Problem 1

1) a) We have $w[n] = (h * v^-)[n] = \sum_m h[n-m]v[-m]$

and $w[-n] = \sum_m h[-n-m]v[-m]$

$$= \sum_p h[-n+p]v[p]$$

$$= h^- * v[n] = [(h^- * h) * x][n]$$

Therefore $h_1 = h^- * h$

where $h^-[n] = h[-n]$

b) The Fourier transform of h^- is given by

$$H^-(e^{jw}) = \sum_n h^-[n]e^{-jwn} = \sum_n h[-n]e^{-jwn}$$

$$= \sum h[n]e^{jwn} = H^*(e^{jw})$$

Therefore $H_1(e^{jw}) = H(e^{jw})H^*(e^{jw}) = A^2(w)$

$H_1(e^{jw})$ is a positive real number and has no phase.

2) Method B.

a) We have $u[n] = \sum_m h[n-m]x[-m]$

$$u[-n] = \sum_m h[-n-m]x[-m] = \sum_p h[-n+p]x[p]$$

$$= (h^- * x)[n]$$

Therefore $h_2 = h^- + h$

b) We have $H_2(e^{j\omega}) = H^-(e^{j\omega}) + H(e^{j\omega})$

$$= H^*(e^{j\omega}) + H(e^{j\omega})$$

Thus $H_2(e^{j\omega}) = 2A(\omega)\cos(\Psi(\omega))$.

$H_2(e^{j\omega})$ is real valued and has no phase.

Problem 2.

From Fig 1 we have the following equation

$$y[n] = h * [2x[n] - (h*x)[n]]$$

$$y[n] = 2(h*x)[n] - (h*h*x)[n]$$

therefore

$$g[n] = 2h[n] - (h*h)[n]$$

Since h is FIR so is g.

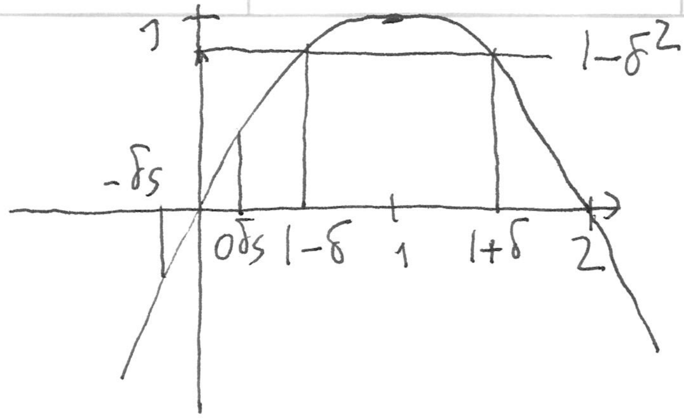
If h is symmetric, then $h*h$ is symmetric.

Therefore g is symmetric.

2. We have

$$G(e^{j\omega}) = 2H(e^{j\omega}) - H^2(e^{j\omega})$$

3) We observe that the graph of the function $2x - x^2$ is as follows



Therefore the interval $[1-\delta_p, 1+\delta_p]$ is mapped to

$$[1-\delta_p^2, 1]$$

The passband of $G(e^{j\omega})$ is therefore defined by

$$1-\delta_p^2 \leq G(e^{j\omega}) \leq 1 \qquad 0 \leq \omega \leq \omega_p$$

Similarly, the interval $[-\delta_s, \delta_s]$ is mapped to

$$[-2\delta_s - \delta_s^2, 2\delta_s - \delta_s^2]$$

Therefore the stopband of $G(e^{j\omega})$ is defined by

$$\omega \in [\omega_s, \pi), \; -2\delta_s - \delta_s^2 \leq G(e^{j\omega}) \leq 2\delta_s - \delta_s^2$$

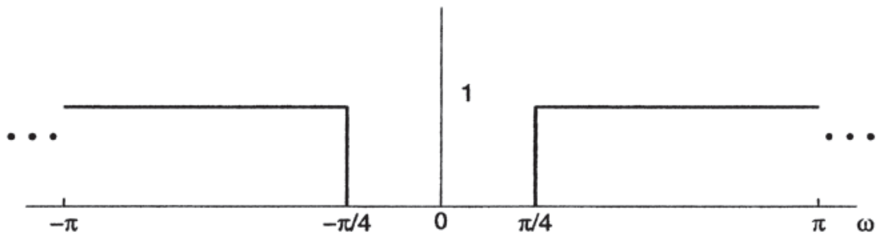
4.) If $\delta_p \ll 1$ we have $1-\delta_p^2 \simeq 1$

If $\delta_s \ll 1$ we have $-2\delta_s - \delta_s^2 \simeq -2\delta_s$.

