The purpose of this project is to derive option price deviation using Kalman Filter and Finite Difference Method. Note that the "deviation" of the option price can be defined as the difference between the "theoretical" or "fair" option price derived from Black-Shoels equation and the observed option price. This definition of price deviation can give you an insight on how much the real price deviated from the true problem.

In this document, I explain how to construct state-space representation. (If you want to refer to how to solve BS equation using FDM, see the following reference).

Let S_t and C_t be prices of underliying asset(e.g. stock price) and derivative(e.g. European Vanila option).

First, we define state transition equation and observation equation as the followings;

$$X_{t+1} = A_t X_t + B\omega_{t+1}$$
$$Y_t = C_t X_t + D\epsilon_t$$

where $X_t := [S_t, C_t]$, and $Y_t := \left[S_t, \tilde{C}_t\right]$, $E\left\{\epsilon_t \epsilon_t^{'}\right\} = R$, and assume that $E\left[\omega_{t+1}\epsilon_s^{'}\right] = 0$ for all $t+1$ and $s \geq 0$. Also, assume that $X_0 \sim N(\hat{X}_0, \Sigma_0)$. Note that tilde notation implies the variable that can be observed in the data set. Note that C_t and $\tilde{C}_t$ are different variables. This is the important distinction because we want to differentiate between "true" and observed values of option. Then,

$$A_t := \begin{bmatrix} S_{t+1}/S_t & 0 \\ \Delta(S_{t+1}) & 0 \end{bmatrix}$$
$$C_t := \begin{bmatrix} 1 & 0 \\ 0 & \gamma_t \end{bmatrix}$$

Note that in this formulation, **we want to measure γ_t(or estimate) that captures the discrepancy between C_t and $\tilde{C}_t$.** Also, note that $\Delta(S_{t+1})$ can be backed out by using the solution we computed from FDM exercise(This means that the value of $\Delta(S_{t+1})$ can be interpolated given the stock price S_{t+1}).

Here is our computer algorithm to back this out.

Input Data: Stock price and corresponding option price

Step 1 : Suppose that the strike price K, interest rate r, and stock volatility σ are constant.(We relax this assumption by allowing the algorithm compute the solution of BK equation at every iteration. For demonstration, we assume that interest rate r_t is time-dependent, and other parameters are assumed to be constant). Solve BK equation using FDM to compute a vector of solution such that

$$\{\Delta(S_{t+1})\}_{S_{t+1}\in S} := \left\{\Delta(s^0), \Delta(s^1), \cdots, \Delta(s^N)\right\}$$

where S is the discretization scheme set of FDM and N is the number of nodes. We can index the vector of solution by r_t. In this case, we have to compute the vector for each value r_t.

Step 2 : Guess the values of $\{\gamma_0, \gamma_1, \cdots, \gamma_T\}$. (Assume that $X_0 = Y_0$, meaning that the theoretical and observed stock and option price are the same at the initial period. This is quite strong assumption, but we will check whether our result $\{\hat{\gamma}_t\}_{t=0,1,\ldots,T}$ can be robustly calculated by taking different value of X_0). Also, construct $\{A_0, A_1, \cdots, A_T\}$ using stock price time-series and BK solutions computed in the step1.

Step 3 : Initiate the first iteration of Kalman Filtering Algorithm. Given $\left(X_0(=\hat{X}_0), Y_0, C_0, S_0, S_1\right)$, Compute Kalman gain K_t, projected value of $\hat{X}_1$, and the variance of X_1, i.e. Σ_1 using the following formula.

$$\begin{aligned}
a_0 &= Y_0 - C_0\hat{X}_0 \\
K_0 &= A_0\Sigma_0 C_0^{'}\left(C_0\Sigma_0 C_0^{'} + R\right)^{-1} \\
\hat{X}_1 &= A_0\hat{X}_0 + K_0 a_0 \\
\Sigma_1 &= C_0 C_0^{'} + K_0 R K_0^{'} + (A_0 - K_0 G)\,\Sigma_0\,(A_0 - K_0 G)^{'}
\end{aligned}$$

Iterate until $t = T$.

Step 4 : Compute the maximum likelihood using the following formula

$$L\left(\{\hat{\gamma}_t\}_{t=0,1,\ldots,T}\right) = \sum_{t=0}^{T} \log\left(|\Omega_t|\right)$$

If the estimated $\{\hat{\gamma}_t\}_{t=0,1,\ldots,T}$ does not attain the maximum value, then update the guess and return back to the step 2.

Step 4 :