W.A.S.S.C.E NOTE PACK

PHYSICS-MECHANICS

Concise, credible and self-explanatory notes for excellent grades at W.A.S.S.C.E.

"A revolutionary approach to quality and affordable education"

This pack is one of many packs comprehensively made to suit the needs of candidates attempting the West African Senior School Certificate Examination. It has been concisely arranged and thoroughly researched to adequately provide students with knowledge sufficing a proficient result at WASSCE.

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REFERENCES

Physics
- A.F. Abbott

Physics GCE
- P.N. Okeke

Physics For a Modern World
- Dean J. Bunn

Advanced Level Physics
- Nelkon and Parker

Mechanics for Advanced Level
- L. Bostock and S. Chandler

Pure and Applied Mathematics
- J.F. Talbert and H.H. Heng

Advanced Level Applied Mathematics
- C.G. Lambe

CONTENTS

PHYSICS

Physics-Mechanics Wassce Syllabus

SCHEME OF EXAMINATION

There will be **three** papers, Papers 1, 2 and 3, all of which must be taken. Papers 1 and 2 will be a composite paper to be taken at one sitting.

PAPER 1: Will consist of **fifty** multiple choice questions lasting 1¼ hours and carrying **50** marks.

PAPER 2: Will consist of **two** sections, Sections A and B, both lasting for 1½ hours and carrying **60** marks.

Section A - Will comprise **seven** short-structured questions. Candidates will be required to answer any **five** questions for a total of **15** marks.

Section B - Will comprise **five** essay questions out of which candidates will be required to answer any **three** for **45** marks.

PAPER 3: Will be a practical test for school candidates **or** a test of practical work for private candidates. Each version of the paper will comprise **three** questions out of which candidates will be required to answer any **two** in 2¾ hours for **50** marks.

Candidates taking the practical test will be allowed additional **15** minutes for reading the paper during which time they are not expected to do any writing.

PRACTICAL PHYSICS

This will be tested by a practical examination based on the syllabus. The objective of the practical examination is to test how well the candidates understand the nature of scientific investigation and their capability in handling simple apparatus in an experiment to determine an answer to a practical question. It is also to determine their competence in demonstrating their understanding of some of the principles involved in a small-scale laboratory experiment.

The practical test will contain enough instructions to enable candidates to carry out the experiment. Even when standard experiments, such as the determination of focal lengths or specific heat capacities are set, candidates will be told what readings to take and how to arrive at the result. Therefore, it should not be necessary for candidates to learn by heart how to perform any experiment.

In addition to experiments on the topics in the syllabus, candidates may be asked to carry out with the aid of full instructions, variants of standard experiments.

Candidates should be trained to take as varied a set of readings as possible and to set out the actual observed readings systematically on the answer sheet. The experiments may require a repetition of readings and an exhibition of results graphically and their interpretation.

DETAILED SYLLABUS

It is important that candidates are involved in practical activities in covering this syllabus. Candidates will be expected to answer questions on the topics set in the column headed ' TOPIC'. The 'NOTES' are intended to indicate the scope of the questions which will be set but they are not to be considered as an exhaustive list of limitations and illustrations.

NOTE: Questions will be set in S.I. units. However, multiples or sub-multiples of the units may be used.

PART 1
INTERACTION OF MATTER, SPACE & TIME

TOPICS	NOTES
1. Concepts of matter	Simple structure of matter should be discussed. Three physics states of matter, namely solid, liquid and gas should be treated. Evidence of the particle nature of matter e.g. Brownian motion experiment, Kinetic theory of matter. Use of the theory to explain; states of matter (solid, liquid and gas), pressure in a gas, evaporation and boiling; cohesion, adhesion, capillarity. Crystalline and amorphous substances to be compared (Arrangement of atoms in crystalline structure to be described e.g. face centred, body centred.
2. Fundamental and derived quantities and units (a) Fundamental quantities and units	Length, mass, time, electric current luminous intensity, thermodynamic temperature, amount of substance as examples of fundamental quantities and m, kg, s, A, cd, K and mol as their respective units.
(b) Derived quantities and units	Volume, density and speed as derived quantities and m^3, kgm^{-3} and ms^{-1} as their respective units.
3. Position, distance and displacement. (a)) Concept of position as a location of point-rectangular coordinates.	Position of objects in space using the X,Y,Z axes should be mentioned.
(b) Measurement of distance	Use of string, metre rule, vernier calipers and micrometer screw gauge. Degree of accuracy should be noted. Metre (m) as unit of distance.
(c)) Concept of direction as a way of locating a point –bearing	Use of compass and a protractor.
(d) Distinction between distance and displacement.	Graphical location and directions by axes to be stressed.

TOPICS	NOTES
4. Mass and weight	Use of lever balance and chemical/beam balance to measure mass and spring balance to measure weight. Mention should be made of electronic/digital balance.
Distinction between mass and weight	Kilogram (kg) as unit of mass and newton (N) as unit of weight.
5. Time (a)) Concept of time as interval between physical events	The use of heart-beat, sand-clock, ticker-timer, pendulum and stopwatch/clock.

(b) Measurement of time	Second(s) as unit of time.
6. Fluid at rest	
(a) Volume, density and relative density	Experimental determination for solids and liquids.
(b) Pressure in fluids	Concept and definition of pressure. Pascal's principle, application of principle to hydraulic press and car brakes. Dependence of pressure on the depth of a point below a liquid surface. Atmospheric pressure. Simple barometer, manometer, siphon, syringe and pump. Determination of the relative density of liquids with U-tube and Hare's apparatus.
(c) Equilibrium of bodies	Identification of the forces acting on a body partially or completely immersed in a fluid.
(i) Archimedes' principle	Use of the principle to determine the relative densities of solids and liquids.
(ii) Law of flotation	Establishing the conditions for a body to float in a fluid. Applications in hydrometer, balloons, boats, ships, submarines etc.

TOPICS	NOTES
7. Motion	
(a) Types of motion: Random, rectilinear, translational, Rotational, circular, orbital, spin, Oscillatory.	Only qualitative treatment is required. Illustration should be given for the various types of motion.
(b) Relative motion	Numerical problems on co-linear motion may be set.
(c) Cause of motion	Force as cause of motion.
(d) Types of force:	
(i) Contact force	Push and pull
(ii) Non-contact force(field force)	These are field forces namely; electric and magnetic attractions and repulsions; gravitational pull.
(e) Solid friction	Frictional force between two stationary bodies (static) and between two bodies in relative motion

	(dynamic). Coefficients of limiting friction and their determinations. Advantages of friction e.g. in locomotion, friction belt, grindstone. Disadvantages of friction e.g reduction of efficiency, wear and tear of machines. Methods of reducing friction; e.g. use of ball bearings, rollers, streamlining and lubrication.
(f) Viscosity (friction in fluids)	Definition and effects. Simple explanation as extension of friction in fluids. Fluid friction and its application in lubrication should be treated qualitatively. Terminal velocity and its determination.
(g) Simple ideas of circular motion	Experiments with a string tied to a stone at one end and whirled around should be carried out to (i) demonstrate motion in a Vertical/horizontal circle.

TOPICS	NOTES
	(i) show the difference between angular speed and velocity. (ii) Draw a diagram to illustrate centripetal force. Banking of roads in reducing sideways friction should be qualitatively discussed.
8. Speed and velocity	
(a) Concept of speed as change of distance with time	
(b) Concept of velocity as change of displacement with time	Metre per second (ms^{-1}) as unit of speed/velocity.
(c) Uniform/non-uniform speed/velocity	Ticker-timer or similar devices should be used to determine speed/velocity. Definition of velocity as $\vec{s}/t$.
(d) Distance/displacement-time graph	Determination of instantaneous speed/velocity from distance/displacement-time graph and by calculation.

9. Rectilinear acceleration	
(a) Concept of Acceleration/deceleration as increase/decrease in velocity with time.	Unit of acceleration as ms^{-2}
(b) Uniform/non-uniform acceleration	Ticker timer or similar devices should be used to determine acceleration. Definition of acceleration as v/t.
(c) Velocity-time graph	Determination of acceleration and displacement from velocity-time graph
(d) Equations of motion with constant acceleration;	Use of equations to solve numerical problems.
Motion under gravity as a special case.	

10. Scalars and vectors	
(a) Concept of scalars as physical quantities with magnitude and no direction	Mass, distance, speed and time as examples of scalars.
(b) Concept of vectors as physical quantities with both magnitude and direction.	Weight, displacement, velocity and acceleration as examples of vectors.
(c) Vector representation	
(d) Addition of vectors	Use of force board to determine the resultant of two forces.
(e) Resolution of vectors	Obtain the resultant of two velocities analytically and graphically.
(f) Resultant velocity using vector representation.	
11. Equilibrium of forces	Torque/Moment of force. Simple treatment of a couple, e.g. turning of water tap, corkscrew and steering wheel.)
(a) Principle of moments	Use of force board to determine resultant and equilibrant forces. Treatment should include resolution of forces into two perpendicular directions and composition of forces
(b) Conditions for equilibrium of rigid bodies under the action of parallel and non-parallel forces.	Parallelogram of forces. Triangle of forces.
	Should ne treated experimentally. Treatment should include stable, unstable and neutral equilibra.
(c) Centre of gravity and stability	Use of a loaded test-tube oscillating vertically in a liquid, simple pendulum, spiral spring and bifilar suspension to demonstrate simple harmonic motion.
12. Simple harmonic motion	
(a) Illustration, explanation and definition of simple harmonic motion (S.H.M)	

TOPICS	NOTES
(b) Speed and acceleration of S.H.M.	Relate linear and angular speeds, linear and angular accelerations.
(c) Period, frequency and amplitude of a body executing S.H.M.	Experimental determination of 'g' with the simple pendulum and helical spring. The theory of the principles should be treated but derivation of the formula for 'g' is not required
(d) Energy of S.H.M	Simple problems may be set on simple harmonic motion. Mathematical proof of simple harmonic motion in respect of spiral spring, bifilar suspension and loaded test-tube is not required.
(e) Forced vibration and resonance	
	Distinction between inertia mass and weight
13. Newton's laws of motion:	
(a) First Law: Inertia of rest and inertia of motion	Use of timing devices e.g. ticker-timer to determine the acceleration of a falling body and the relationship when the accelerating force is constant.
(b) Second Law: Force, acceleration, momentum and impulse	Linear momentum and its conservation. Collision of elastic bodies in a straight line.
	Applications: recoil of a gun, jet and rocket propulsions.
(c) Third Law: Action and reaction	

TOPICS	NOTES
14. Energy:	
(a) Forms of energy	Examples of various forms of energy should be mentioned e.g. mechanical (potential and kinetic), heat, chemical, electrical, light, sound, nuclear etc.
(b) World energy resources	Renewable (e.g. solar, wind, tides, hydro, ocean waves) and non-renewable (e.g. petroleum, coal, nuclear, Biomass). Sources of energy should be discussed briefly.
(c) Conservation of energy	Statement of the principle of conservation of energy and its use in explaining energy transformations.
15. Work, Energy and Power	
(a) Concept of work as a measure of energy transfer	Unit of work as the joule (J)
(b) Concept of energy as capability to do work	Unit of energy as the joule (J) while unit of electrical consumption is kWh.
(c) Work done in a gravitational field.	Work done in lifting a body and by falling bodies.
(d) Types of mechanical energy	Derivation of P.E. and K.E. are expected to be known. Identification of types of energy possessed by a body under given conditions.
(i) Potential energy (P.E.)	
(ii) Kinetic energy (K.E.)	
(e) Conservation of mechanical energy	Verification of the principle

TOPICS	NOTES
(f) Concept of power as time rate of doing work.	Unit of power as the watt (W).
(g) Application of mechanical energy – machines. Levers, pulleys, inclined plane, wedge, screw, wheel and axle, gears.	The force ratio (F.R.), mechanical advantage (M.A.), velocity ratio (V.R.) and efficiency of each machine should be treated. Identification of simple machines that make up a given complicated machine e.g. bicycle. Effects of friction on machines. Reduction of friction in machines.

PHYSICS → INTRODUCTION

Definition, Branches and The Concept of Matter

What is Physics?

The term Physics emanates from an ancient Greek word "Physis" meaning nature. It is a part of natural philosophy and a natural science that involves the study of matter, its motion through space and time in relation to energy.

More broadly, it is the general analysis of nature, conducted in order to understand how the universe behaves.

Branches

Main Divisions

1. **CLASSICAL PHYSICS** – includes the traditional branches and topics that were recognized and well-developed before the beginning of the 20th century. Branches include classic mechanics, wave motion or acoustics, optics, thermodynamics, electricity and magnetism.
2. **MODERN PHYSICS-** is concerned with the behaviour of matter and energy under extreme conditions or on a very large or very small scale. For example, atomic and nuclear physics studies on the smallest scale at which chemical elements can be identified.

MECHANICS- is a branch of physics concerned with the behaviour of physical bodies when subjected to forces or displacement, and the subsequent effects of the bodies on their environments.

The major division of mechanics separates into **classical** and **quantum** mechanics.

Historically, classical mechanics preceded quantum mechanics (i.e. quantum mechanics is a comparatively recent invention). Classical mechanics originated from Isaac Newton's laws of motion in Principia Mathematica, while quantum mechanics did not appear until 1900.

CLASSICAL MECHANICS- deals with a set of physical laws describing the motion of bodies under the action of a system of forces. This branch is further divided into;

STATICS: a branch of mechanics concerned with the study of loads (forces and torque or moment of a force) on physical systems in static equilibrium.

KINEMATICS: describes the motion of points, bodies and system of bodies without considering the causes of the motion.

KINETICS/DYNAMICS: analytically concerned with the relationship between motion of bodies and it causes, namely forces acting on the bodies and the properties of the bodies (particularly mass and moment of inertia).

QUANTUM MECHANICS- also known as quantum theory is a branch of physics concerned with physical phenomena at microscopic scales. It provides a mathematical description of the dual-particle-like and wave-like behaviour and interactions of energy and matter.

WAVE MOTION/ACOUSTICS is the interdisciplinary science that deals with the study of all mechanical waves in gases, liquids, and solids including vibration, sound, ultrasound and infrasound. The application of acoustics can be seen in all aspects of our modern society with the most obvious being the audio and noise control industries.

OPTICS this branch deals with the behaviour and properties of light, including its interactions with matter and the construction of instruments that use or detect it. Optics usually describes the behaviour of visible, ultraviolet and infrared light. Because light is an electromagnetic radiation such as X-rays, microwaves, and radio waves exhibit similar properties.

ELECTRICITY AND MAGNETISM

Deals with the study of electric charges and its relation to magnetic forces.

CONCEPT OF MATTER

1. SIMPLE STRUCTURES OF MATTER

For centuries now scientists believed that matter or all substances are made up of tiny particles called atoms (which are found in) ***elements*** and these elements can form ***molecules*** which in turn can form ***chemical compounds***.

- **ATOMS, ELEMENTS AND MOLECULES**
  - **Atoms** *are the smallest portions of an element which can take part in a chemical change.*
    - (a) Atoms contain particles called charges (proton or electron), which can conduct electricity in motion, thus electric and magnetic forces do exists between them.
    - (b) These forces are hence ***electromagnetic in nature***.
  - **Elements** *are substances which cannot be split into simpler substances.*
  - **Molecules** *are groups of atoms of various elements united in the same simple numerical proportion.*

  NOTE: these electromagnetic forces differ from one kind of atom or molecules to another and even between atoms or molecules of the same kind depending on the state in which the substance exists.

- **Three States or Phases of Matter**
  - **Solid** *substances are substances whose molecules vibrate about their zero resultant force position, alternately attracting and repelling one another.*
    - (a) They are crystalline in structure with atoms arranged in a regular pattern called ***lattice***.
    - (b) They are good conductors of electricity and heat.
    - (c) Solid molecules sustain a fixed position due to their strong intermolecular forces.
  - **Liquids** *are substances which can flow or move freely within their structure and from one point to another. This is due to the molecular nature of the liquid substance.*
    - (a) They have weak intermolecular forces causing them to flow easily.
    - (b) This enable them to take the shape of the vessel in which they are placed.
  - **Gases** *are substances whose molecules move at higher velocities colliding with one another and with walls of their container*.
    - (a) Gases have weaker intermolecular forces than solids and liquids, hence they are perfectly free to expand and completely fill the vessel in which they are place

- **Evidence of the Particle Nature of Matter**

  The idea of molecular motion was examined in 1827 by Robert Brown, who observed in his experiment, the haphazard movement of pollen particles in water. This movement was caused by the impact of moving water molecules. It was later named after him as the **Brownian Motion**.

  - **Brownian Motion** *the motion describes the particle nature of matter suggesting that molecules of solids and gases are in continuous motion. This idea of molecular motion and the mathematical calculations applied to it is called the* ***Kinetic Theory of Matter.***
  - **Kinetic Theory of Matter** *this theory simply states that the mean kinetic energy for all particles is the same.* This theory sensibly explains the states of matter, pressure in gases, evaporation and boiling, diffusion, adhesion and cohesion, and capillarity.

    Assignment: Use the kinetic theory of matter to explain the following phenomenon.
    - (a) States of matter (solids, Liquids and gases)
    - (b) Pressure in gases.
    - (c) Evaporation and boiling
  - **Diffusion** *is the process by which substances mix (intimately) with one another due to the random motion of their molecules.*

    The rate at which substances (or gases) diffuse depend on their ***masses*** and ***temperatures***.
    - *(a) Heavier molecules move more slowly than lighter ones.*
    - *(b) Increase in temperature can cause increase in kinetic energy, which in turn increases the rate of diffusion.*

(Capillarity, cohesion and adhesion are explained in the succeeding chapters)

PHYSICS ⟶ Mechanics

UNITS AND MEASUREMENTS

1. Scientific units and measurements

Unit- A unit is simply a standard of measurement.

Measurement- Is a basic means of communication used for understanding natural phenomenon.

In the 60's, the general conference of weights and measures recommended that everyone should use a metric system of measurement known as The International System of Units (SI-UNITS).

- **Standard/SI Units**- are units of measurement that are understood and accepted by people all over the world.
- **Base/Fundamental Units**- are units from which all other units depend or are derived. They are units of fundamental quantities.
- **Derived Units**- are units of two or more base/fundamental units.

Fundamental and Derived quantities

(a) **Fundamental Quantities**-Are basic quantities independent of others and cannot be derived or define in terms of other quantities.

(b) **Derived Quantities**- Quantities obtained by the combination of two or more fundamental quantities.

- Below are some fundamental quantities, their units and symbols.

NO.	Quantity	SI unit	Symbol
1.	Length	metre	M
2.	mass	Kilogram	Kg
3.	Time	seconds	S
4.	Electric current	Ampere	A
5.	Thermodynamic Temperature	Kelvin	K
6.	Luminous intensity	Candela	Cd
7.	Amount of substance	mole	Mol

- Below are some derived quantities, SI units and their symbols.

NO	Quantity	SI unit	Symbol
1.	Force	Newton	N
2,	Work and energy	Joules	J
3.	Quantity of electricity	Coulombs	C
4	Electric resistance	Ohm	Ω

Scalar and Vector quantities

Scalar Quantities are quantities with magnitude but no direction. E.g. distance,

Vector Quantities are quantities with both magnitude and direction. E.g. Displacement

Dimensional Analysis

- ***A Dimensional Equation*** is an equation which expresses a physical quantity in terms of its dimensions.
- **[ ] means 'dimension of'** e.g. **[**mass**]** means 'dimension of mass'

Uses of dimensional equations

1. To check the validity of an equation
2. To find the units of quantities
3. To derive the equation between quantities

From the three fundamental quantities shown in the table below, we can derive dimensions of other quantities.

Quantity	Dimension
Mass	**M**
Length	**L**
Time	**T**

Worked Example 1.0

Q1. *Find the dimensional equations of the following quantities:*

(a) Area *(d) Acceleration* *(g) Momentum*
(b) Density *(e) Force* *(h) Work*
(c) Velocity *(f) Pressure*

Sol^n

(a) [Area] = [Length]×[Length] = L × L = L^2

(b) [Density]= $\frac{[Mass]}{[Volume]} = \frac{M}{L\times L\times L}$=$ML^{-3}$

(c) [Velocity]=$\frac{[Displacement]}{[Time]} = \frac{L}{T} = LT^{-1}$

(d) [Acceleration]=$\frac{[Velocity]}{[Time]} = \frac{LT^{-1}}{T} = \frac{L}{T\times T} = LT^{-2}$

(d) [Force]= $[Mass] \times [Accleration] = M \times LT^{-2} = MLT^{-2}$

(e) [Pressure]= $\frac{[Force]}{[Area]} = \frac{MLT^{-2}}{L^2} = \frac{ML}{L\times L\times T^2} = ML^{-1}T^{-2}$

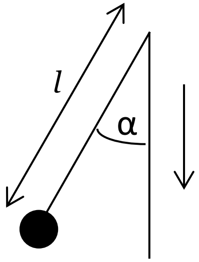
(f) [Momentum]= $[Mass] \times [Velocity] = M \times LT^{-1} = MLT^{-1}$

(g) [Work]= $[Force] \times [Distance] = MLT^{-2} \times L = ML^2T^{-2}$

Q2. *A simple pendulum of length (l) and mass (m) is set into oscillation by displacing it bulb at an angle (α) as shown in the diagram. The Period (T) is given by: T= $cl^w m^x \alpha^y g^z$ where w, x, y, z are Unknowns and (g) as acceleration due to gravity with c=1.*

(a) Find the values of w, x, y, z.

(b) Hence derive an expression relating T, l, m and g.



Sol^n

(a) T= $cl^w m^x \alpha^y g^z$ where α = dimensionless constant and c = 2π

» $[T] = [L]^w [M]^x [\frac{L}{T^2}]^z$

» $[T] = [L]^{w+z} [M]^x [T]^{-2z}$

» Equating powers of dimensions on both sides $1 = -2z$

$0 = x$

» $x = 0, w = {}^1/_2, z = -{}^1/_2$ $\quad 0 = w + z$

(b) »$T = l^{(\frac{1}{2}+0)}\, m^0\, g^{-\frac{1}{2}}$ (Substituting values of unknowns)

»T= $l^{\frac{1}{2}} g^{-\frac{1}{2}}$

»T= $2\pi \frac{l^{\frac{1}{2}}}{g^{\frac{1}{2}}} = 2\pi \frac{\sqrt{l}}{\sqrt{g}}$

∴T= $2\pi \sqrt{\frac{l}{g}}$ *(answer)*

Comparison between quantities

Distance	Displacement
• The interval between two points in a straight line. • Scalar Quantity	• The distance covered in a specified direction. • Vector quantity
Mass	**Weight**
• The quantity of matter a body contains. • SI unit Kilogram (Kg).	• The force with which the earth attract a body towards its centre. • SI unit is Newton(N)
Speed	**Velocity**
• The rate at which a body covers a distance. • Scalar quantity (no specified direction only magnitude)	• The rate of change of displacement. • Vector quantity (direction and magnitude are specified)

1.1 Position, Distance and Displacement

- **Position-** The position of a particle is defined to be the coordinate vector from the origin of a coordinate frame to the particle.
- Mathematically $|\boldsymbol{P}| = \sqrt{(x_2 - x_1)_p^2 + (y_2 - y_1)_p^2 + (z_2 - z_1)_p^2}$
- For example, consider a tower 50 m south from your home, where the coordinate frame is located at your home, such that East is the x-direction and North is the y-direction, then the coordinate vector to the base of the tower is **r**=(0, -50, 0). If the tower is 50 m high, then the coordinate vector to the top of the tower is **r**= (0, -50, 50)**.**

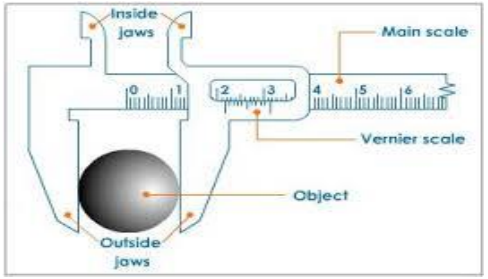
1.2 Measurement of Quantities

QUANTITY	INSTRUMENTS
1. Distance	• Metre rule • Vernier Calliper • Micrometre screw gage • Tape
2. Mass	• Lever balance • Beam balance • Electronic balance
3. Weight	• Spring balance
4. Time	• Sand clock • Ticker-timer • Pendulum • Stop watch

Need –to-know instruments

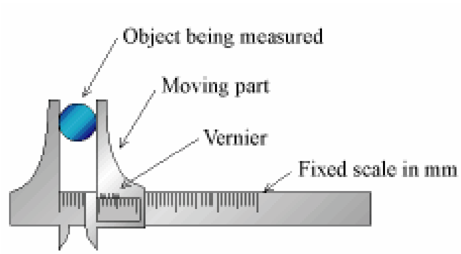
(a) VERNIER CALIPER

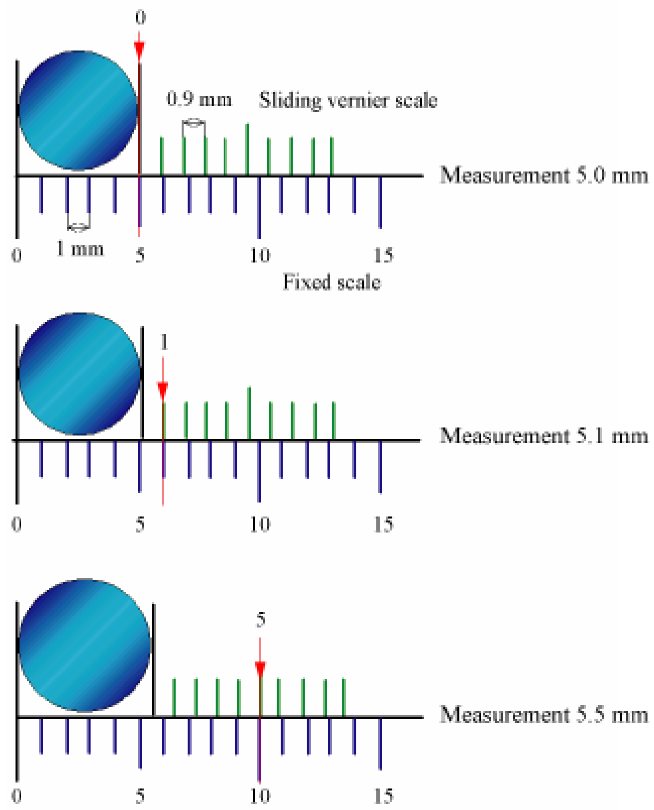
- The main use of the Vernier calliper is to measure the internal and the external diameters of an object (e.g. pipe) using the Vernier scale.



- To measure using a Vernier scale, the user first reads the finely marked "fixed/main" scale (in the diagram). This measure is typically between two of the scale's smallest graduations. The user then reads the finer Vernier scale (see diagram), which measures between the smallest graduations on the fixed scale—providing much greater accuracy. See images on the right for better understanding.
- **Formula:** actual reading = main scale + Vernier scale − (zero error)
- Zero error may arise due to knocks that cause the calibration at the 0.00 mm when the jaws are perfectly closed or just touching each other.

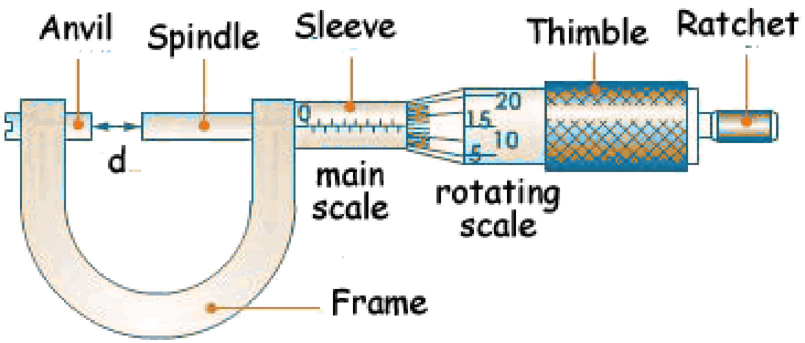
It is also used in measuring an object to its lowest decimal point.





(b) MICROMETER SCREW GUAGE

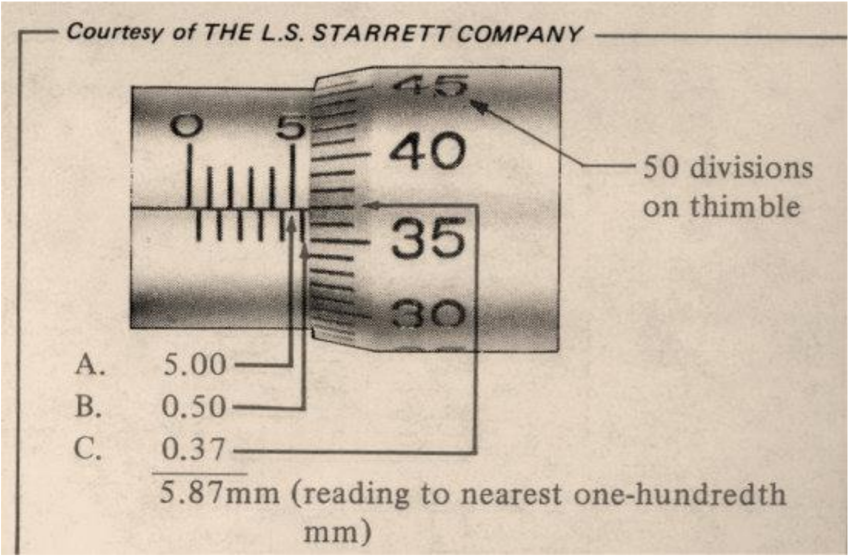
- Measures precisely and accurately than the Vernier calliper. E.g. measures the diameter of a wire more accurately than the Vernier calliper.



Formula: actual reading = main scale reading + rotating/Vernier scale

(c) Main scale reading=upper sleeve reading+ lower sleeve reading

(d) Below is an example where A = upper sleeve of main scale reading B = lower sleeve of main scale reading and C=rotating/Vernier scale reading



FLUID MECHANICS

Fluids at rest: volume, density and relative density

- **Fluid Mechanics**- the branch of mechanics that deals with the behaviour of fluids at rest as well as in motion.
- **Volume-** is the quantity of a three-dimensional space enclosed by a closed surface. SI unit is Metre cube or cubic metre (m^3)
- It is the Three-dimensional space that matter (solid, liquid, gas or plasma) occupies or contains.

Volumes of regular solids/objects

- They are found using mathematical formulas:

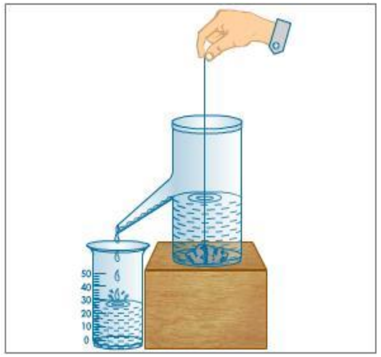
OBJECT	VOLUME FORMULA
Rectangular block (cuboid)	$L \times B \times H$
Cylinder	$\pi r^2 h$
Sphere	$^4/_3\, \pi r^3$
Cone	$^1/_3\, \pi r^2 h$
Square	L^3

Volume of irregular solids/objects

- The solid/object whose volume is to be determined is tied to a thread and then immersed in a eureka about half-filled with water. A calibrated cylinder is placed just beneath the sprout of the eureka for the liquid from the cylinder to flow into.
- See diagram below

NOTE: This is applicable only to non-dissolvable solids or objects.

Volume of Liquids



- *Measured directly using transparent graduated cylinders.*
- ***Measuring cylinder** for measuring or pouring out various volumes of liquids.*
- ***Measuring flask** for swirling liquid mixtures and for getting fixed predetermined volumes.*

DENSITY

- ***The density of a substance is it mass per unit volume.***
  SI unit is Kg/m^3 or Kgm^{-3}.
- *Formula:* $Density = \frac{mass}{volume}$, $\rho = \frac{m}{v}$ where $\rho = rho(greek\ letter)$

Gist for conversion of unit

- Converting $g/cm^3 \longrightarrow Kg/m^3$, multiply g/cm^3 by 1000
- Converting $kg/m^3 \longrightarrow g/cm^3$, divide kg/m^3 by1000
- Density of water=$1g/cm^3$ or $1000kg/m^3$

Density of Liquid

- $\rho_l = \frac{[(m_{(b+l)} + m_b]}{V_L} = \frac{M_L}{V_L}$ where
- $m_{(b+l)} = mass\ of\ beaker + liquid$

m_b= *mass of beaker*

m_L $(m_{(b+l)} - m_b)$ = *mass of liquid*

V_L= *volume read directly from beaker*

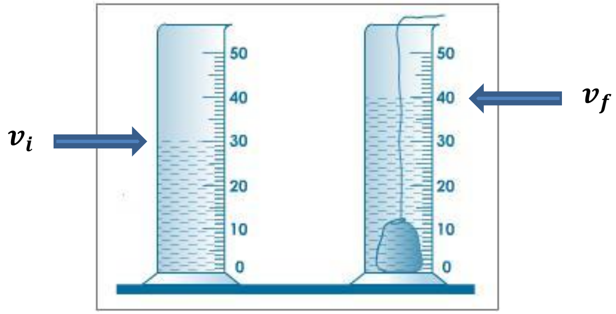
Practical Procedure

- A calibrated beaker is first weighed empty (m_b) and then weighed with the liquid **($m_{(b+L)}$)**
- Mass of the liquid is found by subtraction ***($m_{(b+l)} - m_b$)***
- The volume of the liquid is read directly from the graduated beaker.

