Problem 1

1) We have $y(t+T) = \sum_{n=-\infty}^{\infty} x(t+T-nT)$

$$= \sum_{m=-\infty}^{\infty} x(t-mT) \quad \text{where } m = n-1$$

$$= y(t).$$

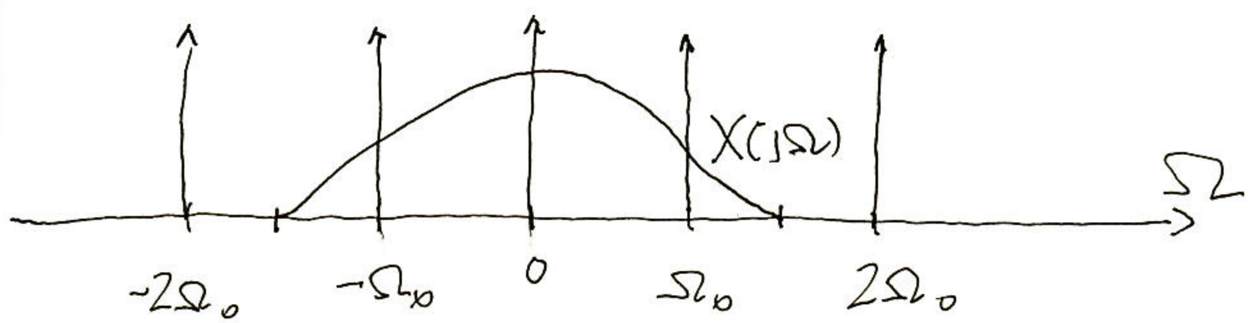
2) The Fourier transform of $y(t)$, $Y(j\Omega)$ is given by

$$Y(j\Omega) = \sum_{n=-\infty}^{\infty} e^{-jn\Omega T} X(j\Omega)$$

$$= \left(\sum_{n=-\infty}^{\infty} e^{-jnT\Omega} \right) X(j\Omega)$$

$$= \frac{2\pi}{T} \left(\sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{T}\right) \right) X(j\Omega).$$

$$= \frac{2\pi}{T} \left(\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0) \right) X(j\Omega)$$



$$Y(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} X(jk\Omega_0)\,\delta(\Omega - k\Omega_0)$$

But since X is band limited on $(-1.5\Omega_0, 1.5\Omega_0)$

$X(jk\Omega_0) = 0$ except for $k = -1, 0, 1$. (see figure).

Hence

$$Y(j\Omega) = \frac{2\pi}{T}\Big\{X(j\Omega_0)\,\delta(\Omega - \Omega_0) + X(-j\Omega_0)\,\delta(\Omega + \Omega_0) + X(0)\,\delta(\Omega)\Big\}$$

If we inverse Fourier transform this equality we get:

$$y(t) = \frac{1}{T}\Big\{X(j\Omega_0)\,e^{j\Omega_0 t} + X(-j\Omega_0)\,e^{-j\Omega_0 t} + X(0)\Big\}$$

Since $x(t)$ is real $X(-j\Omega_0) = \int_{-\infty}^{\infty} x(t)\,e^{-j\Omega_0 t}\,dt$

$$= \left(\int_{-\infty}^{\infty} x(t)\,e^{j\Omega_0 t}\,dt\right)^*$$

If we write $X(j\Omega_0) = |X(j\Omega_0)|\,e^{j\measuredangle X(j\Omega_0)}$

we have

$$y(t) = \frac{1}{T}X(0) + \frac{2}{T}|X(j\Omega_0)|\frac{1}{2}\Big\{e^{j\Omega_0 t + \measuredangle X(j\Omega_0)} + e^{-(j\Omega_0 t + \measuredangle X(j\Omega_0))}\Big\}$$

In other words

$$y(t) = \underbrace{\frac{1}{T}X(0)}_{A} + \underbrace{\frac{2}{T}|X(j\Omega_0)|}_{B} \cos\Big(\underset{\substack{\| \\ \frac{2\pi}{T}}}{\Omega_0} t + \underbrace{\measuredangle X(j\Omega_0)}_{\varphi}\Big)$$

Problem 2

1. Define $x_2(t) = x(t/2)$. The Fourier transform of x_2 is given by

$$X_2(j\Omega) = 2X(2j\Omega)$$

X_2 is band limited on the interval $[-\Omega_0/2, \Omega_0/2]$.

and can be sampled with $\frac{T}{2} = \frac{\pi}{\Omega_0/2} = \frac{2\pi}{\Omega_0} = 2T$ without aliasing.

the sampled signal is given by

$$\sum_{n=-\infty}^{\infty} x_2(2Tn)\,\delta(t - n2T) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - n2T)$$

which is exactly equal to $y(t)$.

