Problem 1

The Fourier transform of the signal $y(t)$ at the output of the analog filter is :

$$Y(j\Omega) = H(j\Omega)X(j\Omega)$$

1. The Fourier transform of the sampled signal $y_s[n]$ is given by the sampling theorem,

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H\left(j\,\frac{\omega - 2\pi k}{T}\right) X\left(j\,\frac{\omega - 2\pi k}{T}\right).$$

2. We have

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\,\frac{\omega - 2\pi k}{T}\right).$$

3. For the second system, with the digital filter, we have

$$G(e^{j\omega}) = \sum_{l=-\infty}^{\infty} H\left(j\,\frac{\omega - 2\pi l}{T}\right).$$

4. In the Fourier domain, the convolution becomes a product,

$$W(e^{j\omega}) = G\left(e^{j\omega}\right) X\left(e^{j\omega}\right) = \frac{1}{T}\left\{\sum_{k=-\infty}^{\infty} H\left(j\,\frac{\omega - 2\pi k}{T}\right)\right\}\left\{\sum_{l=-\infty}^{\infty} X\left(j\,\frac{\omega - 2\pi l}{T}\right).\right\}$$

The two outputs are equal if most of the terms in the product of the sums cancel,

$$H\left(j\,\frac{\omega - 2\pi k}{T}\right) X\left(j\,\frac{\omega - 2\pi l}{T}\right) = 0 \qquad k \neq l$$

This is equivalent to

$$H\left(j\,\frac{\omega}{T}\right) X\left(j\,\frac{\omega - 2\pi k}{T}\right) = 0 \qquad k \neq 0$$

We have

$$X\left(j\,\frac{\omega}{T}\right) = 0 \qquad \text{if } |\omega| > \Omega_1 T$$

and

$$H\left(j\,\frac{\omega - 2\pi k}{T}\right) = 0 \qquad \text{if } |\omega - 2\pi k| > \Omega_2 T$$

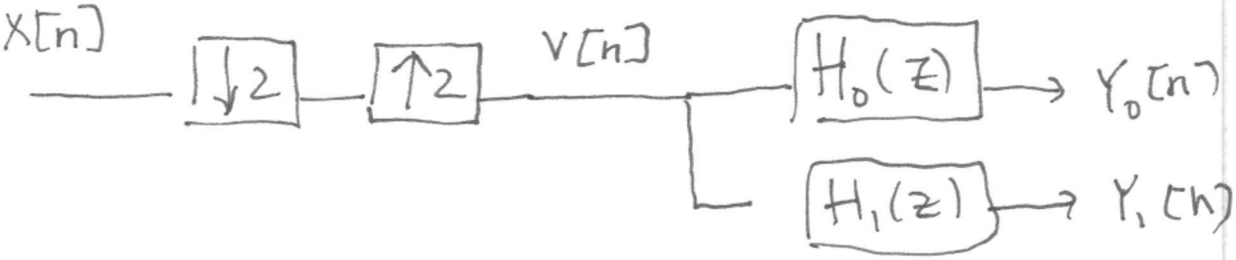
Therefore the condition will be met if and only if

$$\Omega_1 T < 2\pi - \Omega_2 T$$

that is,

$$T < \frac{2\pi}{\Omega_1 + \Omega_2}$$

Problem 2



$$V(z) = \frac{1}{2}\left\{ X(z) + X(-z) \right\}$$

so $Y_0(z) = H_0(z)V(z) = \frac{1}{2}\left\{ H_0(z)X(z) + H_0(z)X(-z) \right\}$

Similarly

$$Y_1(z) = \frac{1}{2}\left\{ H_1(z)X(z) + H_1(z)X(-z) \right\}$$

In the Fourier domain

$$Y_0(e^{j\omega}) = \frac{1}{2}\left\{ H_0(e^{j\omega})X(e^{j\omega}) + H_0(e^{j\omega})X(e^{j(\omega-\pi)}) \right\}$$

$$Y_0(e^{j\omega}) = \frac{1}{2} X(e^{j(\omega-\pi)})$$

$$Y_1(e^{j\omega}) = \frac{1}{2}\left\{ H_1(e^{j\omega})X(e^{j\omega}) + H_1(e^{j\omega})X(e^{j(\omega-\pi)}) \right\}$$

$$Y_1(e^{j\omega}) = \frac{1}{2} X(e^{j\omega})$$

Problem 3

Let $X_s(j\Omega)$ be the Fourier transform of the sampled signal

$$X_s(j\Omega) = \frac{\Omega_0}{2\pi} \sum_k X_c\left(j\left(\Omega - \frac{2\pi}{2\pi} k \Omega_0\right)\right)$$

$$= \frac{\Omega_0}{2\pi} \cdot \frac{\pi}{\Omega_0} \sum_k \left[1 + \cos\left(\frac{\pi}{\Omega_0}(\Omega - k\Omega_0)\right)\right] \mathbb{1}_{[(K-1)\Omega_0, (K+1)\Omega_0]}(\Omega)$$