Irregular Solids

- $\rho = \frac{M_S}{V_F - V_I}$, Where ***$M_S$ = mass of solid, V_F = final volume,***

  V_I = initial volume
- The solid is first weighed with a beam balance and it mass recorded as ***M_S*** , then immersed into a graduated cylinder half-filled with water and of volume ***V_I***. This will lead to a rise in the water level giving it a new volume ***V_F.***



RELATIVE DENSITY

- ***The Relative Density** of a substance is the ratio of the density (mass of a unit volume) of the substance to the density of a given reference substance (e.g. water). It is also known as Specific Gravity. It has no unit.*
- *Mathematically* **R.D =** $\frac{Density\ of\ substance}{Density\ of\ water\ (reference\ substance\}}$

  R.D = $\frac{Mass\ of\ substance}{Mass\ of\ equal\ volume\ of\ water}$

  R.D = $\frac{Weight\ of\ substance}{Weight\ of\ equal\ volume\ of\ water}$

Archimedes Principle

- ***This states that the upward buoyant force (upthrust) that is exerted on a body when fully or partially immersed in a fluid, is equal to the weight of the fluid the body displaces and it acts in the upward direction through the centre of mass of the displaced fluid.***

Practical determination of the relative density of a liquid

1. Using the relative density bottle

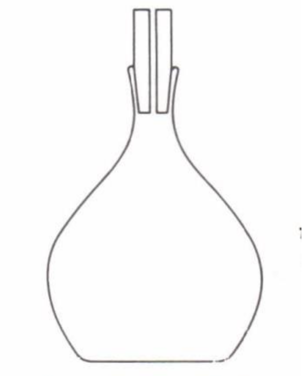
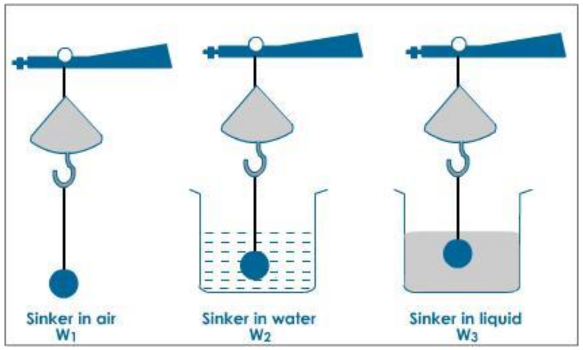


Fig. 9.2. Density bottle

- The bottle is first weighed empty and mass recorded as ***m***
- It is then filled with the liquid, weighed and recorded as m_1
- It is finally emptied, dried and filled with water. This is recorded as m_2
- *Mathematically* $R.D = \frac{m_1 - m}{m_2 - m}$

2. Using Archimedes Principle



- Object is weighed in air and it weight is record as W_1
- Then immersed in water and it weight is recorded as W_2
- Later immersed into a liquid of known density and this is recorded as W_3
- *Mathematically*

$$R.D = \frac{upthrust\ in\ liquid}{upthrust\ in\ water} = \frac{w_3 - w_1}{w_2 - w_1}$$

Density measurement using Archimedes Principle

1. Solid

- A solid is weighed in air using a spring balance and its weight recorded as W_a
- It is then completely immersed in water and the reading on the spring balance is recorded as W_w
- *Calculation* $R.D = \frac{weight\ in\ air}{upthrust\ in\ water} = \frac{W_a}{W_a - W_w}$
- Also $R.D = \frac{Density\ of\ solid}{Density\ of\ water}$

$$\Rightarrow density\ of\ solid = R.D \times density\ of\ water$$

$$= R.D \times 1g/cm^3 \text{ or } 1000\ Kg/m^3$$

2. Liquid

- A solid is first weighed in air (W_a) using a spring balance, then in water (W_w) and finally in a given liquid (W_l)
- *Calculation* $R.D = \frac{upthrust\ in\ liquid}{upthrust\ in\ water}$

$$= \frac{W_a - W_l}{W_a - W_w}$$

- **Also,** $R.D = \frac{Density\ of\ solid}{Density\ of\ water}$

$$\Rightarrow density\ of\ Liquid = R.D \times density\ of\ water$$

$$= R.D \times 1g/cm^3 \text{ or } 1000Kg/m^3$$

Worked Examples 1.0

1. ***The relative density of steel is 7.8.***
   (a) Find the mass of a solid steel cube of side 10m
   (b) What volume of the steel has a mass of 8kg

Solution

(a) DATA: density of water$(\rho_w) = 1000kg/cm^3$*, R.D = 7.8,*

Volume of steel cube $(V_c) = L^3 = 10^3cm^3$ $mass\ of\ steel(m_s) =?$

$$but\ \ R.D = \frac{Density\ of\ steel}{Density\ of\ water}$$

$$7.8 = \frac{Density\ of\ steel}{1g/cm^3}$$

$$density\ of\ steel = 7.8 \times 1g/cm^3 = 7.8g/cm^3$$

$$\Longrightarrow \frac{\mathrm{m_s}}{10^3\mathrm{cm}^3} = 7.8\mathrm{g/cm}^3,\ \Longrightarrow \mathrm{m_s} = 1000cm^3 \times \frac{7.8g}{cm^3} = \mathbf{7.8 \times 10^3}\mathrm{g}$$

(b) $m_S = 8\ Kg \equiv 8 \times 10^3g,$

Using variation, $1000cm^3 \equiv 7.8 \times 10^3$g

$$V_S \equiv 8 \times 10^3g \qquad \Longrightarrow\ V_S = \mathbf{1025.64cm^3\ ans}$$

2. ***What volume of brass of density 8.5*** g/cm^3 ***must be attached to a piece of wood of mass 100g and density 0.2*** g/cm^3 ***so that the two together will just submerge beneath the water?***

Solution

DATA: Let volume of brass = $\boldsymbol{V_B}$*,* $density\ of\ brass(\boldsymbol{\rho_B}) = \mathbf{8.5g/cm^3}$

Mass of brass$(\boldsymbol{m_B}) = \boldsymbol{V_B} \times \mathbf{8.5\ g/cm^3}$*, mass of wood=* $\mathbf{100g}$*,*

$density\ of\ wood\ (\boldsymbol{\rho_w}) = \mathbf{0.2g/cm^3}$*, density of water=* $\mathbf{1g/cm^3}$

Volume of wood = ${}^{100}/_{0.2} = 500cm^3$

For the two to submerge beneath the water, the average density of the whole should be equal to the density of water ($1g/cm^3$)

$$\Longrightarrow \text{Total mass of the whole(wood + brass)} = \text{Total volume of the whole(wood + brass)}$$

$$8.5V_B + 100 = 500 + V_B$$

$$8.5V_B - V_B = 500 - 100$$

$$\frac{7.5V_B}{7.5} = \frac{400}{7.5}$$

$$\Longrightarrow V_B = \mathbf{53.3cm^3\ ans}$$

3. ***An object of mass 60g and density 10g/***$\boldsymbol{cm^3}$ ***is suspended by a thread and lowered until half of its volume is immersed in a liquid of density 0.80g/***$\boldsymbol{cm^3}$***. What is the tension in the thread?***

Solution

DATA: mass of object $(\mathrm{m_o}) = 60$g, density of object$(\rho_o) = 10.0g/cm^3$

Tension in thread = weight of object – upthrust in liquid

weight of object = 0.06kg = 0.6N

Volume of object$(\boldsymbol{v_o})$ = $\frac{m_o}{\rho_o} = \frac{60}{10} = 6.0cm^3$,

half of this volume was displced $(\mathrm{v_1}) = 3.0\mathrm{cm}^3$

upthrust in liquid = weight of liquid displace

$$= \rho_l \times v_1 = 0.8 \times 3 = 2.4g$$

$$= 0.0024kg$$

$$= 0.024\text{N}$$

∴Tension in thread = 0.6 – 0.024 = **0.58N**

4. ***A piece of sealing-wax weighs 0.27 N in air and 0.12 N when immersed in water. Calculate (a) its relative density;***
   (b) its apparent weight in a liquid of density 800 kgm^{-3}***.***

Solution

(a) Weight in air (W_A) = 0.27 N, weight in water (W_W) = 0.12 N

density of liquid (P_L) = 800 kgm^{-3}

Relative density (R.D) $= \frac{upthrust\ in\ air}{upthrust\ in\ water} = \frac{W_A}{W_A - W_L} = \frac{0.27}{0.27-0.12} = \mathbf{1.8}$

Hence the R.D of the sealing-wax is **1.8**.

(b) Apparent weight in liquid = upthrust in liquid

$$= \boldsymbol{V_L \times \rho_L \times \mathrm{g}}$$

Where $\boldsymbol{V_L}$ = **volume of liquid displaced,** $\boldsymbol{\rho_L}$ = **density of liquid** and $\mathbf{g}$ = **acc^n due gravity**

Volume of liquid displaced $V_L = \frac{mass\ of\ liquid\ displaced}{density\ of\ liquid\ displaced} = \frac{0.027}{1440}$

Where: $density\ of\ liquid\ displaced = \rho_L \times R.D = 800 \times 1.8 = 1440$ kgm^{-3}

$mass\ of\ liquid\ displaced = 0.027\ kg$

Thus apparent weight in liquid $= \frac{0.027}{1440} \times 800 \times 10 = \mathbf{0.15\ N}$

5. ***A metal cube of side 2 cm weighs 0.56 N in air.***
Calculate (a) Its apparent weight when immersed in white spirit of density 0.85 gcm^{-3}.
(b) the density of the metal of which it is made.

Solution

Data: volume of cube = $L^3 = 2^3 = 8\ cm^3 = 0.008\ m^3$, weight of cube = 0.56 N

(a) Apparent weight in liquid = weight in air − upthrust in liquid

$= 0.56 - V_L \times \rho_L \times g$

$= 0.56 - 0.008 \times 0.85 \times 10$

$= 0.56 - 0.068$

$= \mathbf{0.492\ N}$

(b) Density of metal $= \frac{m}{v} = \frac{0.056}{0.008} = \mathbf{7\ kgm^{-3}}$

PROBLEM SET OR EXERCISE 1.0

1. ***A piece of anthracite has a volume of 15 cm^3 and a mass of 27 g. what is its density: (a) in g/cm^3 (ans 1.8 g/cm^{-3})***
(b) in kg/m^3? (ans 1800 kg/m^{-3})
2. ***An empty 60 litre petrol tank has a mass of 10 kg. What will be its mass when full of fuel of relative density 0.72? (ans 53.2 kg)***
3. ***The mass of a density bottle is 18.00 g when empty, 44.00 g when full of water and 39.84 g when full of a second liquid. Calculate the density of the liquid. (ans 0.84 g/cm^{-3})***
4. ***A spring balance has a maximum reading of 10 N and the length of the calibrated scale is 20 cm. A rectangular metal block measuring 10 cm by 3 cm by 2cm is hung on the balance and stretches the spring by 15 cm. calculate (i) the weight of the block; (ans 7.5 N)***
(ii) the mass of the block; (ans 0.75 kg)
(iii) the density of the metal from which the block is made. [g = 10ms^{-2}]
(ans 12500 kg/m^{-3})

Floatation

- ***The ability of an object to float in a medium or fluid (i.e. air or water)***
  - An object floats in a fluid when the upthrust acting on it from the fluid equals its own weight.
- Thus, floatation depends on three factors.
  1. **The weight/mass of the object**
  2. **Upthrust**
  3. **The structure of the object (i.e. materials, shape etc)**

The Law of Floatation

This states that, a floating body displaces its own weight in the fluid in which it floats.

Applications of floatation

1. **Hydrometer**: An instrument used to directly determine the density of liquids. Consisting of a narrow glass tube or stem (calibrated in densities) and a bulb or float chamber (loaded with lead), it does it work by **sinking low** in liquids with **less densities** and **rising high** in liquids with **much densities**.
   Uses: 1. To test the concentration of acid in batteries
   2. To test the purity of some liquids with known densities.
2. **Ships:** Made of steel (though of much density than water), they are able to carry heavy loads without sinking because of their structural design- a hollow cavity beneath them containing much volume of air. This creates an upthrust large enough to support the weight of the ship and its loads. This same principle applies to boats and submarines.

ASSIGNMENT: Briefly explain application of floatation in: boats, submarines, balloons etc.

PROBLEM SET OR EXERCISE 2.0

1. ***A river car-ferry boat has a uniform cross-sectional area in the region of its water-line 720 m^2. If sixteen cars of average mass 1100 kg are driven on board, find the extra depth to which the boat will sink in the water. (ans 0.024 m)***

2. ***A cube of wood of volume 0.2 m^{-3} and density 600 kgm^{-3} is placed in a liquid of density 800 kgm^{-3}.*** ***(ans i. 0.75 ii. 400 N)***
   (i) what fraction of the wood would be immersed in the liquid?
   (ii) what force must be applied to cube so that the top face of the cube is on the same level as the liquid surface?
3. ***A solid has volume of 30 cm^{-3} and density 2.4 gcm^{-3}. If this solid is suspended on a spring balance with half its volume immersed in water. What would be the reading on the spring balance?***
   [WASSCE NOV/DEC 2016] (ans 0.57 N)
4. ***A light spiral spring which obeys Hooke's law has an unstretched length of 220 mm. It is attached at its upper end to a fixed support and, when a piece of metal of mass 2 kg is hung from the lower end, the spring extends to a length of 274 mm.***
   (a) Find the force in newtons needed to produce an extension of 10 mm. (ans 3.7 N)
   (b) When the metal is totally immersed in water, the length of the spring becomes 274 mm. What is the upthrust on the metal? (ans 10 N)
   (c) Find the mass of water displaced by the metal. (ans 1 kg)
   (d) Calculate the volume of the piece of metal. (ans 0.001 m^{-3})
   [g = 10 ms^{-2}, density of water = 1000 kgm^{-3}]

Pressure in fluids

- ***Pressure*** *is the perpendicular force applied to the surface of an object or substance per unit area over which that force is distributed.* ***OR***
- ***Pressure*** *is also the thrust (force) acting per unit area. It is measured in Pascal (Pa) or newton per metre squared (N/m^2 or $kg{\cdot}m^{-1}{\cdot}s^{-2}$). Another SI unit is Bar.* **1Pa =1Nm^{-2}, 1bar= 10^5Nm^{-2} = 10^5Pa**
- ***Mathematically:*** $p = \frac{F}{A}$ *where: p= pressure, F= force, A= area*

Pressure in Liquids

The pressure at any point in a liquid depends on the following:

- $\boldsymbol{\rho} = density\ of\ the\ liquid, \boldsymbol{h} = depth\ of\ the\ point\ and\ \boldsymbol{g} = accleration\ due\ to\ gravity\ at\ the\ point.$

- ***Mathematically*** $p = h\rho g$

Characteristics of liquid pressure

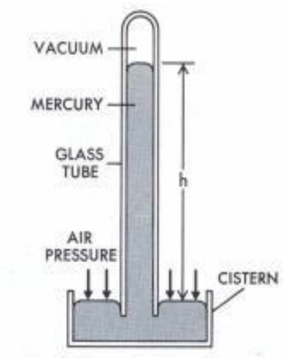
1. The pressure at all points at the same level within a liquid is the same.
2. The pressure in a liquid increases in direct proportion to the depth of the liquid.
3. The pressure in different liquids at the same depth varies directly with density.

Atmospheric pressure

- **Atmospheric Pressure** is the pressure exerted by the weight of air in the atmosphere of Earth (or that of another planet).
- We don't feel this pressure because every cell of our body maintains an internal pressure that just balances the external pressure.
- The same applies to balloons with the exception of tyres- they maintain greater internal than external pressure due to their rigidity.
- Atmospheric pressure is 10Pa at sea level.

Applications of Atmospheric Pressure

1. **Simple Barometer**: is a scientific instrument used in meteorology to measure atmospheric pressure.

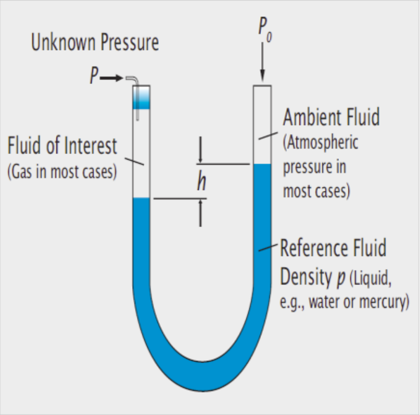


The weight of the mercury creates a vacuum at the top of the tube known as Torricellian vacuum. Mercury in the tube adjusts until the weight of the mercury column balances the atmospheric force exerted on the reservoir.

- Pressure is calculated by using the equation: $p_{atm} = \rho hg$
- Other examples include Fortin, multiple folded, stereometric, and balance barometers

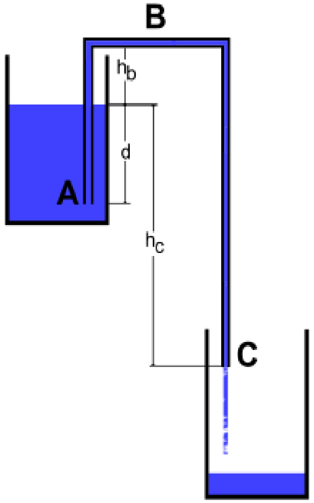
2. **Manometer:** It uses columns of liquid (e.g mercury) to both measure and indicates pressure of other systems (mostly gaseous).

- One side of the tube is connected to the region of interest while the reference pressure (which might be the atmospheric pressure or a vacuum) is applied to the other.
- The difference in liquid level represents the applied pressure.
- The pressure exerted by a column of fluid of height ***h*** and density ***ρ*** is given by the hydrostatic Pressure equation, ***P = hgρ***.



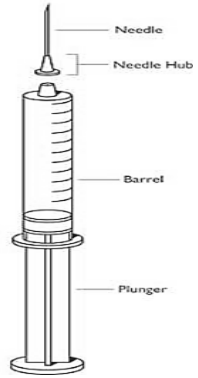
- Therefore, the pressure difference between the applied pressure P_a and the reference pressure P_0 in a U-tube manometer can be found by solving ***$P_a = P_0 + hg\rho$***.

3. Siphon: A Greek word meaning ''Pipe or tube'' is a simple device used to remove liquids from reservoirs which cannot be easily reached (e.g petrol tank).



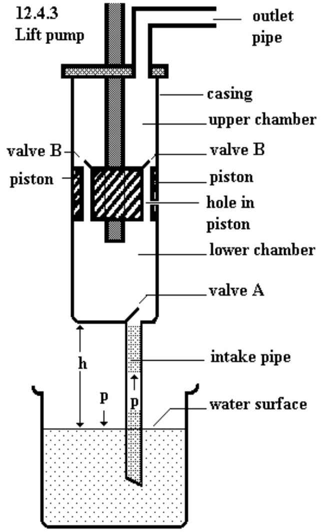
- Gravity pulling the liquid down on the exit side of the siphon, resulted in reduced pressure at the top of the siphon.
- Then atmospheric pressure was able to push the liquid from the upper reservoir, up into the reduced pressure at the top of the siphon into the lower reservoir.
- Today, there is now strong evidence to support the view that **cohesion** between the **liquid molecules** plays an important role in the siphon operation.
- Pressure at ***A*** $= \ p_{atm} - h_b\rho g$
- Pressure at ***B*** $= \ \rho hg$
- And at ***C*** $= \ p_{atm} + h_C\rho g$

6. Syringe: A **syringe** is a simple reciprocating pump consisting of a plunger (actually a piston) that fits tightly fit within a cylindrical tube (called a barrel)



- With the nozzle or nidddle deped into a liquid and the piston pulled up,atmospheric pressure acting on the reservoir forces the liquid up the cylinder and vice versa.
- It is commonly used for injecting.

5. Pump: A **pump** is a device that moves fluids (liquids or gases), by mechanical action.



- Pumps can be classified into three major groups according to the method they use to move the fluid: *direct lift*, *displacement*, and *gravity* pumps.(E.g. lift pump and force pump are examples of displacement pumps)
- **Lift Pump:** In a lift pump, the upstroke of the piston draws water, through a valve, into the lower part of the cylinder.
- On the downstroke, water passes through valves set in the piston into the upper part of the cylinder.
- On the next upstroke, water is discharged from the upper part of the cylinder via a spout or outlet pipe.

- It lifts water no greater than 10 m below ground level because atmospheric pressure cannot support a column of water above 10m high.
- **Force Pump:** In a force pump, the upstroke of the piston draws water due to atmospheric pressure, through an inlet valve, into the cylinder.
    - Unlike lift pump, force pump can lift water to a height above 10m.

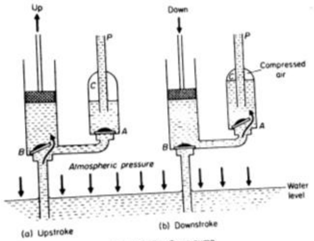


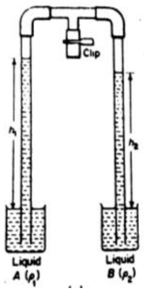
Fig. 11.3 The Force pump

- On the upstroke, atmospheric pressure pushes water through the valve P into the pump. Valve Q is closed.
- On the downstroke, the water is discharged, through an outlet valve, into the outlet pipe.

<u>**NOTE:**</u> Theoretically, there is no limit as to how high the water can be forced. But in practice of course, this will depend on the **force used** and the **quality of the pump**.

Determination of Relative Density using Hare's Apparatus

- The apparatus consists of two vertical wide-bore glass tubes connected at the top by a glass T-piece. These tubes dip into beakers containing two liquids of different densities as shown below.
- Air is sucked out of the tubes through the T-piece and closed.
- This removal of air causes reduction of pressure inside the tube, thereby allowing atmospheric pressure acting on the surface of the liquid outside the tube, to push the liquid inside, up the tubes.
- The liquids then rise until the pressures at the base of each column equals the atmospheric pressure.
- **Calculation**



Pressure at the base of each column consists of:

(a) Pressure, ρ, of air in the tube above the liquid.

(b) The, $h\rho g$, of the liquid column itself.

Hence: $\rho + h_1\rho_1 g = \rho + h_2\rho_2 g$

$h_1\rho_1 g = h_2\rho_2 g$

Or $$\frac{\rho_1}{\rho_2} = \frac{h_2}{h_1}$$

If liquid B is water, then $\frac{\rho_1}{\rho_2}\left(or\ \frac{h_2}{h_1}\right)$ will be the relative density of liquid A.

Worked Examples 1.1

1. ***Two vertical capillary tubes of the same diameter are lowered into beakers situated at the same level containing liquids A and B of densities 9.2 x 10² kgm⁻³ and 1.30 x 10³ kgm⁻³ respectively. A suction pump is used to withdraw air from the top of the liquid columns in the tubes by means of a T-piece arrangement until the liquid in A rises to a height of 26.0 cm. Calculate the height of the liquid in tube B. [WASSCE MAY/JUNE 2010]***

<u>**Solution**</u>

Data: ρ_A = 9.2 x 10^2 or 0.92 x 10^3 kgm^{-3}, ρ_B = 1.30 x 10^3 kgm^{-3}, h_A = 26.0 cm

The question above clearly describes Hare's apparatus, thus applying the required equations:

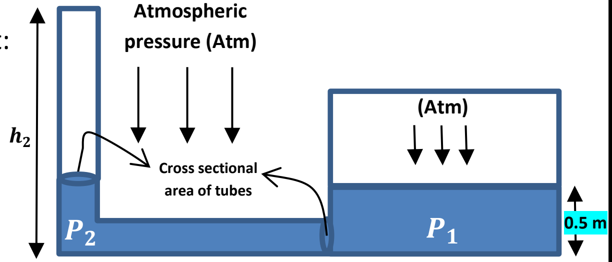
i.e. $$\frac{\rho_A}{\rho_B} = \frac{h_B}{h_A} \Longrightarrow h_B = \frac{26\times 0.92\times 10^3}{1.30\times 10^3} = 18.4\ \text{cm}$$

hence height of liquid B is 18.4 cm.

2. ***A tank with base area of 4 m² is connected at the bottom of a vertical tube of cross sectional area 0.01 m² by a horizontal tube. A liquid of density 1000 kg/m³ is poured into the tank until the depth of the liquid in the tank is 0.5 m.***
    (a) Sketch the arrangement of the tank and the tubes showing clearly the depth of the liquid in the tank and in the vertical tube.
    (b) Calculate
    (i) the pressure due to the liquid on the base of the tank;
    (ii) the pressure due to the liquid at the base of the vertical tube.
    (c) if the atmospheric pressure at the time were 120000 Pa (N/m2), what would be the total pressure on the base of the tank.

<u>**Solution**</u>

(a) Sketch of arrangement:



(b) (i) the pressure due to liquid at the base of tank $P_1 = h_1 \rho_L g$

$$= 0.5 \times 1000 \times 10$$
$$= 5000\ Nm^{-2} \text{ or Pa}$$

Hence pressure due to liquid at the base of the tank is 5000 Pa

(ii) Pressure due to liquid at base of the vertical tube:

$$P_2 = \frac{F}{A} = \frac{mass\ of\ liquid\ displaced \times g}{cross\ sectional\ area\ of\ tube}$$

$$mass\ of\ liquid\ displaced = \rho_L \times V_L$$
$$= 1000 \times 0.005$$
$$= 5\ kg$$

Where
$V_L = cross\ sectional\ area \times height$
$= 0.01 \times 0.5$
$= 0.005\ m^3$

Hence $P_2 = \frac{5 \times 10}{0.01} = 5000\ Nm^{-2}$ or Pa

(c) Total pressure at base of tank $= P_1\ or\ P_2 + atmospheric\ pressure$

$$= 5000 + 120000 = 125000\ Nm^{-2} \text{ or Pa}$$

PROBLEM SET OR EXERCISE 1.0

1. ***The air pressure at the base of a mountain is 75.0 cm of mercury and at the top is 60.0 cm of mercury. Given that the average density of air is 1.25 kg/m^{-3} and the density of mercury is 13600 kg/m^{-3}. Calculate height of the mountain. (ans 1632 m)***
2. ***A solid right circular cylinder of length 10 cm and radius 2 cm has its axis vertical and its top end is 15 cm below the surface of a fluid of relative density 1.3. Calculate the thrust on:***
   - ***(a) the upper; and (ans 2.45 N)***
   - ***(b) lower end of the cylinder due to the fluid. (ans 4.08 N)***
   - ***(c) From results obtained in (a) and (b), deduce the loss of weight of the cylinder on immersion in fluid. (ans 1.63 N)***
3. ***A television tube has a flat rectangular end of size 0.40 m by 0.30 m. calculate the thrust exerted on this end by the atmosphere, if the atmospheric pressure is 1.01 x 10^5 N/m^2. (ans 12120 N)***
4. ***The base of a rectangular vessel measures 10 cm x 18 cm. Water is poured in to a depth of 4 cm.***
   - ***(i) What is the pressure on the base? (ans 400 Nm^{-2})***
   - ***(ii) What is the thrust on the base? (ans 7.2 N)***
5. ***Calculate the mass of air in a room 5 m x 4 m x 2 m given that the density of air is 1.3 kgm^{-3}. (ans 52 kg)***

MOTION

Motion is the change in position or direction of a body with respect to time.

- **KINEMATICS:** Describes the geometry of motion of a particle without considering the cause (or force) of the motion.
- It uses mathematics to describe motion in terms of **position**, **velocity** and **acceleration.**

Types of Motion

There are basically four major types of motion;

1. **LINEAR OR RECTILINEAR MOTION:** This describes the motion of a body in a straight line at constant acceleration.
2. **RANDOM MOTION:** This is the motion of a body with no specific direction of movement. E.g. motion of gas molecules
3. **CIRCULAR MOTION:** Describes the motion of a body in a circle relative to a centre or axis. E.g. a rotating fan
4. **OSCILLATORY MOTION:** This is the back-and-forth motion of a body with reference to a fixed point. E.g. a swinging pendulum.
   Other types of motion include; translational, rotational and spin.

- **ASSIGNMENT:** Write briefly on the other types of motion; translational, rotational and spin motion.

1. Linear or Rectilinear motion *(motion in a straight line)*

In kinematics, four main quantities are considered for a body moving in a straight line or moving with constant acceleration (uniform acceleration). These include:

(a) Displacement (s): The distance travelled in a specified direction. SI unit is metre (m).

(b) Velocity (v): The rate of change of displacement. SI unit is **m/s.**

(c) Acceleration (a): The rate of change of velocity. SI unit is **m/s^2**

(d) **Time (s):** is a measure of the succession of events. SI unit is **seconds (s).**

Other related definitions

1. ***Uniform acceleration:*** implies the constant change of velocity with time (or constant acceleration).
2. ***Retardation or deceleration:*** the decrease in velocity with time (the reverse of acceleration).
3. ***Uniform velocity***: the constant change of displacement with time (or constant velocity).
4. ***Average velocity***: the mean of both initial and final velocities with time.

The Equations of Linear Motion/Algebraic approach

- Suppose a body moves with an initial velocity ***u***, accelerates uniformly in a straight line with acceleration ***a***, covering a distance ***s*** in time ***t*** and then reaches a final velocity ***v***. The distance ***S*** covered is given by

  S = average velocity x time

$$s = \left(\frac{u+v}{2}\right) \times t \qquad \ldots\ldots 1$$

The defining equation for the acceleration is

$$a = \frac{v-u}{t}$$

$$\gg v = u + at \qquad \ldots\ldots 2$$

From **2**, $t = \frac{v-u}{a}$, we can substitute into **1** to eliminate ***t***

$$s = \left(\frac{u+v}{2}\right) \times t$$

$$s = \left(\frac{u+v}{2}\right)\left(\frac{v-u}{a}\right)$$

$$s = \frac{(v+u)(v-u)}{2a}$$

Factors of the difference of squares

$$s = \frac{v^2 - u^2}{2a}$$

$$\gg v^2 = u^2 + 2as \qquad \ldots\ldots 3$$

From 3, $v = u + at$. Substitute this into ***1*** to eliminate ***v***. Thus,

$$s = \left(\frac{u+v}{2}\right) \times t$$

$$s = \left(\frac{u+u+at}{2}\right) \times t$$

$$s = \left(\frac{2u+at}{2}\right) \times t$$

$$\gg s = ut + \frac{1}{2}at^2 \qquad \ldots\ldots\ldots\ldots 4$$

For convenience, the equations for uniform acceleration or straight line motion are summarised in the table below.

Variables involved	General equation
s, v, u, a	$a = \frac{v-u}{t}$
s, u, a, t	$s = ut + \frac{1}{2}at^2$
s, u, v, t	$s = v_{av}t$
s, v, u, a	$v^2 = u^2 + 2as$
s, v, t, u	$s = \left(\frac{u+v}{2}\right) \times t$

Worked Examples 1.10

1. ***A car accelerates uniformly from rest at t = 0.0s, at 4.0m/s². How far does the car travel between 5.0s and 8.0s?***

Solution

Data: **a** = 4.0m/s³, t_1 = 5.0s, t_2 = 8.0s, u = 0 m/s, $S_{2\text{-}1}$ = distance travelled?

Variables: u, a, t, s thus we use $s = ut + \frac{1}{2}at^2$

at time t_1 = 5.0s, $s_1 = (0)t + \frac{1}{2}(4)(5.0)^2$

$s_1 = 50m$

at time t_2 = 8.0s, $s_2 = (0)t + \frac{1}{2}(4)(8.0)^2$

$s_2 = 128m$

∴ **The car travels $S_{2\text{-}1}$ = 128 – 50 =78m between 5.0s and 8.0s**

2. ***A car moves from rest with an acceleration of 0.2m/s². Find its velocity when it has moved a distance of 50m.***

Solution

Data: u = 0 m/s, **a** = 0.2 m/s², **S** = 50m **variables: s, u, v, a**

thus we use $v^2 = u^2 - 2as \gg V = \sqrt{u^2 + 2as}$

$V = \sqrt{0^2 + 2(0.2)(50)}$

$V =$ **4.47m/s**

3. ***A train slows from 108km/h with uniform retardation of 5m/s². How long will it take to reach 18km/h, and what is the distance covered?***

Solution

Data: u = 108km/h = 30m/s, **v** = 18km/h = 5m/s ***(conversion is done using the 'chain-hink' conversion method)*** **i.e.** 108km/h = $\frac{108\times1000m}{60\times60s} = 30m/s$

a = 5m/s 18km/h = $\frac{18\times1000m}{60\times60s} = 5m/s$

Q1: *how long? Time t (unknown)*, **using** $v = u + at$ *(with v, u and a known)*

$\gg 5 = 30 - 5t,$

$\therefore t = 5s$

Q2: distance covered s? *(With all variables/parameters known, we can use any of the distance-equation).* Using $v^2 = u^2 + 2as$

$5^2 = 30^2 + 2 \times (-5)s$

$s = \frac{875}{10} = 87.5m$ *(ans)*

4. ***The driver of a car travelling at 72km/h observes the light 300m ahead of him turning red. The traffic light is timed to remain red for 20s before it turns green. If the motorist wishes to pass the light without stopping to wait for it to turn green.***
Determine (i) the required uniform acceleration of the car.
(ii) the speed with which the motorist crosses the traffic light.

Solution

Data: u = 72km/h = $\frac{72\times1000m}{60\times60s} = 20m/s$, **S** = 300m, t = 20s, **a** = ? **v** = ?

(d) Using $s = ut + \frac{1}{2}at^2$

$300 = (20 \times 20) + \frac{1}{2}a(20)^2$

$a = -0.5m/s^2$ ***(-ve sign indicates retardation of the car)***

(ii) Using $v = u + at$

$v = 20 - 0.5 \times 20$

$v = 10m/s^2$

5. ***A particle travelling in a straight line with constant acceleration 4ms⁻² passes a point O when its velocity is 12ms-1. It passes another point P after a further 3s. Find the velocity of the particle at P and the distance OP.***

Solution

Data: u = 12ms⁻¹, **a** = 4ms⁻², **t** = 3s, distance **OP** = s m, let velocity of particle at P = v ms⁻¹ using $v = u + at$

$v = 12 + 4(3)$

$v = 24ms^{-1}$

Hence the velocity of the particle at P is $24ms^{-1}$

For the distance OP, we use $s = \left(\frac{u+v}{2}\right) \times t$

$$s = \left(\frac{12+24}{2}\right) \times 3$$

$$s = 54m$$

Hence the distance OP is $54m$

PROBLEM SET OR EXERCISE 3.0

1. ***A particle starts with velocity 3ms^{-1} and accelerates at 0.5ms^{-2}. What is the velocity after (a) 3s (b) 10s (c) t s? how far has it travelled in these times?*** *{ans (a) $V_1 = 4.5ms^{-1}, S_1 = 11.25m$ (b) $V_2 = 8ms^{-1}, S_2 = 55m$}*
2. ***If a particle passes a certain point with speed 5ms-1 and is accelerating at 3ms^{-2}, how far will it travel in the next 2s?*** $(ans\ 16m)$ ***how long will it take (from the start) to travel 44m?*** $(ans\ 4s)$
3. ***A car, retarding uniformly, passes over three cables, P, Q, and R set at right angles to the path of the car and 11m apart. It takes 1s between Pand Q and 1.2s between Q and R. Find***
   (a) ***Its retardation,*** $(ans\ 1.67ms^{-2})$
   (b) ***its velocity when it crosses P,*** $(ans\ 11.83ms^{-1})$
   (c) ***its distance beyond R when it comes to rest.*** $(ans\ 20m)$
4. ***A particle starting from rest moves with constant acceleration x ms^{-1} for 10s; tavels with constant velocity for a further 10s, and then retards at 2x ms^{-2} to come to rest 300m from its starting point. Find the value of x.*** $(ans\ 1.71ms^{-1})$
5. ***A body, decelerating at 0.8ms^{-2}, passes a certain point with a velocity of 30ms^{-1}. Find its velocity after 10s, the distance it covered in that time and how much further the body will go until it stops?*** $(ans\ 22ms^{-1}, 260m, 302, 5m)$
6. ***A body travelling along a straight line with constant acceleration of a ms^{-2} passes a point O with velocity u ms^{-1}. Between 4s and 5s after passing O, the body travels 10m. between 6s and 7s after passing O, the body tarvels 12m. calculate the values of u and a.*** $(ans\ u = 5.5ms^{-2}, a = 1ms^{-2})$

2. UNIFORM (CONSTANT) ACCLERATION MOTION *(Graphical approach)*

Now, let us glance at the graphical approach in solving uniform acceleration motion problems. This can be done as mentioned in our syllabus, by use of a **velocity-time graph and displacement-time graph.** They are both straight-line graphs.

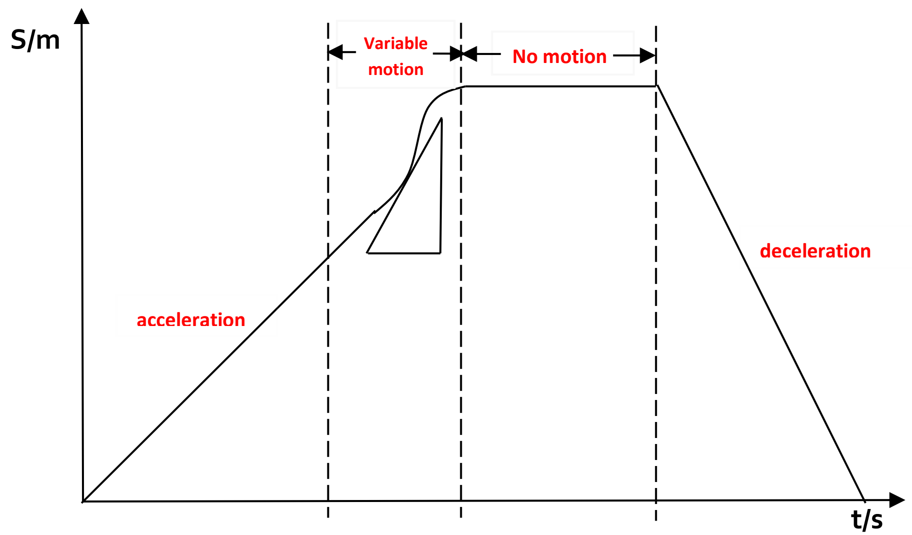
Facts to note

Displacement-time graph

- The **slope/gradient** of the graph at any instant gives the **velocity** of the particle at that instant.

$$slope = \frac{\Delta s}{\Delta t} = velocity$$

- Variable to determine is velocity (final and initial).
- **Flat sections (-------)** imply no motion or the body stopped moving.
- **Downhill sections (\\)** mean body is returning back to its **starting point.**
- **Curves** represent **acceleration** or **deceleration.**
- **Steepening curve ( )** means the body is **speeding up/increasing gradient.**
- **Levelling off curve ( ) means** the body is **slowing down.**
- The **steeper** the graph the **faster** the body is moving.