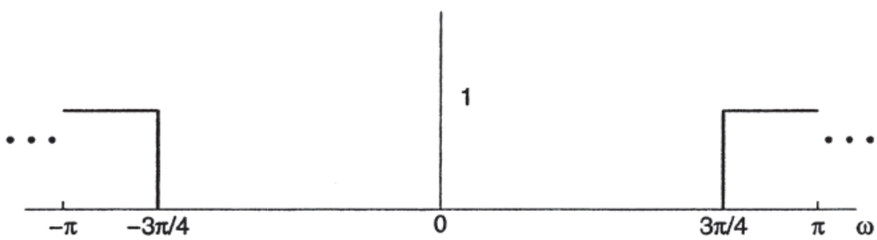
The tollerance in the stopband is twice as lage.

5.21. $h_{lp}[n]$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$

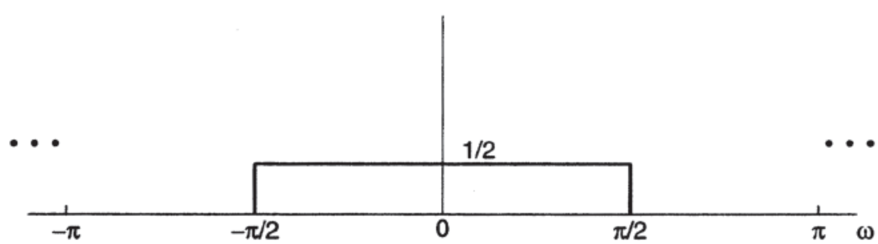
(a) $y[n] = x[n] - x[n] * h_{lp}[n] \Rightarrow H(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$
This is a highpass filter.



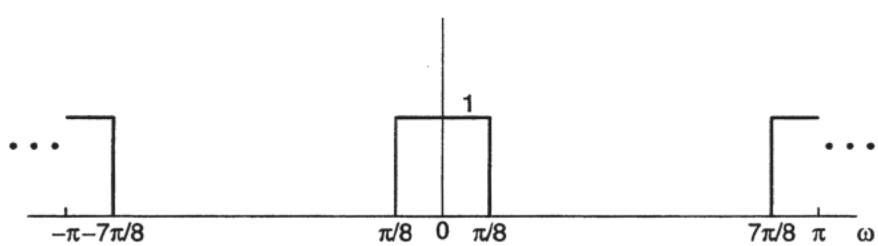
(b) $x[n]$ is first modulated by π, lowpass filtered, and demodulated by π. Therefore, $H_{lp}(e^{j\omega})$ filters the high frequency components of $X(e^{j\omega})$.
This is a highpass filter.



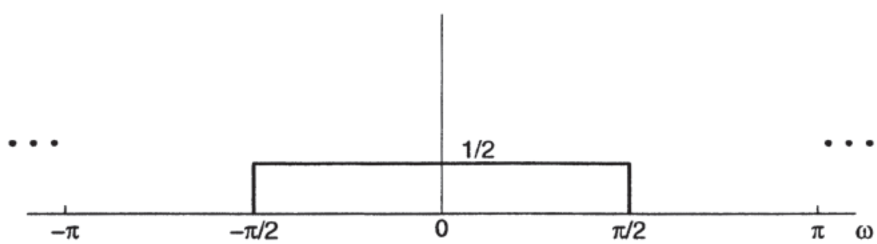
(c) $h_{lp}[2n]$ is a downsampled version of the filter. Therefore, the frequency response will be "spread out" by a factor of two, with a gain of $\frac{1}{2}$.
This is a lowpass filter.



(d) This system upsamples $h_{lp}[n]$ by a factor of two. Therefore, the frequency axis will be compressed by a factor of two.
This is a bandstop filter.



