ECEN 4632 Problem set 1

Problem 1

Define $x_1[n] = \frac{1}{2^n} u[n]$ and $h_1[n] = u[n]$

then. $x[n] = x_1[n-2]$ and $h[n] = h_1[n+2]$

Because the convolution $y[n] = h * x[n]$ is time invariant $y[n] = h_1 * x_1[n+2-2] = h_1 * x_1[n]$

Now

$$h_1 * x_1[n] = \sum_{k=-\infty}^{\infty} h_1[k]\, x_1[n-k]$$

$$= \sum_{k=0}^{\infty} x_1[n-k] = \sum_{k=0}^{\infty} \frac{1}{2^{n-k}} u[n-k]$$

$$= \sum_{k=0}^{n} \frac{1}{2^{n-k}} = \frac{1}{2^n} \sum_{k=0}^{n} 2^k$$

$$= \frac{1}{2^n}\left(2^{n+1} - 1\right) = 2\left(1 - \frac{1}{2^{n+1}}\right)$$

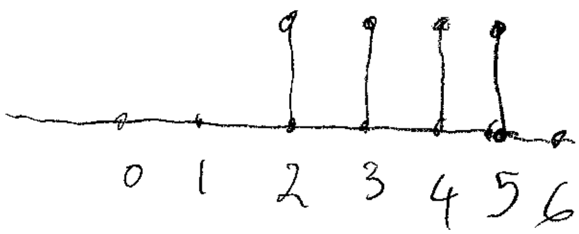
Also $h_1 * x_1[n] = 0$ if $n < 0$ since the previous sum is empty. We conclude:

$$x * h[n] = 2\left(1 - \frac{1}{2^{n+1}}\right) u[n].$$

Problem 2

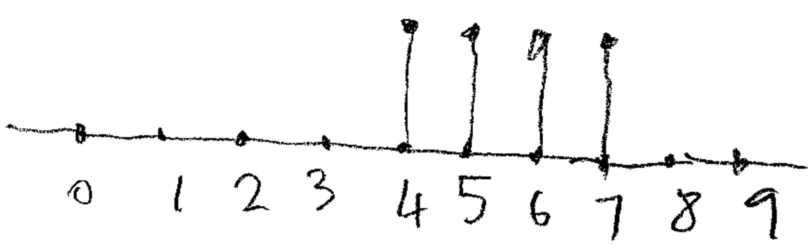
1. If $x[n] = \delta[n-1]$

$$y[n] = g[n-2] = u[n-2] - u[n-6]$$



2) If $x[n] = \delta[n-2]$.

$$y[n] = g[n-4] = u[n-4] - u[n-8]$$



3) If the system were to be time-invariant then we would have

$$\delta[n-2] \longrightarrow g[n-3]$$

which is not the case.

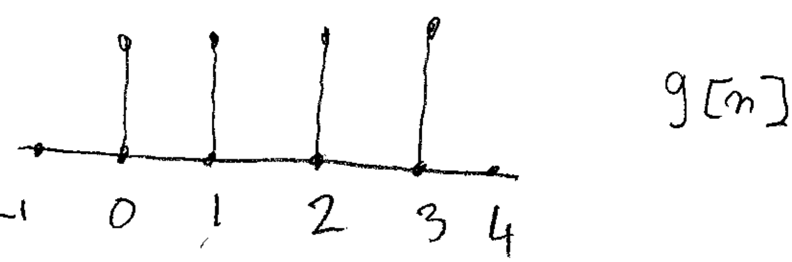
The system is not time invariant.

4). If $x[n] = u[n]$ then

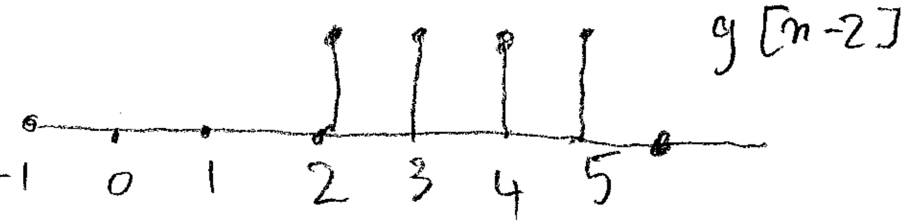
$$y[n] = \sum_{k=0}^{\infty} g[n-2k].$$

Now it helps to visualize all the terms in this sum:

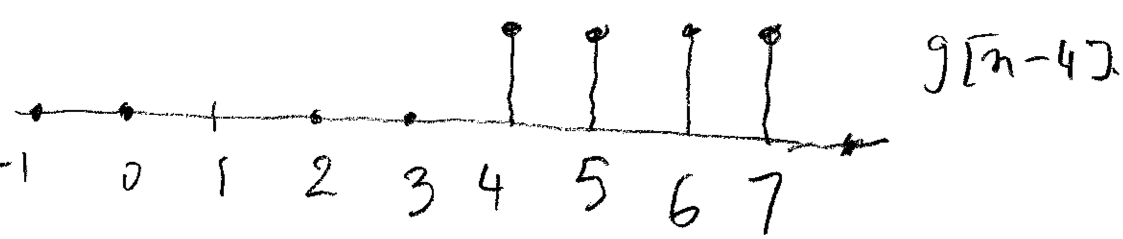
$k=0$



$k=1$



$k=2$



etc...

We conclude that

for $n=0,1$ only $g[n]$ contributes to sum

and $y[n]=1$

for $n \geq 2$, there are only two values of k that contribute to any output value. And thus

$$y[n]=2$$

Formally, if $n=2m$, $m \geq 1$

$$y[n] = \sum_{k=m-1}^{n} g[2m-2k]$$

$$= g[2]+g[0) = 2$$

if $n=2m+1$, $m \geq 1$

$$y[n] = \sum_{k=m-1}^{m} g[2m+1-2k] = g[3]+g[1] = 2$$

Finally, if $n=0$ $y[0] = \sum_{k=0}^{0} g[0-2k] = g[0] = 1$

$n=1$

$$y[1] = \sum_{k=0}^{0} g[1-2k] = g[1] = 0$$

Also $y[n]=0$ if $n<0$

2.1. (a) $T(x[n]) = g[n]x[n]$

- Stable: Let $|x[n]| \le M$ then $|T[x[n]| \le |g[n]|M$. So, it is stable if $|g[n]|$ is bounded.
- Causal: $y_1[n] = g[n]x_1[n]$ and $y_2[n] = g[n]x_2[n]$, so if $x_1[n] = x_2[n]$ for all $n < n_0$, then $y_1[n] = y_2[n]$ for all $n < n_0$, and the system is causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= g[n](ax_1[n] + bx_2[n] \\ &= ag[n]x_1[n] + bg[n]x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

So this is linear.

- Not time-invariant:

$$\begin{aligned} T(x[n-n_0]) &= g[n]x[n-n_0] \\ &\ne y[n-n_0] = g[n-n_0]x[n-n_0] \end{aligned}$$

which is not TI.

- Memoryless: $y[n] = T(x[n])$ depends only on the n^{th} value of x, so it is memoryless.

(b) $T(x[n]) = \sum_{k=n_0}^{n} x[k]$

- Not Stable: $|x[n]| \le M \to |T(x[n])| \le \sum_{k=n_0}^{n} |x[k]| \le |n-n_0|M$. As $n \to \infty$, $T \to \infty$, so not stable.
- Not Causal: $T(x[n])$ depends on the future values of $x[n]$ when $n < n_0$, so this is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= \sum_{k=n_0}^{n} ax_1[k] + bx_2[k] \\ &= a\sum_{k=n_0}^{n} x_1[n] + b\sum_{k=n_0}^{n} x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

The system is linear.

- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= \sum_{k=n_0}^{n} x[k-n_0] \\ &= \sum_{k=0}^{n-n_0} x[k] \\ &\ne y[n-n_0] = \sum_{k=n_0}^{n-n_0} x[k] \end{aligned}$$

The system is not TI.

- Not Memoryless: Values of $y[n]$ depend on past values for $n > n_0$, so this is not memoryless.