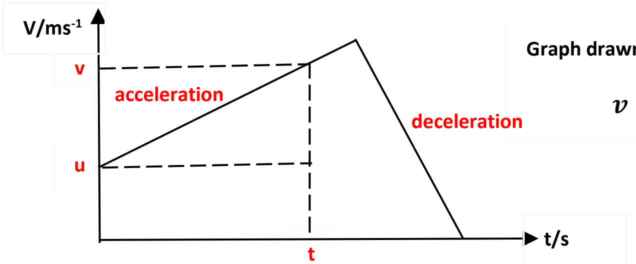


For curves, the speed at any time t is given by the tangent to the curve at that time.

Velocity/Speed-time graph

- The **area** under the graph between the given intervals of time gives the displacement **(S)** or distance travelled by the particle during the same interval.
- The **slope** at any instant gives the **acceleration (a)** of the particle.
- **Flat sections** represent **steady/constant** speed/velocity.
- **Uphill sections (/)** represent **acceleration.**
- **Downhill sections (\)** represent **deceleration.**
- **Curve** means **changing acceleration/non-uniform acceleration.**
- The **steeper** the graph, the **greater** the **acceleration or deceleration**.

$$slope = \frac{\Delta v}{\Delta t} = acceleration$$



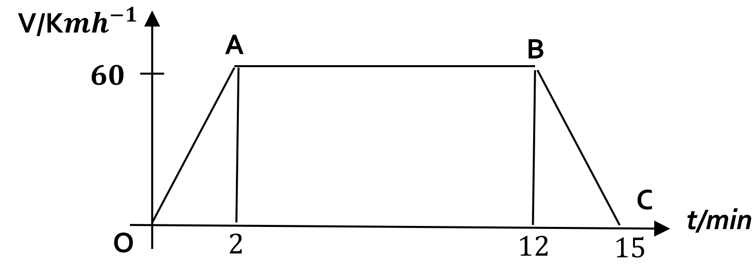
Graph drawn using the equation:

$$v = u + at$$

Worked Examples 1.20

1. ***A train starts from rest from station P and accelerates for 2min reaching a speed reaching a speed of 60kmh^{-1}. It maintains this for 10min and then retards uniformly for 3min to come to rest at station Q. Sketch a velocity-time graph for the motion. Using the graph Find***
   (a) The distance PQ in in Km,
   (b) The average speed of the train in ms-1,
   (c) The acceleration in ms^{-2}.

Solution



(a) The area under the graph represents the distance travelled from P to Q.

Area under graph = Area of the trapezium OABC

$= \frac{1}{2}(AB + OC) \times 60$

$= \frac{1}{2}(\frac{10}{60} + \frac{15}{60}) \times 60$ = **12.5Km**

Hence the distance PQ is **12.5Km.**

(b) The average speed = $\frac{total\ distance\ travelled}{time\ taken}$

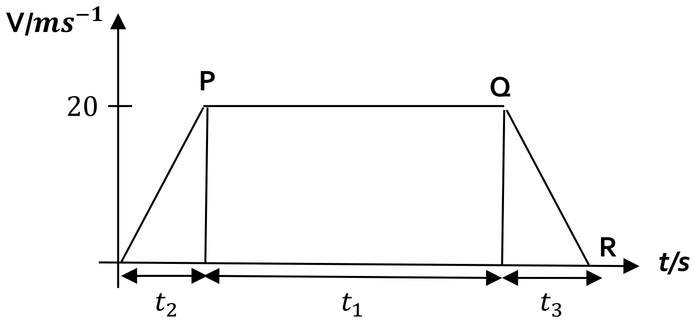
$= \frac{12.5 \times 1000m}{15 \times 60s}$ = **13.89ms^{-1}**

(c) The acceleration $= \frac{speed/velocity}{time}$ speed = $\frac{60 \times 1000m}{60 \times 60s}$ = 16.67ms-1

$= \frac{16.67\ m}{120s}$ = **0.14ms^{-2}**

2. ***An MRT train starts from station X and accelerates uniformly to a speed of 20ms^{-1}. It maintains this speed then retards uniformly until it comes to rest at station Y. The distance between the stations is 2km. The total time taken is 2min. if the retardation is twice the acceleration in magnitude. Find using a sketched velocity time graph,***
   (a) The time for which the train is travelling at constant speed,
   (b) The acceleration of the train.

Solution



Let t1, t2 and t3 represent the time for the 3 stages of the motion as indicated in the graph. Data: total time taken $(t_1 + t_2 + t_3)$ = 120s, distance travelled between stations = 2000m

(a) The train travels at constant speed for t_2 seconds

The area under the graph = distance travelled between the stations (area of a trapezium)

Hence $\frac{1}{2}[t_2 + (t_1 + t_2 + t_3)] \times 20 = 2000$

$$\frac{1}{2}[t_2 + (120)] \times 20 = 2000$$

$$t_2 + 120 = 200$$

$$t_2 = 80s$$

Therefore the train travels at constant speed for **80s.**

(b) With acceleration as $a\ ms^{-2}$

Then retardation is $2a\ ms^{-2}$

The gradient of OA gives the acceleration

$$\Rightarrow \textit{acceleration is given by } \frac{20}{t_1} = a \quad or \quad t_1 = \frac{20}{a}$$

$$\textit{similarly retardation is given by } \frac{20}{t_3} = 2a \text{ or } t_3 = \frac{10}{a}$$

Using $(t_1 + t_2 + t_3) = 120$ (total time taken)

$\frac{20}{a} + 80 + \frac{10}{a} = 120$ (substituting for t1, t2 and t3)

$\frac{2}{a} + \frac{1}{a} = 4$ (simplifying)

$$\frac{3}{a} = 4$$

Hence the acceleration of the train $a = \frac{3}{4}ms^{-2}$

Q 3. A bus leaves a bus stop and accelerates uniformly for 10s over a distance of 100m. It then moves uniformly with the speed it has attained for 30s and finally retards uniformly to rest at the next stop. If the two bus stops are 1km apart, find using a sketched v-t graph

(a) the maximum velocity,

(b) the acceleration,

(c) the total time taken between the two stops.

Solution

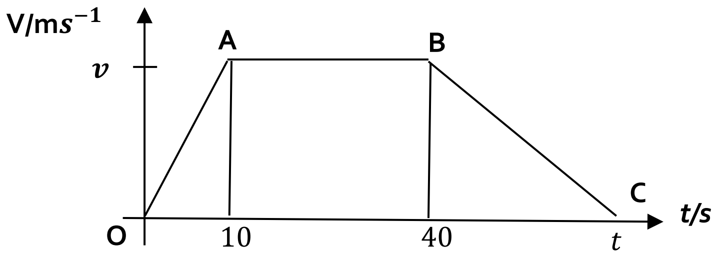
(a) **Using** $s = \left(\frac{u+v}{2}\right) \times t$ where v ms^{-1} is the maximum velocity, u = 0, s = 100m. $100 = \left(\frac{0+v}{2}\right) \times 10$ giving v = **$20ms^{-1}$**

(b) using $v = u + at$ where a ms^{-2} is the acceleration, v = $20ms^{-1}$, u = 0, t =10s $20 = 0 + a \times 10$ giving **a = $2ms^{-2}$**

(c) With total time taken as t s.

Using Area of trapezium OABC = distance travelled

$\frac{1}{2}(30 + t) \times 20 = 1000$ **giving t = 70s**



PROBLEM SET OR EXERCISE 4.0

Q1. A body starts from rest and travels distances of 120m, 300m and 180 m in successive equal time intervals of 12 s. During each interval the body is uniformly accelerated.

(i) Calculate the velocity of the body at the end of each successive time interval ***(ans $V_1 = 20ms^{-1}, V_2 = 30ms^{-1}, V_3 = 0ms^{-1}$)***

(ii) Sketch a velocity-time graph for the motion. ***(WASSCE MAY/JUNE 2010)***

Q2. A car of mass M starts from rest and moves with a constant acceleration, a. it attains a velocity, v after time t, and maintains this velocity until time t2. It is then brought to rest with uniform retardation, d at time t3.

(i) Draw and label a velocity-time graph for the motion.

(ii) Deduce an expression for the force on the car during the period of retardation. ***(WASSCE MAY/JUNE 2010)***

Q3. A car accelerate uniformly from rest to reach a speed of V ms^{-1} in 5 seconds. It travels at this speed for 20 seconds when it decelerates uniformly to come to rest in a further t seconds.
(a) Sketch the v-t graph for the motion.
(b) If the distance travelled while decelerating is 4/5 of the distance travelled while accelerating, find the value of t. ***(ans $t = 4s$)***
(c) Given that the total distance travelled was 637m, find the value of V.
(ans $V = 26ms^{-1}$)

Vertical Motion Under Gravity/Free Fall

This special case of motion describes a body moving freely in space under the influence of a gravitational force. This force produces an acceleration which act towards the centre of the earth. The following facts so far have been stated describing this case of motion;

1. ***The earth (or any other planet) attracts bodies near it with a force. This force of attraction is called the gravitational force. It produces an acceleration towards the centre of the earth on any body free to move.***
2. ***The magnitude of the acceleration is about $9.8ms^{-2}$ but varies from place to place due to***
   (a) ***The distance/altitude of the body from the centre of the earth.***
   (b) ***The earth not being a perfect sphere. It is slightly less near the equator than near the earth poles.***
3. ***The symbol g is used to denote the acceleration due to gravity. Ignoring air resistance, a piece of paper and a stone will would each fall with the same acceleration g, and theoretically if they are released simultaneously from the same height they would hit the ground at the same time. This was experimented by Galileo (1564-1642).***

In our exercises we will ignore air resistance and approximate ***g*** as ***$10ms^{-2}$*** unless otherwise stated. The table below will help with the equations to use.

	Vertical motion under gravity	
Constant acceleration motion equations	**Upward**	**Downward**
$v = u + at$ $v^2 = u^2 + 2as$ $s = ut + \frac{1}{2}at^2$	$v = u - gt$ $V = u^2 - 2gh$ $h = ut - \frac{1}{2}gt^2$	$v = u + gt$ $V = u^2 + 2gh$ $h = ut + \frac{1}{2}gt^2$

Worked Examples 1.30

1. ***A stone is dropped from the top of a building of height 20m. Find the time it takes to reach the ground and the velocity with which it hits the ground.***

   Solution
   Data: u = 0, h = 20m, g =$10ms^{-2}$, t =?, v =?
   Time t: using $h = ut + \frac{1}{2}gt^2$ *(motion is downward in favour of gravity)*
   $20 = 0 + \frac{1}{2} \times 10 \times t^2$
   $t^2 = 4$
   $t = 2s$
   Hence the stone took **2s** to reach the ground.

   Velocity v: using $v = u + gt$
   $v = 0 + 10 \times 2$
   $v = 20ms^{-1}$
   Hence the stone hit the ground with velocity **$20ms^{-1}$.**
2. ***A particle is projected vertically upwards with a velocity of $30ms^{-1}$ from point O. Find***
   (a) the maximum height reached
   (b) time taken for it to return back to point O.
   (c) the time taken for it to be 35m below O.

Solution

(a) **Maximum height:** at maximum height V = 0, g = -10ms^{-2}, u = 30ms^{-1}, h =?

Using $v^2 = u^2 - 2gh$

$0^2 = 30^2 - 2 \times 10 \times h$

$\boldsymbol{h} = \mathbf{45m}$

Hence the maximum height reached is **45m.**

(b) **Time taken to return:** on return to point O, the displacement h = 0m, u = 30ms^{-1}, g = -10ms^{-2}, t = ?

Using $h = ut - \frac{1}{2}gt^2$

$$0 = 30t - \frac{1}{2} \times 10 \times t^2$$

$5t^2 - 30t = 0$

Giving $t = 0$ (at start) and $t = 6s$ (when particle return to O)

Hence the time taken to return to O is **6s.**

(c) Time taken to be 35m below O: 35m below O means displacement h = -35m, u = 30ms^{-1}, g = -10ms^{-2}, t =?

Using $h = ut - \frac{1}{2}gt^2$

$-35 = 30t - \frac{1}{2} \times 10 \times t^2$

$t^2 - 6t - 7 = 0$

$(t+1)(t-7) = 0$

Giving $t = -1$ (not admissible) and $t = \mathbf{7s}$

Hence the particle will be 35m below O after **7s.**

3. ***A ball is projected horizontally from a height with a velocity of 40 ms^{-1}. Calculate the drop in height after travelling a horizontal distance of 30m. [Neglect air resistance; g = 10 ms^{-1}] (WASSCE Nov/Dec 2010)***

Solution

Data: v = 40ms^{-1}, u = 0, s = 30m, g =10ms^{--2}

Using $h = ut + \frac{1}{2}gt^2$

$h = 0 + \frac{1}{2}gt^2$

but from $v = \frac{s}{t} \Rightarrow t = \frac{s}{v}$, $t = \frac{30}{40} = 0.75s$

therefore $h = 0 + \frac{1}{2} \times 10 \times 0.75^2$

$h = \mathbf{2.81m}$

4. ***A particle P is projected vertically upwards from O with velocity 40ms^{-1}. One second later, another particle Q is projected from O with the same vertical velocity. After what time and at what height will the two particles collide?***

Solution

If the time of collision be ***t s*** after the projection of P,
Then it will be ***(t -1)*** after the projection of Q, since Q was projected 1s later.

Let ***h*** be the height above O during collision.
Considering the motion of P,
At the time of collision, the displacement is ***h m*** and the time **t s**, u =40ms^{-1}, g = 10ms^{-1}

Using $h = ut - \frac{1}{2}gt^2$

$h = 40t - \frac{1}{2} \times 10 \times t^2$

$h = 40t - 5t^2$ ………… (1)

Similarly, for the motion of Q,
At the time of collision, displacement is ***h m*** with time as **(t – 1)s since it was thrown 1s later,** u =40ms^{-1} g = 10ms^{-2}

We have $h = 40(t-1) - \frac{1}{2} \times 10 \times (t-1)^2$

Giving $h = 40(t-1) - 5(t-1)^2$

$h = 50t - 5t^2 - 45$ ………… (2)

Equating (1) and (2)

$40t - 5t^2 = 50t - 5t^2 - 45$

$10t = 45$

$t = \mathbf{4.5s}$

Substituting ***t*** for ***h*** into (1) we get $h = \mathbf{78.75m}$

Hence the particles collide **4.5s** after the projection of P at a height of **78.75m**

PROBLEM SET OR EXERCISE 5.0

1. ***A ball is dropped from a height, at the same time as another ball is projected horizontally from the same height.***

   i. ***would the balls hit the ground at the same time?***
   ii. ***Explain your answer in (i). (WASSCE Nov/Dec 2010)***

2. ***From a height 120m above ground level, a stone is projected vertically upwards with u ms^{-1}. If the stone rises to a height of 5m above the point of projection, calculate***
   (a) the value of u. (ans 10ms^{-1})
   (b) the time the stone takes from projection until it reaches ground level. (ans 6s)
   (c) the speed of the stone at ground level. (ans 50ms^{-1})

3. ***A particle is projected vertically upwards from the ground with a speed of 36ms^{-1}. Calculate***
   (a) the time for which it is above a height of 63m, (ans 1.2s)
   (b) the speed which it has at this height on its way down, (ans 6ms^{-1})
   (c) the total time of flight. (ans 7.2s)

4. ***A ball is projected vertically upwards with velocity u ms^{-1} from ground level. Between 3 and 4 seconds after leaving the ground, it rises 10m. Calculate***
   (a) the value of u. (ans 45ms^{-1})
   (b) the maximum height reached by the ball. (ans 101.25m)

5. ***A stone is projected vertically upwards from the top of a building 80m high with a speed of 30ms^{-1}. Calculate***
   (a) the height it reaches above the top of the building, (ans 45m)
   (b) the time it takes to reach the ground. (ans 8s)
   A second stone is dropped from the top of the building t seconds after the first one was projected. Both stones reach the ground at the same time. (c) find the value of t. (ans 4s)

SCALARS AND VECTORS

Scalars are physical quantities with magnitude but no direction. E.g. distance, mass, speed and time.

Vectors are physical quantities with both magnitude and direction. E.g. Displacement, weight, velocity, acceleration etc.

Vector representation

- **Graphical Representation:** A vector can be represented graphically by a line, drawn so that
  (a) the length of the line denotes the magnitude of the quantity, according to some stated vector scale.
  (b) the direction of the line indicated by an arrow denotes the direction in which the vector quantity acts.

e.g.

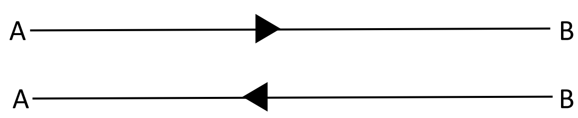


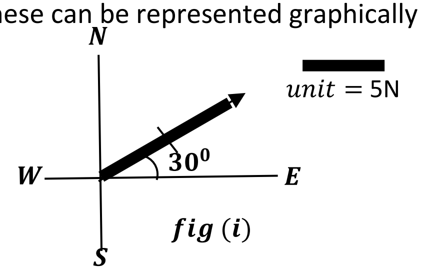
fig (a)

fig (b)

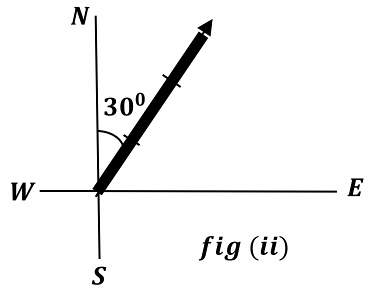
- **Algebraic Representation:** The vector quantity AB (fig. a) above, is algebraically represented as $\overrightarrow{AB}$ or $\overline{a}$ and the magnitude is represented as $|\overline{AB}|$ or $|\overline{a}|$. The same vector can take an opposite sense (direction) and be represented as $\overrightarrow{BA}$ or - $\overline{a}$ which is also known as a **negative vector** and is graphically represented in fig (b). The directions can also be written in degrees with the cardinal point or a bearing indicated.

E.g. (i) A force of 10N in a direction 30° north of east.
(ii) A force of 15N in a direction 30° east of north.

These can be represented graphically as



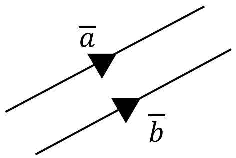
$fig\ (i)$



$fig\ (ii)$

- **Equal vectors:** Two vectors are said to be equal when they have the same magnitude and the same direction.

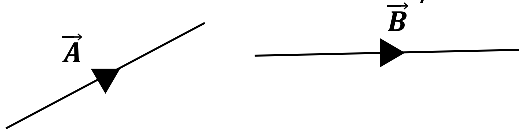
e.g.

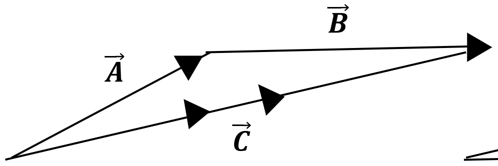


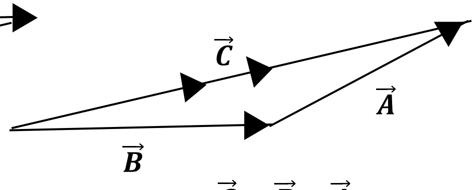
The two vectors $\overline{a}$ and $\overline{b}$ are equal because
(i) a = b (in magnitude)
(ii) the direction of a = the direction of b
(i.e. the vectors are parallel and act in the same Sense)

Vector addition

To add two vectors $\vec{A}$ and $\vec{B}$ graphically, we place them 'end-on' and drawing a third vector known as the ***resultant vector*** of $\vec{A}$ and $\vec{B}$. This forms a triangle or a parallelogram if the vectors are a placed adjacent to each other starting from a point. The order of addition does not matter just that the addition makes sense only for the same kinds of vectors,







$\vec{C} = \vec{A} + \vec{B}$ $\qquad$ $\vec{C} = \vec{B} + \vec{A}$

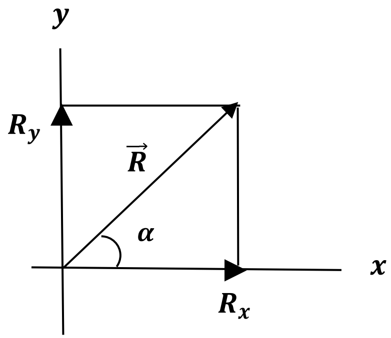
This implies addition of vectors is commutative (i.e. $\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$)

- <u>NOTE:</u> ***In order to algebraically find the sum of two vectors which are inclined to each other we use two geometric methods:***
  - ***(a) The parallelogram law of vector addition***
  - ***(b) The triangle law of vector addition***

Detail analysis on the two methods will be discussed next, under vector resolution.

Vector Resolution

Vector resolution is a process which involves the combination of two or more vector components into a single vector or it is the process of splitting a vector into its components-***magnitude*** and ***direction***. This can be done using three methods;



Resultant magnitude $\vec{R}$

$$\vec{R} = \sqrt{{R_x}^2 + {R_y}^2}$$

Direction (α)

$$\tan\alpha = \frac{R_y}{R_x} \Longrightarrow \alpha = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

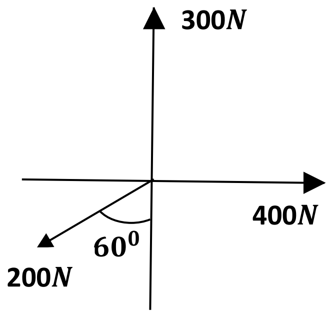
where $R_x = \vec{R}\cos\theta$

$R_y = \vec{R}\sin\theta$

Appropriate Sign convention must be adopted to avoid sign accidents. In the worked examples we shall be using the Cartesian sign convention;

$\textbf{North}: +y \quad \textbf{South}: -y \quad \textbf{East}: +x \quad \textbf{West}: -x$

<u>Example:</u> ***determine the resultant of the system of coplanar forces shown in the figure below, giving its magnitude and the angle it makes with the 400N force.***



$x-$component	$y-$component
400 $-200cos30$	300 $-200sin30$
226.8	**200**

Using $\vec{R} = \sqrt{{R_x}^2 + {R_y}^2}$

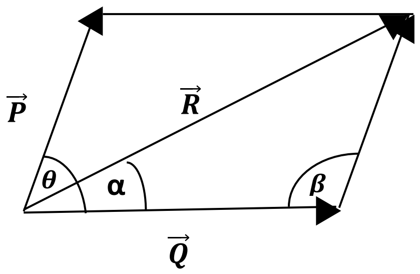
$$\vec{R} = \sqrt{226.8^2 + 200^2} = \mathbf{302.4N}$$

$$\alpha = \tan^{-1}\left(\frac{200}{226.8}\right) = \mathbf{41.4^0}$$

Hence the resultant has magnitude $\mathbf{302.4N}$ and direction of $\mathbf{41.4^0}$ north of east.

(a) Parallelogram law of vector addition- which states that *when two vectors are represented in magnitude and in direction as the adjacent sides of a parallelogram, the resultant vector represented in magnitude and direction is the diagonal of the parallelogram.*

In this law we apply the cosine rule to get the magnitude and the sine rule to determine the direction of the resultant vector.



Resultant magnitude $\vec{R}$

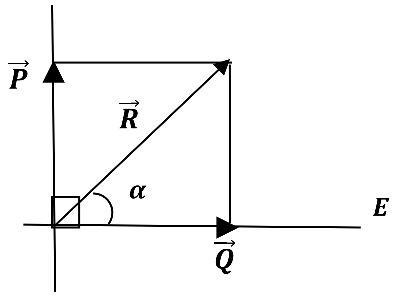
$$\vec{R} = \sqrt{\vec{P}^2 + \vec{Q}^2 - 2\vec{P}\vec{Q}\cos\theta}$$

Direction (α)

$$\frac{\sin\alpha}{\vec{P}} = \frac{\sin\beta}{\vec{R}} \Rightarrow \alpha = \sin^{-1}\left(\frac{P\sin\beta}{\vec{R}}\right)$$

$Where\ \alpha = direction, \quad \theta = angle\ between\ the\ vectors, \quad \beta = 180 - \theta$

When the two vectors act perpendicular (at 90^0) to each other, we apply the Pythagoras' theorem;



Resultant magnitude $\vec{R}$

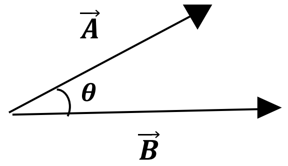
$$\vec{R} = \sqrt{\vec{P}^2 + \vec{Q}^2}$$

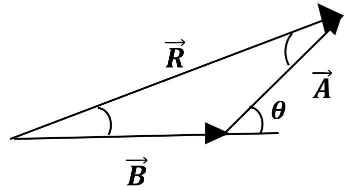
Direction (α)

$$\tan\alpha = \frac{\vec{P}}{\vec{Q}} \Rightarrow \alpha = \tan^{-1}\left(\frac{\vec{P}}{\vec{Q}}\right)$$

(b) Triangle law of vectors: *this states that if two vectors are represented in magnitude and in direction as the two adjacent sides of a triangle, taken in order, then the resultant vector is the closing side of the triangle taken in the reverse order.*

For example to find the resultant and magnitude of the vectors $\vec{A}$ and $\vec{B}$





Resultant magnitude $\vec{R}$

$$\vec{R} = \sqrt{\vec{P}^2 + \vec{Q}^2 - 2\vec{P}\vec{Q}\cos(180 - \theta)}$$

or

$$\vec{R} = \sqrt{\vec{P}^2 + \vec{Q}^2 + 2\vec{P}\vec{Q}\cos\theta}$$

Direction (α)

$$\frac{\sin\alpha}{\vec{A}} = \frac{\sin\beta}{\vec{B}} \Rightarrow \alpha = \sin^{-1}\left(\frac{\vec{A}\sin\beta}{\vec{B}}\right)$$

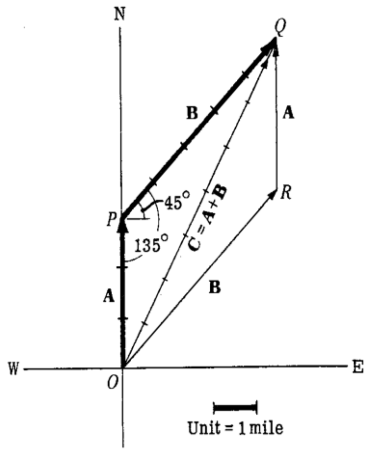
Worked Examples 1.40

1. ***An automobile travels 3 miles due north then 5 miles northeast. Represent these displacements graphically and determine the resultant displacement (a) graphically (b) algebraically or analytically***

Solution

(a) Graphical determination of resultant displacement:

The is done using a pencil, ruler and a protractor as shown in fig



$vector\ \overrightarrow{OP} =$ $displacement\ of\ 3\ miles\ due\ north$

$vector\ \overrightarrow{PQ} =$ $displacement\ of\ 3\ miles\ due\ northeast$

The resultant can be found by joining OQ then laying off 1 unit starting from O. The magnitude is approximately **7.4 miles** and the direction which is the angle OQ makes with the x-axis $(\angle EOQ)$ is approximately **61.5^0** North of East.

(b) Algebraic determination of resultant displacement:

For magnitude, Using $\vec{R} = \sqrt{A^2 + B^2 - 2AB\cos(180 - \theta)}$

$$\vec{R} = \sqrt{3^2 + 5^2 - 2(3)(5)\cos(135)}$$

$$\vec{R} = \mathbf{7.43\ miles}$$

For direction, using sine rule $\frac{\sin\angle OQP}{A} = \frac{\sin\angle OPQ}{B} \Longrightarrow \angle OQP = \sin^{-1}\left(\frac{A\sin\angle OPQ}{B}\right)$

$\angle OQP = \sin^{-1}\left(\frac{3\sin 135}{5}\right)$ $\therefore \angle OQP = 16.35^0$

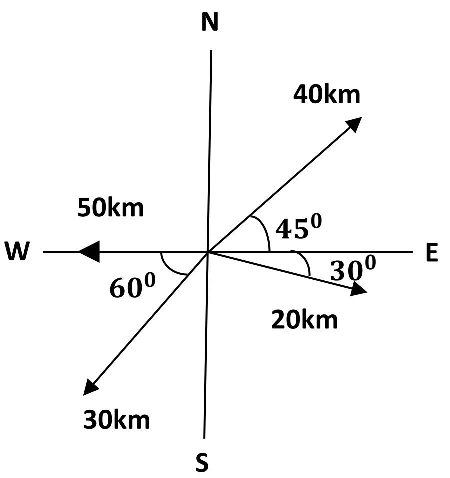
$$therefore, direction = 45^0 + 16.35^0 = 61.35^0$$

Hence the resultant displacement has magnitude of **7.43 miles** and a direction of **61.35^0** due north of east.

2. ***Find the resultant of the following displacements:***
   (a) 20Km 30^0 south of east. (b) 50km due west (c) 40km northeast (d) 30km 60^0 south of west

Solution

We first represent the following statements on the catesian plane then we resolve each component to get the resultant dislacement.



$x - component$	$y - component$
$20\cos 30$ $40cos45$ -30$cos45$ -50	$40\sin 45$ $-20sin30$ $-30\sin 60$
-19.395	-7.696

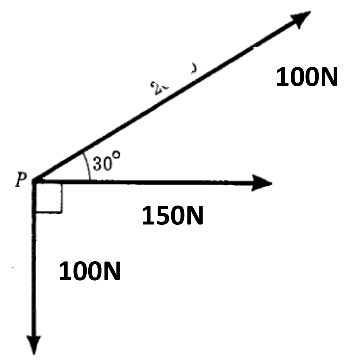
Using $\vec{R} = \sqrt{(-19.395)^2 + (-7.696)^2}$

$$\vec{R} = \mathbf{20.9N}$$

$$\alpha = \tan^{-1}\left(\frac{-7.696}{-19.353}\right) = \mathbf{21.64^0}$$

Hence the resultant displacement has magnitude **$20.9N$** and direction **21.64^0** south of west.

3. ***An object P is acted upon by three coplanar forces as shown in fig.(a) below. Determine the force needed to prevent P from moving.***



Solution

$x - component$	$y - component$
150 **$200cos30$**	$200\sin 30$ -100
323.2	**0**

Using $\vec{R} = \sqrt{{R_x}^2 + {R_y}^2}$

$$\vec{R} = \sqrt{323.2^2 + 0^2} = \mathbf{323.2N}$$

Hence the force needed to prevent P from moving is **323.2N** and it acts opposite in direction to the 150N force.

PROBLEM SET OR EXERCISE 6.0

1. ***Two forces P N and Q N include an angle of 120° and their resultant is 19 N. If the included angle between the forces were 60°, their resultant would be 7 N. Find P and Q. (ans P = 5 N, Q = 3 N)***

2. ***Find the magnitude and direction of the resultant of the following coplanar forces acting at a point 0: 10 N in direction 000°; 5 N in direction 090°; 20 N in direction 135°; 10 N in direction 225°. (ans 116.5 N in direction 132.9°)***

3. ***The resultant of two forces X N and 3 N is 7 N. If the 3 N force is reversed, the resultant is 19 N. Find the value of X and the angle between the two forces; (ans 5 N, 60o)***

4. ***Two forces of 13 N and 5 N act at a point. Find the angle between the forces when their resultant makes the largest possible angle with the 13 N force. Find also the magnitude of the resultant when the angle between the forces has this value. (ans 112.6°, 12 N)***

Relative velocity/Resolution of velocities

Relative velocity is the rate at which the distance between two bodies moving relative to each other, is increasing or decreasing with time.

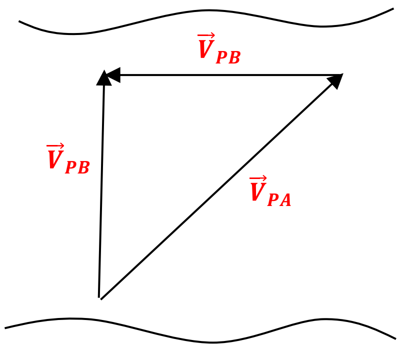
The relative velocity of a body (or a reference frame) is also the velocity observed by another body (or from another reference frame) moving relative to the other. This is algebraically found by the vector addition of the velocities of both frames involved. Now,

- ***For any two bodies A and B moving along a straight line and in the same direction with velocities V_A and V_B respectively, the distance between the bodies is increasing at the rate $(V_A - V_B)$, this is therefore the velocity of A relative to B.***
- ***If B is moving opposite to A, the relative velocity of A relative to B will be $(V_A + V_B)$.***

NOTE:** **the relative velocity is independent of the positions of the bodies involved.

for example,

When a body ***P*** moves relative to a body (or reference frame) ***B***, and ***B*** moves relative to ***A***, (relative velocity along a line).



We denote the velocity of ***P*** relative to ***B*** by $\vec{V}_{PB}$

Where $\vec{V}_{PB} = \vec{V}_P - \vec{V}_B$

The velocity of ***P*** relative to ***A*** by $\vec{V}_{PA}$

Where $\vec{V}_{PA} = \vec{V}_P - \vec{V}_A$

And the velocity of ***B*** relative to ***A*** by $\vec{V}_{BA}$

Where $\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$

This leads to the triangle law of vector addition *(or The Galilean vector transformation)*

Giving $\vec{V}_{PA} = \vec{V}_{PB} + \vec{V}_{BA}$

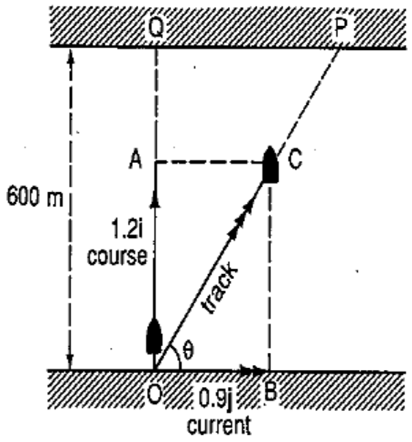
Hence, for any two bodies A and B, the relative velocity of A with respect to B maybe found by adding to A's velocity B's velocity reverse.

To find the relative velocity it is not necessary to know the positions of the bodies, but only the magnitude and direction of their velocities.

Worked Examples 1.50

1. ***A river flowing at $0.9ms^{-1}$. A man sets out, at right angles to the banks, to row across. He can row at $1.2ms^{-1}$ in still water.***
   (a) What is the actual velocity across the river?
   (b) If the river is 600m wide, how long does he take to cross the river?

Solution



Data:

Velocity of boat in still water *(the course vector $\overrightarrow{OA}$)*

$\vec{V}_B = 1.2\ ms^{-1}$

Velocity of river flowing *(the river current vector $\overrightarrow{AC}$)*

$\vec{V}_w = 0.9\ ms^{-1}$

Direction of boat downstream = **θ**

Actual/resultant velocity *(the track vector $\overrightarrow{AC}$)* = $\vec{V}_R$

(a) As shown in above, the boat moves 1.2 ms^{-1} perpendicular to the bank; but the water moving 0.9 ms^{-1} simultaneously carries the boat along, parallel to the bank. The boat therefore moves crabwise across pointing at right angle to the bank. Since this forms a right-angled triangle, we apply pythagoras' theorem to get the actual velocity across the river.

$$\vec{V}_R^{\,2} = \vec{V}_B^{\,2} + \vec{V}_W^{\,2} \Longrightarrow \vec{V}_R = \sqrt{1.2^2 + 0.9^2} = 1.5ms^{-1}$$

Hence the actual velocity has magnitude **$1.5ms^{-1}$** and a direction of

$\tan\theta = \frac{1.2}{0.9} \Longrightarrow \theta = \tan^{-1}(1.3) = 53^0$ ***downstream to bank.***

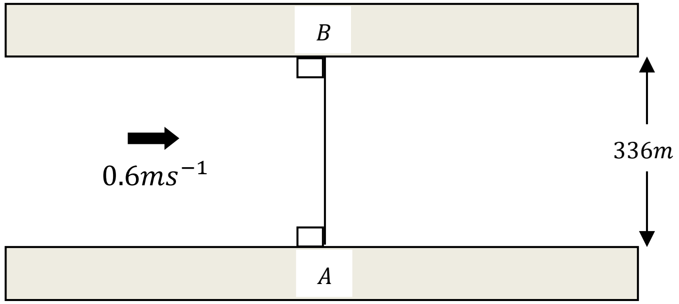
(b) The boat covers a distance **OP** after reaching the opposite bank at P

$$Time\ taken = \frac{distance\ OP}{speed\ along\ track} \qquad Using\ similar\ triangles, \frac{OP}{1.5} = \frac{600}{1.2}$$

$$\therefore OP = 750m$$

$$= \frac{750}{1.5} = 500s$$

2.