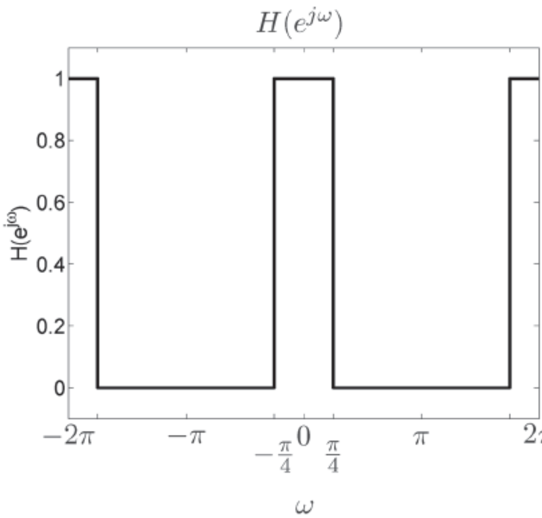
(e) This system upsamples the input before passing it through $h_{lp}[n]$. This effectively doubles the frequency bandwidth of $H_{lp}(e^{j\omega})$.
This is a lowpass filter.

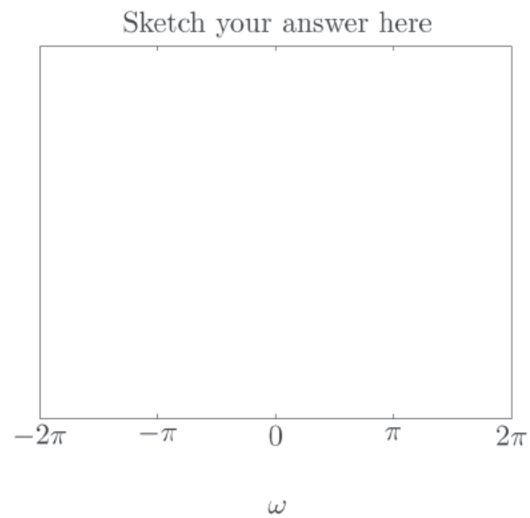


5.25. Problem 1 in Spring2005 Final exam.

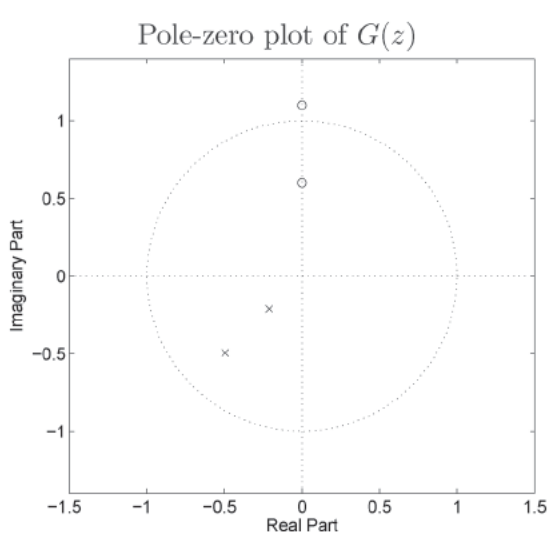
Problem

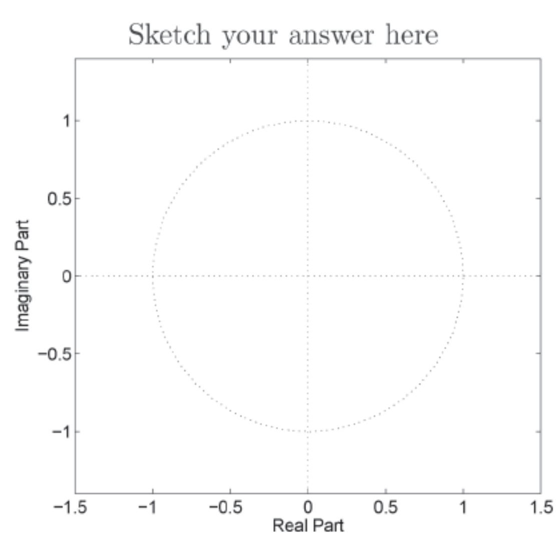
(a) An ideal lowpass filter $h[n]$ is designed with zero phase, a cutoff frequency of $\omega_c = \pi/4$, a passband gain of 1, and a stopband gain of 0. ($H(e^{j\omega})$ is shown below on the left.) Sketch the discrete-time Fourier transform of $(-1)^n h[n]$.