(c) $T(x[n]) \sum_{k=n-n_0}^{n+n_0} x[k]$

- Stable: $|T(x[n])| \le \sum_{k=n-n_0}^{n+n_0} |x[k]| \le \sum_{k=n-n_0}^{n+n_0} x[k]M \le |2n_0 + 1|M$ for $|x[n]| \le M$, so it is stable.
- Not Causal: $T(x[n])$ depends on future values of $x[n]$, so it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= \sum_{k=n-n_0}^{n+n_0} ax_1[k] + bx_2[k] \\ &= a\sum_{k=n-n_0}^{n+n_0} x_1[k] + b\sum_{k=n-n_0}^{n+n_0} x_2[k] = aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

This is linear.

- TI:

$$
\begin{aligned}
T(x[n-n_0] &= \sum_{k=n-n_0}^{n+n_0} x[k-n_0] \\
&= \sum_{k=n-n_0}^{n} x[k] \\
&= y[n-n_0]
\end{aligned}
$$

This is TI.

- Not memoryless: The values of $y[n]$ depend on $2n_0$ other values of x, not memoryless.

(d) $T(x[n]) = x[n-n_0]$

- Stable: $|T(x[n])| = |x[n-n_0]| \leq M$ if $|x[n] \leq M$, so stable.
- Causality: If $n_0 \geq 0$, this is causal, otherwise it is not causal.
- Linear:

$$
\begin{aligned}
T(ax_1[n] + bx_2[n]) &= ax_1[n-n_0] + bx_2[n-n_0] \\
&= aT(x_1[n]) + bT(x_2[n])
\end{aligned}
$$

This is linear.

- TI: $T(x[n-n_d] = x[n-n_0-n_d] = y[n-n_d]$. This is TI.
- Not memoryless: Unless $n_0 = 0$, this is not memoryless.

(e) $T(x[n]) = e^{x[n]}$

- Stable: $|x[n]| \leq M$, $|T(x[n])| = |e^{x[n]}| \leq e^{|x[n]|} \leq e^M$, this is stable.
- Causal: It doesn't use future values of $x[n]$, so it causal.
- Not linear:

$$
\begin{aligned}
T(ax_1[n] + bx_2[n]) &= e^{ax_1[n]+bx_2[n]} \\
&= e^{ax_1[n]}e^{bx_2[n]} \\
&\neq aT(x_1[n]) + bT(x_2[n])
\end{aligned}
$$

This is not linear.

- TI: $T(x[n-n_0]) = e^{x[n-n_0]} = y[n-n_0]$, so this is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of x only, so it is memoryless.

(f) $T(x[n]) = ax[n] + b$

- Stable: $|T(x[n])| = |ax[n] + b| \leq a|M| + |b|$, which is stable for finite a and b.
- Causal: This doesn't use future values of $x[n]$, so it is causal.
- Not linear:

$$
\begin{aligned}
T(cx_1[n] + dx_2[n]) &= acx_1[n] + adx_2[n] + b \\
&\neq cT(x_1[n]) + dT(x_2[n])
\end{aligned}
$$

This is not linear.

- TI: $T(x[n-n_0]) = ax[n-n_0] + b = y[n-n_0]$. It is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of $x[n]$ only, so it is memoryless.

(g) $T(x[n]) = x[-n]$

- Stable: $|T(x[n])| \leq |x[-n]| \leq M$, so it is stable.
- Not causal: For $n < 0$, it depends on the future value of $x[n]$, so it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[-n] + bx_2x[-n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= x[-n-n_0] \\ &\neq y[n-n_0] = x[-n+n_0] \end{aligned}$$

  This is not TI.
- Not memoryless: For $n \neq 0$, it depends on a value of x other than the n^{th} value, so it is not memoryless.

(h) $T(x[n]) = x[n] + u[n+1]$

- Stable: $|T(x[n])| \leq M + 3$ for $n \geq -1$ and $|T(x[n])| \leq M$ for $n < -1$, so it is stable.
- Causal: Since it doesn't use future values of $x[n]$, it is causal.
- Not linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[n] + bx_2[n] + 3u[n+1] \\ &\neq aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is not linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0] &= x[n-n_0] + 3u[n+1] \\ &= y[n-n_0] \\ &= x[n-n_0] + 3u[n-n_0+1] \end{aligned}$$

  This is not TI.
- Memoryless: $y[n]$ depends on the n^{th} value of x only, so this is memoryless.