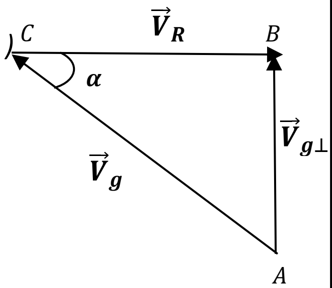
A girl swims in still water at 1 m $^{-1}$. She swims across a river which is 336 m wide and is flowing at 0.6 m $^{-1}$. She sets off from a point A on one bank and lands at a point B, which is directly opposite A, on the other bank as shown in Fig. 1. Find

(a) the direction, relative to the earth, in which she swims,

(b) the time that she takes to cross the river.

Solution

Velocity of girl in still water *(the course vector $\overrightarrow{AC}$)*
$\vec{V}_g = 1\ ms^{-1}$
Velocity of river flowing *(the river current Vector $\overrightarrow{CB}$)* $\vec{V}_R = 0.6\ ms^{-1}$
Velocity of girl perpendicular to bank $\vec{V}_{g\perp} =?$



(a) Direction of girl relative to the earth α

$\cos\alpha = \frac{0.6}{1} \Rightarrow \alpha = \cos^{-1}(0.6) = \mathbf{51.3^0\ upstream\ to\ bank}.$

(b) $Time\ taken = \frac{width\ of\ river}{speed\ along\ track} = \frac{AB}{\vec{V}_{g\perp}}$

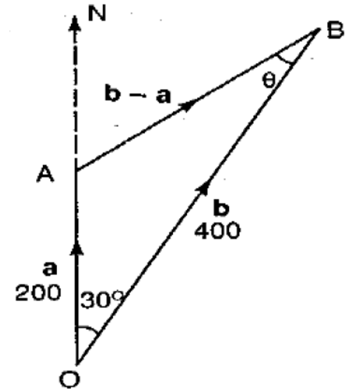
$= \frac{336}{0.8}$ $\quad$ **but** $\vec{V}_{g\perp}{}^2 = \vec{V}_g{}^2 - \vec{V}_R{}^2$

$= \mathbf{420s}$ $\quad$ $\vec{V}_{g\perp} = \sqrt{1^2 - 0.6^2} = 0.8ms^{-1}$

3. Two aeroplanes A and B flying at 200kmh-1 North and B at 400kmh-1 on a course 030^0. Find the velocity of B relative to A and the direction of course.

Solution

We first sketch the vector course of the problem as shown in fig.
where the velocity of B relative to A is ***(b – a) =*** $\boldsymbol{V_{BA}}$
and this can be found by applying the cosine rule



$$V_{BA} = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

$$V_{BA} = \sqrt{200^2 + 400^2 - 2(200)(400)\cos 30}$$

$$V_{BA} = \mathbf{248kmh^{-1}}$$

As for the direction $\angle NAB = 30^0 + \theta$ $(exterior\ angle)$, we use the sine rule to find θ

$$\frac{\sin\theta}{200} = \frac{\sin 30}{248} \Rightarrow \theta = 23.8^0 \qquad \therefore \angle NAB = 30^0 + 23.8^0 = 53.8^0$$

Hence the velocity of B relative to A is $248\ kmh^{-1}$ in direction of N $53.8^0 E$.

PROBLEM SET OR EXERCISE 7.0

1. ***Rain is falling vertically at 5 kmh^{-1}. A man is sitting by the window of a train travelling at 40 kmh^{-1}. In what direction do the raindrops appear to cross the windows of the train? (Ans 7.1^0 to the horizontal)***
2. ***Aeroplane A is flying due North at 150 kmh^{-1}. Aeroplane B is flying due Eat 200 kmh^{-1}. Find the velocity of B relative to A. (ans 250 kmh^{-1}, 126.9^0)***
3. ***Two cars A and B are travelling on roads which cross at right angles. Car A is travelling due east at 60 km h-1, car B is travelling at 40 km h-1 due north, both going towards the crossing. Find the velocity of B relative to A. [The magnitude and direction must be given]. (ans 72.1 kmh^{-1}, 303.7^0)***
4. ***A passenger is on the deck of a ship sailing due east at 25 kmh^{-1}. The wind is blowing from the north-east at 10 kmh^{-1}. What is the velocity of the wind relative to the passenger? (ans 32.8 kmh^{-1}, 077.6^0)***

5. *A road (running north-south) crosses a railway line at right angles. A passenger in a car travelling north at 60 km h-1 and 600 m south of the bridge, sees a train, travelling west at 90 kmh^{-1}, which is 800 m east of the bridge. Find the velocity of the train relative to the car. (ans 108.2 kmh^{-1}, 236.3^0)*
6. *Two particles, A and B, are 60m apart with B due west of A. Particle A is travelling at 9ms^{-1} in a direction 300^0 and B is travelling at 12ms^{-1} in a direction 030^0. Find*
   (a) The magnitude and direction of the velocity of B relative A, (ans 15ms-1, 066.9^0)
   (b) The time taken for B to be due north of A. (ans 4.55s)
7. *A wind is blowing from the direction 320 at 30kmh^{-1}. Find by drawing or by calculation, the magnitude and direction of the velocity of the wind relative to a man who is cycling due east at 18kmh^{-1}. (ans 23.02 kmh^{-1}, 356.8^0)*

PROJECTILES

A projectile is a body which when given an initial velocity, follows a path determined entirely by the effects of gravitational acceleration and air resistance.

- The path travelled by the projectile is called its **trajectory**.

Examples of projectiles include: A batted baseball, a thrown football, a package dropped from an airplane, and a bullet shot from a rifle.

Applications: Projectiles have numerous applications some of which include a javelin used in **sport,** a missile used in **warfare,** rockets or probes used in **astronomy etc.**

Projectile motion is an example of a two dimensional motion, which means the motion has two components $\boldsymbol{the\ horizontal\ (x)\ and\ vertical\ (y) components}$

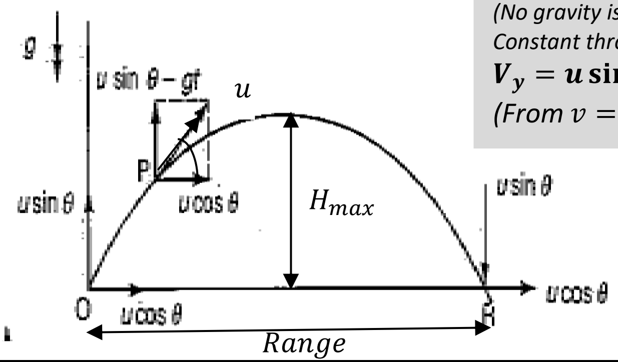
Therefore at any point **P** of the path, reached after **t** seconds, the projectile describes two components of its initial velocities:

$$\boldsymbol{V_x = u\cos\theta}$$

(No gravity is acting therefore V_x is Constant throughout the motion)

$$\boldsymbol{V_y = u\sin\theta - gt}$$

(From $v = u + at$)



- **Coordinates**

The coordinates of **P** at time **t** as referred to the figure above will be

$$\boldsymbol{x = u\cos\theta t}$$

$$\boldsymbol{y = u\sin\theta t - \tfrac{1}{2}gt^2}$$ *(Vertical displacement from $s = ut + \frac{1}{2}gt^2$)*

- **Greatest/Maximum Height**

At maximum height $\boldsymbol{H_{max}}$, the vertical component of the velocity $\boldsymbol{V_y = 0}$

Using $\boldsymbol{V_y = u\sin\theta - gt}$

$$\boldsymbol{0 = u\sin\theta - gt}$$

$$\boldsymbol{t = \frac{u\sin\theta}{g}}$$ *(Time taken to reach H_{max})*

Substituting **t** into $\boldsymbol{y = u\sin\theta t - \frac{1}{2}gt^2}$ where $\boldsymbol{y = H_{max}}$

$$\boldsymbol{H_{max} = u\sin\theta\left(\frac{u\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2}$$

$$\boldsymbol{H_{max} = \frac{u^2\sin^2\theta}{2g}}$$ *(Maximum height)*

- **Time of flight (T)**

This is the time taken for the projectile to travel along its path from initial to final position. At this time **T,** the vertical displacement **y** at **R** is zero

Using $\boldsymbol{y = u\sin\theta t - \frac{1}{2}gt^2}$

$$\boldsymbol{0 = u\sin\theta t - \tfrac{1}{2}gt^2},$$

Giving $\boldsymbol{t = 0\ (time\ at\ initial\ position\ O)\ or\ t = T\ at\ R}$

$$\boldsymbol{0 = u\sin\theta T - \tfrac{1}{2}gT^2}$$

$$\boldsymbol{T = \frac{2u\sin\theta}{g}}$$ *(Time of flight)*

Note that this is twice the time taken to reach $\boldsymbol{H_{max}}$

- **Horizontal Range (R)**

The horizontal range is the distance covered by the projectile along its x-component or x-coordinate. This is given by

$$R_x = V_x \times T \quad \text{Where}$$

V_x *is the horizontal velocity component*

and **T** *is the time of flight*

Using $x = u\cos\theta t$, *where* $V_x = u\cos\theta$, $T = t = \left(\frac{2u\sin\theta}{g}\right)$

$$R_x = (u\cos\theta)\left(\frac{2u\sin\theta}{g}\right)$$

$$R_x = \frac{2u^2}{g}\sin\theta\cos\theta$$

$$R_x = \frac{u^2}{g}\sin 2\theta \quad \textit{(Since } 2\sin\theta\cos\theta = \sin 2\theta\textit{)}$$

From this equation one can determine the angle of projection for which the range is **maximum.**

Since the maximum value of $\sin 2\theta = 1$ **(taking sine inverse of 1)**

$$2\theta = 90^o$$
$$\theta = 45^o$$

Hence the range is maximum when the angle of projection $\theta = 45^o$

Worked Examples 1.60

1. ***A particle is projected from O with speed 30ms⁻¹ at an angle of 30⁰ to the horizontal. Find***
   - ***(a) The time of flight***
   - ***(b) The range on the horizontal ground***
   - ***(c) The maximum height reached.***

Solution

Data: u = 30ms^{-1}, $\theta = 30^o, g = 10ms^{-2}$ $T = ?, R_x = ?, H_{max} = ?$

(a) $T = \frac{2u\sin\theta}{g} = \frac{2(30)\sin 30}{10} = 3s$

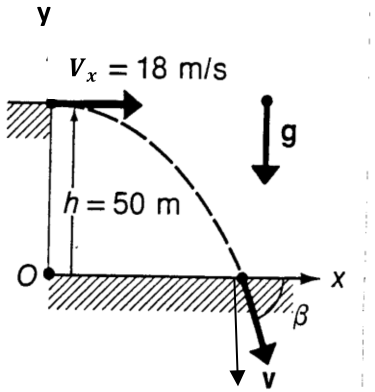
(b) $R_x = \frac{u^2}{g}\sin 2\theta = \frac{30^2 \times \sin(2\times 30)}{10} = 77.9s$

(c) $H_{max} = \frac{u^2\sin^2\theta}{2g} = \frac{30^2\sin^2 30}{2\times 30} = 11.25\ m$

2. ***A ball is kicked horizontally at a speed of 18ms⁻¹ off a 50m high cliff. Find***
   - ***(a) the time at impact,***
   - ***(b) the speed at impact,***
   - ***(c) the impact point (horizontal range),***
   - ***(d) angle at impact.***

Solution

Data: as shown in fig $V_x = 18ms^{-1}$, $h = 50m$, $g = 10ms^{-1}$



(a) the time at impact is the time of flight **T,** considering our coordinate system,

Horizontal coordinate

$$x = u\cos\theta t, \quad but\ \theta = 0$$

$\Rightarrow x = u \times t$ -------- **(1)**

Vertical coordinate

$y = y_o + u\sin\theta t - \frac{1}{2}gt^2$ *(where* y_o *is the initial vertical position of the ball)*

$y = h - \frac{1}{2}gt^2$------- **(2)** *(where* $h = y_o$ *and* $u\sin(0)t = 0$*)*

At impact, $y = 0$ *and* $t = T$*, hence from* ***(2)***

$$\Rightarrow T = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 50}{10}} = 3.16\ s$$

(b) Speed/velocity ***V*** at impact is the resultant of the two velocity components **(V_x *and* V_y).** Where $V_x = 18ms^{-1}$

$V_y = u\sin\theta - gT$ *(from* $v = u + at$*, where* $u\sin(0) = 0$*)*

$\Rightarrow V_y = -gT$, $\therefore V_y = -10 \times 3.16 = -31.6\ ms^{-1}$. Since the velocities are perpendicular, we use Pythagoras' theorem to get the resultant velocity at impact. $\Rightarrow V = \sqrt{V_x{}^2 + V_y{}^2} = \sqrt{(18)^2 + (-31.6)^2} = \mathbf{36.4\ ms^{-1}}$

(c) The point at impact is the horizontal range R_x, this can be found using equation **(1)** $x = u \times t$

$$\Rightarrow R_x = V_x \times T \qquad \textit{(at impact remember)}$$

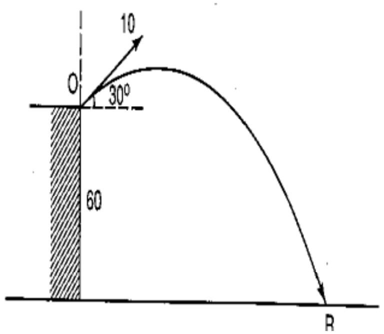
$$R_x = 18 \times 3.16 = \mathbf{56.88\ m}$$

(d) The angle at impact β is the angle which the resultant velocity makes with the ground as shown in fig. This is found by evaluating the direction of the resultant velocity or velocity at impact.

That is, $\tan\beta = \frac{V_y}{V_x} = \frac{-31.6}{18} = -1.76, \Rightarrow \beta = \tan^{-1}(-1.76) = -60.3^0$

3. From the edge of a cliff 60m high, a stone is thrown into the air with speed 10ms^{-1} at an angle of 30^o to the horizontal. Find with a sketch of the trajectory of the stone

(a) the time of flight,
(b) how far from the foot of the cliff it strikes the sea,
(c) the velocity with which it strikes the sea.



Solution

Data: $h = 60m, u = 10ms^{-1}, \theta = 30^0$

(a) Time of flight T,

Referring to the coordinates at o, we get

$$x = 10\cos 30t \Rightarrow x = 5\sqrt{3}\ t$$

$$y = y_o + 10\sin 30t - \tfrac{1}{2} \times 10t^2 \text{ (where } h = y_o = 60)$$

$$\Rightarrow 0 = 60 + 5T - 5T^2 \textit{ (when the stone reaches sea } y = 0, t = T)$$

$$\Rightarrow T^2 - T - 12 = 0$$

$(T-4)(T+3) = 0$ **Giving** $T = \mathbf{4s}$ ***(T = - 3** is invalid)*

(b) This is the horizontal range R_x

$$\textit{and } R_x = V_x \times T \Rightarrow R_x = 10\cos 30 \times 4$$

Hence $R_x = \mathbf{36.64\ m}$

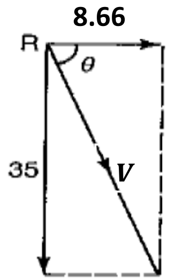
(c) **This the resultant velocity *V* of the two velocity components** (V_x *and* V_y).

Where $V_x = 10\cos 30 = 8.66ms^{-1}$ and $V_y = 10\sin 30 - 10 \times 4$

$V_y = -35ms^{-1}$ *(From $v = u + at$ and the minus sign indicates downward or $-y$)*

Therefore magnitude of $V = \sqrt{V_x{}^2 + V_y{}^2} = \sqrt{(8.66)^2 + (-35)^2} = \mathbf{36\ ms^{-1}}$

and its direction $\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right) = \tan^{-1}\left(\frac{8.66}{35}\right) \approx 76^0$



Hence the stone strikes the sea with a velocity of $36\ ms^{-1}$ at 76^0 to the horizontal.

PROBLEM SET OR EXERCISE 8.0

1. ***If a ball is thrown upwards with speed 12ms^{-1} at an angle 30^o to the horizontal, at what time after the start is it 1m above the ground? (ans 0.2s and 1s) What is the maximum height reached? (ans 1.8 m)***
2. ***A stone is thrown with speed 12ms^{-1} at an angel of 60^0 to the horizontal but hits a wall 9m away. How high above the ground does it hit the wall? (ans 4.3 m)***
3. ***A stone is thrown at angle θ $\left(\tan\theta = \frac{4}{3}\right)$ to the horizontal so that it just passes over the top of a wall 3m away horizontally. If it takes 0.5s to reach the top of the wall, find (a) the speed of projection, (ans 10 ms^{-1})***
(b) the height of the wall. (ans 2.75 m)

4. *A projectile has an initial speed of $84ms^{-1}$ and rises to a maximum height of 40m the level horizontal ground from which it was projected. Calculate*
   (a) *The angle of projection, (ans 19.7^0)*
   (b) *The time of flight, (ans 5.66s)*
   (c) *The horizontal range. (ans 447.4 m)*
5. *An arrow is fired at an apple 60m up a tree, at the same instant it drops from the tree. How must the arrow be aimed so as to hit the apple 30m below its height, if the man firing the arrow is 40m away from the vertical position of the apple? (ans $36.87^o \approx 37^0$)*
6. *A particle is projected from a point O on the horizontal ground and moves freely under gravity. When the particle is at a horizontal distance of 12m from O, it achieves its greatest vertical height which is 8m. Calculate;*
   (a) *The angle to the horizontal at which the particle was projected, (ans 53.1^0)*
   (b) *The speed of projection, (ans 15.8 ms^{-1})*
   (c) *The time from leaving O which before the particle again reaches ground level, (ans 2.535 s)*
   (d) *The horizontal range. (ans 2.4 m)*

FORCES AND FRICTION

Force: ***A force is the quantitative measure of the interaction between two bodies. It can also be defined as a physical quantity which brings about change in a body's state of rest or uniform motion in a straight line. It is a vector quantity and it is measured in Newton (N).***

One newton is the amount of net force that gives an acceleration of 1 meter per second squared to a body with a mass of 1 kilogram.

Forces bring about motion or it is the cause of motion.

Types of Forces

Forces are classified based on how they interact with bodies. This led to two major classifications;

1. **Contact Force:** *this force occurs when there is direct contact between the bodies involved. It can result to a change in shape, position or motion of the bodies involved*. Examples: tension, upthrust, friction, thrust, drag, weight, reaction etc.
2. **Field Force:** *this is the force experienced by bodies in a field involving no physical contact between the source and the bodies. The force acts along a line of action.*
   Examples: magnetic force, electrostatic force, gravitational force etc.

Basic Definitions

- **Tension:** *the force resulting from the compression or extrusion of elastic materials (strings, springs etc.).*
- **Thrust:** *the action and reaction resulting from a push.*
- **Drag:** *the opposing force experienced due to flow of a displaced medium.*
- **Magnetic Force:** *the force experienced by a body in a magnetic field.*
- **Electrostatic Force:** *the existing force between charged bodies.*
- **Centripetal Force:** *the force acting on a body describing or moving in a circular path.*

Friction

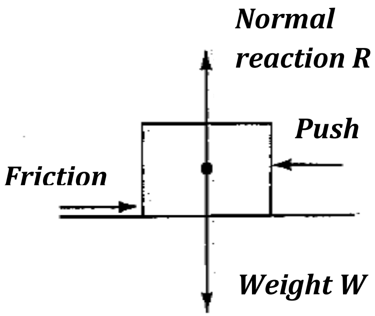
This is the force preventing motion between two or more bodies sliding over each other. It is further classified into two, based on the **inertia** of the bodies involved.

Inertia *is the reluctance of a body to motion and to stop moving once in motion.*

(a) **Dynamic/kinetic Friction:** *the reluctance existing between two bodies in relative motion.*
(b) **Static Friction:** *the reluctance existing between two stationary (not-moving) bodies.*

- **Coefficients of limiting friction**

Consider the block resting on the surface in fig. Friction is experienced when the block is at rest ***(static friction)*** and when a push occurs on one side of the block ***(dynamic friction).*** *The maximum value of the frictional force is called* ***limiting friction.*** This frictional force is affected by the nature of the surfaces to motion. This is quantify as the ***Coefficient of limiting friction*** *represented by* ***(μ)*** *(mu).*



The frictional force existing between the body and the surface depends on:

1. *The normal reaction* ***(R)*** *which results from the weight of the body and*
2. *The nature of the surface* ***(μ)***

Mathematically $F \propto R$

$F = \mu R$ ***(Limiting frictional force)***

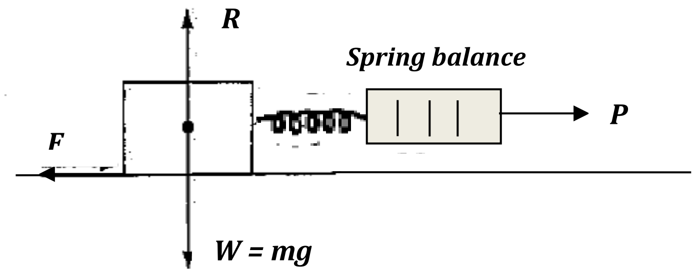
$\mu = \frac{F}{R}$ ***(Coefficient of limiting friction)***

This equation is applicable to both ***dynamic*** and ***static friction.*** Hence there exists two coefficients of limiting friction namely

(a) **Coefficient of kinetic friction** (μ_k) $F_k = \mu_k R$
Acts between contact surfaces in relative motion. (where $\mu_k \geq 0$*)*
(b) **Coefficient of static friction** (μ_s) $F_s = \mu_s R$
Acts between contact surfaces at rest. (where $\mu_s > \mu_k$*)*

- **Practical Determination of the coefficient of limiting friction** ***(μ)***

1. **Using a spring balance**
This is done simply by horizontally pulling a spring balance attached to a body of known mass, resting on a horizontal surface. The spring balance reading is taken at the instant when the body starts moving. This reading is the value of the ***limiting friction.***



Where $F = limiting\ friction,\ R = normal\ reaction,\ W = weight$

From, $\mu = \frac{F}{R}$ $and\ R = w = mg$

Hence $\mu = \frac{F}{w} \Longrightarrow \boldsymbol{\mu = \frac{F}{mg}}$

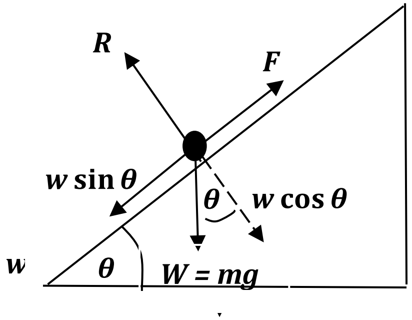
2. **Using an inclined plane**
This can be done by gradually increasing the angle of inclination of an inclined plane, having a body resting on its inclined surface. The angle is measured at the instant when the body is just about to slide down the plane. From the figure below**,** $\boldsymbol{R = w\cos\theta}$ *and* $\boldsymbol{F = w\sin\theta}$ *(Body is in limiting equilibrium)*

Using

$$\mu = \frac{F}{R}$$

$$\mu = \frac{F}{R} = \frac{w\sin\theta}{w\cos\theta}$$

$$\therefore \mu = \tan\theta$$



- **Laws of Solid Friction**

Experimental results on solid friction are summarised in the laws of friction, which state:

1. ***The frictional force between two surfaces opposes their relative motion.***
2. ***The frictional force is independent of the area of contact of the given surfaces when the normal reaction is constant.***
3. ***The limiting frictional force is proportional to the normal reaction for the case of static friction. Whereas the frictional force is proportional to the normal reaction for the case of kinetic (dynamic) friction and is independent of the relative velocities of the surfaces.***

Advantages of Friction	Disadvantage of Friction
1. It brings about locomotion. 2. It supports the working action of friction belts in engines. 3. Brings about the braking action of vehicles. 4. Enables the easy removal and tightening of nails, screws, nuts etc. 5. Brings about the motion of parachutes, aeroplanes, balloons etc.	1. Reduces the efficiency of machines. 2. Causes damage or wear and tear of machine parts. 3. Can cause heat or fire outbreak which can lead to damage. 4. Decreases fuel economy. 5. Slows down locomotion.

- **Methods of Reducing Friction**
  1. Use of ball bearings and rollers.
  2. Lubrication- use of lubricants e.g. oil, grease etc.
  3. Streamlining of moving parts.
  4. Powdering or polishing of surfaces.

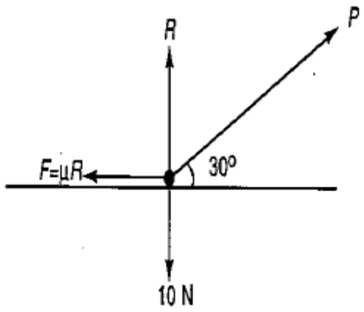
Worked Examples 1.70

1. ***A particle of mass 1kg rests on a horizontal floor. The coefficient of friction between the particle and the floor is $\frac{1}{2}$. What force is required just to make the particle move when***
   (a) Pulling horizontally
   (b) Pulling at an angle of 30^0 to the horizontal?

Solution

Data: $w = 1 \times 10 = 10N, \quad \mu = \frac{1}{2}, \quad \theta = 30^0$

(a) **Pulling horizontally**, the normal reaction ***R = mg = 10N and F = P,*** since the particle is in equilibrium vertically.



Therefore, using

$$F = \mu R$$

$$F = \frac{1}{2} \times 10 = 5N$$

Hence the horizontal force required just to make the particle move is **5N.**

(b) Pulling at 30^0 to the horizontal,

At equilibrium, we resolve the forces acting on the particle

Resolving horizontally,

$$Pcos30 = \frac{1}{2} \times R$$
$$\Rightarrow R = \sqrt{3}P$$

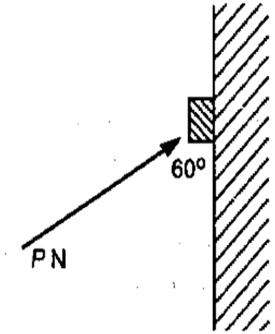
Resolving vertically,

$$R + Psin30 = 10$$
$$\sqrt{3}P + \frac{1}{2}P = 10 \textit{ (Substituting for R)}$$

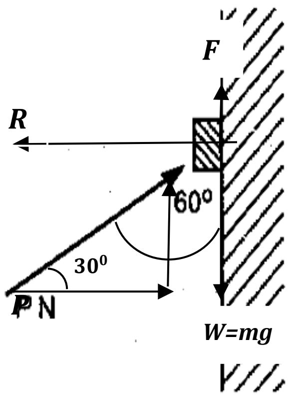
$$\Rightarrow \boldsymbol{P = 4.5N}$$

Hence the required force is **4.5N.**

2. ***The figure shows a small block of mass 5kg held against a rough vertical wall by a force P N inclined at angle of 60^0 to the wall. If the coefficient of friction is 0.4, calculate the value of P when the block is***
   (a) Just prevented from slipping down,
   (b) About to move up the wall.



Solution



We first scheme out the forces acting on the block as shown in the figure.

Data: w = mg = 5×10 = 50N, $\mu = 0.4$, $\theta = 30^0$

(a) As the block is held at rest, it is already on the point of slipping down. Hence the frictional force $\boldsymbol{F}$ together with the vertical component of $\boldsymbol{P}$ act upwards preventing the block from slipping down. ***Note that the components resolution of P is with respect to the angle it makes with the horizontal and not the vertical, this is the hard nut in the question.***

Resolving horizontally,

$$R = Pcos30$$
$$R = \frac{\sqrt{3}}{2}P$$

Resolving vertically,

$$F + Psin30 = w$$
$$\frac{2}{5} \times R + \frac{1}{2}P = 50$$
$$\frac{2}{5} \times \frac{\sqrt{3}}{2}P + \frac{1}{2}P = 50 \textit{ (substituting for R)}$$
$$\Rightarrow P = \frac{500}{(5+2\sqrt{3})} = \boldsymbol{59.10N}$$

Hence P is **59.1N** when the block is just prevented from slipping down.

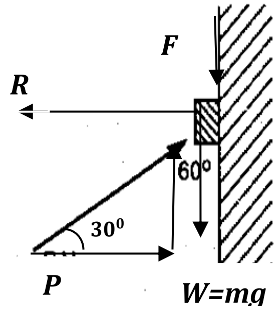
(b) As the block is pulled up, friction tends to oppose the motion and hence it will act downward.

Resolving horizontally,

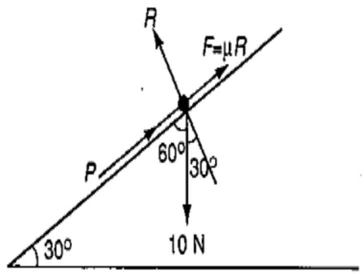
$$R = Pcos30$$
$$R = \frac{\sqrt{3}}{2}P$$

Resolving vertically,

$$F = w + Psin30$$
$$\frac{2}{5} \times R = 50 + \frac{1}{2}P$$
$$\frac{2}{5} \times \frac{\sqrt{3}}{2}P - \frac{1}{2}P = 50 \textit{ (substituting for R)}$$
$$\Rightarrow P = \frac{500}{(5-2\sqrt{3})} = \boldsymbol{326N}$$



3. ***A particle of mass 1kg is placed on a rough plane inclined at 30^0 to the horizontal. The coefficient of friction is $\frac{2}{5}$. Find the least force parallel to the plane that is required***
   (a) To hold the particle at rest,



(b) To make the particle slide up the plane.

Solution

Data: w = 1×10 = 10N, $\mu = \frac{2}{5}$, $\theta = 30^0$

(a) When held at rest, the particle is on the point of slipping down. Therefore, friction acts upwards. Let **P** represents the force (push).

Resolving perpendicular (normal) to the plane,

$$R = 10cos30$$
$$R = 5\sqrt{3}$$

Resolving parallel to the plane,

$$P + \mu R = 10sin30$$
$$P + \frac{2}{5} \times 5\sqrt{3} = 5 \textit{ (substituting for R)}$$

Giving $P = \mathbf{1.54}N$

Hence the least force parallel to the plane to hold the particle at rest is **1.54N**

(b) As the particle is made to slide up the plane, friction tends to oppose its motion and hence acts downwards along the plane.
Resolving perpendicular (normal) to the plane,

$$R = 10cos30$$
$$R = 5\sqrt{3}$$

Resolving parallel to the plane,

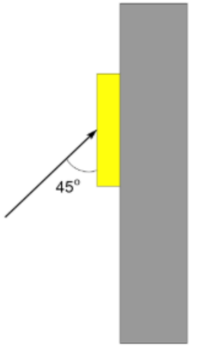
$$P = 10sin30 + \mu R$$
$$P = 5 + \frac{2}{5} \times 5\sqrt{3} \textit{ (substituting for R)}$$

Giving $P = \mathbf{8.46}N$

Hence the least force required to make the particle slide up the plane is **8.46N**

PROBLEM SET OR EXERCISE 9.0

1. ***A block of mass 300kg is just pulled along rough hor izontal ground by two equal forces P N inclined at 30^0 to the line of motion. If the coefficient of friction is 0.6. find the value of P.*** *(ans 772N)*
2. ***A horizontal force of 10N just prevents a mass of 2kg from sliding down a rough plane inclined at 45^0 to the horizontal. Find the coefficient of friction.*** *(ans $\mu = \frac{1}{3}$)*
3. ***A block of mass 5.2kg is placed on a rough plane inclined at an angle α to the horizontal where sinα = 0.6. The coefficient of friction between the block and the plane is 0.4. The block is just prevented from sliding down the plane by a horizontal force P N.***
   (a) ***Draw a diagram showing all the forces acting on the block,***
   (b) ***Calculate the value of P.*** *(ans 14N)*
4. ***Find the resultant in magnitude and in direction of the following coplanar forces acting at a point; 10N in direction 000^0, 20N in direction 060^0 and 5N in direction 090^0.*** *(ans 30N, 048.1^0)*
   If the forces are attached to a particle of mass 8kg on rough horizontal ground and the body is just about to move, calculate the coefficient of friction. *(ans $\mu = \frac{3}{8}$)*
5. ***A block of mass 10kg is held at rest against a vertical wall by a force F exerted at an angle of 45^0 as shown in fig. The coefficient of static friction between the wall and the block is 0.41. Determine***
   (a) ***The minimum magnitude of the force needed to prevent the block from falling,*** *(ans 141.4 N)*
   (b) ***The minimum magnitude of force required to just move it up the wall.*** *(ans 239.8 N)*



6. ***Distinguish between static friction and dynamic friction***
7. ***Why do tyres have treads?***

Viscosity *(Fluid Friction)*

Viscosity is a quantitative measure of a fluid's resistance to flow.

More specifically, it is the ***internal friction*** that exists in fluids. We can easily move through air, due to its very low viscosity but movement is more difficult in water, which has 50 times higher viscosity than air. The more viscous a fluid, the harder it is for it to flow or the more resistance it is to motion.

In liquids, viscosity depends significantly on the temperature of the fluid. Hence, ***A liquid is viscostatic if its viscosity does not change (appreciably) with change in temperature.***

We can compare roughly the viscosity of two liquids by filling two measuring cylinders with each of them, and allowing identical small steel ball-bearings to fall through each liquid. The sphere falls more slowly through the liquid with high viscosity.

The effect of viscosity in gases is called ***air resistance***. This force acts between falling objects and air. It increases with decrease in temperature of the air molecules. As objects fall freely in air, the air resistance increases until it equals the weight of the object where the resultant force on the object tends to zero. At this instant, no acceleration acts on the object and the velocity stays constant resulting into floatation. This velocity is known as the objects ***Terminal Velocity.***

Terminal Velocity is the steady or constant speed with which a body moves freely downward through a fluid when the resultant force on it is zero.

Examples of substances with high and low viscosities are shown in table below

Low viscosity substances	High viscosity substances
• water • kerosene • ethanol • methylated spirits.	• palm oil • engine oil • glycerine • glue

- **Effects of viscosity**

  Viscous effects act
  1. in the flow of fluid in pipes,
  2. the flow of blood,
  3. the lubrication of engine parts, and many other situations
- **Velocity gradient**

  This is the difference in velocity between adjacent layers of a fluid.
  As fluids flow, their adjacent layers experience a force that tends to shear each layer over the other- this force is called ***shearing stress (F).***
- **Coefficient of viscosity** η

  This is defined as the property of a fluid's tendency to resist flow.

  Or

  This is the ratio of the shearing stress to the velocity gradient of a fluid.

  Mathematically
  $$\eta = \frac{F}{A \times \text{velocity gradient}}$$
- **Surface Tension**

  This is defined as the force per unit length acting on the surface at right angles to one side of a line drawn on the surface of a liquid OR the property of a liquid that makes it behave as if its surface is an elastic skin.

  Surface tension arises because the molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule inside the volume of the liquid, but a surface molecule is drawn into the volume. Thus the liquid tends to minimize its surface area, just as a stretched membrane does.

Surface tension explains why
- freely falling raindrops are spherical (not teardrop shaped):
- A sphere has a smaller surface area for its volume than any other shape.
- Hot, soapy water is used for washing.

- **Example: Why hot, soapy water is suitable for washing?**

  To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers of the clothe. To do so, requires increasing the surface *area of the water, which is difficult to achieve because of surface tension. Th job is made easier by increasing the temperature of the water and adding soap, both of which decrease the surface tension.*

- **How the addition of detergents to water reduces surface tension?**

Surface tension is due to the intermolecular forces among the surface molecules of a liquid. Addition of detergents to water reduces/weakens theses forces; hence the reduction in its surface tension.

Assignment: explain why

- Freely falling raindrops are spherical (not teardrop shaped)?
- A sphere has a smaller surface area for its volume than any other shape
- Do tyres have treads?
- **Methods of Reducing Surface Tension**

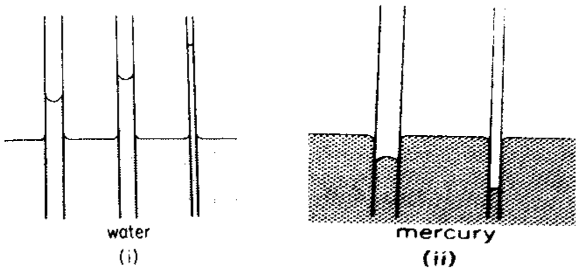
This can be done by

1. adding detergent.
2. adding soap.
3. adding alcohol.
4. adding oil to the liquid.
5. increasing the temperature of the liquid.

