c(nT)$.

Problem 4

We consider a type I generalized linear phase filter $h(n)$ with real coefficients and order $2M$. Let $H(\omega)$ be the frequency response of $h(n)$. We assume that H is a low pass filter with the following specifications :

$$\begin{cases} 1-\delta_p \le H(\omega) \le 1+\delta_p & \text{if } \omega \in [0, \omega_p), \\ -\delta_s \le H(\omega) \le \delta_s & \text{if } \omega \in [\omega_s, \pi), \end{cases} \tag{2}$$

with $\delta_s < 1 - \delta_p$.

1. We define a new filter $g(n)$ as follows

   $$g(n) = \begin{cases} h(n) & \text{if } n \ne M, \\ h(M) + \delta_s & \text{if } n = M. \end{cases} \tag{3}$$

   Prove that $G(\omega)$, the frequency response of $g(n)$, can be written as

   $$G(\omega) = P(\omega)e^{-jM\omega}, \tag{4}$$

   where $P(\omega)$ is real and positive : $P(\omega) \ge 0$.

2. Prove that there exists a minimum phase filter $Q(z)$ such that

   $$P(z) = Q(z)Q(1/z). \tag{5}$$

Problem 5

Let H_1 and H_2 be two first order stable all-pass filters. Prove that if

$$H(z) = H_1(z) + H_2(z) \tag{6}$$

is all-pass then $H_1 = H_2$.