2.3. We desire the step response to a system whose impulse response is

$$h[n] = a^{-n}u[-n], \text{ for } 0 < a < 1.$$

The convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The step response results when the input is the unit step:

$$x[n] = u[n] = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

Substitution into the convolution sum yields

$$y[n] = \sum_{k=-\infty}^{\infty} a^{-k}u[-k]u[n-k]$$

For $n \leq 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} a^{-k} \\ &= \sum_{k=-n}^{\infty} a^{k} \\ &= \frac{a^{-n}}{1-a} \end{aligned}$$

For $n > 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{0} a^{-k} \\ &= \sum_{k=0}^{\infty} a^{k} \\ &= \frac{1}{1-a} \end{aligned}$$

2.10. (a)

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} a^k u[-k-1]u[n-k] \\ &= \begin{cases} \sum_{k=-\infty}^{n} a^k, & n \le -1 \\ \sum_{k=-\infty}^{-1} a^k, & n > -1 \end{cases} \\ &= \begin{cases} \frac{a^n}{1-1/a}, & n \le -1 \\ \frac{1/a}{1-1/a}, & n > -1 \end{cases} \end{aligned}$$

(b) First, let us define $v[n] = 2^n u[-n-1]$. Then, from part (a), we know that

$$w[n] = u[n] * v[n] = \begin{cases} 2^{n+1}, & n \le -1 \\ 1, & n > -1 \end{cases}$$

Now,

$$\begin{aligned} y[n] &= u[n-4] * v[n] \\ &= w[n-4] \\ &= \begin{cases} 2^{n-3}, & n \le 3 \\ 1, & n > 3 \end{cases} \end{aligned}$$

(c) Given the same definitions for $v[n]$ and $w[n]$ from part(b), we use the fact that $h[n] = 2^{n-1}u[-(n-1)-1] = v[n-1]$ to reduce our work:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * v[n-1] \\ &= w[n-1] \\ &= \begin{cases} 2^n, & n \le 0 \\ 1, & n > 0 \end{cases} \end{aligned}$$

(d) Again, we use $v[n]$ and $w[n]$ to help us.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= (u[n] - u[n-10]) * v[n] \\ &= w[n] - w[n-10] \\ &= (2^{n+1}u[-(n+1)] + u[n]) - (2^{n-9}u[-(n-9)] + u[n-10]) \\ &= \begin{cases} 2^{(n+1)} - 2^{(n-9)}, & n \le -2 \\ 1 - 2^{(n-9)}, & -1 \le n \le 8 \\ 0, & n \ge 9 \end{cases} \end{aligned}$$

2.22. For an LTI system, we use the convolution equation to obtain the output:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Let $n = m + N$:

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m+N-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} x[(m-k)+N]h[k] \end{aligned}$$

Since $x[n]$ is periodic, $x[n] = x[n+rN]$ for any integer r. Hence,

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m-k]h[k] \\ &= y[m] \end{aligned}$$

So, the output must also be periodic with period N.

2.23. (a) Since $\cos(\pi n)$ only takes on values of +1 or -1, this transformation outputs the current value of $x[n]$ multiplied by either ± 1. $T(x[n]) = (-1)^n x[n]$.

- Hence, it is stable, because it doesn't change the magnitude of $x[n]$ and hence satisfies bounded-in/bounded-out stability.
- It is causal, because each output depends only on the current value of $x[n]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = \cos(\pi n)x_1[n]$, and $y_2[n] = T(x_2[n]) = \cos(\pi n)x_2[n]$. Now

$$T(ax_1[n] + bx_2[n]) = \cos(\pi n)(ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = (-1)^n x[n]$, then $T(x[n-1]) = (-1)^n x[n-1] \neq y[n-1]$.

(b) This transformation simply "samples" $x[n]$ at location which can be expressed as k^2.