- **Capillarity**

Capillarity is defined as a phenomenon of the rise or fall of a liquid surface in a small tube relative to the adjacent general level of the liquid when the liquid is held vertically in the tube.

Different liquids exhibit different capillarity. For example a ***capillary rise*** is seen in water while a ***capillary fall*** is seen in mercury as shown below.



This varies significantly with the force acting between the glass molecules ***(cohesion)*** and the force acting between the glass and the liquid ***(adhesion).***

Cohesion – the force of attraction between molecules of the same substance.
Adhesion - the force of attraction between molecules of different substances.

Difference between solid friction and viscosity (fluid friction)

Solid friction	Viscosity
• Independent of surface area in contact.	• Depends on surface area in contact
• Depends on normal reaction (of one solid surface on the other)	• Independent of normal reaction
• Occurs between solids surfaces in contact	• Occurs between fluid layers/solid and fluid in contact/in fluids
• Independent of relative velocity (between solid surfaces in contact)	• Depends on relative velocity (between fluid layers)

Elastic Properties of Materials -Elasticity

Elasticity is the ability of a material or substance to resume its original size and shape after being distorted (compressed or stretched).

- When a substance or material is stretched or compressed deformation occurs, this thereby produces an extension or compression of the natural length or size of the material.
- The strength of the material depends mainly on how appreciably it responds to forces of distortion (compression or extension).
  e.g.
  - **Brittle** substances or materials (such as masonry and cast iron) can withstand large forces of compression, but will break easily if stretching **(tensile)** forces are applied.

- **Ductile** substances or materials (copper and lead) on the other hand can lengthen or stretch considerably and undergo **plastic deformation** until they break.

- ## Hooke's Law

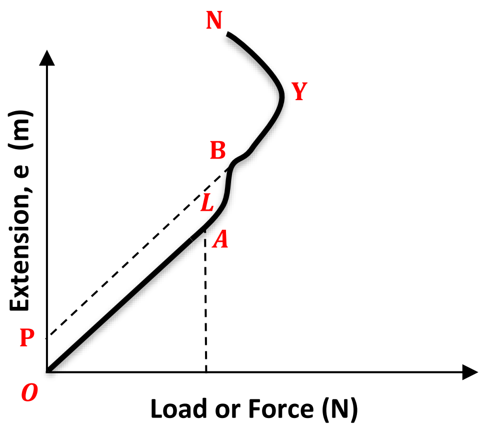
Hooke's law defines the relationship between the ***deformation*** of a material and the ***force applied*** to it- named after Robert Hooke.

This law states that the extension of a material is directly proportional to the applied force, provided the elastic limit is not exceed.

Mathematically

$$F = ke$$

This law can be verified in the lab *(by using an extensible wire with load(s) and a scale to measure the extension)* . A graph of extension against load (force) like the one shown will be obtained.



- Along **OA** and up to **L** just beyond **A**, the wire will regain its original length when the load is removed (i.e. it will obey Hooke's law)
- The force at **L** is called the ***elastic limit***.
- Along **OL** the wire is said to undergo ***elastic deformation***.
- Beyond L the wire gain permanent extension such as OP when the load is removed at B (i.e. wire is no longer elastic).
- Along the curve **ABY**, a small load or force will produce a large extension.
- At **N** the wire breaks.

- ## Tensile Stress, Tensile Strain and Young's Modulus

- **Tensile stress** is the elastic force applied per unit area $= \frac{F}{A}$
  SI unit **Nm^{-2}**.
- **Tensile strain** is the extension produced per unit length
  No SI unit.
- **Young's Modulus** is the ratio of the tensile stress to the tensile strain.
  SI unit **Nm^{-2}**.
  Mathematically: $E = \frac{F/A}{e/l}$

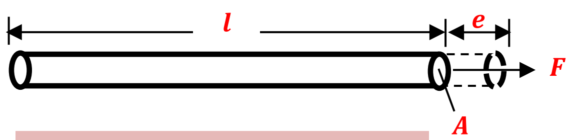
- ## Experimental Determination of Young's Modulus

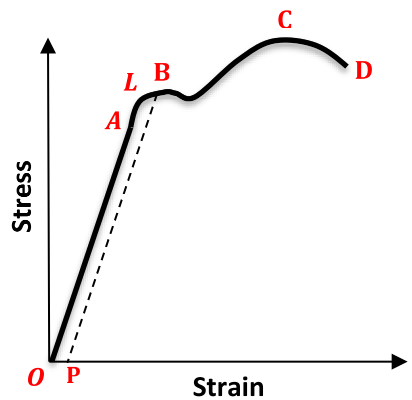
Apparatus: two thin extensible wires, loads and Vernier scale and micrometre screw gauge..

Procedure:
1. Measure the diameter of wire Q with the screw guage.
2. Suspend the wires beside each other with their respective weights from a rigid support B as shown in the diagram.
3. **Q** has a Vernier scale which reads the extension produced.
4. **P** is kept taut by a weight **A** attached to its end.
5. Attach the first load **W** to end of **Q** and read the Vernier scale for the extension produced.
6. Remove the load **W** and check if the wire returns to it initial Length.
7. Repeat steps **5** and **6** for five other loads.

Calculation:

$$E = \frac{F/A}{e/l} = \frac{F}{e} \times \frac{l}{A} \quad \textit{Where } A = \frac{\pi d^2}{4}$$





A = proportional limit
L = elastic limit
B = yield point
C = breaking (max) stress
D = wire breaks
OL = elastic deformation
BC = plastic deformation

- Workdone = average force x extension

$$W = \frac{1}{2}Fe = \frac{1}{2}ke^2$$

$$W = \frac{1}{2}EA\frac{e^2}{l} \quad (\textit{where } F = EA\frac{e}{l})$$

Equilibrium of Forces

- ***Equilibrium*** *is a state of balance resulting from opposing forces acting on a rigid body or particle.*

A rigid body is a body whose change in distance between any two of its particles is negligible for the purpose at hand.

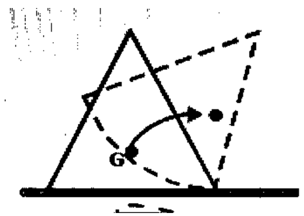
- **Types of equilibrium**

1. **Static Equilibrium:** this defined is as the state of balance a body attains when at rest.
2. **Dynamic equilibrium:** the state of balance attained by a body when the body moves with uniform linear or angular velocity.

- **States of Equilibrium**

1. **Stable equilibrium:** this is defined as the state experienced by a body when the body after slight displacement returns back to its original position.

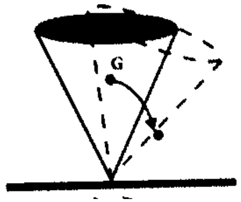
   This state results from the resultant weight of the body *(centre of gravity C.G)* acting at its base or bottom. The displacement produces an increase in potential energy due to rise of the C.G.



   E.g. a body resting on its broad side or base.

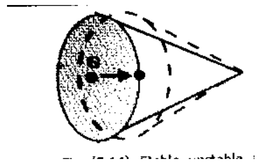
2. **Unstable equilibrium:** this state is achieved when a body does not return to its original position after slight displacement.

   At this state, the body's resultant weight acts way off its base or broader side. The displacement results into decrease in the body's potential energy as the C.G is lowered when displaced E.g. Bodies resting on their sharp point or apex.



3. **Neutral equilibrium:** this state is experienced when a body neither returns to its initial position, nor does it rest at another position after displacement. It rather rolls after the displacement.

   During this state, the body's centre of gravity **(C.G)** remains unchanged and the displacement results into no change in its potential energy as the C.G is at a fixed height or position.



   E.g. a marble or coin rolling on a flat surface.

- **Centre of Gravity (C.G)**

The centre of gravity is the point where the resultant force of gravitational attraction (or weight) of a body acts.

C.G. Position of shapes or figures, Examples:

- **Square & Rectangular planes** – C.G is at the point of intersection of the diagonals.
- **Triangular plane** – C.G is at the point of intersection of the medians.
- **Uniform right Solid Cone** – C.G is three-quarters along the axis from the apex.
- **Hollow & uniform solid cylinder** – C.G acts at the midpoint of the cylinder axis

- **Experimental determination of the position of Centre of Gravity of an irregular plane shape (or an irregular shaped card board)**

PROCEDURE

1. Make at least 3 well-spaced pin holes round the edge of the cardboard.
2. Clamp the pin horizontally and suspend the cardboard on it through one of the pin-holes such that the cardboard can swing freely.
3. Hang the simple pendulum on the same pin and let its string be very close to the cardboard.
4. When the whole system is at rest (or in equilibrium) trace the plumbline on the cardboard. Repeat the procedure for each of the two other pin holes.

CONCLUSION

The point at which the (three) traced lines intersect is the centre of gravity of the cardboard.

PRECAUTIONS

- Repeat procedure
- Pin rigidly and firmly held by retort stand and clamp
- Allow the simple pendulum to rest before tracing the shadow of the plumbline on the cardboard.
- The string to be close to the cardboard.

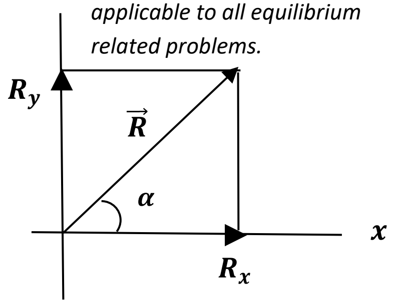
- **Condition for rigid bodies under the action of non- parallel forces to be in equilibrium**

A rigid body or particle under the action of two or more forces is said to be in equilibrium if the resultant of the forces (or the vector sum of all the applied forces) equals zero.

As force is a vector quantity, we hereby apply four methods of resolving forces in equilibrium, three of which have been discussed earlier.

1. ***Resolution of the forces in two perpendicular directions***

This method is generally applicable to all equilibrium related problems.



Resultant magnitude $\vec{R}$

$$\vec{R} = \sqrt{{R_x}^2 + {R_y}^2}$$

Direction (α)

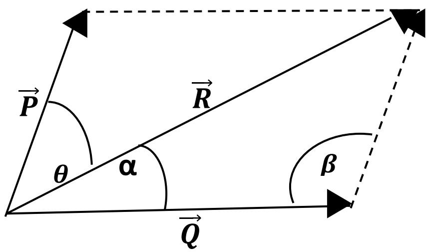
$$\tan\alpha = \frac{R_y}{R_x} \Longrightarrow \alpha = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

$$\text{where } R_x = \vec{R}\cos\theta$$
$$R_y = \vec{R}\sin\theta$$

2. ***Parallelogram law of forces***

Which states that ***when two vectors are represented in magnitude and in direction as the adjacent sides of a parallelogram, the resultant vector represented in magnitude and direction is the diagonal of the parallelogram.***

In this law we apply the cosine rule to get the magnitude and the sine rule to determine the direction of the resultant vector.



Resultant magnitude $\vec{R}$

$$\vec{R} = \sqrt{\vec{P}^2 + \vec{Q}^2 - 2\vec{P}\vec{Q}\cos\theta}$$

Direction (α)

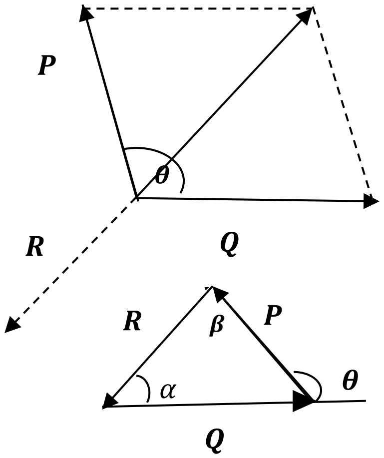
$$\frac{\sin\alpha}{\vec{P}} = \frac{\sin\beta}{\vec{R}} \Longrightarrow \alpha = \sin^{-1}\left(\frac{P\sin\beta}{\vec{R}}\right)$$

3. ***Triangle law of forces***

This states ***that if two vectors are represented in magnitude and in direction as the two adjacent sides of a triangle, taken in order, then the resultant vector is the closing side of the triangle taken in the reverse order.***

For example consider a particle acted upon by three forces P, Q and R.

The particle will be in equilibrium if ***R*** is equal in magnitude but opposite in direction to the resultant of ***P*** and ***Q***.



Resultant magnitude $\vec{R}$

$$\vec{R} = \sqrt{\vec{P}^2 + \vec{Q}^2 - 2\vec{P}\vec{Q}\cos(180-\theta)}$$

or

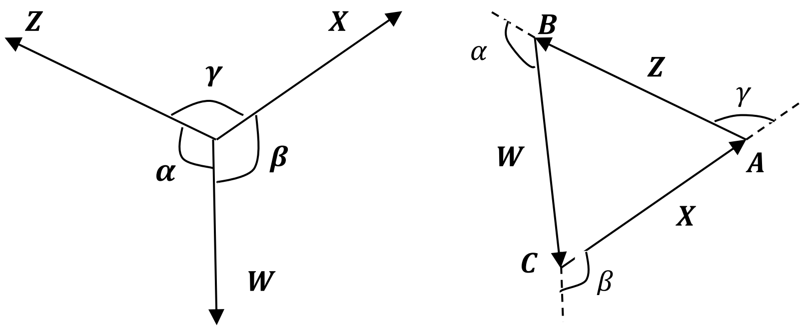
$$\vec{R} = \sqrt{\vec{P}^2 + \vec{Q}^2 + 2\vec{P}\vec{Q}\cos\theta}$$

Direction (α)

$$\frac{\sin\alpha}{P} = \frac{\sin\beta}{Q} = \frac{\sin(180-\theta)}{R}$$

4. ***Lami's theorem*** *(not in syllabus but useful to know)*

This theorem relates forces in equilibrium to the angles between their directions. This is applicable to three forces acting at a point and in equilibrium. Consider for example three forces ***W, X*** and ***Z*** acting at a point on a particle as shown below.



as shown in fig, the three forces can be arranged in order, forming a triangle of forces. Hence using the sine rule,

$$\frac{AB}{\sin(180-\beta)} = \frac{BC}{\sin(180-\gamma)} = \frac{CA}{\sin(180-\alpha)}$$

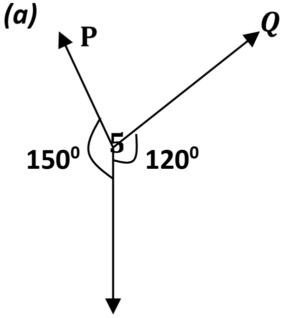
Giving

$$\boldsymbol{\frac{Z}{\sin\beta} = \frac{W}{\sin\gamma} = \frac{X}{\sin\alpha}}$$

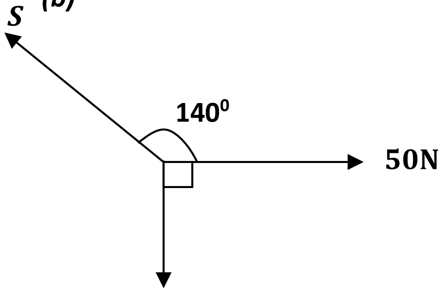
Worked Examples 1.80

1. ***Find the unknown forces and angles of the equilibrium forces in the figures below,***

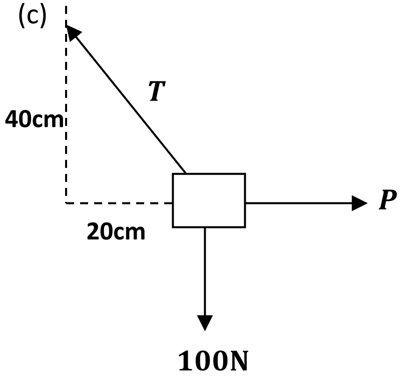
(a)



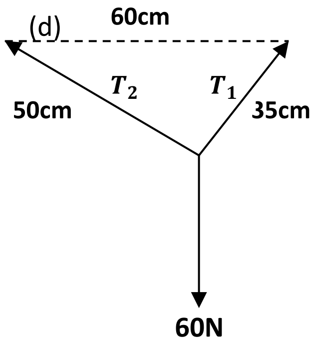
(b)



(c)



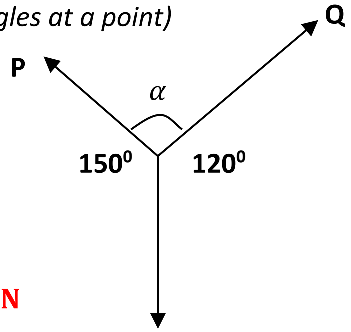
(d)



Solutions

(a) From fig, $\alpha = 360^o - (150^o + 120^o)$ *(angles at a point)*

$$\alpha = 90^o$$



Using therefore Lami's theorem,

$$\frac{50}{\sin 90} = \frac{P}{\sin 120} = \frac{Q}{\sin 150}$$

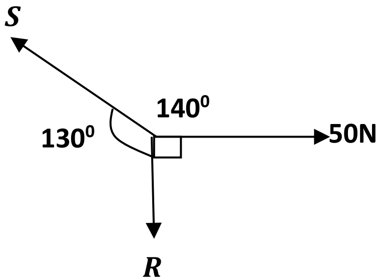
Taking, $\frac{50}{\sin 90} = \frac{P}{\sin 120} \Rightarrow P = \frac{50 \sin 120}{\sin 90} = \mathbf{43.3N}$

Also, $\frac{50}{\sin 90} = \frac{Q}{\sin 150} \Rightarrow Q = \frac{50 \sin 150}{\sin 90} = \mathbf{25N}$

Hence $P = \mathbf{43.3N}$ and $Q = \mathbf{25N}$

(b) from fig, $\beta = 360^o - (90^o + 140^o)$

$$\beta = 130^o$$



Using Lami's theorem,

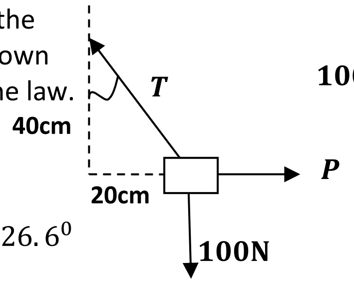
$$\frac{80}{\sin 130} = \frac{S}{\sin 90} = \frac{R}{\sin 140}$$

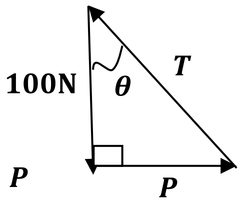
Taking, $\frac{80}{\sin 130} = \frac{S}{\sin 90} \Rightarrow S = \frac{80 \sin 90}{\sin 130} = \mathbf{104.4N}$

Also, $\frac{80}{\sin 130} = \frac{R}{\sin 140} \Rightarrow R = \frac{80 \sin 140}{\sin 130} = \mathbf{67.13N}$

Hence $\mathrm{S} = \mathbf{104.4N}$ and $\mathrm{R} = \mathbf{67.13N}$

(c) from fig, we draw the triangle of forces as shown in fig and then apply the law.





$$\mathrm{T}^2 = 100^2 + \mathrm{P}^2$$

But $\tan\theta = \frac{20}{40} \Rightarrow \theta = 26.6^0$

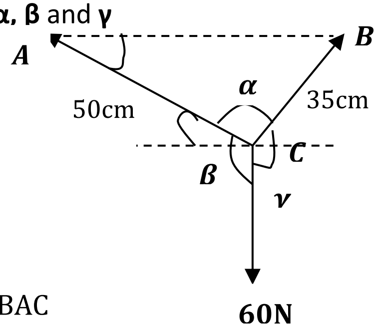
Where $\cos 26.6 = \frac{100}{T} \Rightarrow T = \mathbf{111.84N}$

Thus, $\mathrm{P}^2 = 111.84^2 - 100^2$

$\mathrm{P} = \sqrt{2507.63} = \mathbf{50.076N}$

Hence $\mathrm{T} = \mathbf{111.84N}$ and $\mathrm{R} = \mathbf{50.076N}$

(d) As shown in fig, we first find the angles between the forces. Since we are given values for all sides of ΔABC, we use the cosine rule to get, let the angles be **α, β** and **γ**



$\overrightarrow{AB}^2 = \overrightarrow{AC}^2 + \overrightarrow{BC}^2 - 2(\overrightarrow{AC} \times \overrightarrow{BC}) \cos\angle ACB$

$60^2 = 50^2 + 35^2 - 2(50 \times 35) \cos\angle ACB$

$\cos\angle ACB = \alpha = \frac{125}{3500} \Rightarrow \angle ACB = 87.95^0$

Also, $\overrightarrow{BC}^2 = \overrightarrow{AB}^2 + \overrightarrow{AC}^2 - 2(\overrightarrow{AB} \times \overrightarrow{AC}) \cos\angle BAC$

$35^2 = 60^2 + 50^2 - 2(60 \times 50) \cos\angle BAC$

$\cos\angle BAC = \frac{4875}{6000} \Rightarrow \angle BAC = 35.66^0, \quad \beta = 35.66^0 + 90^0 = 125.66^0$

Then, $\gamma = 360^0 - (35.66^0 + 90^0) = 146.39^0$

Hence applying Lami's theorem,

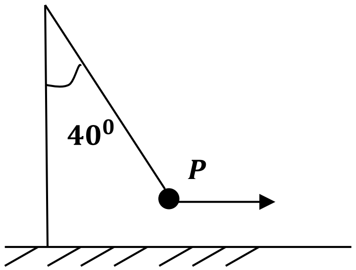
$$\frac{60}{\sin 87.95} = \frac{T_1}{\sin 125.66} = \frac{T_2}{\sin 146.39}$$

Taking, $\frac{60}{\sin 87.95} = \frac{T_1}{\sin 125.66} \Rightarrow T_1 = \frac{60 \sin 125.66}{\sin 87.95} = \mathbf{48.78N}$

Also, $\frac{60}{\sin 87.95} = \frac{T_2}{\sin 146.39} \Rightarrow T_2 = \frac{60 \sin 146.39}{\sin 87.95} = \mathbf{33.2N}$

Therefore, $T_1 = \mathbf{48.78N}$ and $T_2 = \mathbf{33.2N}$

PROBLEM SET OR EXERCISE 10.0



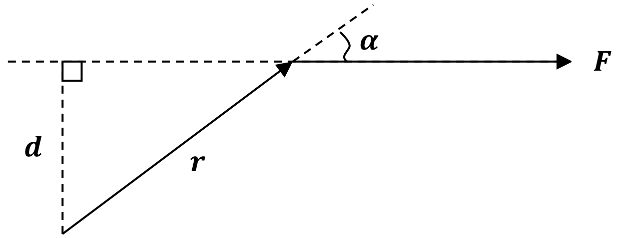
1. *A tennis ball P is attached to one end of a light inextensible string, the other end of the string being attached to the top of a fixed vertical pole. A girl applies a horizontal force of magnitude 50 N to P, and P is in equilibrium under gravity with the string making an angle of 40^0 with the pole, as shown in Fig. 1.*
   By modelling the ball as a particle find, to 3 significant figures,
   (a) the tension in the string, (ansa 77.8 N)
   (b) the weight of P. (ansa 59.6N)
2. *A body of weight 40N hangs from a string attached to a point on a vertical wall. The string will break when its tension exceeds 50N. If the body is pulled away from the wall by a horizontal force P N,*
   (a) what angle does the string make with the vertical when it breaks and (ansa 36.9^0)
   (b) what is the value of P (ansa 30N)
3. *A particle M of weight 20N is supported by two strings, MA and MB making angles of 30^0 and 45^0 on opposite sides with the vertical through M. Find, by drawing or calculation, the tensions in the strings. (ansa MA: 14.6N, MB: 10.4N)*

- ## Moment of a Force

The moment of a force about a point O, is the product of the force and its perpendicular distance from the fixed point O to the line of action of the force.

Mathematically: $M = F \times r$

Where r is the vector from O to where the force is applied. The moment of a force does not change if the force is moved along the straight line of its action or application. This is shown in fig below



From the fig, the magnitude of the moment is given by

$$M = Fr\sin\alpha$$
$$= Fd \quad \text{(where } d = r\sin\alpha\text{)}$$

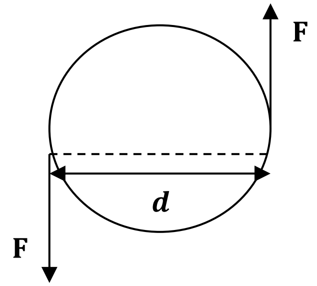
Where d is the distance between the point O and the line of action of the force.

- ## Torque T

A torque is a measure of the ability of a force to cause the object upon which it acts to rotate about an axis. Torque is also regarded as the ***moment of a force*** but it acts in the case where we have two equal but opposite forces acting perpendicular to and on the same line of action. This system is known as a ***Couple.*** Hence torque is also defined as the moment of a couple.

Mathematically,

$$\text{torque} = \text{one force} \times \text{perpendicular distance between forces}$$



$$\mathbf{T = F \times d}$$

- ## Couple

A couple is a system comprising of two opposite but equal forces acting perpendicular to and on the same line of action.

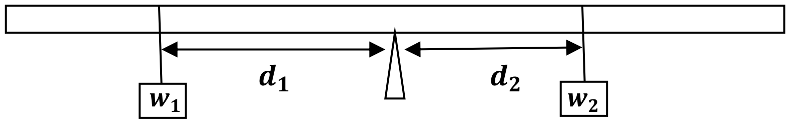
Examples: turning of water tap, corkscrew,

- ## Principle of moment

This states that ***when a beam is in equilibrium under the action of several forces, the sum of the clockwise moments of all the forces about a point must***

equal the sum of the anti-clock wise moments of all the forces about that same point.

- **Experimental verification of the principle of moment**



Apparatus: uniform metre rule (100m), two known masses, string or thread, knife edge.

Procedures:

1. First balance the metre rule freely on the knife edge to get the pivot point.
2. Hang the masses on opposite ends of the rule respectively.
3. Adjust the positions of the masses until the entire system balances horizontally (equilibrium).
4. Note and record the distances of the masses from the pivot point.

Calculation & conclusion

From the fig above, with the system in static equilibrium we apply the principle of moment.

sum of clockwise moments = sum of anti-clock wise moments

$$\sum w_1 d_1 = \sum w_2 d_2$$

Upon calculating, one will find that the sum of the clockwise moments equalled the sum of anti-clock wise moments; hence the principle of moment is valid.

- **Conditions for equilibrium of rigid bodies under the action of Parallel forces**

1. **The sum of all external forces acting on the body must equal zero or the sum of the upward forces must equal the sum of the downward forces.**

2. **The sum of the clockwise moments about a point must equal the sum of the anti-clockwise moments about that same point.**

Note: *In the case of a couple, it is the sum of the torques due to all external forces acting on the body, with respect to any specified point, that must be zero.*

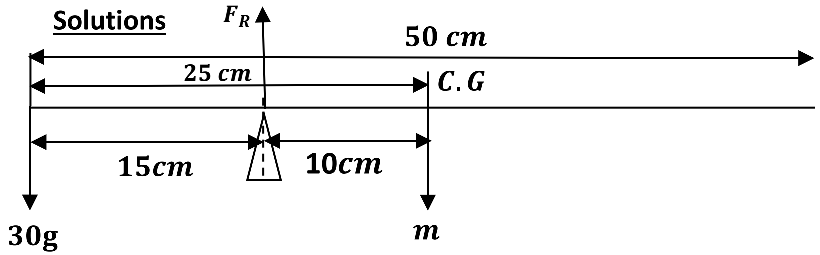
Worked Examples 1.80

1. A uniform half-metre rule AB is balanced horizontally on a knife edge placed 15cm from A, with a mass of 30g at A. Calculate the

(i) Mass of the metre rule

(ii) Force exerted on the rule by the knife edge

Solutions



Let ***m*** be the mass of the half-meter ***(50 cm)*** rule and since the rule is uniform its mass act at the midpoint **(25 cm)** of the rule (i.e where the centre of gravity is)

(i) Now at equilibrium and taking moment about knife edge:

$$30 \times 15 = m \times 10$$

$$m = 45g$$

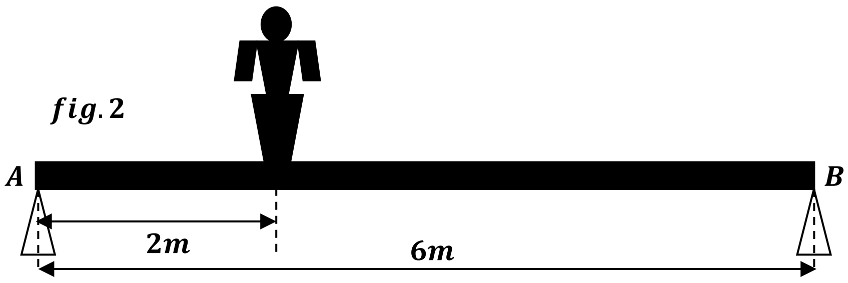
Thus mass of rule = 45g

(ii) Also at equilibrium and using condition **2:**

$$F_R = (30 + 45)g$$

$$F_R = 75 \times 10$$

Thus force exerted on rule by knife edge = **750 N** *fig*. 2



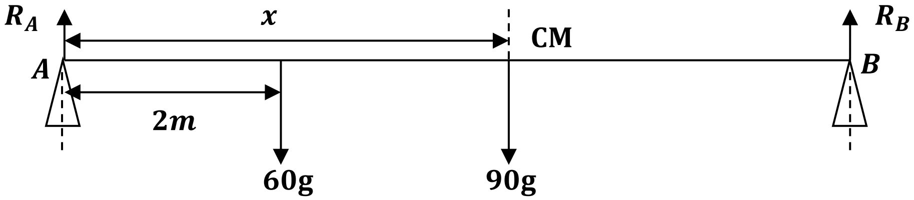
2. A non-uniform plank of wood AB has length 6 m and mass 90 kg. The plank is smoothly supported at its two ends A and B, with A and B at the same horizontal level. A woman of mass 60 kg stands on the plank at the point C, where AC = 2 m, as shown in Fig. 2. The plank is in equilibrium and the magnitudes of the reactions on the plank at A and B are equal. The plank is modelled as a non-uniform rod and the woman as a particle. Find

(a) the magnitude of the reaction on the plank at B,
(b) the distance of the centre of mass of the plank from A

$$Take\ g = 9.8ms^{-2}$$

Solutions

We first draw a free-body diagram of the system;



Note that the plank is non uniform; therefore the centre of mass is not at the midpoint of the plank.

(iii) At equilibrium, sum of upward forces = sum of downward forces

$$R_A + R_B = 60g + 90g$$
$$2R = 150g \text{ (where } R_A = R_B = R\text{)}$$
$$R = 75g$$
$$R = 735N$$

Hence the reaction at B is **735N**

(iv) The distance of the centre of mass from A is unknown, ***x***

Also at equilibrium, sum of clockwise moment = sum of anti-clockwise moment;

Taking moment about A,

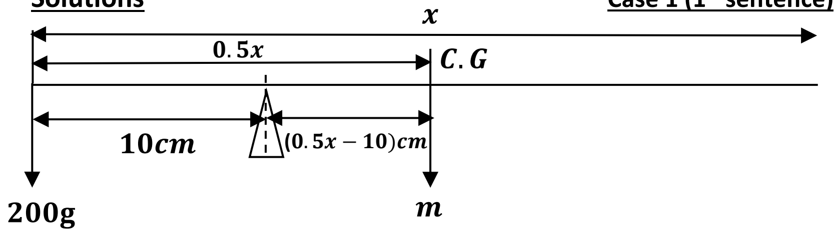
$$M(A) \quad 60g \times 2 + 90g \times x = 75g \times 6$$
$$90x = 450 - 120$$
$$90x = 330$$
$$x = 3.67$$

Hence the distance of the centre of mass from A is 3.67m

3. A uniform stick can be balanced on a knife edge 10cm from one end when a mass of 200g is hung from that end. When the knife edge is moved 5cm further from that end, the 200g mass has moved to a point 8.75cm from the knife edge to obtain a balance. Find

(a) length of the stick
(b) mass of the stick

Solutions **Case 1 (1st sentence)**



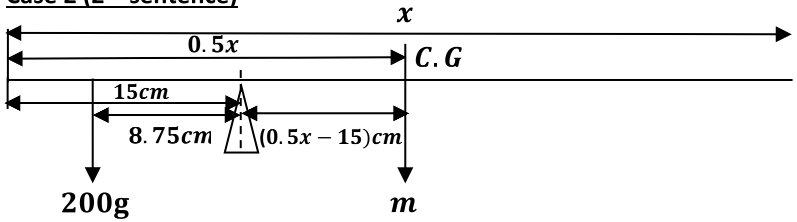
Let length of stick = ***x* cm**

Since stick is uniform, its mass will act at the midpoint of the stick ***(0.5x)cm***

Thus at equilibrium and taking moment about knife edge, we get:

$$200 \times 10 = m(0.5x - 10)$$
$$2000 = m(0.5x - 10) \quad \text{—— Eqn 1}$$

Case 2 (2nd sentence)



Also at equilibrium with moment about knife edge,

$$200 \times 8.75 = m(0.5x - 15)$$

$$1750 = m(0.5x - 15) \quad \text{——— } \textbf{Eqn 2}$$

(a) From 2,

$$m = \frac{1750}{0.5x-15} \quad \text{———— } \textbf{Eqn 3}$$

Eqn 3 into **Eqn 1**,

$$2000 = \frac{1750}{0.5x-15}(0.5x - 10)$$

$$1000x - 30000 = 875x - 17500$$

$$125x = 12500$$

$$x = 100\ cm$$

Hence the length of stick = 100cm

(b) Now $x = 100\ cm$ into 1

$$2000 = m(0.5 \times 100 - 10)$$

$$2000 = 40m$$

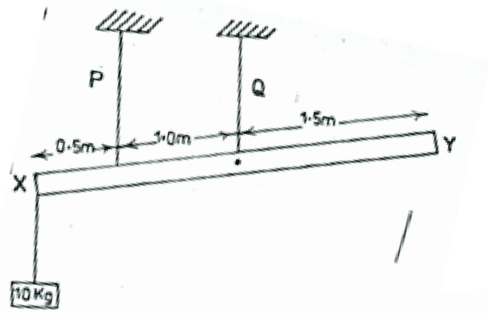
$$m = 50g$$

Hence mass of stick = 50 g

4. *A uniform bar XY of mass 5.0 kg and length 300 cm is suspended by two inextensible wires P and Q. P is attached 50 cm from the end X and Q to the centre of mass C of the bar, such that they are vertical. A mass of 10 .0 kg is attached to the end X of the bar.*
   (a) *Draw and label a diagram of the arrangement.*
   (b) *Determine the distance from the centre of mass C where another mass of 10.0 kg should be hung on the bar to keep the bar horizontal and to make the tension in Q four times the tension in Q. (g = 10 ms^{-2}) [WASSCE NOV/DEC 2016]*

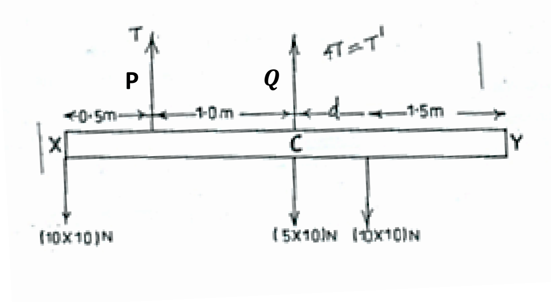
Solutions

(a) As shown in the diagram, the bar should be slightly tilted and not horizontally balanced.



(b) We are given that, the tension at Q is four times that at P: $T_Q = 4T_P$

At equilibrium, sum of upward forces = sum of downward forces



$\boldsymbol{T + 4T = 5T = (10 + 5 + 10) \times 10}$ Hence $\boldsymbol{T = 50\ \mathrm{N}}$ and $\boldsymbol{T' = 200\ \mathrm{N}}$

Let distance from centre of mass C for the 10.0 kg **= d**

Taking moment about C: $\boldsymbol{100d + 50 \times 1 = 100 \times 1.5}$

$$d = \frac{(100\times1.5)-50}{100} = 1.0\ \mathrm{m}$$

Hence the 10.0 kg mass will be 1.0 m from the centre of mass C of the bar.