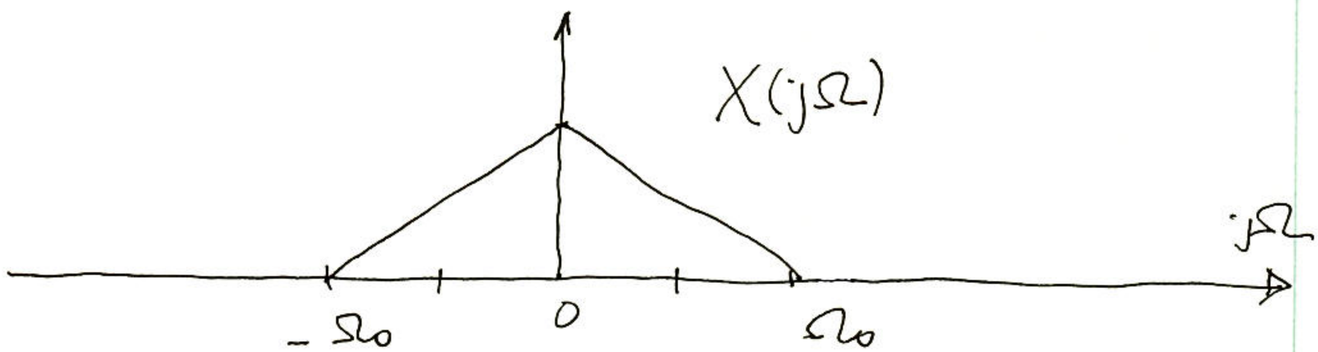
The Fourier transform of $y(t)$ is therefore the Fourier transform of the sampled signal $x_2(t)$ with period $2T$

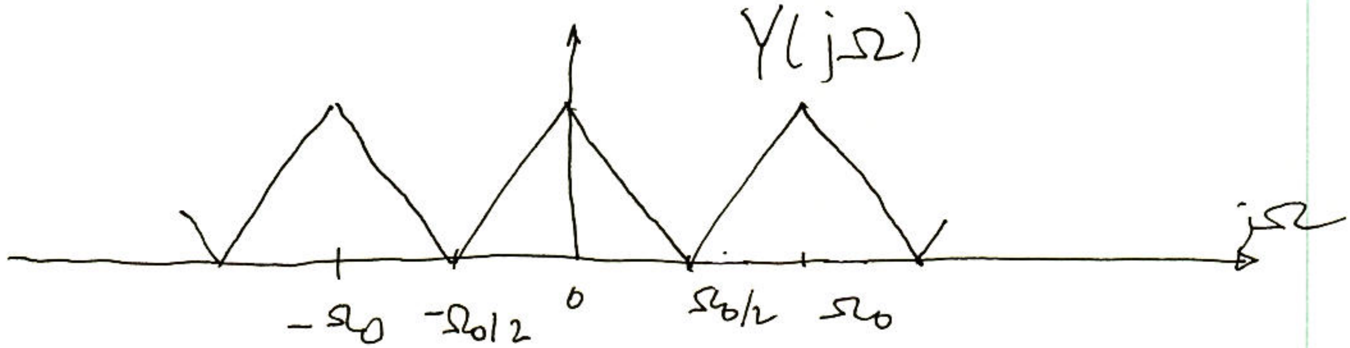
$$Y(j\Omega) = \frac{1}{2T}\sum_{k=-\infty}^{\infty} X_2\left(j\left(\Omega - \frac{2\pi k}{2T}\right)\right)$$

$$Y(j\Omega) = \frac{1}{2T} \sum_{k=-\infty}^{\infty} X_2(j(\Omega - k\frac{\pi}{T})) = \frac{1}{2T} \sum_{k=-\infty}^{\infty} 2X(2j(\Omega - k\frac{\pi}{T}))$$

$$Y(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j2(\Omega - k\Omega_0))$$

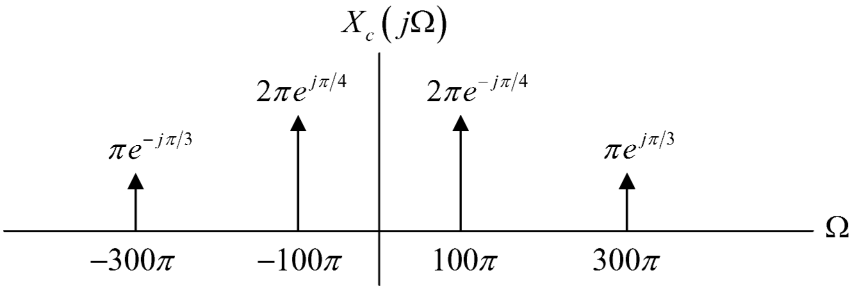
2) As explained before there is no aliasing. This is also clear from the Fourier transform of $y(t)$, which is the same as the Fourier transform of the sampled $x(t)$ with sampling period T when the frequency is scaled by a factor 2.