- The system is stable, since if $x[n]$ is bounded, $x[n^2]$ is also bounded.
- It is not causal. For example, $Tx[4] = x[16]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = x_1[n^2]$, and $y_2[n] = T(x_2[n]) = x_2[n^2]$. Now

$$T(ax_1[n] + bx_2[n]) = ax_1[n^2] + bx_2[n^2]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = x[n^2]$, then $T(x[n-1]) = x[n^2-1] \neq y[n-1]$.

(c) First notice that

$$\sum_{k=0}^{\infty} \delta[n-k] = u[n]$$

So $T(x[n]) = x[n]u[n]$. This transformation is therefore stable, causal, linear, but not time-invariant.

To see that it is not time invariant, notice that $T(\delta[n]) = \delta[n]$, but $T(\delta[n+1]) = 0$.

(d) $T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$

- This is not stable. For example, $T(u[n]) = \infty$ for all n ¿ 1.
- It is not causal, since it sums *forward* in time.
- It is linear, since

$$\sum_{k=n-1}^{\infty} ax_1[k] + bx_2[k] = a\sum_{k=n-1}^{\infty} x_1[k] + b\sum_{k=n-1}^{\infty} x_2[k]$$

- It is time-invariant. Let

$$y[n] = T(x[n]) = \sum_{k=n-1}^{\infty} x[k],$$

then

$$T(x[n-n_0]) = \sum_{k=n-n_0-1}^{\infty} x[k] = y[n-n_0]$$

2.27. Problem 1 from Fall 2003 Background exam
Note: There is a very similar problem in Fall06 background exam, worth checking out

Problem

For each of the systems below, choose the strongest valid conclusion of the following:

1. The system must be LTI and is uniquely specified by the information given.
2. The system must be LTI, but cannot be uniquely determined from the information given.
3. The system could be LTI, and if it is the information given uniquely specifies the system.
4. The system could be LTI, but cannot be uniquely determined from the information given.
5. The system could not possibly be LTI.

$$\delta[n] \longrightarrow \boxed{\text{System (a)}} \longrightarrow 2e^{j\frac{n}{4}}u[n]$$

$$\left(\tfrac{1}{3}\right)^n u[n] \longrightarrow \boxed{\text{System (b)}} \longrightarrow \delta[n]$$

$$\begin{matrix} x[n]+\alpha y[n] \\ \forall \alpha, x[n], y[n] \end{matrix} \longrightarrow \boxed{\text{System (c)}} \longrightarrow T(x[n]) + \alpha T(y[n])$$

$$\cos\left(\tfrac{\pi}{3}n\right) \longrightarrow \boxed{\text{System (d)}} \longrightarrow 3\cos\left(\tfrac{\pi}{3}n\right) + \tfrac{1}{2}\sin\left(\tfrac{\pi}{2}n + \tfrac{\pi}{5}\right)$$

$$x[n] \longrightarrow \boxed{\text{System (e)}} \longrightarrow y[n] = 0.2y[n+1] + x[n]$$

Solution from Fall 2003 background exam

(a) To prove that a system is LTI, it must show linearity and time invariance *for all* inputs. In this case we are only given one input signal, so we cannot prove that the system is LTI. However, we cannot disprove it either.

However, if it is LTI then it is completely characterized, since the input is the delta function.

The system could be LTI, and if it is the information given uniquely specifies the system. (3)

(b) Similarly we cannot show that the system is LTI, but cannot disprove it either. The input signal $\left(\frac{1}{3}\right)^n u[n]$ does not have any spectral nulls, so if the system were LTI it would be completely characterized.

The system could be LTI, and if it is the information given uniquely specifies the system. (3)

(c) Again, we cannot show that the system is LTI. We have linearity but not necessarily time invariance. In this case it is not said whether $T(\cdot)$ is known or not. If it is know, then with the assumption of LTI it will be completely specified, otherwise it is not.

The system could be LTI. (3) or (4)

(d) First note that $\cos(\pi/3n)$ is an eigenfunction of an LTI system, and so is $\sin(\pi/2n)$. Therefore if the system were LTI it is not possible to get a $\sin(\pi/2n)$ term in the output if there is no such term in the input.

The system is not LTI. (5)

(e) A system described by a difference equation need to have zero initial conditions to be LTI.

The system could be LTI, but cannot be uniquely determined from given information. (4)