PROBLEM SET OR EXERCISE 11.0

1. ***A uniform metal tube of length 5 cm and mass 9 kg is suspended horizontally by two vertical wires attached at 50 cm and 150 cm respectively from the ends of the tube. Find the tensions in each wire. (ans 30 N, 60 N)***
2. ***It is found that a uniform lath 100 cm long and of mass 95 g can be balanced on a knife-edge when a 5 g mass is hung 10 cm from one end . How far is the knife-edge from the centre of the lath? (ans 2 cm)***
3. ***A uniform half metre rule is freely pivoted at the 15 cm mark and it balances horizontally when a body of mass 40 g is hung from the 2 cm mark. Draw a clear force-diagram of the arrangement and determine the mass of the metre rule. (ans 52 g)***

Newton's Laws of Motion

In 1678 Sir Isaac Newton published a work called *principia,* in which he stated out clearly the Laws of Mechanics. He gave three fundamental Laws of Motion.

LAW I – ***A body will continue in a state of rest, or uniform motion in a straight line, unless it is acted upon by a resultant external force.***

This law is also known as the law of **inertia-** ***which is a body's reluctance to move if at rest or to stop moving once in motion.***

Deductions from Newton's First Law

- When a body is at rest, or moving with constant velocity, there is no resultant force acting on it and any other forces that do act must achieve a state of balance i.e. must be in equilibrium.

- If the speed of a moving object is changing, then there must be a resultant force acting it.

- If also, the direction of motion is changing, i.e. not moving in a straight line), there must also be a resultant force acting on it. *Thus there is always a force acting on a body that is moving in a curve, even if the speed is constant.*

From Newton's first law we can classically define

Force- **as a quantity which when acting on a body, changes the velocity of the body.**

Finally, since when the velocity of a body changes, there is an acceleration, we thus wisely say:

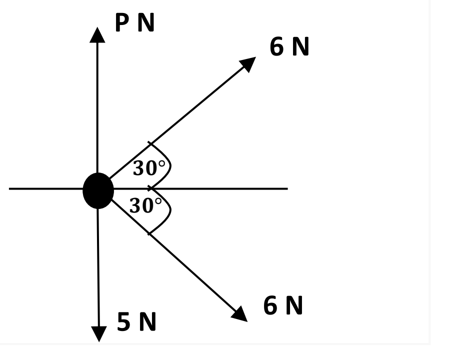
If a body has an acceleration there is a resultant force acting on it.

AND

If a body has no acceleration there is no resultant force acting on it.

WORKED EXAMPLE 1.90

1. **The diagram shows the forces acting on a particle. Determine whether the resultant acceleration of the particle is horizontal, vertical or in some other direction, if (a) P = 5 (b) P = 8.**



Solution: The resultant acceleration will act in a direction for which the forces do not balance.

(a) At P = 5, this happens horizontally, where the two inclined forces have no balance or no equal opposite force(s). Unlike vertically, the forces balance in pairs, which implies no vertical acceleration.

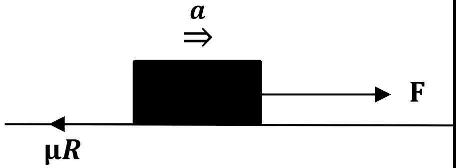
Hence the resultant acceleration is horizontal when p = 5.

(b) At P = 8, the vertical components of the 6 N forces balance but the 5 N and 8 N do not, this implies a vertical acceleration component.

The resultant acceleration is a combination of the two components and it is neither horizontal nor vertical but in some other direction.

2. The diagram above illustrates a body of mass 5.0 kg being pulled by a horizontal force F. If the body accelerates at 2.0 ms^{-2} and experiences a frictional force of 5 N, calculate the:
(i) net force on it;
(ii) magnitude of F:
(iii) coefficient of kinetic friction
[g = 10 ms-2] *(WASSCE MAY/JUNE 2014)*

Solution



(i) Net force = $F - F_R = ma$ 1

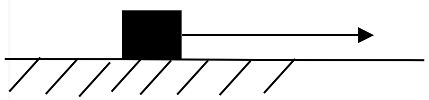
$\therefore \quad 5.0 \times 2.0 = \mathbf{10.0\ N}$

(ii) From 1, $F = ma + F_R = 10.0 + 5.0 = \mathbf{15.0\ N}$

(iii) Now $F_R = \mu \times R = \mu \times mg$ $thus\ 1 \Rightarrow \mu = \frac{F-ma}{m\times g} = \frac{15-10}{5\times 10} = \mathbf{0.1}$

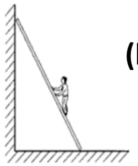
PROBLEM SET OR EXERCISE 12.0

1. The diagram shows a block in rough contact with a horizontal surface. It is being pulled along a horizontal ring.



   (a) Make a copy of the diagram and on it mark all the forces acting on the block.
   (b) What can you say about the tension in the string compared with the frictional force if the block
      (i) is accelerating?
      (ii) moves with constant velocity?

2. The diagram shows a ladder, with its foot on the rough ground, leaning against a smooth wall. The weight of the ladder, W, acts through the midpoint; a man, also of weight W, is standing on the ladder as shown.



   (a) Mark all the forces that act on the ladder, on a copy of the diagram (represent the man by a particle)
   (b) Write down two expressions involving the forces, that you could use if you were asked to find out whether the ladder is stationary.

LAW II – ***The rate of change of linear momentum is proportional to the impressed force, and takes place in the direction of the straight line along which the force acts.***

Mathematically: μR

$F \propto \frac{m(v-u)}{t}$ where *m (v-u)* is the change in momentum.

$\frac{m(v-u)}{t} = rate\ of\ change\ of\ momentum$

$F = km\frac{(v-u)}{t}$ Where k is a constant.

$F = kma$ Since $\frac{(v-u)}{t} = acceleration\ a$

Now if **m = 1**, and **a = 1** then **F = k**, thus the amount of force needed to give a mass of 1kg an acceleration is **1m/s²** is given by k.

If this amount is chosen as the unit of force, we have **k = 1** and

$$F = ma$$

The unit of force is Newton defined as the amount of force that gives 1kg of mass an acceleration of 1m/s^2.

Newton's first law defines the relationship between force, mass and acceleration. Thus, it seems reasonable to accept the following deductions:

(a) Deductions from Newton's Second Law

- For a body of a particular mass, the bigger the force is, the bigger the acceleration will be.
- The larger the mass is, the larger will be the force needed to produce a particular acceleration.
- Lastly, the resultant force acting on a body of constant mass is equal to the product of its mass and acceleration.

F(N) = m(kg) × a(m/s^2)

Where the resultant force and the acceleration act in the same direction.

WORKED EXAMPLE 2.0

1. ***A force F newton acts on a particle of mass 3kg.***
   (a) If the particle accelerates uniformly from 2 m/s to 8 m/s in 2 seconds, find the value of F.
   (b) If F = 6, find the displacement of the particle 4 seconds after starting from rest.

Solution

Data: u = 2 m/s, v = 8 m/s, t = 2s, m = 3kg

(a) First we determine the acceleration, $a = \frac{(v-u)}{t} = \frac{8-2}{2} = 3m/s^2$

Thus F = ma =3×3 = 9. The force is therefore **9 N.**

(b) *After 4s u =0 m/s, and with a = 2m/s^2, we can determine the displacement using* $s = ut + \frac{1}{2}at^2$

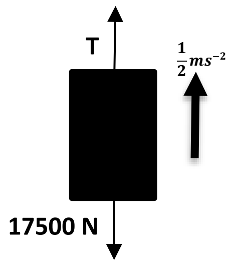
$$s = 0 + \frac{1}{2}(2)(4)^2$$

$s = 16$ *The displacement of the particle is* ***16m***.

2. ***A goods lift with a mass of 750kg can be raised and lowered by a cable. The maximum load it can hold is 1000kg.***
   Find the tension in the cable when
   (i) raising the fully-loaded lift with an acceleration of $\frac{1}{2}\boldsymbol{ms^{-2}}$
   (ii) lowering the empty lift with an acceleration of $\frac{3}{4}\boldsymbol{ms^{-2}}$

Solution

(i) Upon raising the lift, the acceleration of the lift acts upwards hence the resultant force acts upwards as well.



Where: weight of fully-loaded lift = 17500 N

Resultant force upward = $(T - 17500)$ N

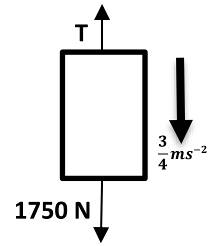
Using $\mathbf{F} = \boldsymbol{ma}$

$$T - 17500 = 17500 \times \frac{1}{2}$$

$\boldsymbol{T} =$

Hence tension in cable is

(ii) Acceleration of the empty lift is downwards when lowering it and the resultant force thus acts downwards.



Where: weight of empty lift = 750 N

Resultant force downward = $(1750 - T)$ N

Using $\mathbf{F} = \boldsymbol{ma}$

$$1750 - T = 1750 \times \frac{3}{4}$$

$\boldsymbol{T = 437.5}$ **N**

Hence tension in cable is $\mathbf{437.5}$ **N**

3. ***A man of mass 80kg stands on a lift. What is the reaction from the floor of the lift if the lift***
   (a) moves upward in with steady speed.
   (b) moves upwards with acceleration 0.5 ms^{-2},
   (c) moves downwards with acceleration 0.4 ms^{-2}. (g = 10 ms^{-2})

Solution

(a) When moving upward with steady speedy, no acceleration acts on the lift or man, therefore reaction from floor $\boldsymbol{R = 80g = 800}$ **N**

(b) Acceleration is upward, meaning resultant force is upward.

Where: Resultant force = $(T - 800)$ N

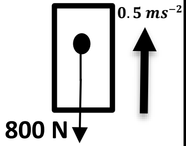
Weight of man = 80g = 800 N

Using $\mathbf{F} = \boldsymbol{ma}$

$$R - 800 = 80 \times 0.5$$

$$\boldsymbol{R = 840\ \mathrm{N}}$$

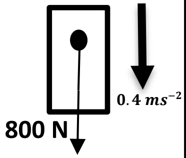
Hence reaction from lift floor is 840 N



(c) Acceleration acts downwards, thus resultant force acts in the same direction as well.

Where: Resultant force = $(800 - R)$ N

Weight of man = 80g = 800 N



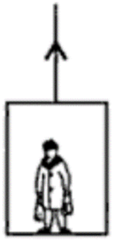
Using $\mathbf{F} = \boldsymbol{ma}$

$$800 - R = 80 \times 0.4$$

$$\boldsymbol{R = 768\ \mathrm{N}}$$

Hence reaction from lift floor is 768 N

PROBLEM SET OR EXERCISE 12.0



1. ***A lift of mass 800kg is operated by a cable as shown in the diagram. A passenger of mass 70kg is standing in the lift. Find, stating what object you can use to represent the passenger,***
   - ***(I)*** ***the force exerted by the passenger on the floor of** the lift,* *(ans 700 N)*
   - ***(II)*** ***the tension in the cable when the lift is accelerating***
     - ***(a) upwards at 0.8 ms^{-2}*** *(ans 18600 N)*
     - ***(b) downwards at 0.5 ms^{-2}*** *(ans 7950 N)*
2. ***A car of mass 1000 kg is pulling a caravan of mass 800 kg along a level straight road. The total resistance force to motion is 450 N and the individual resistances on the car are in the ratio of their masses. If the combination accelerates uniformly from rest at 20ms^{-1} in 12.5 s.***

   Find ***(a) the tension in the tow-bar*** *(ans 148 N)*

   (c) The driving force exerted by the car's engine. *(ans 3330 N)*
3. ***A truck of mass 600kg is pulling a trailer of mass 200kg up an inclined plane of angle to the horizontal. The resistance to motion on either of the vehicle is 0.2 N per kg of mass. Calculate***
   - ***(a) the driving force of the engine and*** *(ans 392 N)*
   - ***(b) the tension in the tow-bar when the vehicles are acceleration at 0.25 ms^{-2}.*** *(ans 98 N)*

(b) Linear Momentum

The linear momentum of a body in a straight line is the product of its mass and velocity.

$$\boldsymbol{momentum = \vec{p} = m \times \vec{v}}$$

It is a **vector** quantity with SI unit ***Ns***.

This expression derives its way from Newton's second law. It tells us that as the force $\boldsymbol{F}$ increases on a given body, the change in velocity **($\boldsymbol{\Delta\vec{v} = v - u}$)** will increase for a given time **($\boldsymbol{\Delta t}$)** i.e. the acceleration will increase in direct proportion.

Mathematically $\boldsymbol{F = \frac{m\Delta v}{\Delta t}}$ **or** $\boldsymbol{F\Delta t = m\Delta v}$

$$\boldsymbol{\therefore F\Delta t = m(v - u)}$$

*Where $\boldsymbol{m}$ is the proportionality constant known as **inertia mass**, $\boldsymbol{F\Delta t}$ is the **impulse** and $\boldsymbol{m\Delta v}$ the change in momentum.*

The Impulse ($J = F\Delta t$) of this force is the product of the (large) force acting on a body and the (short) time during which it acts.
This impulse is equal to the change in momentum as shown in the eqn above.

$$\text{Hence} \quad \boldsymbol{J = mv - mu}$$

Example 1

A ball of mass 4.0 kg hits a smooth platform vertically with a speed of 3 ms^{-1} and rebounds with a speed of 2 ms^{-1}. Calculate the impulse experienced by the ball. (WASSCE Nov/Dec 2010)

Solution

Data: mass (m) = 4 kg, initial speed (u) = 3 ms^{-1}, final speed (v) = 2 ms^{-1}

Impulse experienced is $\boldsymbol{J = Ft = mv - mu}$

$$\Rightarrow \boldsymbol{J = 4 \times 2 - (-4 \times 3)}$$

$$\boldsymbol{J = 20\ Ns}$$

Example 2

A body of mass 4 kg, is moving in a straight line with a speed of 5 ms^{-1}. A force of 10 N acts on the body for 6 s, in the direction of the motion.

Find (a) the magnitude of the impulse exerted on the body.

(b) the speed of the body at the end of this time.

Solution

Data: mass (m) = 4 kg, initial speed (u) = 5 ms^{-1}, force (F) = 10 N and time (t) = 6s

(a) Impulse exerted is $\boldsymbol{J = F \times t \quad \Longrightarrow \quad J = 10 \times 6 = 60\ Ns}$

(b) We are required to determine the final speed $\boldsymbol{v}$

Using $\boldsymbol{J = mv - mu}, \quad \Longrightarrow \quad \boldsymbol{60 = 4v - 4 \times 5}$

Hence $\boldsymbol{v = 40\ ms^{-1}}$

Example 3

A jet of water strikes a wall, at right angles to the jet, with a speed of 20 ms^{-1}. The water does not bounce off the wall. Given that average force exerted by the wall in stopping the flow is 360N, find the mass of water being delivered per second.

Solution

Data: initial speed (u) = 20 ms^{-1}, final speed (v) = 0,

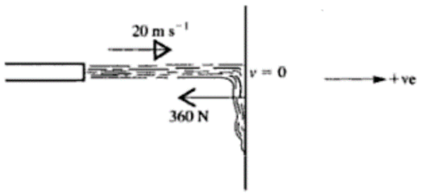
Average force (F) = 360 N, time t = 1 s

Let the mass of the water = m

Using $\boldsymbol{J = Ft = mv - mu}$

$$\boldsymbol{-360 \times 1 = m \times 0 - 20m}$$

$$\boldsymbol{\Longrightarrow \quad m = \frac{-360}{-20} = 18\ \mathrm{kg}}$$



Example 3

A bullet of mass 0.2 kg strikes a wall at right angles with a speed of 8 ms-1. After rebounding it has 25% of its initial kinetic energy.

(a) Find its speed after rebounding.

(b) Find the impulse it exerts on the wall.

(c) If it is in contact with the wall for 0.01s, find the average force it exerts on the wall.

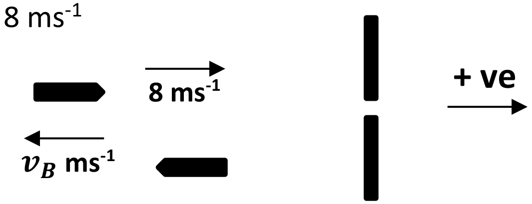
Solution

Data: mass of bullet **(m_B)** = 0.2kg, time (t) = 0.02s

speed of bullet **(u_B)** = 8 ms^{-1}

Before collision

After collision



(a) Now initial kinetic energy = $\frac{1}{2}m_B u_B{}^2 = \frac{1}{2} \times 0.2 \times (8)^2 = 6.4\,J$

K.E after rebound = 25% of initial kinetic energy= $\boldsymbol{\frac{25}{100} \times 6.4 = 1.6\,J}$

i.e. $\quad \frac{1}{2}m_B v_B{}^2 = 1.6 \ \Longrightarrow v_B{}^2 = \frac{2\times1.6}{0.2} = 16 \ \Longrightarrow v_B = \sqrt{16}$

Thus $\quad \boldsymbol{v_B = 4ms^{-1}}$

Hence speed after rebounding is $\boldsymbol{4ms^{-1}}$

(b) Impulse $-ft = m_B v_B - m_B u_B = -0.2 \times 4 - 0.2 \times 8 = \mathbf{2.4\ Ns}$

Hence impulse exerted on the wall is $\mathbf{2.4\ Ns}$.

(c) Average force exerted $\boldsymbol{-\mathrm{F} = m\frac{(-v_B)-u_B}{t} = 0.2 \times \frac{-8-4}{0.01} = 240\mathrm{N}}$

PROBLEM SET OR EXERCISE 13.0

1. *A block of mass 6kg is pulled along a smooth horizontal surface by a horizontal string. If the block reaches a speed of 20m/s in 4 seconds from rest, find the tension in the string. (ans 30 N)*

2. *A ball of mass 0.10kg is projected horizontally onto a vertical wall with a speed of 17ms^{-1}. The ball makes contact with the wall for 0.15s and rebounds horizontally with a speed of 13ms^{-1}.*
   Calculate the: (a) change in momentum of the ball; (ans 13 kgm/s or 13 Ns)
   (b) Average force exerted on the ball during its collision with the wall. (WASSCE May/June 2012) (ans 20 N)

3. *A dart of mass 40 g hits the dartboard at a speed of 16 ms^{-1}. If the dart comes to rest in the board in 0.02s, find the average force exerted by the board on the dart. (ans 32 N)*

4. *A high pressure hose is being used to clean the wall of a town hall. The hose delivered a horizontal stream of water which hits the wall at a speed of 20 ms^{-1}. Find the average force exerted on the wall, assuming that the water does not bounce back off the wall,*
   If (a) 8 kg of water is delivered per second. (ans 160 N)
   (B) The cross sectional area of the hose pipe is 0.5 cm^2.
   (Take density of water as 1000 kgm^{-3}) (ans 20 N)

5. *A tennis ball of mass 30 g travelling horizontally at 20 ms-1 is hit straight back at 30 ms^{-1}. If the impact lasted for 0.04s, find the average force exerted on the ball. (ans 37.5 N)*

6. *A sphere of mass 3 kg is dropped from on to a horizontal plane from a height of 2 m above the plane. [g = 10 ms^{-2}]*
   (a) Find the speed of the sphere when it hits the plane. (ans 6.3 ms^{-1})
   (b) If the sphere does not bounce find the impulse it exerts on the plane. (ans 19 Ns)
   If the sphere rebounds to a height of 1.4 m, find
   (c) The speed at which the sphere rises off the plane. (ans 5.3 ms^{-1})
   (d) The impulse exerted by the plane on the sphere. (ans 35 Ns)

(c) Collision

A body is said to be in collision when it exerts an equal and opposite impulse on another body during contact.

When two bodies are in contact, they exert equal and opposite impulses on each other. Provided neither of the bodies is fixed, these ***equal and opposite impulses*** produce equal and opposite ***changes in momentum***, thus the total change in momentum caused by the collision is zero, provided no external force acts on the bodies.

This property is known as ***The Principle of Conservation of Momentum (PCM)***, which states that:

The total momentum of a system of colliding bodies in a specified direction is constant, provided no external force acts on the motion of the system.

OR

In a system of colliding objects the total momentum is always conserved provided that there is no net external force acting on the system.

OR

If two or more bodies collide in a closed system, the total momentum after the collision is equal to the total momentum before the collision.

Mathematically

$$\boldsymbol{m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2}$$

Where m_1, m_2 = respective masses of bodies in collision
u_1, u_2 = respective velocities before collision
v_1, v_2 = respective velocities after collision

(d) Types of Collision

A collision can either be perfectly elastic or inelastic

- **Perfectly Elastic Collision**: is a collision system in which the kinetic energy of the system is conserved (it increases).
- **Perfectly Inelastic Collision:** is a collision system in which the kinetic energy of the system decreases (or is not conserved).

Differences

Perfectly elastic collision	Perfectly inelastic collision
Kinetic energy is conserved	- Kinetic energy decreases
Colliding bodies do not stick stogether after collision	- Colliding bodies stick together and move as a unit after collision
In head-on collision directions of the bodies are reversed	- Direction of both bodies are not necessarily reversed.

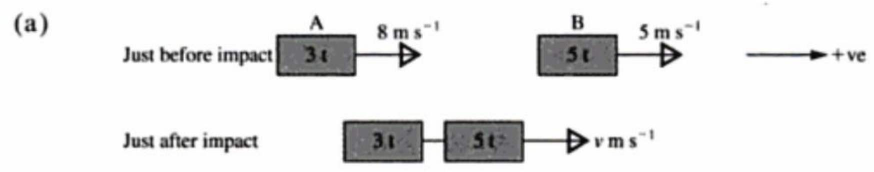
Example 1

A 3000 kg truck is moving along a track at 8 ms^{-1} towards a 5000 kg truck travelling at 5 ms^{-1} on the same track. If the truck become coupled at impact, find the velocity at which they will continue to move if they are travelling (a) in the same direction

(b) in opposite directions.

Solution

Let v be the velocity of the coupled (joined) trucks



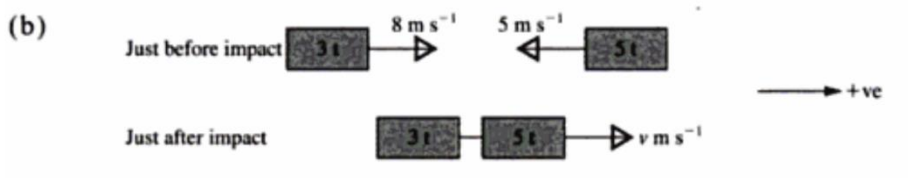
Using the **PCM**

$$m_A u_A + m_B u_B = (m_A + m_B)v$$

$$3000 \times 8 + 5000 \times 5 = (3000 + 5000)v$$

$$v = \frac{49000}{8000}$$

$$v = 6.125\ ms^{-1}$$



In this case, the velocity of the 5000kg truck is reversed (opposite to our positive coordinate)

Thus $\quad 3000 \times 8 + 5000 \times (-5) = (3000 + 5000)v$

$$v = \frac{-1000}{8000}$$

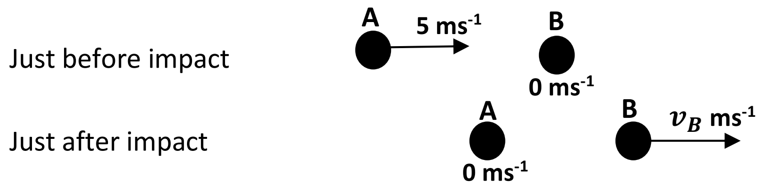
$$v = -0.125\ ms^{-1}$$

Hence the velocity of the coupled trucks is $0.125\ ms^{-1}$ in the direction of the heavier truck before impact.

Example 2

A particle A of mass 5 kg travelling with speed 6 ms^{-1}, collides directly with a stationary particle B of mass 10kg. If A is brought to rest by the impact, find the speed with which B begins to move.

Solution



From the PCM: $\quad m_A u_A + m_B u_B = m_A v_A + m_B v_B$

$$5 \times 6 + 10 \times 0 = 5 \times 0 + 10 \times v_B$$

$$v_B = 3\ ms^{-1}$$

Example 3

A gun of mass 10 kg fires a bullet of mass 90g horizontally at a speed of 500 ms^{-1}.

(a) With what initial speed does the gun recoil?

(b) If the bullet then hit a stationary block of wood of mass 1kg resting on a smooth horizontal surface and remains embedded in it, find the final speed of the block of wood.

Solution

Data: mass of gun **(m_G)** = 10kg, mass of wood **(m_W)** =1kg,
mass of bullet **(m_B)** = 90g = 0.09kg, speed of bullet **(u_B)** = 500 ms^{-1}
Initial speed of wood **(u_G)** = 0,

(a) 1st Case: Gun and Bullet

Taking the direction of the bullet as positive for momentum, we have by PCM,

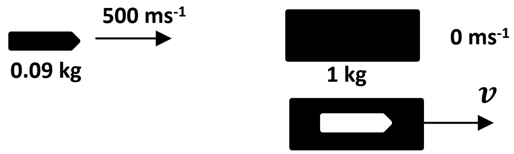
$$0.09 \times 500 + 10 \times (-u_B) = 0$$
$$u_B = \frac{45}{10}\ ms^{-1}$$
$$u_B = 4.5\ ms^{-1}$$

Hence the gun will recoil with an initial speed of $4.5\ ms^{-1}$**.**

(b) 2nd Case: Bullet and wood

Before collision

After collision



Again taking the direction of the bullet as positive momentum, and applying PCM,

$$m_B u_B + m_W u_W = (m_B + m_W)v$$
$$0.09 \times 500 + 1 \times 0 = (0.09 + 1)v$$
$$v = 41.28\ ms^{-1}$$

Hence the speed of the wood is $41.28\ ms^{-1}$**.**

PROBLEM SET OR EXERCISE 10.2

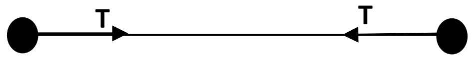
1. ***Two masses of 30 kg and 15kg, travelling at 4 ms^{-1} and 2.5 ms^{-1} respectively in opposite directions, collide inelastically. Find their common speed after collision.***
   (ans 3.5 ms^{-1})
2. ***A gun of mass 500kg fires a shell of mass 4kg horizontally at a speed of 350 ms^{-1}.***
   (a) Find the initail recoild velocity of the gun (ans 2.8 ms^{-1})
   (b) if the gun (moving horizontally) comes to rest in 10s, find the average resisting force. (ans 140 N)
3. ***A spacecraft of mass 450kg is moving in space with a speed of 3 x 10^3 ms^{-1}. A rocket is fired straight ahead by the spacecraft, emmiting 1.5kg of gas at a speed of 2 x 10^4 ms^{-1}. Ignoring the slight redcution in the mass of the spacecraft, find its new speed. (ans 977.67 ms^{-1})***
4. ***A bullet of mass 0.1kg is fired at 80 ms^{-1}, into a stationary block of wood that is free to move on a horizontal smooth plane. The wooden block with the bullet embedded in it then moves off with speed 5 ms^{-1}. Find the mass of the block. (ans 1.5 kg)***

LAW III – ***Action and reaction are always equal and opposite.***

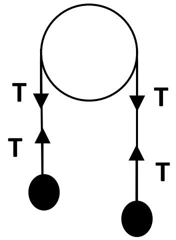
This simply means that if one body A exerts a force on another body B, then B exert an equal force in the opposite direction on A.

This law holds valid whether the two bodies are in contact or are some distance apart, and whether they are moving or are stationary.
If we consider two particles connected by a taut (stretched) string. Though the particles are not in contact but they do exert equal and opposite forces

on each other by means of the equal tensions in the string which act inwards at each end.



The law holds valid also if the string passes round a smooth body, such as a pulley, which will change the direction of the string.



The tensions in the two portions of the string are the same and each portion exerts an inward pull at each end.

All tensions are equal.

WORKED EXAMPLE 2.10

1. ***A rocket of mass 5000 kg carrying 4000 kg of fuel is to be launched vertically. The fuel is consumed at a steady rate of 50 kgs^{-1}, calculate the least velocity of the exhaust gases if the rocked will just lift off the launching pad immediately after firing [g = 10 ms^{-1}](WASSCE Nov/Dec 2010)***

Solution

Data: mass of rocket ***(m_R)*** = 500 kg, mass of fuel ***(m_F)*** = 4000 kg

rate of fuel consumption = 50 kgs^{-1}, g = 10 ms^{-2}

Weight of rocket (**W_R**) = weight of empty rock + weight of fuel

W_R = (m_R + m_F) x g = (5000 + 4000) x 10

W_R = 90,000 N

From Newton's third law,

Downward action = upward reaction

$$90{,}000 = 50 \times \frac{(V-0)}{1s} \implies V = \frac{90{,}000}{50}$$

Hence $\quad V = \mathbf{1.8 \times 10^3\ ms^{-1}}$

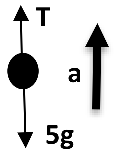
• Connected Bodies

2. ***An inextensible string passes over a smooth fixed pulley and carries particles of masses 5kg and 7kg, one at each end. If the system is moving freely, find in terms of g (gravity)***
   (a) the acceleration of each particle,
   (b) the tension in the string,
   (c) the force exerted on the pulley by the string.

Solution

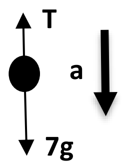
(a) The acceleration will act in the direction of the heavier mass (or weight) 7 kg and against the lighter one 5kg.

Analysing net force for each mass using Newton's 2nd and 3rd law,



5 kg mass: $\quad \sum F_y = T = ma \uparrow$

$$T - 5g = 5a \text{ ---------- eqn 1}$$

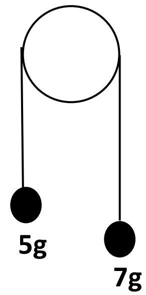


7 kg mass: $\quad \sum F_y = T = ma \downarrow$

$$7g - T = 7a \text{ ---------- eqn 2}$$

Solving eqn 1 and 2 simultaneously,

$$T - 5g = 5a \text{ ---------- eqn 1}$$
$$\underline{7g - T = 7a} \text{ ---------- eqn 2}$$
$$a = \frac{2}{12}g = \mathbf{1.7\ ms^{-2}}$$

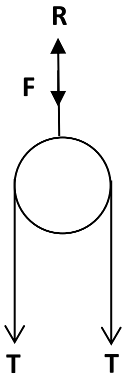


Hence the acceleration of each particle is $\mathbf{1.7\ ms^{-2}}$

(b) The tensions in the string are equal, thus using eqn 1

i.e.
$$T - 5g = 5a$$
$$T - 5 \times 10 = 5 \times 1.7$$
$$T = \mathbf{58.5\ N}$$

Hence Tension in strings is $\mathbf{58.5\ N}$



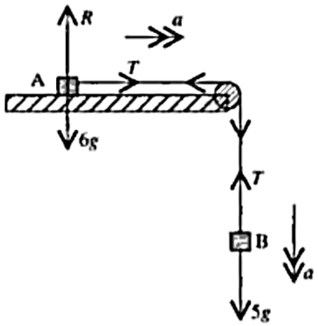
(c) The force $\boldsymbol{F}$ exerted on the pulley equals the reaction on the pulley

i.e.
$$R = F = T + T = 2T$$
$$F = 2 \times 58.5$$
$$F = \mathbf{117\ N}$$

3. ***A small block of mass 6kg rests on a table top and is connected by a light inextensible string that passes over a smooth pulley, fixed on the edge of the table to another small block of mass 5kg which is hanging freely. Find the acceleration of the system and the tension in the string if***
   (a) the table is smooth,
   (b) the table is rough and exerts a frictional force of 2g N.

Solution

Let the blocks be A and B respectively as shown in the diagram.



(a) Analysing each block in the direction of motion,

Block A: → $\sum F_x = T = ma$

$T = 6a$ ------ 1

Block B: ↓ $\sum F_y = T = ma$

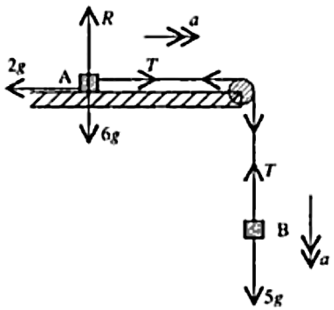
$5g - T = 5a$ ------ 2

[1+2] $5g = 11a$

$\Rightarrow a = \frac{5}{11}g = \mathbf{4.55\ ms^{-2}}$

Tension: from 1 $T = 6 \times 4.55 = \mathbf{27.27\ N}$

(b) In this case there's a frictional resistance to the motion of block A.



Block A: → $\sum F_x = T = ma$

$T - 2g = 6a$ ------ 1

Block B: ↓ $\sum F_y = T = ma$

$5g - T = 5a$ ------ 2

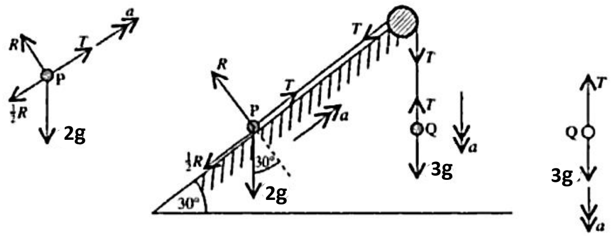
[1+2] $3g = 11a$

$\Rightarrow a = \frac{3}{11}g = \mathbf{2.73\ ms^{-2}}$

Tension: from 1 $T = 6 \times 2.73 + 2 \times 10 = \mathbf{36.38\ N}$

4. ***A particle P of mass 2kg rests on a rough inclined plane at 30º to the horizontal. The frictional force is equal to one-half the normal reaction. P is attached to one end of a light inelastic string which passes over a smooth pulley fixed at the top of the plane and carries a particle of mass 3kg hanging freely at the other end.***
   Find (a) the normal reaction between P and the plane.
   (b) the acceleration of P,
   (c) the force exerted by the string on the pulley. [g = 10 ms-2]

Solution



(a) There is no acceleration acting perpendicular to the plane, so resultant force in that direction is zero. Resolving for P

Perpendicular to plane: ↖ $R - 2g\cos 30 = 0$

$R = \mathbf{10\sqrt{3} = 17.32\ N}$

Hence normal reaction between P and the plane is $\mathbf{17.32\ N}$

F

(b) Acceleration acts up the plane and frictional force is half the normal reaction. **Resolving along plane for:**

P: ↗ $T - \frac{1}{2} \times 10\sqrt{3} - 2g\sin 30 = 2a$

$T - 5\sqrt{3} - 10 = 2a$-------1

Q: ↓ $3g - T = 3a$-------2

[1+2] $3g - 10 - 5\sqrt{3} = 5a \Rightarrow a = \frac{3\times10-10-5\sqrt{3}}{5}$

$\mathbf{a = 2.27\ ms^{-2}}$

Hence acceleration of P is $\mathbf{2.27\ ms^{-2}}$

(c) Two equal tension forces acts on the pulley each making an angle of 30^0 to the resultant force F_R.

From 2: $T = 3g - 3a = 3 \times 10 - 3 \times a = 21.84\ N$

Resolving in the direction of the resultant force:

$$F_R = 2T\cos 30 = 2 \times 21.84 \times \frac{\sqrt{3}}{2} = \mathbf{37.83\ N}$$

Hence force exerted by string is $\mathbf{37.83\ N}$

PROBLEM SET OR EXERCISE 14.0 *(take g = 10 ms^{-2})*

1. ***A rocket of total mass 7000 kg, of which 5000 kg is propellent fuel is to be launched vertically. If the fuel is consumed at a rate of 60 kgs^{-1}. What is the least velocity of the exhaust gases if the rocket will just lift off the launching pad immediately after firing?***
   (2 x 10^3 ms^{-1})
2. ***A man whose mass is 70kg stands on a spring weighing machine inside a lift. When the lift starts to ascend its acceleration is 2.5ms^{-2}. What is the reading of the weighing machine? (ans 875 N)***
   What will it read :
   - ***(a) when the velocity of the lift is uniform; (ans 700 N)***
   - ***(b) as it comes to rest with a retardation of 5.0 ms^{-2}? (ans 350 N)***
3. ***Two particles of masses 4kg and 8kg hang one at each end of a light inextensible string which passes over a fixed smooth pulley.***
   - ***(a) The acceleration of the system when the particles are released from rest. (ans 3.33 ms^{-2})***
   - ***(b) The distance that each particle moves during the first 5 seconds. (ans 41.63 m)***
4. ***A particle of mass 4kg rests on a rough inclined plane at 60^o to the horizontal. The frictional force on the plane is two-third of the normal reaction. The particle is attached to one end of a light inextensible string which passes over a smooth pulley at the top of the plane and carries a particle of mass 2kg at the other end.***
   Find (a) the acceleration of the system; (4.66 ms^{-2})
   (b) the tension in the string; (ans 29.32 N)
   (c) the force exerted by the string on the pulley. (ans 29.32 N)

CIRCULAR MOTION

Circular motion is the motion that describes a body moving in a circular path.