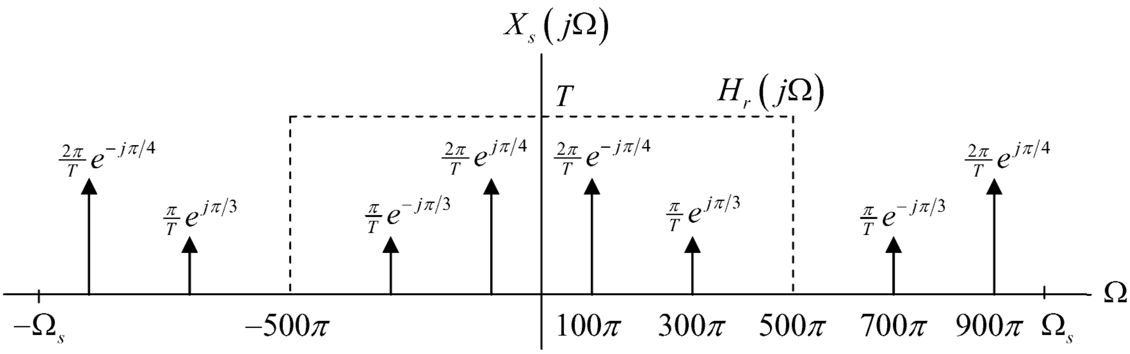


4.28. A.

$$X_c(j\Omega) = 2\pi e^{-j\pi/4}\delta(\Omega-100\pi) + 2\pi e^{j\pi/4}\delta(\Omega+100\pi) + \pi e^{j\pi/3}\delta(\Omega-300\pi) + \pi e^{-j\pi/3}\delta(\Omega+300\pi).$$



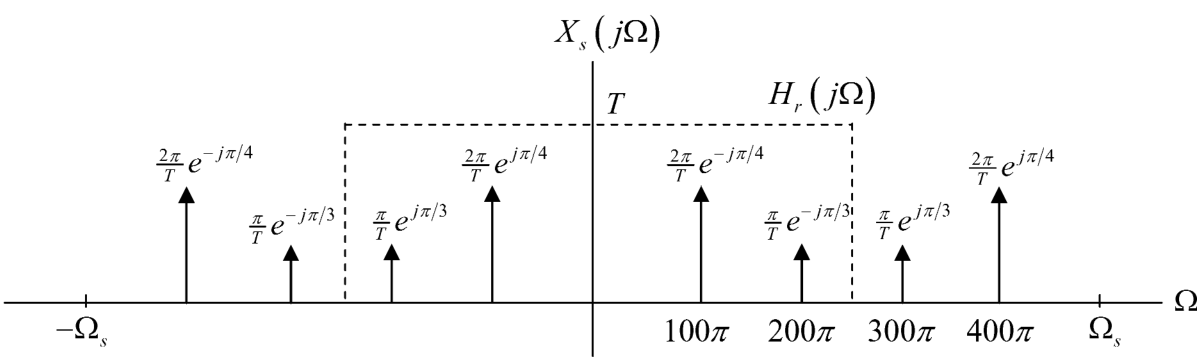
B. If $f_s = 1/T = 500$ samples/s then $\Omega_s = 2\pi/T = 1000\pi$ rad/s.



There is no aliasing, so $x_r(t) = x_c(t)$; that is,

$$x_r(t) = 2\cos(100\pi t - \pi/4) + \cos(300\pi t + \pi/3).$$

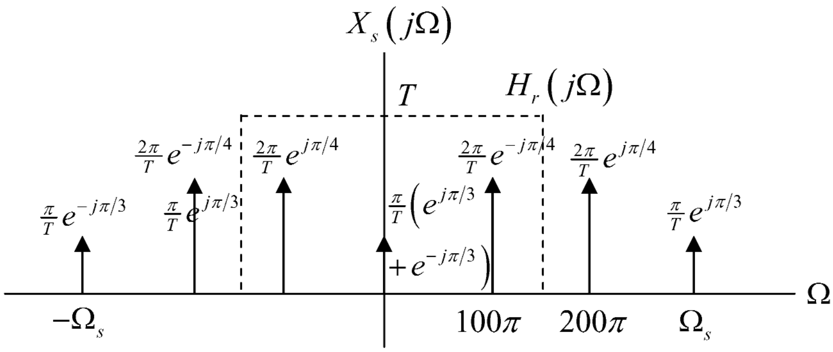
C. If $f_s = 1/T = 250$ samples/s then $\Omega_s = 2\pi/T = 500\pi$ rad/s.



