Problem: 1

1) $X(z) = \frac{2}{3}\left\{\frac{1}{1-2z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}\right\}$

Because $x[n]$ is even, the ROC is $\frac{1}{2} < |z| < 2$, and the sequence is

$$x[n] = -\frac{2}{3}\,\frac{1}{2^{|n|}}$$

2) If $z_0 \in ROC(Y)$ then $\sum_n y[n] z_0^{-n}$ converges

But since y is even

$$\sum y[n] z_0^{-n} = \sum y[-n] z_0^{-n} = \sum y[n] z_0^{n}$$

$$= \sum y[n]\left(\frac{1}{z_0}\right)^{-n}$$

or $\boxed{Y(z_0) = Y\left(\frac{1}{z_0}\right)}$

Hence $\frac{1}{z_0}$ belongs to the ROC

Because $y[n]$ is even $ROC(Y) = \{R_1 < |z| < R_2\}$

Now $Y(z_0) = Y\left(\frac{1}{z_0}\right) \Rightarrow ROC = \phi$ if $R_2 < 1$

or $ROC = \{z \mid \frac{1}{R_2} < |z| < R_2\}$

Problem 2

$$G(z) = \sum h[2n] z^{-n}$$

$$G(z^2) = \sum h[2n] z^{-2n}$$

$$\frac{1}{2}\left\{ H(z) + H(-z) \right\} = \sum h[2n] z^{-2n} = G(z^2)$$

$$\frac{z^{-1} - a^*}{1 - az^{-1}} + \frac{-z^{-1} - a^*}{1 + az^{-1}} = \frac{(1+az^{-1})(z^{-1}-a^*) - (1-az^{-1})(z^{-1}+a^*)}{1 - a^2 z^{-2}}$$

$$= \frac{z^{-1} - a^* + az^{-2} - |a|^2 z^{-1} - (z^{-1} + a^* - az^{-2} - |a|^2 z^{-1})}{1 - a^2 z^{-2}}$$

$$= \frac{-2a^* + 2az^{-2}}{1 - a^2 z^{-2}} = 2\left(\frac{-a^* + az^{-2}}{1 - a^2 z^{-2}} \right)$$

$$G(z) = \frac{-a^* + az^{-1}}{1 - a^2 z^{-1}} = -a^* \frac{(1 - a/a^* z^{-1})}{(1 - a^2 z^{-1})}$$

$$|G(e^{j\omega})|^2 = 1 \Rightarrow |a|^2 \left|1 - \frac{a}{a^*} e^{-j\omega}\right|^2 = |1 - a^2 e^{-j\omega}|^2$$

$$a = \rho e^{j\theta} \quad \frac{a}{a^*} = e^{j2\theta} \quad a^2 = \rho^2 e^{j2\theta}$$

$$\rho^2 |1 - e^{2j\theta} e^{-j\omega}|^2 = |1 - \rho^2 e^{2j\theta} e^{-j\omega}|$$

$\omega = 2\theta$ yields $\quad 0 = |1 - \rho^2| \quad$ or $\rho = 1$

Going back to $H(z)$, we have

$$H(z) = \frac{z^{-1} - e^{-j\theta}}{1 - e^{j\theta} z^{-1}} = -e^{-j\theta} \frac{(1 - e^{+j\theta} z^{-1})}{1 - e^{j\theta} z^{-1}}$$

So $\boxed{h[n] = -e^{-j\theta} \delta[n]}$

3.30. $$H(z)=\frac{1-z^{-1}}{1-0.25z^{-2}}=\frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.5z^{-1})}$$

A. Given $x[n]=u[n]$, we have $X(z)=\frac{1}{1-z^{-1}}, \quad 1<|z|$. Then

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.5z^{-1})}\frac{1}{1-z^{-1}} \\ &= \frac{1}{(1-0.5z^{-1})(1+0.5z^{-1})} \\ &= \frac{\frac{1}{2}}{(1-0.5z^{-1})}+\frac{\frac{1}{2}}{(1+0.5z^{-1})}, \quad 0.5<|z|. \end{aligned}$$

(The ROC for $Y(z)$ includes the intersection of the ROC of $H(z)$ with the ROC of $X(z)$.)