This type of motion repeats itself at regular intervals and can either be uniform (moving with constant magnitudes of velocity and acceleration) or non-uniform (moving with variable magnitudes of velocity and acceleration).

Our focus however according to our syllabus, will be with ***uniform circular motion.***

Uniform Circular Motion (UCM)

Facts to notes

- **As an example of periodic motion, a body is said to uniformly move in a circular path (or a circle) only:**
  - **(a) when it is acted upon by a *force of constant magnitude acting towards the centre of the circle,***
  - **(b) when this force is directed towards the centre of the circle (*centripetal force*).**
  - **(c) when the magnitudes of the acceleration and velocity are constant,**
  - **(d) when the direction of the *instantaneous velocity* is perpendicular to the direction of the centripetal force.**

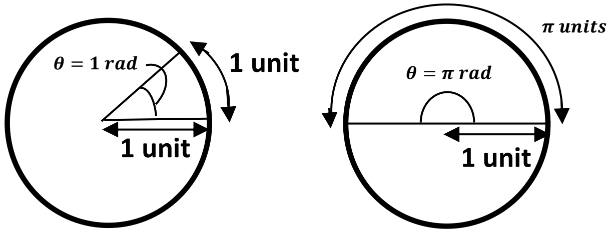
Note that it is only the magnitudes of the acceleration and velocity that are constant. The acceleration and velocity themselves are not constant as their directions are continuously changing.

Describing Uniform Circular Motion by Equations

(a) Mathematical tools for circular motion

- **Radian**

  A radian is the measure of an angle subtended at the centre of a unit circle by a unit arc. This is shown in ***fig (a)*** below



In the fig (b) we see that there are π radians (θ^r = *radian measure*) to half a revolution (θ° = *degree measure*) ***(i.e. 180^0),*** this implies therefore:

That $360^\circ = 2\pi\ rad$

$180^\circ = \pi\ rad$

$90^\circ = {}^{\pi}/_{2}\ rad$

$45^\circ = {}^{\pi}/_{4}\ rad$

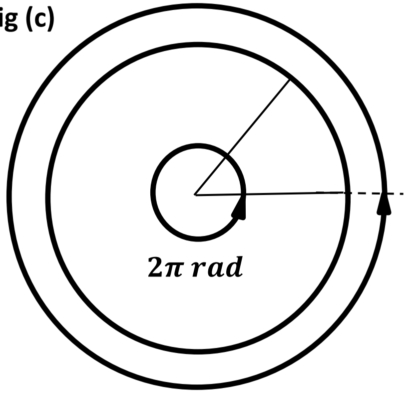
Conversion equation

$$\frac{\theta^\circ}{360^\circ} = \frac{\theta^r}{2\pi}$$

- **Arc Length**

  In fig (c) we can find the length of the arc by using ratios and radian measure. That is

Fig (c)



C $= 2\pi r$

Conversion equation

$$\frac{\theta^r}{2\pi} = \frac{l}{2\pi r} = \frac{\theta^\circ}{360^\circ}$$

Thus,

$$\frac{\theta^r}{2\pi} = \frac{l}{2\pi r} \Longrightarrow \theta^r = \frac{l}{r}$$

Hence: $\boxed{l = r\theta}$ ***where θ must be in radians.***

(b) Equations of Circular Motion

5. Angular Velocity (ω)

$$\omega = \frac{\Delta\theta}{\Delta t}$$

It is measured in $rad\ s^{-1}$

$$\omega = 2\pi f$$

The angular velocity or angular frequency is the rate at which the angle θ is changing with respect to the centre O.

Thus, for a body moving in this circular path this velocity is the ratio of the angular displacement $\Delta\theta$ to that of the time taken Δt. It is constant in the case of UCM.

6. Linear Velocity (v)

$$v = r\omega$$

It is measured in $m\ s^{-1}$

Now from the arc length equation earlier derived, we can derive the equation for the linear velocity of a body undergoing uniform circular motion.

If an object is moving round the perimeter of the circle in fig (c), its linear velocity around the circle will be given by

$$v = \frac{\Delta l}{\Delta t}$$

But $\Delta l = r\Delta\theta$, therefore

$$\frac{\Delta l}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

Hence,

$$v = r\omega$$

7. Angular acceleration *(a)*

$$a = \frac{\Delta\omega}{\Delta t}$$

It is measure in $rad\ s^{-2}$

This is the change in angular velocity with time.

8. Linear or Centripetal Acceleration (a_c)

$$a_c = \frac{v^2}{r} = \omega^2 r$$

It is measure in $m\ s^{-2}$

It is the change in linear velocity with time and it acts towards the centre of the circular path or circle.

Proof

- Consider the body ***A*** in fig (d) moving in the circular path of radius ***r*** around O and from position **P** to position ***Q***.

Fig (d)

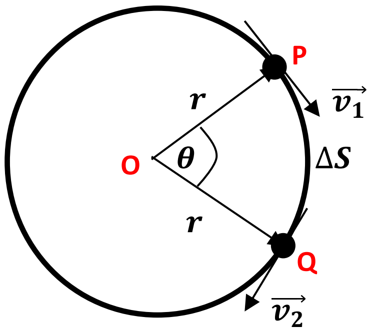
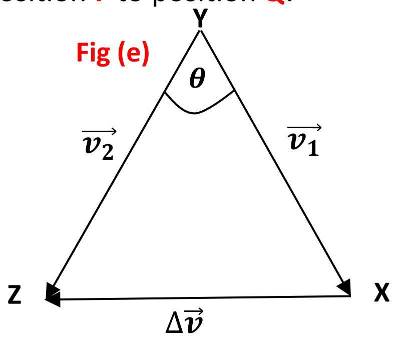


Fig (e)



- Since the direction of the velocity changes, there will be a change in the velocity (i.e. Δv) from **P** to ***Q*** . This is found by vector analysis from the triangle ΔXYZ in fig (e)

Thus: $\Delta\vec{v} = \vec{v_2} - \vec{v_1}$ and similarly $\Delta t = t_2 - t_1$

The resultant acceleration is thus given by $\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$

- Now from geometry, ΔXYZ and ΔPQO are similar, therefore the ratio of their sides must be equal.

i.e. $\frac{OP}{YX} = \frac{PQ}{XZ}$ Or $\frac{r}{V} = \frac{\Delta S}{\Delta\vec{v}}$

Where, $OP = r$
$YX = v_1 = v_2 = v$
$PQ = \Delta S$
$XZ = \Delta\vec{v}$

Thus $\frac{r}{V} = \frac{\Delta S}{\Delta\vec{v}} \implies \Delta\vec{v} = \Delta S\left(\frac{V}{r}\right) = v\Delta t\left(\frac{v}{r}\right)$ where $\Delta S = v\Delta t$

(Note that for very small values of θ, ΔS becomes a straight line)

$\implies \frac{\Delta\vec{v}}{\Delta t} = \frac{v^2}{r}$ Since $\frac{\Delta\vec{v}}{\Delta t} = a_c$ and $v = \omega r$

The centripetal acceleration $a_c = \frac{v^2}{r} = \omega^2 r$

9. Centripetal Force

$$F_c = \frac{mv^2}{r}$$

It is measured in **Newton (N)**

It is the force that tends to pull the moving body towards the centre of the circular path or circle.

Hence with the centripetal acceleration a_c known, the centripetal force can be given by

$$F_c = ma_c = \frac{mv^2}{r}$$

6. Frequency *(f)*

$$f = \frac{1}{T}$$

It is measured in **Henry (H)**

It is the number of oscillations made in 1s.

7. Period *(T)*

$$T = \frac{1}{f} \text{ Or } T = \frac{2\pi}{\omega}$$

It is measured in **seconds (s)**

It is the time taken to make one complete oscillation.

Worked Example A (Horizontal Circle)

1. ***A mass 5kg is attached to the end of a string and whirled in a horizontal circular path at 12.3 ms^{-1}. The radius of the path is 183.4 cm.***
   Determine (a) the acceleration of the mass.
   (b) the angular speed.
   (c) the resultant force acting on the body.
   (d) the displacement after 0.25s at a frequency 2 Hz.

Solution

(a) The acceleration is centripetal $a_c = \frac{v^2}{r} = \frac{12.3^2}{1.834} = 6.7\ ms^{-2}$

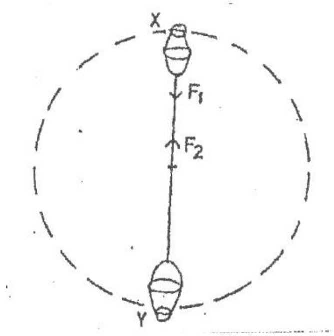
(b) The force is also centripetal $F_c = \frac{mv^2}{r} = 5 \times 6.7 = \mathbf{33.5\ N}$

(c) Displacement is instantaneous $\theta = \omega t = 2\pi f t = 2 \times 3.14 \times 2 \times 0.25$

$\theta = 3.14 \text{ radians} \approx 180^o$

Worked Example B (Vertical Circle)

1. ***A bucket filled with water is tied to a piece of string and whirled in a vertical plane as illustrated in the diagram on the right.***
   (i) Identify the forces F$_1$ and F$_2$.
   (ii) At which position X or Y will the bucket be heavier? Explain.
   (iii) Give the reason why the water does not pour out at X. *(WASSCE Nov/Dec 2013)*



Solution

(i) **F$_1$ = centripetal force**
F$_2$ = reactional/centrifugal force

(ii) **The bucket is heavier at position Y**.
This is due to the direction of the centripetal force acting.

At position X, weight $= \frac{mv^2}{r} - mg$

At position Y, weight $= \frac{mv^2}{r} + mg$

r

(iii) ***This is because the weight acts in opposite direction to the tension and the weight of the water is less than the centrifugal force.***

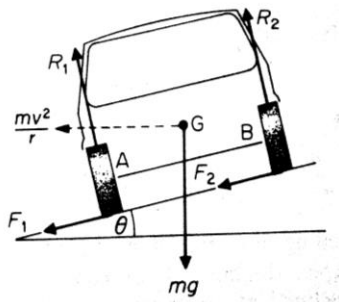
• Banked Tracks (Roads)

Banked roads are mainly constructed to reduce sideways friction especially on curved tracks for vehicles and trains.

- The road is banked because at high speeds automobiles tend to move towards a part of the track which is steeper and sufficient to prevent side-slip or skidding.

Now suppose a car moving round a banked road in a circular path of horizontal radius r (as shown). Provided the reactions at the wheels are normal to the slope, the component of the reaction force towards the centre will equal to the centripetal force of the circular path.

i.e. $(R_1 + R_2)\sin\theta = \frac{mv^2}{r}$ ---------- eqn 1



resolving normal to the plane

$(R_1 + R_2)\cos\theta = mg$ ---------- eqn 2

Dividing eqn 1 by 2

$\tan\theta = \frac{v^2}{rg}$ ---------- eqn 3

Thus for a given velocity v and radius r, the angle of inclination of the track for no side-slip must be

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

Hence as the speed increases, the angle θ, increases.

SIMPLE HARMONIC MOTION

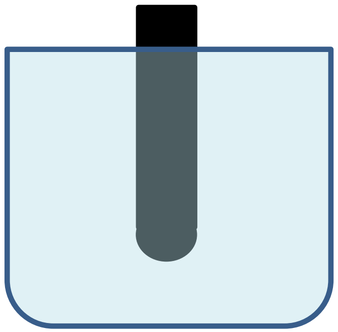
A simple harmonic motion is the motion of a body about a fixed (or equilibrium) position such that its acceleration is directly proportional to its displacement from that position and it is always directed towards that position.

Mathematically

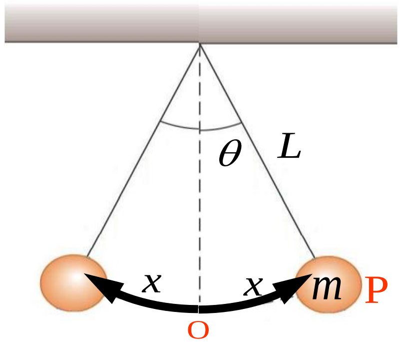
$$a = -\omega^2 x$$

Our syllabus restricts this type motion as an associative to circular motion.

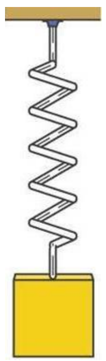
(a) Examples of systems undergoing Simple Harmonic Motion



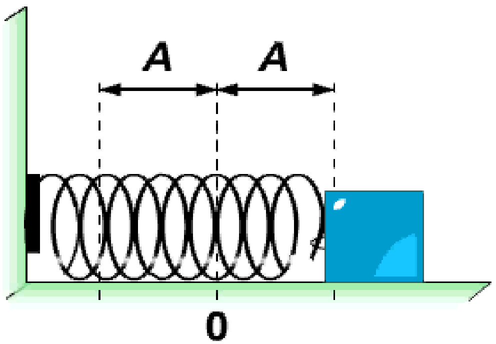
1. Loaded Oscillating test-tube



3. Simple Pendulum



2. Vertical spiral mass-spring oscillation



4. Horizontal spiral mass-spring oscillation

(b) Facts to Note (Explanation)

A body undergoing SHM repeats its motion at regular intervals with important observations on the following parameters:

- **Position:** The body moves back and forth around a point called the ***equilibrium position***. The maximum distance it moves from this position is called the ***amplitude A.***
- **Velocity:** The velocity of the body is continuously changing. It **slows down (or decreases)** as the body **moves away** from the equilibrium position, it increases as the body **moves towards** the equilibrium position and it is **maximum** at the **equilibrium position.**
- **Acceleration:** The body's acceleration is also continuously changing. The body experiences its **greatest (or maximum) acceleration** when it is at the **amplitude position**. This is because at this position, the body experiences the greatest force exerted by the medium or material used to displace it (i.e. spring, string or rope and liquid).
- **Restoring Force:** This is the force acting on the body in order to always **restore** the body back to its equilibrium position.

(c) Describing Simple Harmonic Motion by Equations

1. Instantaneous Displacement *(X)*

$$X = A\sin(\omega t)$$

It is measured in **metres (m)**

It is the distance covered by the body (or oscillator) in a specified direction at time t.

- **Proof**

  Consider the body moving circularly with an angular velocity ω and a radius of **A** as shown in fig. Let **X** be the displacement as seen by the observer.

From the right-angle triangle,

$$\sin\theta = \frac{X}{A}$$
$$\therefore\; X = A\sin\theta$$
$$\text{But } \omega = \frac{\Delta\theta}{\Delta t}$$
$$\therefore\; \theta = \omega t$$

Hence $\boxed{X = A\sin(\omega t)}$

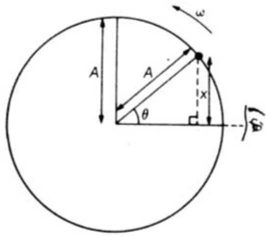


Figure 10.16: The observer is actually a longer distance from the circle than is shown and detects only a vertical shift in position.

Example 1

An object is oscillating upwards and downwards in SHM with a period of 2.0s and an amplitude of 2.0cm. Determine the position of the object 1.3s after displacement.

Solution

Data: amplitude A = 2.0 cm = 0.02 cm, time t = 1.3 s, period T = 2.0 s

Using $X = A\sin(\omega t)$ where $\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi\; rads^{-1}$

$X = 0.02\sin(\pi \times 1.3)$ (put calculator in rad mod)

$\boldsymbol{X = -0.0162\; m}$

Hence the body will be **0.0162 m** or **1.62 cm** below the equilibrium position.

2. Instantaneous Velocity *(v)*

$\boxed{v = \omega A\sin(\omega t)}$ It is measured in **m/s**

It is the change in displacement of the oscillator with respect to time at any instant during the motion.

- **Proof**

  Similarly, consider fig which shows the same body as in fig. $\boldsymbol{v_T}$ represents the tangential velocity of the body in motion and an observer will only see the component $\boldsymbol{v}$ in the diagram

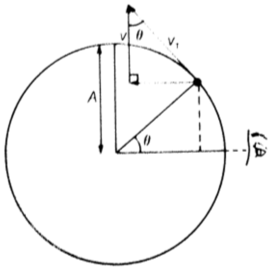


Figure 10.17: The observer is actually a longer distance from the circle than is shown and detects only a vertical change in velocity.

From the triangle of velocities,

$$\cos\theta = \frac{v}{v_T}$$
$$\therefore\; v = v_T\cos\theta$$
$$\text{But } \theta = \omega t \text{ and } v_T = \omega r = \omega A$$

Example 2

Determine velocity of the object in example 1 when the object is 1.3s into its motion.

Solution

Data: amplitude A = 2.0 cm = 0.02 cm, time t = 1.3 s, period T = 2.0 s

Using $V = \omega A\cos(\omega t)$ where $\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi\; rads^{-1}$

$V = \pi \times 0.02 \times \sin(\pi \times 1.3)$ (put calculator in rad mod)

$\boldsymbol{V = -0.0369\; ms^{-1}}$

Hence the body will be moving downwards at **0.0369 ms^{-1}** or **3.69 cms^{-1}**.

3. Instantaneous Acceleration (a)

$\boxed{a = -\omega^2 A\sin(\omega t)}$ It is measured in **m/s^2**

It is the change in instantaneous velocity with respect to time.

- **Proof**

  Consider the diagram in fig, it shows the components of the centripetal acceleration and the acceleration component seen by the observer is $\boldsymbol{a}$.

From the acceleration triangle,

$$\sin\theta = \frac{-a}{a_c}$$
$$\therefore\; a = -a_c\sin\theta$$
$$\text{But } \theta = \omega t \text{ and } v_T = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \frac{A^2\omega^2}{A}$$
$$\therefore\; a_c = A\omega^2$$

$$\boxed{\therefore\; a_c = -\omega^2 A\sin(\omega t)}$$

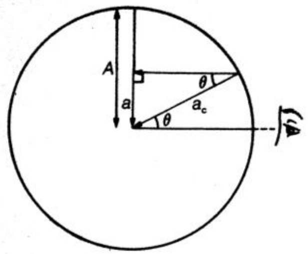


Figure 10.18: The observer is actually a longer distance from the circle than is shown and detects only a vertical change in acceleration.

Example 3

Determine acceleration of the object in example 1 when the object is 1.3s into its motion.

Solution

Data: amplitude A = 2.0 cm = 0.02 cm, time t = 1.3 s, period T = 2.0 s

Using $a = -\omega^2 A\sin(\omega t)$ where $\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi\ rads^{-1}$

$a = -\pi^2 \times 0.02 \times \sin(\pi \times 1.3)$ (put calculator in rad mod)

$a = -0.1591\ ms^{-2}$

Hence the body will be accelerating upwards at **0.1597 ms⁻²** or **15.97 cms⁻²**.

4. Acceleration (a)

$$a = -\omega^2 x$$

It is the change in velocity of the body with time when the body is x units from the equilibrium position

- **Proof**

  This equation is easily obtained by combining $X = A\sin(\omega t)$ and $a_c = -\omega^2 A\sin(\omega t)$

5. Velocity (v)

$$v = \omega\sqrt{A^2 - x^2}$$

SI unit **m/s**

It is the rate of change of displacement.

- **Proof**

  This equation is similar to the second equation ($v^2 = u^2 + 2as$) we covered in uniform linear motion. Now using the equations

  $X = A\sin(\omega t)$ ------- eqn 1

  $v = A\omega\cos(\omega t)$ ------- eqn 2

  ≫ Dividing eqn 2 by ω

  $\frac{v}{\omega} = \frac{A\omega\cos(\omega t)}{\omega} = A\cos(\omega t)$ ---- eqn 3

  ≫ Squaring and adding eqn 1 and eqn 3

  i.e. $X^2 = A^2\sin^2(\omega t)$ and $\frac{v^2}{\omega^2} = A^2\cos(\omega t)$

  Adding and Equating both sides of the equations

  $X^2 + \frac{v^2}{\omega^2} = A^2\sin^2(\omega t) + A^2\cos^2(\omega t)$

  $X^2 + \frac{v^2}{\omega^2} = A^2\{\sin^2(\omega t) + \cos^2(\omega t)\}$

  But from trigonometric identity $\sin^2(\omega t) + \cos^2(\omega t) = 1$

  $\Rightarrow X^2 + \frac{v^2}{\omega^2} = A^2$

  Hence $v = \omega\sqrt{A^2 - x^2}$

6. Energy in SHM (Potential and Kinetic Energy)

Kinetic Energy $\mathrm{K.E} = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega A)^2 = \frac{1}{2}m(2\pi f A)^2$

Elastic or Gravitational Potential Energy $\mathrm{E.P.E} = \frac{1}{2}kx^2$ or $\mathrm{GPE} = mgh$

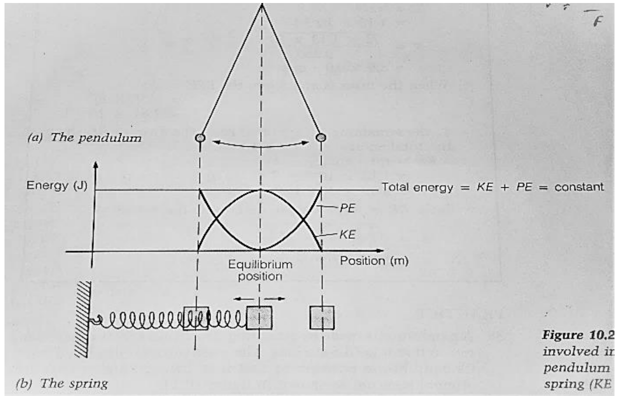
Total Energy or Work Done $= \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ or $+\frac{1}{2}mv^2 + mgh$

The energy of an object or a body undergoing simple harmonic motion is the work done by the body before (GPE/EPE), during (KE) and after (KE+EPE) oscillation.

Facts to Note

- **When a body is undergoing SHM, its velocity changes (i.e. it increases and decreases). This provides a corresponding change in its Kinetic energy.**
- **When the body is pulled beyond its mean or equilibrium position, the energy is stored as elastic potential energy (EPE) if the body is attached to a spring or gravitational potential energy (GPE) if it is attached to a string or thread (simple pendulum).**

- **When the body is released, this energy is stored as kinetic energy. The maximum speed occurs as the body passes the equilibrium position and this point is where the EPE or GPE takes complete transfer into kinetic energy.**



Example 3

A body of mass 20g performs a simple harmonic motion at a frequency of 5Hz. At a distance 10cm from the mean position it velocity is 200 cms^{-1}. Calculate its (i) maximum displacement from the mean position.

(ii) Maximum velocity.

(iii) Maximum potential energy. (g = 10 ms^{-2}, $\pi = 3.14$)

(WASSCE May/June 2016)

Solution

Data: mas (m) = 20 g = 0.02 kg, velocity (V) = 200 cms^{-2} = 2ms^{-2}, g = 10ms^{-2}

distance from mean position (x) = 10 cm = 0.1 m, frequency (f) = 5Hz

(i) Maximum displacement means the amplitude (A) and this can be found by using $v = \omega\sqrt{A^2 - x^2}$

$$v = 2\pi f\sqrt{A^2 - x^2}$$

$$2 = 2 \times 3.14 \times 5\sqrt{A^2 - 0.1^2}$$

$$\Rightarrow A = \sqrt{\frac{2^2}{(2\times3.14\times5)^2} + 0.1^2}$$

Hence $A = 0.12\ m$

(ii) Maximum velocity $v = \omega A = 2\pi f A$

$$= 2 \times 3.14 \times 5 \times 0.12$$

Hence $v = 3.77\ ms^{-1}$

(iii) Maximum Potential energy $P.E = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}m(2\pi f)^2A^2$

$$\Rightarrow P.E = \frac{1}{2} \times 0.02 \times (2 \times 3.14 \times 5)^2 \times 0.12^2$$

Hence $\mathbf{P.E = 1.42 \times 10^{-1}\ J}$

(d) Classical Examples of Simple Harmonic Motion

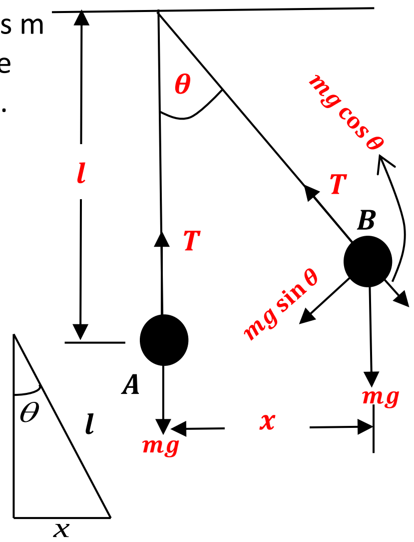
There are numerous examples of SHM in our everyday life but our syllabus limits us to the study of three examples;

1. Simple Pendulum

The simple pendulum is a simple oscillating system of a string with a mass attached to one of its ends and the other held fixed.

It has long been used as an instrument for measuring time in clocks (pendulum clocks). This system is also used to determine the value for acceleration due to gravity in the lab.

Now consider the pendulum of mass m been displaced a distance x from the equilibrium position A at an angle θ.



Resolving for positions A and B

Vertically

A: $T = mg$ **B:** $T = mg\cos\theta$

Horizontally

A: $F = ma$ **B:** $F = -mg\sin\theta$

Thus $ma = -mg\sin\theta$

$$a = -g\sin\theta$$

$$a = -g\left(\frac{x}{l}\right)$$

$$a = -\left(\frac{g}{l}\right)x \quad \textit{(Comparing with } a = -\omega^2 x\textit{)}$$

$$\omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

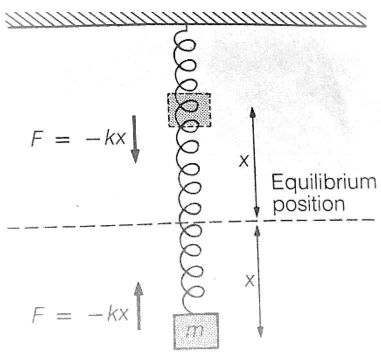
Hence $\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$ *(period of oscillation)*

Rearranging, $\boxed{g = \frac{4\pi^2 l}{T}}$ *(acceleration due to gravity)*

2. Mass on a spiral spring

This system has a mass attached to a spiral spring of known constant k, moving about a fixed or equilibrium point O.

If the mass m attached to a spring of constant k, is extended beyond its equilibrium position and then released, it will oscillate in SHM.



Analysing the elastic forces acting on the mass, it can be found that there is a restoring force F applied and it has a magnitude of $F = -kx$.

Applying Newton's second law, $F = -kx$ and $F = ma$

Thus $ma = -kx$

$$a = -\frac{k}{m}x$$

$$\omega^2 = \frac{k}{m} \quad \textit{(i.e. Comparing with } a = -\omega^2 x\textit{)}$$

$$\text{i.e. } \omega = \sqrt{\frac{k}{m}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

Hence $\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$ *(period of oscillation)*

and $\boxed{k = \frac{4\pi^2 m}{T}}$ *(Spring constant)*

- **The conditions for the spring-mass system to executes SHM**
  1. The **elastic limit of the spring is not exceeded when the spring is being pulled**.
  2. The spring is **light and obeys Hooke's law**.
  3. **No air resistance and surface friction**

3. Loaded oscillating test-tube

This system consists of a loaded test-tube vertically oscillating in a liquid. It is similar to the oscillation of a liquid in a U-tube.

The loaded test-tube is slightly displaced at a distance x below its equilibrium position (see diagram below)

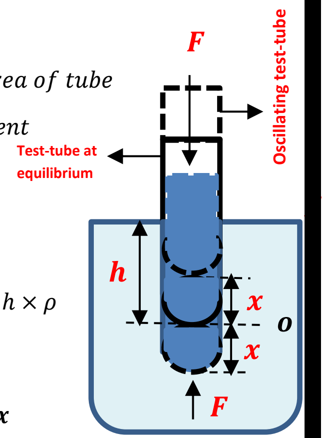
(Upthrust on tube)

$$\textit{Force on liquid} = \textit{pressure} \times \textit{cross} - \textit{sectional area of tube}$$

Where $\textit{pressure} = \textit{liquid density} \times g \times \textit{displacement}$

$$= \rho \times g \times x$$

Thus $F = -\rho \times g \times x \times A$



From Newton's 2nd law, $F = ma = -\rho \times g \times x \times A$

and $m = \textit{mass of liquid in test tube} = V \times \rho = A \times h \times \rho$

Thus $Ah\rho a = -\rho g x A$

$\therefore$ $a = -\frac{g}{h}x$ comparing with $a = -\omega^2 x$

Thus $\omega^2 = \frac{g}{h} \quad \Longrightarrow \quad \frac{2\pi}{T} = \sqrt{\frac{g}{h}}$

Hence $$T = 2\pi\sqrt{\frac{h}{g}}$$ *(Period of oscillation)*

Example 1

The length of a simple pendulum is 75.0 cm and it is released at an angle 8° to the vertical. Calculate

a) the period of the oscillation,

b) the pendulum's bob speed and acceleration when it passes through the lowest point of the swing. (Given g = 9.81 m s⁻²)

Solution

(a) Period of oscillation for a simple pendulum $T = 2\pi\sqrt{\frac{L}{g}} = 2 \times 3.14\sqrt{\frac{0.75}{9.81}}$

Hence $\mathbf{T = 1.74\ s}$

(b) When th bob passes through lowest point it velocity is maximum and is given by $\boldsymbol{v = \omega x}$ Where $(x = l \sin\theta = 0.75 \sin 8 = 0.1\ m)$ is the linear displacement

And $\omega = \frac{2\pi}{T} = \frac{2\times3.14}{1.74} = 3.61\ rads^{-1}$ hence $v = 3.61 \times 0.1 = \mathbf{0.38\ ms^{-1}}$

Example 2

A small mass 0.2 kg is attached to one end of a helical spring and produces an extension 0.15 mm. The mass is now set into vertical oscillation of amplitude 10 mm.

What is (a) the period of oscillation?

(b) the maximum kinetic energy of the mass?

(c) the potential energy of the spring when the mass is 5 mm below the centre of oscillation?

Solution

(a) For a vertical mass-spring, period $T = 2\pi\sqrt{\frac{m}{k}}$ where from Newton's law $ke = mg \quad \Longrightarrow k = \frac{0.2\times10}{0.015} = 133.33\ Nm^{-1}$

Hence $T = 2\pi\sqrt{\frac{0.2}{133.33}} = \mathbf{0.24s}$

(b) Maximum KE is attained at the maximum velocity, i.e. $KE = \frac{1}{2}mv^2{}_{max}$

Where $v_{max} = \omega A \quad \Longrightarrow KE = \frac{1}{2}m\,\omega^2A^2$ and $\omega^2 = \frac{g}{e} = \frac{10}{0.015}$

$$\Longrightarrow KE = \frac{1}{2} \times 0.2 \times \frac{10}{0.015} \times 0.01^2$$

$$\therefore \quad \mathbf{KE = 6.7 \times 10^{-3}\ J}$$

(c) PE at a displacement of 5 mm (or 0.05 m) below equilibrium position is given by $PE = \frac{1}{2}kx^2$ where x is the extension produced from original length, which is a total sum of (15 + 5) mm = 20 mm = 0.02 m

Hence $PE = \frac{1}{2} \times 133.33 \times 0.02^2 = \mathbf{2.7 \times 10^{-2}\ J}$

Example 3

An object of mass 2.1 kg is executing simple harmonic motion, attached to a spring with spring constant k = 280 N m⁻¹. When the object is 0.020 m from its equilibrium position, it is moving with a speed of 0.55 m s⁻¹. Calculate

(a) the amplitude of the motion.

(b) the maximum velocity attained by the object.

Solution

(a) Using $v^2 = \omega^2(A^2 - x^2)$ from $(v = \omega\sqrt{A^2 - x^2})$

$$v^2 = \left(\frac{k}{m}\right)(A^2 - x^2)$$

$$0.55^2 = \frac{280}{2.1}(A^2 - 0.02^2)$$

$$\Rightarrow A = \sqrt{\frac{(0.55)^2\times2.1}{280} + 0.02^2}$$

Hence $\mathbf{A = 0.02\ m}$

(b) $v = \omega A = \frac{k}{m}A = \frac{280}{2.1} \times 0.02 = \mathbf{2.67\ ms^{-1}}$

Practice Problems

1. *An object moving with SHM has an amplitude of 0.02 m and a frequency of 20 Hz.*
   Calculate (i) the period of oscillation (ans 0.05 s)
   (ii) the acceleration at the middle and at the end of an oscillation. (ans 0 ms^{-2} at middle, 31.6 ms^{-2} at end)
   (iii) the velocities at the corresponding instants. (ans 0.25 ms^{-1})

2. *A body of mass 0.2 kg is executing SHM with an amplitude of 20 mm. The maximum force which acts upon it is 0.064 N.*
   Calculate (a) its maximum velocity. (ans 0.08 ms^{-1})
   (b) its period of oscillation. (ans 1.57 s)

3. *A simple pendulum of length 1.5 m has a bob of mass 2.0 kg. If the string is taut and the bob is displaced aside a horizontal distance of 0.15 m from the mean position and released from rest. State the formula for the period of small oscillations and evaluate it in this case. (ans 2.43 s)*
   Determine (a) the kinetic energy and the speed with which it passes the mean position. (ans 0.150 J and 0.387 ms^{-1})
   (b) After 50 complete swings, the maximum horizontal displacement of the bob has become 0.10 m. What fraction of the initial energy has been lost? (ans 0.56)

4. *A mass X of 0.1 kg is attached to the free end of a vertical helical spring whose upper end is fixed and the spring extends by 0.04 m. X is now pulled down a small distance 0.02 m and then released.*
   Find (i) its period. (ans 0.4s)
   (ii) the maximum force acting on it during oscillation. (ans 0.5N)
   (iii) its kinetic energy when X passes through its mean position. (ans 0.005 J)

5. *A mass of 0.1 kg oscillates in simple harmonic motion with its amplitude of 0.2 m and a period of 1.0 s.*
   (a) Calculate its maximum kinetic energy. (ans 7.9 x 10^{-2} J)
   (b) Draw a sketch showing how the mechanical energy varies with the displacement.

MECHANICAL ENERGY

1. Concept of Energy

Energy is the ability or capacity of a body to do work.

- Anything that is able to do work is said to possess energy. Energy is measured in ***Joules (J).*** In mechanics there are basically two kinds of energy: **1. Kinetic (motion) energy**
  2. Potential (position) energy
- Both of which will be discussed later.

(a) Forms of Energy

Energy exists in many different forms and its utilization is of great importance. The table below lists 7 forms of energy

Forms of energy	Comments
Mechanical (KE & PE) energy	Energy of motion and of position.
Heat/Thermal energy	Results from the motion of particles (or molecules) at high temperature.
Chemical energy	Stored in materials such as food, battery and fuel.
Electrical energy	Results from the forces of repulsion or attraction of charged particles.
Light energy	Stored in rays or beams of light produced by sun, light bulb, fire etc.
Sound energy	Stored as waves in the air, allowing us to hear vibrations.
Nuclear energy	Stored in the nuclei of atoms.

(b) World Energy Resources

We categorize our energy resources today based on usability (i.e. whether they are reusable or non-reusable)

(a) **Renewable Energy:** they are energy resources which can be reused after being used. Solar, wind, tides, hydro and ocean waves energy are all examples of renewable energy.

(b) **Non-renewable Energy:** they are energy resources which once used, cannot be reused again. E.g. petroleum, coal, nuclear and biomass.

(c) Conservation of Energy

This simply explains how best we can utilize our energy resources for useful gain and less harmful effects.

Principle of conservation of energy

This states that the total quantity of energy in a body or the universe is constant and can neither be created nor destroyed.

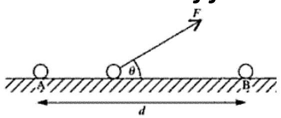
- Albert Einstein made this principle simple by showing that *the mass of a body is a measure of the quantity of energy it contained*. Thus, we now take the view that the sum total of **mass** plus **energy** in the universe is **fixed or constant**.