Now there is aliasing and

$$x_r(t) = 2\cos(100\pi t - \pi/4) + \cos(200\pi t - \pi/3).$$

D. We want to sample the component at 300π rad/s exactly once per cycle, so that all the samples have the same value. At $\Omega_s = 300\pi$ rad/s we have

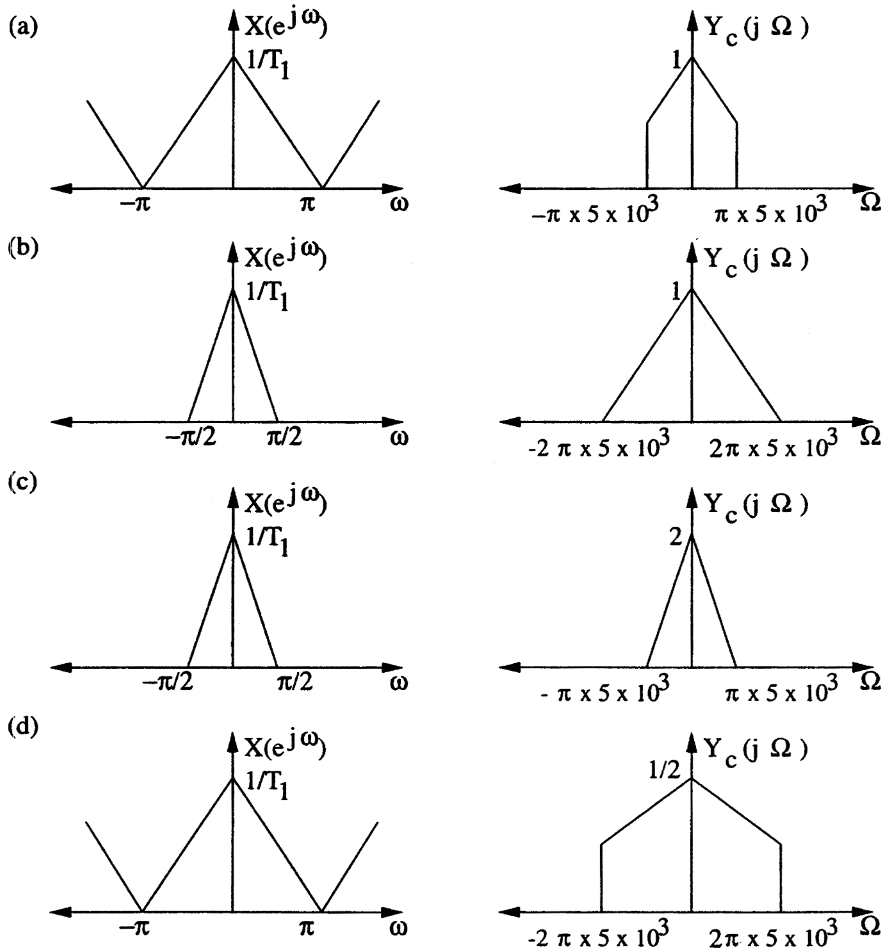


Now

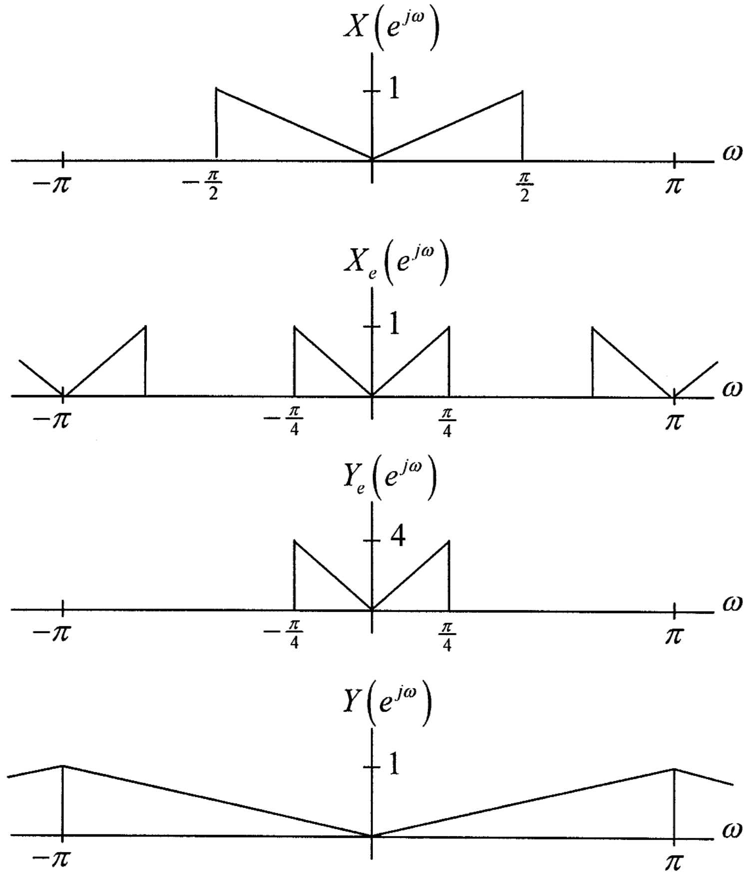
$$x_r(t) = \cos(\pi/3) + 2\cos(100\pi - \pi/4)$$
$$= 1/2 + 2\cos(100\pi - \pi/4).$$