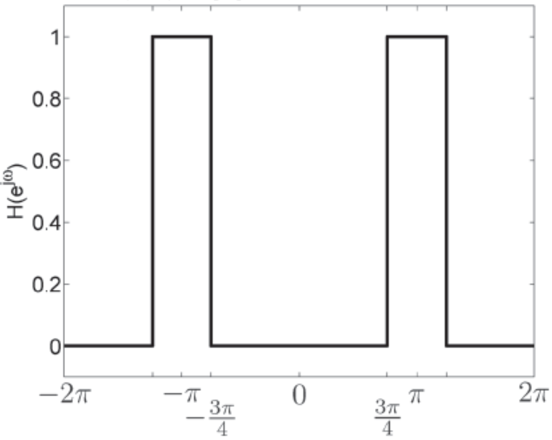
(b) A (complex) filter $g[n]$ has the pole-zero diagram shown below. Sketch the pole-zero diagram for $(-1)^n g[n]$. If there is not sufficient information provided, explain why.



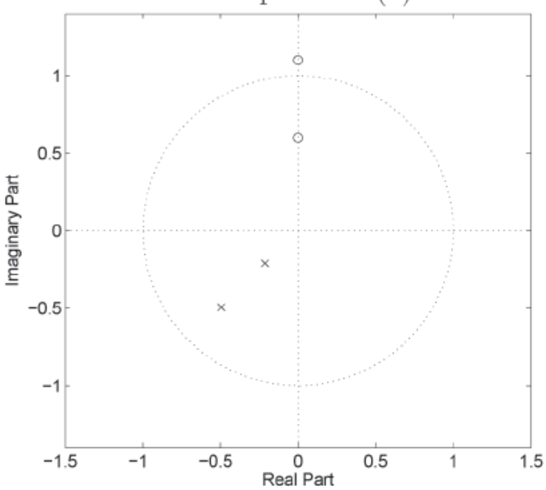


Solution from Spring2005 Final

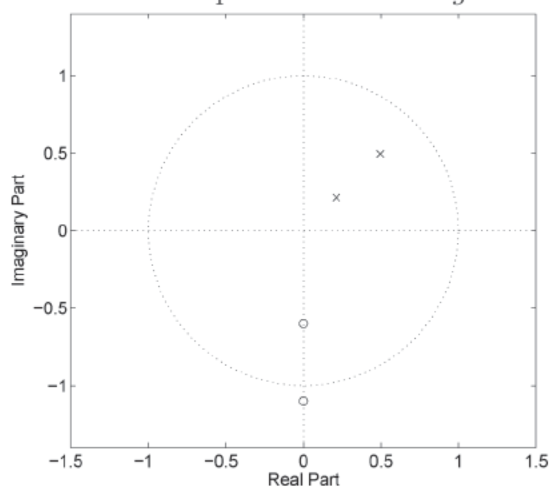
DTFT of $(-1)^n h[n]$



Pole-zero plot of $G(z)$



Pole-zero plot of modulated g



5.28. A.

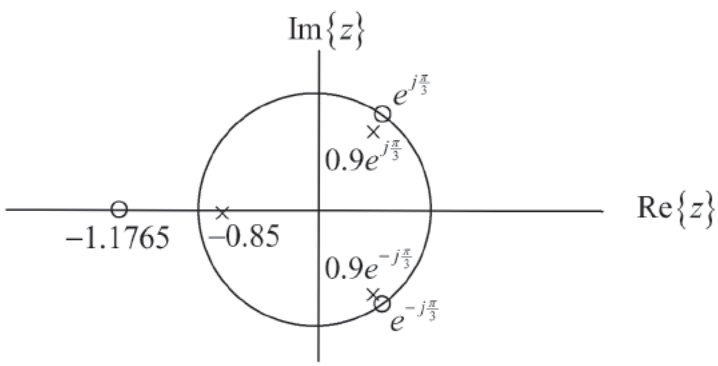
$$H(z)=\frac{\left(1-e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-e^{-j\frac{\pi}{3}}z^{-1}\right)\left(1+1.1765z^{-1}\right)}{\left(1-0.9e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-0.9e^{-j\frac{\pi}{3}}z^{-1}\right)\left(1+0.85z^{-1}\right)}$$