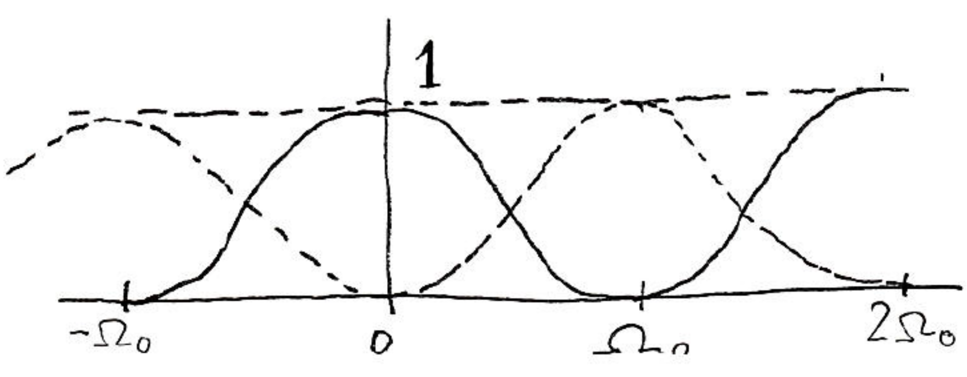
$$X_s(j\Omega) = \frac{1}{2} \sum_k \left[1 + \cos\left(\frac{\pi}{\Omega_0}\Omega - k\pi\right)\right] \mathbb{1}_{[(K-1)\Omega_0, (K+1)\Omega_0]}(\Omega)$$

$X_s(j\Omega)$ is Ω_0 periodic. On $[0, \Omega_0]$, we have

$$X_s(j\Omega) = \frac{1}{2}\left\{2 + \cos\left(\frac{\pi}{\Omega_0}\Omega\right) + \cos\left(\frac{\pi\Omega}{\Omega_0} - \pi\right)\right\}$$

$$= \frac{1}{2}\left\{2 + \cos\left(\frac{\pi\Omega}{\Omega_0}\right) - \cos\left(\frac{\pi\Omega}{\Omega_0}\right)\right\} = \frac{2}{2} = 1$$

So $X_s(j\Omega) = 1$ for all Ω, and $x[n] = \delta[n]$



Problem 4.

1) We have $G(\theta) = \sum_{0}^{2M} g(n) e^{-jn\theta} = \sum_{\substack{n=0 \\ n \neq M}}^{2M} h(n) e^{-jn\theta} + h(M) e^{-jM}$

$$G(\theta) = \sum_{n=0}^{2M} h(n) e^{-jn\theta} + \delta_s e^{-jM\theta}$$

$$G(\theta) = A(\theta) e^{-jM\theta} + \delta_s e^{-jM\theta} = P(\theta) e^{-jM\theta}$$

where $H(\theta) = A(\theta) e^{-jM\theta}$

and $P(\theta) = A(\theta) + \delta_s \geq 0$. since $A(\theta) \geq -\delta_s$.

2) G is a type I linear phase filter, since $P(\theta)$ is even.
We have

$$G(z) = P(z) z^{-M}.$$

Also G has real coefficient. Therefore if β_k is a zero of G, different from 0, we have:

β_k is a zero of P,

$\bar{\beta}_k$ ———— P,

β_k^{-1} ———— P,

$\bar{\beta}_k^{-1}$ ———— P.

We conclude that $P(z) = Q(z) Q(1/z)$ with

$$Q(z) = \prod_{|\beta_k|<1} (1 - \beta_k z^{-1})(1 - \bar{\beta}_k z^{-1}).$$

Because all the zeros of Q are inside the unit circle, Q is minimum phase.

Problem 5

$$H_1 = \frac{z^{-1} - a^*}{1 - az^{-1}} \qquad H_2 = \frac{z^{-1} - b^*}{1 - bz^{-1}}$$

$$H(z) = (H_1 + H_2)(z) = \frac{(1 - bz^{-1})(z^{-1} - a^*) + (1 - az^{-1})(z^{-1} - b^*)}{(1 - az^{-1})(1 - bz^{-1})}$$

- We can simplify by $(1 - az^{-1})$ the denominator if a is a zero in the numerator; This implies

$$(1 - ba^{-1})(1/a - a^*) = 0$$

We either have $a = b$ or $|a| = 1$.

If $a = b$ we are finished.

If $|a| = 1$ $\quad H_1(z) = -a^*$

and $1/b^*$ is a zero of $H(z)$, which implies:

$$(1 - bb^*)(b^* - a^*)$$

Again, we have $a = b$, or $|b| = 1$

If $|b| = 1$ we have $H_2(z) = -b^*$