Some remarks on the set of injections and the set of surjections on a set

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Abstract

We provide sufficient condition for a set A which implies that the set of bijections on A is equinumerous to the set of injections on A or the set of surjections on A. This work extends some results in [2].

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1 Introduction

One of the most widely used foundation of mathematics is the Zermelo–Fraenkel set theory (ZF) with the Axiom of Choice (AC). There are many equivalent forms of AC. One of them is the cardinal comparability. Without AC, the cardinal comparability does not hold in general.

For any sets A and B, we say that $|A| = |B|$ or $A \approx B$ if there is a bijection between A and B and $|A| \leq |B|$ or $A \preceq B$ if there is an injection from A to B. The cardinal comparability states that for any sets A and B, $|A| \leq |B|$ or $|B| \leq |A|$.

For any set A, let A^A be the set of all functions on A, $\mathcal{P}(A)$ be the power set of A and $S(A) = \{f \in A^A : f \text{ is bijective}\}$. Dawson and Howard showed in [1] that $|\mathcal{P}(A)| = |S(A)|$ for any infinite set A in ZF with AC. Moreover, without AC, we cannot conclude any relationship between $|\mathcal{P}(A)|$ and $|S(A)|$ for an arbitrary infinite set A. In this paper we consider the set $I(A) = \{f \in A^A : f \text{ is injective}\}$ and $J(A) = \{f \in A^A : f \text{ is surjective}\}$. It is clear that $S(A) \subseteq I(A)$ and $S(A) \subseteq J(A)$.

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Definition 1.1. A set A is said to be Dedekind-finite if there is no injection from ω to A, and is said to be Δ_5 if there is no surjection from A to $A \cup \{b\}$ for any set $b \notin A$.

These are some variants of finiteness which are studied in set theory without AC. Some properties of them are similar to those of finite sets. For example, any subset of a Dedekind-finite set is also Dedekind-finite, and any subset of a Δ_5 set is also Δ_5. Moreover, any finite set is Δ_5, and any Δ_5 set is Dedekind-finite.

We have shown in [2] that if A is a union of a Dedekind-finite set and an infinite ordinal then $S(A) \approx I(A)$. In this paper, we prove that $S(A) \approx I(A)$ under a weaker condition on the set A. Also, we show that if a set A is a union of a Δ_5 set and an infinite ordinal, then $S(A) \approx J(A)$.

Let us recall some definitions and facts about ordinals. An *ordinal* is a set α which is transitive (that is, if $X \in \alpha$ then $X \subseteq \alpha$) and $(\alpha, \in)$ is a well-ordered set. Some primary examples are natural numbers

$$0 = \emptyset,\ \ 1 = \{0\},\ \ 2 = \{0, 1\},\ \ 3 = \{0, 1, 2\},\ \text{and so on.}$$

The set of natural numbers $\omega = \{0, 1, 2, ...\}$ is also an ordinal. Note a fact that any well-ordered set $(X, \leq)$ is isomorphic to an ordinal. So ordinals can be regarded as representatives of well-ordered sets. Moreover, some cardinal computations are easy for well-ordered sets: For any well-ordered sets A and B such that A or B is infinite,

$$|A| + |B| = |A| \cdot |B| = \max\{|A|, |B|\}$$

where $|A| + |B| = |A \cup B|$, provided that $A \cap B = \emptyset$, and $|A| \cdot |B| = |A \times B|$. Another equivalent form of AC states that every set can be well-ordered. So, without AC, the above equations do not hold in general.

Finally, let us list some facts which will be used in the next section.

Theorem 1.2. *For any sets A and B such that $A \cap B = \emptyset$, $S(A) \times S(B) \preceq S(A \cup B)$. As a result, for any sets A and B, $S(A) \times S(B) \preceq S((A \times \{0\}) \cup (B \times \{1\}))$.*

Proof. Define a function $F : S(A) \times S(B) \to S(A \cup B)$ by $F(p, q) = p \cup q$. F is injective, since $p = F(p, q){\restriction}A$ and $q = F(p, q){\restriction}B$. Hence $S(A) \times S(B) \preceq S(A \cup B)$. □

Theorem 1.3. *For an arbitrary set A, A is Dedekind-finite if and only if $S(A) = I(A)$, and A is Δ_5 if and only if $S(A) = J(A)$.*

Proof. Cf. [2, Theorem 3.2 and 3.3]. □

2 Results

In our proofs, we write id_A for the identity function on a set A.