Inverse z-transforming gives

$$y[n]=\tfrac{1}{2}(0.5)^n u[n]+\tfrac{1}{2}(-0.5)^n u[n].$$

B. If $y[n]=\delta[n]-\delta[n-1]$, then $Y(z)=1-z^{-1}, \quad 0<|z|$. We have

$$\begin{aligned} X(z) &= \frac{Y(z)}{H(z)} \\ &= \frac{1-z^{-1}}{\left(\frac{1-z^{-1}}{1-0.25z^{-2}}\right)} \\ &= 1-0.25z^{-2}, \quad 0<|z|. \end{aligned}$$

Inverse z-transforming gives

$$x[n]=\delta[n]-0.25\delta[n-2].$$

C. Now $x[n]=\cos(0.5\pi n), \quad -\infty<n<\infty$. At $\omega=0.5\pi$ we have

$$\begin{aligned} H\left(e^{j0.5\pi}\right) &= \frac{1-e^{-j0.5\pi}}{1-0.25e^{-j\pi}} \\ &= 1.13e^{j\frac{\pi}{4}}. \end{aligned}$$

Then

$$y[n]=1.13\cos\left(0.5\pi n+\tfrac{\pi}{4}\right).$$

3.32. (a)

$$
\begin{aligned}
X(z) &= \frac{1}{(1+\frac{1}{2}z^{-1})^2(1-2z^{-1})(1-3z^{-1})} \qquad \frac{1}{2} < |z| < 2 \\
&= \frac{\frac{1}{35}}{(1+\frac{1}{2}z^{-2})^2} + \frac{\frac{88}{1225}}{(1+\frac{1}{2}z^{-1})} - \frac{\frac{1568}{1225}}{(1-2z^{-1})} + \frac{\frac{2700}{1225}}{(1-3z^{-1})}
\end{aligned}
$$

Therefore,

$$
x[n] = \frac{1}{35}(n+1)\left(\frac{-1}{2}\right)^{n+1} u[n+1] + \frac{58}{(35)^2}\left(\frac{-1}{2}\right)^{n} u[n] + \frac{1568}{(35)^2}(2)^n u[-n-1] - \frac{2700}{(35)^2}(3)^n u[-n-1]
$$

(b)

$$
X(z) = e^{z^{-1}} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \ldots
$$

Therefore, $x[n] = \frac{1}{n!}u[n]$.

(c)

$$
X(z) = \frac{z^3 - 2z}{z-2} = z^2 + 2z + \frac{2}{1-2z^{-1}} \qquad |z| < 2
$$

Therefore,

$$
x[n] = \delta[n+2] + 2\delta[n+1] - 2(2)^n u[-n-1]
$$

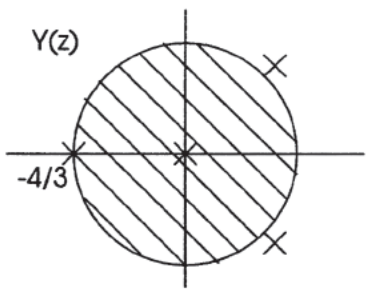
3.37. From the pole-zero diagram

$$X(z) = \frac{z}{(z^2 - z + \frac{1}{2})(z + \frac{3}{4})} \qquad |z| > \frac{3}{4}$$

$$\begin{aligned} y[n] &= x[-n+3] = x[-(n-3)] \\ &\Rightarrow \; Y(z) = z^{-3}X(z^{-1}) = \frac{z^{-3}z^{-1}}{(z^{-2} - z^{-1} + \frac{1}{2})(z^{-1} + \frac{3}{4})} \\ &= \frac{8/3}{z(2 - 2z + z^2)(\frac{4}{3} + z)} \end{aligned}$$

Poles at $0, -\frac{4}{3}, 1 \pm j$, zeros at ∞

$x[n]$ causal $\Rightarrow x[-n+3]$ is left-sided $\Rightarrow$ ROC is $0 < |z| < 4/3$.



3.39.

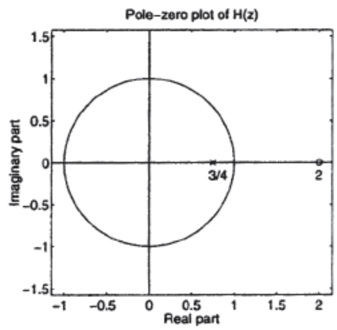
$$
\begin{aligned}
H(z) &= \frac{1 - z^3}{1 - z^4} = z^{-1}\left(\frac{1 - z^{-3}}{1 - z^{-4}}\right) \qquad |z| > 1 \\
u[n] &\Leftrightarrow \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \qquad |z| > 1 \\
U(z)H(z) &= \frac{z^{-1} - z^{-4}}{(1 - z^{-4})(1 - z^{-1})} \\
&= \frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-4}}{1 - z^{-4}} \qquad |z| > 1 \\
u[n] * h[n] &= u[n-1] - \sum_{k=0}^{\infty} \delta[n - 4 - 4k]
\end{aligned}
$$

3.45. (a)

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}}, \qquad \frac{1}{2} < |z| < 2$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{\frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})}}{\frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}} \\ &= \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4} \end{aligned}$$



(b)

$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$$

(c)

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$$

(d) The system is stable because the ROC includes the unit circle. It is also causal since $h[n] = 0$ for $n < 0$.