$$=\frac{1+0.1765z^{-1}-0.1765z^{-2}+1.1765z^{-3}}{1-0.05z^{-1}+0.045z^{-2}+0.6885z^{-3}}$$

$$=\frac{Y(z)}{X(z)}.$$

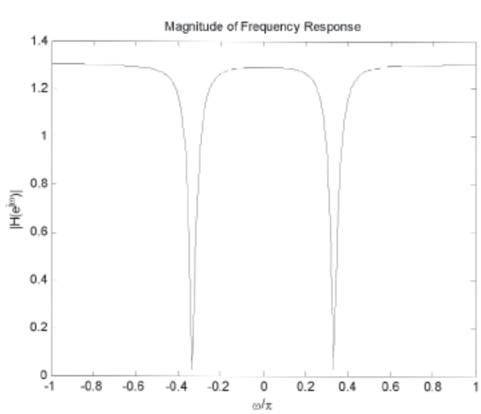
$$y[n]=0.05y[n-1]-0.45y[n-2]-0.6885y[n-3]$$
$$+x[n]+0.1765x[n-1]-0.1765x[n-2]+1.1765x[n-3].$$

B.



Since the system is causal, the ROC is the region outside the outermost pole. That is, $|z|>0.9$.

C.



The zeros on the unit circle null the frequency response at $\omega = \pm\pi/3$. The sharpness of the nulls depend on how close the nearby poles are to the zeros. The factor

$$\frac{1+1.1765z^{-1}}{1+0.85z^{-1}} = 1.1765\frac{z^{-1}+0.85}{1+0.85z^{-1}}$$ is allpass and does not affect the magnitude response.

D. 1. True. The system is stable because the ROC contains the unit circle.

2. False. The impulse response must approach zero for large n because the system is stable.

3. False. The system function has a zero on the unit circle at $\omega = \pi/3$. This negates the effect of the pole, and since the pole is not on the unit circle, the pole does not cancel the zero. Instead, the sharpness of the notch depends on how close the pole is to the zero.

4. False. There is a zero outside the unit circle.

5. False. The system is not a minimum-phase system so it does not have a causal and stable inverse.

5.29. A. For the inverse system $H\left(e^{j\omega}\right)H_i\left(e^{j\omega}\right)=1$. This means

$$H_i\left(e^{j\omega}\right)=\frac{1}{H\left(e^{j\omega}\right)}$$

$$=\frac{1}{1+2e^{-j\omega}}.$$

The ROC of $H_i\left(e^{j\omega}\right)$ must include the unit circle if the inverse system is to be stable. That is, $|z|<2$. Taking the inverse Fourier transform,

$$h_i[n]=-(-2)^n u[-n-1].$$

The inverse system is not causal.

B. For this part

$$H_i\left(e^{j\omega}\right)=\frac{1}{1+\alpha e^{-j\omega}}.$$

For the inverse system to be causal and stable we require $|\alpha|<1$. Then the ROC of a stable $H_i\left(e^{j\omega}\right)$ will be $|z|>|\alpha|$, and the impulse response will be

$$h_i[n]=(-\alpha)^n u[n]$$

corresponding to a causal system.

5.42. A. From the pole-zero diagram,

$$H(z) = K\frac{(z-2)}{z\left(z-\frac{1}{2}\right)(z+3)}$$
$$= K\frac{z^{-2}\left(1-2z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)}.$$

We are given that the system is BIBO stable, and this implies that the ROC must include the unit circle. We therefore have the ROC $\frac{1}{2}<|z|<3$.