Theorem 2.1. *For any set A, if $A = B \cup C$ where B is Dedekind-finite and C is a set such that $C \times \omega \approx C$, then $S(A) \approx I(A)$.*

Proof. Without loss of generality, let us assume that C and $\{0,1\} \times \omega \times C$ are disjoint. As B is Dedekind-finite, so is $B \setminus \big(C \cup (\{0,1\} \times \omega \times C)\big)$. So, without loss of generality, we assume that B and $\big(C \cup (\{0,1\} \times \omega \times C)\big)$ are disjoint.

Consider any injection $p : A \to A$ and any $b \in B \setminus \text{ran}(p)$. Since p is injective and $p(b) \neq b$, elements in the sequence $p(b), p^2(b), ...$ are all distinct. Since B is Dedekind-finite, it is impossible that all elements in the sequence are in B. We denote by $b_p \in C$ the first appearance of an element of C in the sequence. Note that for each distinct b and b' in $B \setminus \text{ran}(p)$, $b_p \neq b'_p$ by the injectivity of p. Let $L = \{(i, n, x) : i \in \{0,1\},\ n \in \omega \text{ and } x \in (C \setminus \text{ran}(p)) \cup \{b_p : b \in B \setminus \text{ran}(p)\}\}$. Define a function $F : I(A) \to S\big(A \cup (\{0,1\} \times \omega \times C)\big)$ by

$$\begin{aligned} F(p) =& p \cup \{((0, n+1, b_p), (0, n, b_p)) : b \in B \setminus \text{ran}(p), n \in \omega\} \cup \{((0,0,b_p), b) : b \in B \setminus \text{ran}(p)\} \\ & \cup \{((1, n+1, c), (1, n, c)) : c \in C \setminus \text{ran}(p), n \in \omega\} \cup \{((1,0,c), c) : c \in C \setminus \text{ran}(p)\} \cup id_L. \end{aligned}$$

We will show that F is well-defined. For each $a \in A$, if $a \in \text{ran}(p)$ then $a \in \text{ran}(F(p))$ since $p \subseteq F(p)$. If $a \notin \text{ran}(p)$ then $F((0,0,a)) = a$ or $F((1,0,a)) = a$. Note that $\{0,1\} \times \omega \times C$ is a disjoint union of L, $\{(0, n, x) : x = b_p$ for some $b \in B \setminus \text{ran}(p)\}$ and $\{(1, n, x) : x \in C \setminus C \setminus \text{ran}(p)\}$. The two latter sets are image of second and fourth sets in the definition of $F(p)$. Thus $F(p)$ is surjective. Moreover, each function in the definition of $F(p)$ is injective and have disjoint images, so $F(p)$ injective. Hence $F(p)$ is a bijection on $A \cup (\{0,1\} \times \omega \times C)$.

For each $p, q \in I(A)$ such that $F(p) = F(q)$, we have $p = F(p){\restriction}A = F(q){\restriction}A = q$. Hence, F is an injection. So we have $I(A) \preceq S\big(A \cup (\{0,1\} \times \omega \times C)\big)$. Since $C \times \omega \approx C$, $B \cup C \cup (\{0,1\} \times \omega \times C) \approx B \cup (\omega \times C) \approx B \cup C = A$. Therefore, $I(A) \preceq S(A)$. We clearly have $S(A) \preceq I(A)$. So $I(A) \approx S(A)$. □

Lemma 2.2. *For any sets C and D, if D is Δ_5, $C \subseteq D$ and $p : C \to D$ is a surjection, then $C = D$ and p is a bijection on D.*

Proof. Since D is Δ_5, so is C. Assume that $C \neq D$. Pick some $d \in D \setminus C$. Define a function $q : C \to C \cup \{d\}$ by $q(c) = p(c)$ if $p(c) \in C$ and $q(c) = d$ otherwise. Obviously q is a surjection from C onto $C \cup \{d\}$. Thus C is not Δ_5 which is impossible. So $C = D$. Now, $p \in J(D)$ and we know that $J(D) = S(D)$ since D is Δ_5. Hence, $p \in S(D)$. □

Theorem 2.3. *For any set A, if $A = B \cup \alpha$ where B is Δ_5 and α is an infinite ordinal, then $S(A) \approx J(A)$.*

Proof. As B is Δ_5, so is $B \setminus \alpha$. So, without loss of generality, we assume that B and α are disjoint. For each $p \in J(A)$ and each $a \in A$, let $p_a^{-\infty} = \{x \in A : p^n(x) = a$ for some $n \in \omega\}$. Let $B_p = \{b \in B : p_b^{-\infty} \subseteq B\}$. Let $D = \{d_i : i \in \omega\}$ be a countably infinite set such that $d_i \neq d_j$ for any $i \neq j$, which is disjoint from α. Fix an injection $F : \mathcal{P}((\alpha \cup D) \times (\alpha \cup D)) \to S(\alpha)$. This is possible since $\alpha \cup D \approx \alpha$ and α is an infinite well-ordered set.