We have $A = 1/2$.

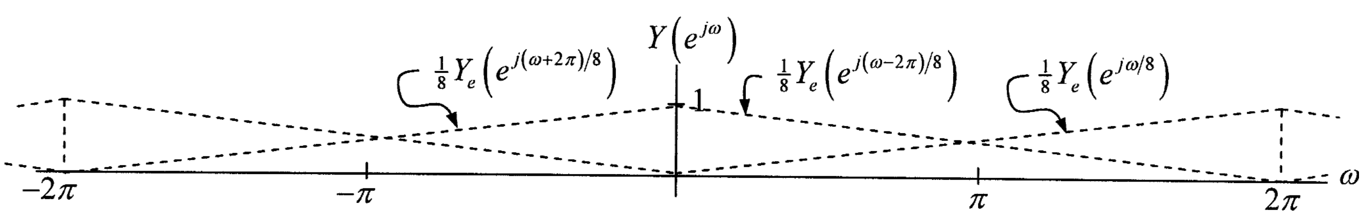
4.30. The Fourier transform of $y_c(t)$ is sketched below for each case.



4.32. A. With $L=2$ and $M=4$,



B. With $L=2$ and $M=8$, $X_e\left(e^{j\omega}\right)$ and $Y_e\left(e^{j\omega}\right)$ remain as in part A, except that $Y_e\left(e^{j\omega}\right)$ now has a peak value of 8. After expanding we have



We see that $Y\left(e^{j\omega}\right)=1$ for all ω. Inverse transforming gives $y[n]=\delta[n]$ in this case.

4.33.

$$x[n] \rightarrow \boxed{\uparrow 5} \rightarrow \boxed{\downarrow 3} \rightarrow \boxed{\downarrow 5} \rightarrow \boxed{\uparrow 3} \rightarrow y[n]$$

Since 3 and 5 are relatively prime, the order of the two operations in the center can be interchanged. This gives

$$x[n] \rightarrow \boxed{\uparrow 5} \rightarrow \boxed{\downarrow 5} \rightarrow \boxed{\downarrow 3} \rightarrow \boxed{\uparrow 3} \rightarrow y[n]$$

Expanding by 5 and immediately compressing by 5 produces no net effect. We have

$$x[n] \rightarrow \boxed{\downarrow 3} \xrightarrow{x_1[n]} \boxed{\uparrow 3} \rightarrow y[n]$$

Compressing by 3 produces

$$x_1[n] = x[3n].$$

Expanding by 3 now gives

$$y[n] = \begin{cases} x_1[n/3], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$y[n] = \begin{cases} x[n], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

(a) There is one output sample generated for every pair of input samples. Even input samples require 3 multiplies and odd input samples require 2 multiplies. Thus each pair requires 5 multiplies.

(b) Applying the compressor identity to the previous structure results in:

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2).$$

From the difference equations in the previous part we have:

$$H_0(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-1}}{1 - \frac{1}{4}z^{-1}},$$

and

$$H_1(z) = \frac{\frac{1}{12}}{1 - \frac{1}{4}z^{-1}}.$$

Thus,

$$H(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-2} + \frac{1}{12}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{-\frac{1}{3}(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{-\frac{1}{3} + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

Therefore, $a = 1/2, b = -1/3$ and $c = 1/4$.

(c) In this implementation 3 multiplies are required for every input sample. For every output sample we need to calculate 2 values of $v[n]$. Altogether we need 6 multiplies per output sample.