Since the ROC includes the unit circle, the system has a frequency response given by

$$H\left(e^{j\omega}\right) = K\frac{e^{-j2\omega}\left(1-2e^{-j\omega}\right)}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+3e^{-j\omega}\right)}.$$

The frequency reponse is the Fourier transform of the impulse response. That is,

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}.$$

At $\omega=\pi$ this is

$$H\left(e^{j\pi}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\pi n} = \sum_{n=-\infty}^{\infty} (-1)^n h[n].$$

Now

$$H\left(e^{j\pi}\right) = K\frac{e^{-j2\pi}\left(1-2e^{-j\pi}\right)}{\left(1-\frac{1}{2}e^{-j\pi}\right)\left(1+3e^{-j\pi}\right)}$$
$$= K\frac{(1+2)}{\left(1+\frac{1}{2}\right)(1-3)}$$
$$= -K$$
$$= -1.$$

Therefore $K=1$ and $H(z)$ is given by

$$H(z) = \frac{z^{-2}\left(1-2z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)}.$$

B. Now $g[n]=h[n+n_0]$. That is,

$$G(z) = z^{n_0}H(z)$$
$$= \frac{z^{n_0-2}\left(1-2z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)}$$
$$= \frac{z^{n_0}(z-2)}{z\left(z-\frac{1}{2}\right)(z-3)}.$$

We will have $G(z)\big|_{z=0}=0$ if $n_0\geq 2$. In addition, we will have $\lim_{z\to\infty} G(z)<\infty$ only if $n_0\leq 2$. Therefore $n_0=2$.

Applying the inverse Fourier transform,

$$g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})e^{j\omega n}d\omega.$$

Then

$$\begin{aligned} g[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-2e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+3e^{-j\omega}\right)}d\omega \\ &= -\frac{3}{7}. \end{aligned}$$

C. Let $f[n] = h[n] * h[-n]$. Then $F(z) = H(z)H(1/z)$. That is,

$$\begin{aligned} F(z) &= \frac{z^{-2}\left(1-2z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)} \frac{z^{2}\left(1-2z\right)}{\left(1-\frac{1}{2}z\right)\left(1+3z\right)} \\ &= \frac{4}{3}\frac{z}{(z+3)\left(z+\frac{1}{3}\right)} \\ &= \frac{4}{3}\frac{z^{-1}}{\left(1+3z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}. \end{aligned}$$

Since $H(z)$ has ROC $\frac{1}{2} < |z| < 3$, then $H(1/z)$ has ROC $\frac{1}{3} < |z| < 2$. The ROC of $F(z)$ must include the intersection of these two regions. Therefore $F(z)$ has ROC $\frac{1}{3} < |z| < 3$.

D. We want $e[n] * h[n] = u[n]$. Then $E(z)H(z) = \frac{1}{1-z^{-1}}$. That is,

$$\begin{aligned} E(z) &= \frac{1}{H(z)}\frac{1}{1-z^{-1}} \\ &= \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)}{z^{-2}\left(1-2z^{-1}\right)\left(1-z^{-1}\right)} \\ &= \frac{z^{2}\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)}{\left(1-2z^{-1}\right)\left(1-z^{-1}\right)}. \end{aligned}$$

The ROC of $\frac{1}{H(z)}$ can be $|z| < 2$ or $2 < |z|$. We will choose the latter to obtain a right-sided sequence. Then the ROC of $E(z)$ is also $2 < |z|$. Thus there is a right-sided sequence with the desired property.

The factor of z^2 in the numerator of $E(z)$ determines that $e[n]$ will take non-zero values at $n = -1$ and $n = -2$. Thus $e[n]$ is a right-sided sequence, but not a causal sequence.

5.44. Assume that each impulse response corresponds to at most one frequency response.

A. $h_1[n]$ is a Type I FIR filter. Frequency reponse C corresponds to a Type I filter, as $\left|H_C\left(e^{j\omega}\right)\right| \neq 0$ for $\omega = 0$ or $\omega = \pi$.

B. $h_2[n]$ is a Type II FIR filter. Frequency response B corresponds to a Type II filter, as $\left|H_B\left(e^{j\omega}\right)\right| = 0$ for $\omega = \pi$.

C. $h_3[n]$ is a Type III filter. The frequency response must be D, as D is the only frequency response for which $\left|H_D\left(e^{j\omega}\right)\right| = 0$ for $\omega = 0$ and $\omega = \pi$.

D. $h_4[n]$ is a Type IV filter. Frequency response A corresponds to a Type IV filter, as $\left|H_A\left(e^{j\omega}\right)\right| = 0$ for $\omega = 0$.