For each $\theta \in \alpha$, $p(\theta) \notin B_p$ by the definition of B_p. Also, for each $b \in B \setminus B_p$, there are some ordinal $\gamma \in \alpha$ and some $n \in \omega$ such that $p^n(\gamma) = b$; so $p^{n+1}(\gamma) = p(b) \notin B_p$. So $p^{-1}[B_p] \subseteq B_p$. Since B is Δ_5, so is B_p. Observe that $p\restriction p^{-1}[B_p]$ is a surjection from $p^{-1}[B_p]$ to B_p. Thus $p^{-1}[B_p] = B_p$ and $p\restriction B_p \in S(B_p)$ by Lemma 2.2. This also implies that $p\restriction(\alpha \cup (B \setminus B_p)) \in (\alpha \cup (B \setminus B_p))^{(\alpha \cup (B \setminus B_p))}$.

Note that $\omega \times \alpha$ is well-ordered by the lexicographic ordering. We define a function from $B \setminus B_p$ to $\omega \times \alpha$ by sending b to the least $(n, \gamma) \in \omega \times \alpha$ such that $p^n(\gamma) = b$. Clearly this map is injective. So $B \setminus B_p$ is well-ordered by the induced ordering. Moreover, since B is Dedekind-finite, so is $B \setminus B_p$. As a well-ordered Dedekind-finite set must be finite, we can write $B \setminus B_p = \{c_0, ..., c_{k-1}\}$ in an increasing order for some $k \in \omega$. Let

$$\hat{p} = \left(\bigcup_{i<k} \{(i, c_i), (c_i, i)\} \right) \cup id_{\omega \setminus k} \cup (p\restriction B_p) \tag{2.1}$$

Define an injection $H : B \setminus B_p \to D$ by $H(c_i) = d_i$ for all $i < k$. Let

$$\begin{aligned} \tilde{p} = &\{(x, y) : x, y \in \alpha \text{ and } p(x) = y\} \\ &\cup \{(x, H(c)) : x \in \alpha,\ \ c \in B \setminus B_p \text{ and } p(x) = c\} \\ &\cup \{(H(c), y) : c \in B \setminus B_p,\ y \in \alpha \text{ and } p(c) = y\} \\ &\cup \{(H(c), H(c')) : c, c' \in B \setminus B_p \text{ and } p(c) = c'\} \end{aligned} \tag{2.2}$$

Note that $\hat{p} \in S(B \cup \omega)$ and $\tilde{p} \in \mathcal{P}((\alpha \cup D) \times (\alpha \cup D))$.

Define a function $G : J(A) \to S(\alpha) \times S(B \cup \omega)$ by $G(p) = (F(\tilde{p}), \hat{p})$. To see that G

is injective, let $p, q \in J(A)$ be such that $G(p) = G(q)$. Then $\hat{p} = \hat{q}$, and as F is injective, $\tilde{p} = \tilde{q}$. Consider (2.1) for p and q. Recall that $B \cap \alpha = \emptyset$. So, members of B which are sent by $\hat{p}$ to natural numbers are exactly those in $B \setminus B_p$. Similarly for $\hat{q}$. As $\hat{p} = \hat{q}$, we obtain $B \setminus B_p = B \setminus B_q$; thus $B_p = B_q$. Then by (2.1) again, we can see that $p \restriction B_p = q \restriction B_q$. Next, consider (2.2) for p and q. Recall that $B \cap \alpha = \emptyset$, $H : B \setminus B_p \to D$ is injective, and $B_p = B_q$. We can see that $\tilde{p} = \tilde{q}$ implies $p \restriction (\alpha \cup (B \setminus B_p)) = q \restriction (\alpha \cup (B \setminus B_q))$. Hence, $p = q$.

Since $S(\alpha) \times S(B \cup \omega) \preceq S((\alpha \times \{0\}) \cup ((B \cup \omega) \times \{1\})) \approx S(A)$, we have that $J(A) \preceq S(A)$. We clearly have $S(A) \preceq J(A)$. Hence, $J(A) \approx S(A)$. $\square$

3 Discussions

We have shown in [2] that $I(A) \preceq J(A)$ for any set A, and it is not provable in ZF that "$S(A) \approx J(A)$ for any infinite set A" (provided that ZF is consistent). However, it is still open that the statement "$S(A) \approx I(A)$ for any infinite set A" is provable in ZF or not.

References

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