This principle can be used to explain energy transformations.

2. Work, Energy and Power

These three quantities explain the useful applications of mechanics.

➢ ***Work is said to be done when an applied force acting on a body, moves the body in the direction of the force.***
Work is also a measure of the transference of energy. It is measured in Joules (J).
Mathematically: $\boldsymbol{W = F \times d}$
Work = component of force in direction of motion x distance moved in direction of force



So if as shown above, a constant force moves a body form **A** to **B**, the amount of work done is given by

$$\boldsymbol{W = F\cos\theta \times d}$$

- **Work done in a Gravitational Field**
This means the work done under the influence of gravity. Such work includes: (a) lifting a body and
(b) falling a body.

(a) Work done by a force lifting a body

This work is done against gravity (i.e. the force acts in the upward direction while gravity acts downwards) and it is given by

$$\boldsymbol{W = F \times d = -m \times g \times h}$$

(b) Work done by falling a body

In this case, work is done in favour of gravity (i.e. both the force and gravity act in the same direction).

$$\boldsymbol{W = F \times d = m \times g \times h}$$

➢ ***Energy is the ability or capacity of a body to do work****. It is measured in* ***Joules (J)*** *as well*. This is because ***energy and work are interchangeable***. As mentioned earlier, this chapter will focus mainly on mechanical energy.

- **Types of Mechanical Energy**
  1. **Kinetic Energy:** this is the energy a body possess when in motion. This energy is equal to the amount of **work** needed to bring the body from rest to the velocity **V**.

$$\boldsymbol{KE = \frac{1}{2}mv^2}$$

- **Proof**
Suppose the body of mass $\boldsymbol{m}$ moves from rest and reaches a speed $\boldsymbol{v}$ after moving through a distance $\boldsymbol{s}$ under the action of a constant force $\boldsymbol{F}$.
*The acceleration of the body is given from the equation
$v^2 = u^2 - 2as \quad \Rightarrow a = \frac{v^2-0}{2s} = \frac{v^2}{2s}$
*From newton's law, the force is given by

$$F = ma$$
$$F = \frac{mv^2}{2s}$$
$$\Rightarrow F \times s = \frac{1}{2}mv^2$$

Hence $\boldsymbol{KE = workdone = \frac{1}{2}mv^2}$

Note: that both $\boldsymbol{m}$ and $\boldsymbol{v^2}$ are always positive, showing that **KE** is always positive and it does not depend on the direction of motion (i.e. it is a scalar quantity)

- **Work-Energy Theorem (The Principle of Work and Energy)**

The link between energy and work can be expressed more precisely by this theorem or principle which states that:

The work done by external forces acting on a body is equal to the change in mechanical energy of the body.

Mathematically:

$$\textbf{work done} = \textbf{Final ME} - \textbf{Initial ME}$$

➢ ***If the external forces acts in the direction that helps promote the motion of the body, the mechanical energy increases. Whereas opposing external forces cause a decrease in mechanical energy.***

2. **Potential Energy:** is the energy a body possess by virtue of its position. This energy is equal to the work needed to **raise** or **fall** the body through a vertical height h.

$$\mathbf{PE} = mgh$$

- Proof

Suppose the body of mass m, falls through a distance h.
*Force on the body $F = mg$
*Distance covered $= h$
Thus work done is given by

$$PE = workdone = mgh$$

If the body falls from rest and reaches a velocity v at the bottom then, using $v^2 = u^2 - 2as \implies v^2 = 2gh$ and

$$h = \frac{v^2}{2g}$$

Thus $$PE = mg \times \left(\frac{v^2}{2g}\right) = \frac{1}{2}mv^2$$

Hence kinetic and potential energy are convertible and this is given by

$$\frac{1}{2}mv^2 = mgh \implies \frac{1}{2}v^2 = gh$$

➢ **Power** ***is the rate of energy transfer or the rate at which work is done.*** It is measured in ***Watt (W).***

Mathematically:

$$\text{average power} = \frac{\textbf{work done}}{\textbf{time taken}}$$

One unit of power is produced when work is done at the rate of 1 Joule per second.

- The rate of electrical consumption is kilo watt hour (KWh) = 1000 Wh

Worked Examples

(a) Work

1. ***A body resting in smooth contact with a horizontal plane, moves 3.6 m along the plane under the action of a force of 30 N. find the work done by the force if it is applied (a) horizontally***
(b) at 60º to the plane.

Solution

(a) Horizontally means the whole force acts in the direction of motion i.e. $work\ done = Fd = 30 \times 3.6 = \mathbf{108\ J}$

(b) The component of the force acting in the direction of motion is $\boldsymbol{F}\cos \mathbf{60}$. And the $work\ done = 30\cos 60 \times 3.6 = \mathbf{54\ J}$

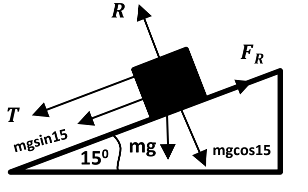
2. ***A wardrobe is lowered by a rope at a steady speed from the balcony of a fifth-floor flat to the ground, 12 m below. Given that the mass of the wardrobe is 37 kg, find the work done by the rope during descent.***

Solution

The Tensional force on the rope acts upwards (against gravity)
thus $work\ done = -mgh = -37 \times 10 \times 12 = \mathbf{-4440\ J}$

3. ***A crate of mass 40kg is pulled by a rope at constant speed 1.5 m/s down a rough slope inclined at 15º to the horizontal. If the coefficient of frictional force is 0.7.***
Find
(a) the frictional force
(b) the tension in the rope
(c) the work done by the rope per seconds
(d) the work done by gravity while the crate moves down the slope for 6 s.

Solution



(a) frictional force $F_R = \mu R = \mu mg \cos 15 = 0.7 \times 40 \times 10 \times \cos 15$

$\boldsymbol{F_R = 270.44 \approx 270}$ **N**

(b) Since the crate is moving with constant speed in the direction of motion, then it is not accelerating.

Resolving along slope: $T + mg \sin 15 - F_R = 0$

$\Rightarrow T = 270 - 40 \times 10 \times \sin 15 = \boldsymbol{166.5}$ **N**

(c) At steady speed, the pulling force equals the tensional force on the rope and for every one second the crate moves 1.5 m. Hence

$work\ done\ in\ 1\ s = 166.5 \times 1.5 = \boldsymbol{249.75 \approx 250}$ **J**

(d) For 6s, distance travelled down slope $= 1.5 \times 6 = 9\ m$

Thus work done by gravity $= mg \times h$ where $h = 9\ sin\ 15$

$= 40 \times 10 \times 9\ sin\ 15$

$=$ **920 J**

4. ***A stone falls vertically through a tank of viscous oil. The speed of the stone as it enters the oil is 2 m/s and at the bottom of the tank is 3 m/s. Given that the oil is of depth 2.4 m, find the resistance force that it exerts on the stone whose mass is 4 kg. [take g = 10 m/s²]***

Solution

Work done by resistance force $= change\ in\ ME = Initial\ ME - Final\ ME$

$Initial\ ME = \frac{1}{2}mu^2 + mgh = \frac{1}{2} \times 4 \times 2^2 + 4 \times 10 \times 2.4 = 104\ J$

$Final\ ME = \frac{1}{2}mu^2 + 0 = \frac{1}{2} \times 4 \times 3^2 = 18\ J$

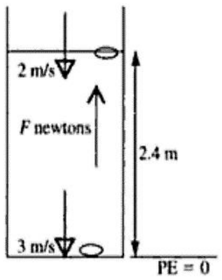
(Since $h = 0$ at the bottom of tank PE $= 0$)

Hence $change\ in\ ME = 104 - 18 = 86$ J

Work done $= F \times d = Final\ ME$

$\Rightarrow F = \frac{86}{2.4} = \boldsymbol{35.83}$ **N**

Therefore the resistance exerted on stone is $\boldsymbol{35.83}$ **N**



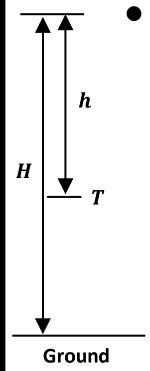
(b) Energy

➢ **Potential energy | Kinetic energy | Principle of work and energy**

1. ***A ball of mass M raised to a height H above the ground is allowed to fall freely towards the ground. Show that at a point T when it has fallen through a distance h, the total mechanical energy remains unchanged. (WASSCE Nov/ Dec 2011)***

Solution

At the top H, total energy is potential $= \boldsymbol{mgH}$



- As the ball passes point T, total mechanical energy is both potential and kinetic $= PE + KE = mg(H - h) + \frac{1}{2}mv^2$

Assuming the ball passes T, velocity v from rest at H,

Using $v^2 = u^2 + 2gh$ hence $v^2 = 2gh$

Therefore at T, total mechanical energy $= mg(H - h) + \frac{1}{2}m2gh$

$= mgH - mgh + mgh$

$= \boldsymbol{mgH}$

Since total mechanical energy at the top H is the same as that at T, it shows that the total mechanical energy remains unchanged.

2. ***A window cleaner of mass 82 kg climbs up a lather to a second-floor window, 5 m above ground level. Assuming that the window cleaner can be treated as a particle,***

find (a) his potential energy relative to the ground.

He then descends 4 m to a first-floor window.

(b) find how much potential energy he has lost. [g = 10 m/s²]

Solution

(a) At the second floor, A, $PE = mgh$

Thus $PE = 82 \times 10 \times 5 =$ **4100 J**

(b) As he descends to the first-floor at B,

the reduction in height is 4 m

Hence loss in $PE = 82 \times 10 \times 4 =$ **3280 J**

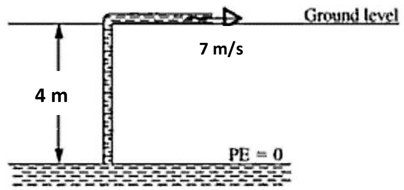
3. ***Water is being raised by a pump from a storage tank 4 m below ground level and ejected at ground level through a pipe at 7 m/s. if the water is delivered at the rate of 520 kg each second,***

(a) Draw a diagram for the arrangement.

(b) Find the total mechanical energy supplied by the pump in one second in lifting and ejecting the water. [g = 10 m/s²]

Solution

(a)



(b) Total mechanical energy = KE gained by water + PE gained by water

Thus in 1 s $KE\ gained\ by\ water = mgh$

$$= 520 \times 10 \times 4 = 20800\,J$$

$$PE\ gained\ by\ water = \frac{1}{2}mv^2$$
$$= \frac{1}{2} \times 520 \times 7 = 1820\,J$$

The total energy gained by the pump is then supplied by the pump.
Hence to ME supplied by the pump = $20800 + 1820 = \mathbf{22620\,J}$

4. *A body of mass 50 kg is raised to a height of 2 m above the ground. (i) What is its potential energy?*
If the body is allowed to fall. Find its kinetic energy:
(ii) when half-way down.
(iii) just before impact with the ground.
What has become of the original energy when the body has come to rest?

Solution

(i) $\mathbf{PE}\ at\ height\ 2\,m = 50 \times 10 \times 2 = \mathbf{1000\ J}$

(ii) When half-way down, the $KE\ gained = \frac{1}{2}mv^2$

But $v^2 = 2gh$ *(from* $v^2 = u^2 + 2gh$*) and* $h = \frac{1}{2} \times 2 = 1\,m$

Hence $KE = \frac{1}{2}mv^2 = \frac{1}{2}m \times 2gh = 50 \times 10 \times 1 = \mathbf{500\ J}$

(ii) When the body comes to rest, it looses its kinetic energy and gains potential energy relative to the 2 m height.
i.e. $\mathbf{PE} = 50 \times 10 \times 2 = \mathbf{1000\ J}$
Therefore the original energy remains the same when the body comes to rest.

➢ **Power**

1. *A pump is used to raise water from a depth of 20 m to fill a reservoir of volume 1800 m³ in 5 hours. Calculate the power of the pump. (Density of water = 1000 kg m⁻³; g = 10 ms⁻²)* *(WASSCE)*

Solution

The $power\ of\ the\ pump = \frac{work\ done\ by\ pump}{time\ taken} = \frac{PE\ at\ depth}{time\ taken}$

$PE\ at\ depth = mgh$ but $m = \rho \times V = 1000 \times 1800 = 1800000$ kg

$$= 18000000 \times 10 \times 20$$
$$= 360{,}000{,}000\,J$$

$$Thus\ power\ of\ pump = \frac{360{,}000{,}000}{5 \times 60 \times 60} = \mathbf{20{,}000\ W}$$

2. *In a certain house, three ceiling fans each of 80 W, an air-conditioner rated 1500 W are operated for 6 hours each day. Seven lamps each rated 40 W are switched on for 10 hours each day. The home theatre 100 W and television set 80 W are also operated each day for 5 hours after which they are kept in the standby mode. In the standby mode, the power consumption is 5% of the power rating. Calculate the:*
(a) Total energy consumed in the house for 30 days in kWh;
(b) Cost of operating all the appliances for 30 days at N10.00 per kWh. (WASSCE Nov/Dec 2012)

Solution

In order to analyse the problem we need to tabulate our given data as shown below

Appliance	Quantity	Power rating / W	Standby power / W	Total Watt	KW	KWh
Ceiling fan	3	80	-	240	0.24	1.44
Air conditional	1	1500	-	1500	1.50	1.50
Lamps	7	40	-	280	0.28	2.80
Home theatre	1	100	5	100 5	0.100 0.005	0.5 0.095
Television set	1	80	4	80 4	0.080 0.004	0.4 0.076
Total power consumed per day						14.311

Note: standby mode was for 19 hours (i.e 24 -5 = 19 hours)

(i) Energy consumed per day = 14.311 kWh
Energy consumed in 30 days = 14.311 x 30
= **429.33 kWh**

(ii) cost of operating per kWh = N 10.00
Cost of operating for 429.33 kWh = 429.33 x 20
= **N 4293.30**

Problem sets or exercise

1. *A body of mass 0.5 kg is thrown vertically upwards with a speed of 25ms^{-1}. Calculate the potential energy of the body at the maximum height reached.[g =10 m/s^2] (WASSCE) (ans 156.25 J)*

2. *A machine drives a conveyor belt which lifts 100 bottles per minutes through a vertical height of 2 m and pushes them forward with a speed of 3 ms^{-1}. The mass of each bottle is 1.2 kg. Calculate the amount of work done per second by the machine. (ans 49 J)*

3. *A constant force acts on a body of mass 2 kg and does 45 J of work. The effect on the body is that its final velocity is 2 ms^{-1} more that its initial velocity. Find the initial velocity of the body. (ans 10.25 ms-1)*

4. *A fire hose delivers water horizontally at a speed of 20 m/s through a nozzle of cross-sectional area 10 cm2. Find the power of the pump if it is only 70% efficient. (ans 5.7 kW)*

5. *A train of total mass 3000 kg is moving up an inclined plane of angle to the horizontal. If the resistance to motion is 3000 N and the train is accelerating at 0.2 ms-2,*
   Find (a) the driving force of the engine. (ans 78 000 N)
   (b) the power exerted at the moment when the speed is 10 m/s. (ans 780 kW)

6. *A bullet of mass 12 g strikes a solid surface at a speed of 400 m/s. If the bullet penetrates to a depth of 3 cm, calculate the average net force on the bullet while it is being brought to rest. (ans 32000 N)*

7. *A body of mass is projected up a board inclined at 30^o to the horizontal with an initial velocity of 6 m/s. If the frictional force opposing its motion is 4.5 N,*
   Find (a) the distance it travels before coming to rest. (ans 3.1 m)
   (b) its increase in potential energy at the end of the run. (ans 76 J)

MACHINES

(APPLICATIONS OF MECHANICAL ENERGY)

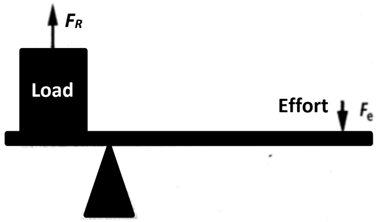
A machine is a device in which an effort applied at one end is used to overcome a greater load/resistance at another point.
Simply, a machine makes work easier.

Machines are divided into two categories:
Simple Machines: are simple systems designed to perform one specific task. (e.g. levers, pulleys, screw, wedge, inclined plane etc)

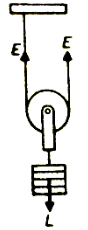
Compound Machines: consists of two or more simple machines linked so that the resistance force of one machine becomes the effort of the other. (e.g. bicycle, car etc)

- **Examples of machines**

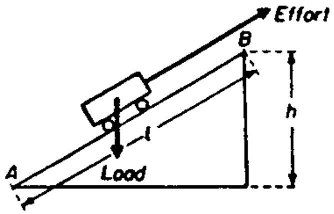
The syllabus limits our study of machines to 7 different examples:



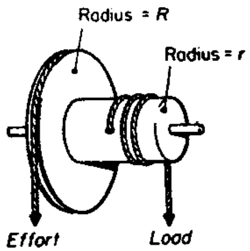
1. The lever



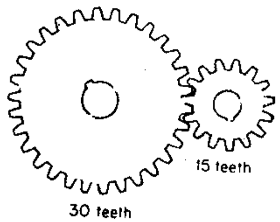
2. Pulley



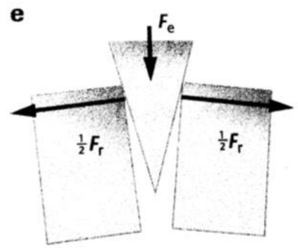
3. Inclined plane



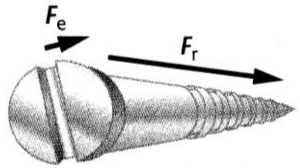
4. Wheel and axle



5. Gears



6. Wedge



7. Screw

- **Basic Machine Relationships**

Machines operate under mechanical principles and these principles form mathematical relationships used to solve problems.

(a) Mechanical Advantage (M.A.)

***The mechanical advantage of a machine is defined as the ratio of the load to the effort applied*. It has no unit.**

$$M.A. = \frac{Load}{Effort}$$

- Machines used to overcome a load much greater than the effort have their M.A. greater than 1. (e.g. a spanner or a screw jack)
- Machines with M.A. less than 1 on the other hand, have the effort greater than the load.(e.g. bicycle)

(b) Velocity Ratio (V.R.)

***This is defined as the ratio of the distance moved by the effort to the distance moved by the load in the same time.* It has no unit as well.**

$$V.R. = \frac{distance\ moved\ by\ effort}{distance\ moved\ by\ load}$$

(c) Efficiency (work done by a machine)

This is the ratio of the work done by the machine to the ratio of the total work put into the machine. It is measured in percentage (%)

$$efficiency = \frac{work\ output}{work\ input} \times 100\% \ \ or\ \ efficiency = \frac{M.A.}{V.R.} \times 100\%$$

- **Relationship between mechanical advantage, velocity ratio, and efficiency | Proof**

Since

$$Work = force\ x\ distance$$

It means also that

$$efficiency = \frac{work\ output}{work\ input} \times 100\% = \frac{load \times distance\ moved\ by\ load}{effort \times distance\ moved\ by\ effort} \times 100\%$$

$$= M.A. \times \frac{1}{velocity\ ratio} \times 100\%$$

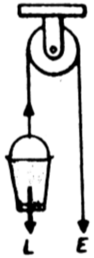
Hence $efficiency = \frac{M.A.}{V.R.} \times 100\%$

1. Lever

- A lever machine uses the principle of moment to overcome its load.
- An effort applied at one point on the lever is used to overcome a load at some other point. This is known as the lever principle and it has many applications
- Its mechanical advantage $M.A. = \frac{Load}{Effort}$

2. Pulleys

- A pulley is a wheel with a grooved rim and there may be several of these mounted in a framework called block.
- The effort is applied to a rope which passes through the pulley.



(a) The single fixed pulley

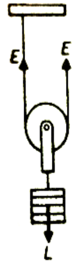
This type of pulley has its grooved rim fixed.

- The tension is the same throughout the rope, and so neglecting the weight of the rope and any friction in the pulley bearings, we have

$$Load = effort$$

and

Its mechanical advantage $M.A. = \frac{Load}{Effort} = 1$



(b) Single moveable pulley

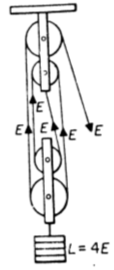
Here, the grooved rim is allowed to move as the effort is being applied.

- The tension in the string or rope equals the effort applied and this effort acts on both sides of the rope and in the same direction.

Thus $load = twice\ the\ effort$

and

Its mechanical advantage $M.A. = \frac{Load}{Effort} = 2$



(c) Block and tackle pulley

In this pulley, two blocks are used containing two to eight pulleys in each, according to the required mechanical advantage required.

- For better understanding, a two-pulley each block system is shown on the left.

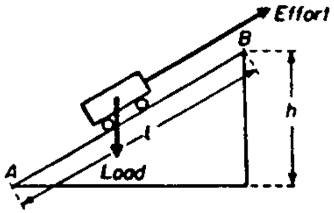
Here, $load = 4 \times effort$

and

Its mechanical advantage $M.A. = \frac{Load}{Effort} = 4$

3. Inclined plane

As simple as it looks, the inclined has been one of the oldest machines in history. It was used in the construction of the Egyptian pyramids.



As shown, a load is being pulled from **A** to **B**, a distance ***l*** of the slope and height ***h*** of the inclined plane.

Velocity ratio $V.R. = \frac{distance\ moved\ by\ effort}{distance\ moved\ by\ load} = \frac{load\ of\ plane}{height\ of\ plane} = \frac{l}{h}$

To find its M.A., we apply the law of conservation of energy (work-energy principle) i.e.

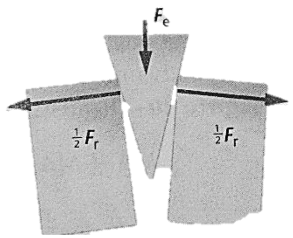
$$work\ done\ on\ load\ = work\ done\ by\ effort$$
$$load \times\ distance\ moved\ by\ load\ = effort\ \times distance\ moved\ by\ effort$$

Therefore, $M.A. = \frac{distance\ moved\ by\ effort}{distance\ moved\ by\ load} = \frac{load\ of\ plane}{height\ of\ plane} = \frac{l}{h}$

Hence $M.A. = V.R.$

4. Wedge

A wedge is a machine which uses its effort or an applied force to separate the load on which it acts ***equally*** *apart*. The work done by the wedge or effort produces two-halves of the reaction force or load.



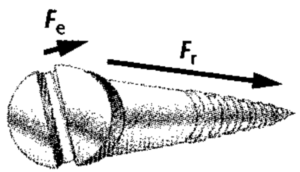
Using the work-energy principle,

$$effort = \left(\frac{1}{2}+\frac{1}{2}\right)\ of\ load = load$$

Hence its mechanical advantage $\mathbf{M.A.} = \frac{Load}{Effort} = \mathbf{1}$

5. Screw

The screw is used for holding together parts or joints. Applications of the screw includes an engineer's vice and a car lifting screw jack. It has wound helical ***threads*** and the distance between each thread is called the ***pitch.***



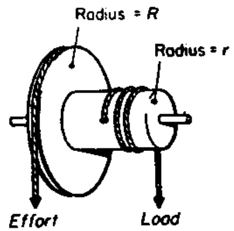
By applying the work-energy principle,

$$work\ done\ on\ load\ = work\ done\ by\ effort$$
$$load \times\ screw\ pitch = effort\ \times circumference\ of\ circle\ traced\ out\ by\ effort$$

Therefore, $M.A. = \frac{distance\ moved\ by\ effort}{distance\ moved\ by\ load} = \frac{2\pi R}{p}$

6. Wheel and axle

This machine works under the wheel and axle principle which depicts *that for one complete turn of the wheel, the load and effort move through distances equal to the circumferences of the wheel and axle respectively.*



Velocity ratio $V.R. = \frac{distance\ moved\ by\ effort}{distance\ moved\ by\ load}$

$$= \frac{2\pi \times radiu\ of\ wheel}{2\pi \times radius\ of\ axle} = \frac{R}{r}$$

From the work-energy principle

$$load \times\ radius\ of\ axle = effort\ \times radius\ of\ wheel$$

Therefore

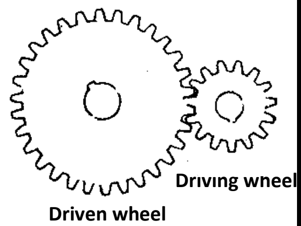
Mechanical advantage $M.A. = \frac{loadt}{effort} = \frac{radius\ of\ wheel}{radius\ of\ axle} = \frac{R}{r}$

Hence $M.A. = V.R.$

Applications of the wheel and axle principle is seen when using a screw driver, a box spanner, a brace, gears etc

7. Gears

Gears operate by the principle of the wheel and axle. The wheels (driven and driving wheel) on the gear have teeth which enable them to perform work more efficiently.



The effort and load are applied to the shaft of the gear and as used before, it can be shown that the velocity ratio

$$V.R. = \frac{number\ of\ teeth\ in\ driven\ wheel}{number\ od\ teeth\ in\ driving\ wheel}$$

➢ **Effects of friction on machines**

1. Causes parts of the machine to wear out.
2. Reduces the efficiency of the machine.
3. Generates heat .

➢ **Reduction of friction in machines**
1. Lubrication of moveable parts.
2. Use of ball bearing and rollers between moving parts.

Worked Examples

1. ***A system of levers with a velocity ratio of 25 overcomes a resistance of 3300 N when an effort of 165 N is applied to it, calculate:***
   (a) The mechanical advantage of the system
   (b) Its efficiency.

Solution

(a) The mechanical advantage for the system

$$M.A = \frac{load}{effort} = \frac{3300}{165} = \mathbf{20}$$

(b) $\boldsymbol{Efficiency} = \frac{M.A.}{V.R.} \times 100\% = \frac{20}{25} \times 100\% = \mathbf{80\%}$

2. ***By using a block and tackle a man can raise a load of 720 N by an effort of 200 N. Find***
   (a) The mechanical advantage of the method
   (b) The man's useful power output if he raises the load through 10 m in 90 s.

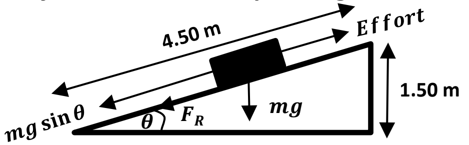
Solution

(a) Mechanical advantage of method $M.A = \frac{load}{effort} = \frac{720}{200} = \mathbf{3.6}$

(b) Power rating $= \frac{work\ done\ by\ load}{time\ taken} = \frac{load \times distance\ moved}{time\ taken} = \frac{720\times10}{90} = \mathbf{80\ W}$

3. ***A man uses rope to haul a packing case of weight 750 N up an inclined wooden plank of effective length 4.50 m and to a platform 1.50 m high. The frictional force between the case and the plank is 200 N. Find***
   (a) the effort he must exert on the rope;
   (b) the velocity ratio;
   (c) the mechanical advantage;
   (d) the useful work done on packing the case.

Solution



(a) if we resolve along the plane, the effort needed will be

$$effort = L\sin\theta + F_R$$
$$effort = 750 \times \frac{1.5}{4.5} + 200$$
$$effort = \mathbf{450\ N}$$

(b) velocity ratio $\boldsymbol{V.R.} = \frac{distance\ moved\ by\ effort}{distance\ moved\ by\ load} = \frac{4.5}{1.5} = \mathbf{3}$

(c) mechanical advantage $\boldsymbol{M.A.} = \frac{Load}{effort} = \frac{750}{450} = \mathbf{1.67}$

(d) useful work done $= load\ \times distance\ moved\ by\ load$
$= 750 \times 1.5$
Hence work done $= \mathbf{1125\ J}$

4. ***A screw jack is used to raise a car of weight 5000 N. The screw is turned by hand using a short steel rod of length 24 cm known as the tommy bar. If the pitch of the screw is 2 mm, calculate the effort applied by the hand on the tommy bar. [take = 3.14]***

Solution

The effort can be found by applying the principle of work

i.e. *work done by effort = work done by load*

effort x circumference of circle traced out by effort = load x screw pitch

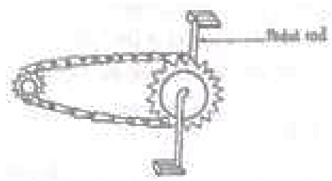
$$\Rightarrow effort\ \times 2\pi L\ = load\ \times pitch$$
$$effort \times 2 \times 3.14 \times 0.24 = 5000 \times 0.002$$

Hence

$$effort = \frac{5000\times0.002}{2\times3.14\times0.24} = \mathbf{6.6\ N}$$

Therefore the effort applied by the is $\mathbf{6.6\ N}$

5. ***The diagram illustrates the gears system of a bicycle.***
   (a) Determine its velocity ratio
   (b) If the bicycle has an efficiency of 90%, calculate the effort required to overcome a load of 70 N.



Why is the calculated effort less than the actual effort required?

Solution

(a) Velocity ratio of gear system $= \frac{number\ of\ teeth\ on\ driven\ wheel}{number\ of\ teeth\ on\ driving\ wheel}$

Thus by counting from the diagram respectively

Velocity ratio $\boldsymbol{V.R.} = \frac{18}{12} = \mathbf{1.5}$

(b) Effort required can be obtained from this relationship

$$efficiency = \frac{M.A.}{V.R.} \times 100\% = \frac{load}{effort} \times \frac{1}{V.R.} \times 100\%$$

$$\Rightarrow effort = \frac{load}{efficiency} \times \frac{1}{V.R.} \times 100\%$$

$$effort = \frac{70}{90} \times \frac{1}{1.5} \times 100\%$$

$$\therefore\ \boldsymbol{effort = 116.7\ N}$$

The calculated effort is less than the actual effort required because some of the applied effort were used to overcome friction.

Practice problems

1. ***A box of mass 12 kg is pulled up a straight smooth slope inclined at 30° to the horizontal, by a force of 10 N for a distance of 5 m. Calculate (a) the work done; (ans 300 J)***
   (b) the velocity ratio if the height of the plane is 4 m; (ans 1.25)
   (c) the efficiency of the slope. (ans 100% assuming plane is "perfect")

2. ***A driving gear wheel having 25 teeth engages with a second wheel with 100 teeth. A third wheel with 30 teeth on the same shaft as the second, engages with a fourth having 60 teeth. Find***
   (a) the velocity ratio; (ans 8)
   (b) the mechanical advantage of the gear system if its efficiency is 85% . (ans 6.8)

3. ***Give a diagram of a single-string pulley having a velocity ratio of 5. Calculate***
   (i) the efficiency of it if an effort of 1000 N is required to raise a load of 4500 N. (ans 75%)
   (ii) the energy wasted when a mass of 500 kg is lifted through 2 m. (ans 3333 J)

4. ***An electric pump raises 9.1 m^3 of water from a reservoir whose water-level is 4 m below ground level to a storage tank above ground. If the discharge pipe outlet is 32 m above ground level and the operation takes 1 hr.***
   (a) Draw a diagram of the arrangement
   (b) Find the minimum power rating of the pump if its efficiency is 70 %. (density of water = 1000 kg/m^3) (ans 1.3 Kw)

5. ***A nut, threaded on a bolt fixed to a bicycle frame, is tightened by means of a spanner. The perpendicular distance between the axis of the nut and the line of action of the force F applied to the free end of the spanner is 0.15 m. F is also perpendicular to the axis of the nut.***
   (a)
   (i) What magnitude of F produces a couple of 10.5 N m about the axis of the nut? (ans 70 N)
   (ii) Where does the second force making up the couple act?
   (iii) Draw a diagram to illustrate your answer.

   (b) Taking the spanner, bolt and nut as a simple machine, what is the velocity ratio if the pitch of the thread is 1 mm? (ans 943)

   (c) Assuming that the efficiency of the machine is 10%, what tension is applied to the bolt when the couple of 10. 5 N acts? (ans 6601 N)

SUMMARISED USEFUL FACTS IN PHYSICS (Courtesy of SparkCharts)

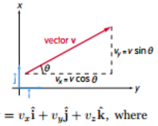
VECTORS

SCALARS AND VECTORS

- A **scalar** quantity (such as mass or energy) can be fully described by a (signed) number with units.
- A **vector** quantity (such as force or velocity) must be described by a number (its magnitude) and direction.

In this chart, vectors are bold: **v**; scalars are italicized: v.

VECTORS IN CARTE-

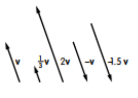


The vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the **unit vectors** (vectors of length 1) in the x-, y-, and z-directions, respectively.

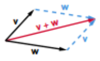
- In Cartesian coordiantes, a vector **v** can be writted as $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, where $v_x\hat{\mathbf{i}}$, $v_y\hat{\mathbf{j}}$, and $v_z\hat{\mathbf{k}}$ are the **components** in the x-, y-, and z-directions, respectively.
- The **magnitude** (or length) of vector **v** is given by
$$v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

OPERATIONS ON VECTORS

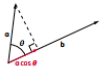
1. **Scalar multiplication**: To multiply a vector by a scalar c (a real number), stretch its length by a factor of c. The vector $-\mathbf{v}$ points in the direction opposite to **v**.



2. **Addition and subtraction**: Add vectors head to tail as in the diagram. This is sometimes called the **parallelogram method**. To subtract **v**, add $-\mathbf{v}$.



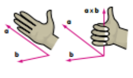
3. **Dot product** (a.k.a. **scalar product**): The dot product of two vectors gives a scalar quantity (a real number):
$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta;$$
θ is the angle between the two vectors.



- If **a** and **b** are **perpendicular**, then $\mathbf{a}\cdot\mathbf{b} = 0$.
- If **a** and **b** are **parallel**, then $|\mathbf{a}\cdot\mathbf{b}| = ab$.
- Component-wise calculation:
$$\mathbf{a}\cdot\mathbf{b} = a_xb_x + a_yb_y + a_zb_z.$$

4. **Cross product**: The cross product $\mathbf{a}\times\mathbf{b}$ of two vectors is a vector perpendicular to both of them with magnitude
$$|\mathbf{a}\times\mathbf{b}| = ab\sin\theta.$$

- To find the direction of $\mathbf{a}\times\mathbf{b}$, use the **right-hand rule**: point the fingers of your right hand in the direction of a; curl them toward b. Your thumb points in the direction of $\mathbf{a}\times\mathbf{b}$.



Right-hand rule

- Order matters: $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.
- If **a** and **b** are **parallel**, then $\mathbf{a}\times\mathbf{b} = 0$.
- If **a** and **b** are **perpendicular**, then $|\mathbf{a}\times\mathbf{b}| = ab$.
- Component-wise calculation:
$$\mathbf{a}\times\mathbf{b} = (a_yb_z - a_zb_y)\,\hat{\mathbf{i}} + (a_zb_x - a_xb_z)\,\hat{\mathbf{j}} + (a_xb_y - a_yb_x)\,\hat{\mathbf{k}}.$$

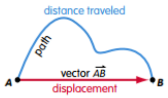
This is the determinant of the 3 × 3matrix
$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \end{vmatrix}.$$

KINEMATICS

Kinematics describes an object's motion.

TERMS AND DEFINITIONS

1. **Displacement** is the change in position of an object. If an object moves from position s_1 to position s_2, then the displacement is $\Delta s = s_2 - s_1$. It is a vector quantity.



2. The **velocity** is the rate of change of position.
   - Average velocity: $v_{\text{avg}} = \frac{\Delta s}{\Delta t}$.
   - Instantaneous velocity: $v(t) = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}$.
3. The **acceleration** is the rate of change of velocity:
   - Average acceleration: $a_{\text{avg}} = \frac{\Delta v}{\Delta t}$
   - Instantaneous acceleration:
$$a(t) = \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

EQUATIONS OF MOTION: CONSTANT a

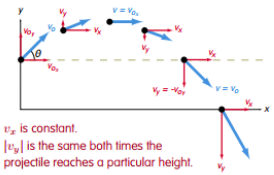
Assume that the acceleration a is constant; s_0 is initial position; v_0 is the initial velocity.

$$v_f = v_0 + at \qquad s = s_0 + v_0t + \tfrac{1}{2}at^2$$
$$v_{\text{avg}} = \tfrac{1}{2}(v_0 + v_f) \qquad = s_0 + v_ft - \tfrac{1}{2}at^2$$
$$v_f^2 = v_0^2 + 2a(s_f - s_0) \qquad = s_0 + v_{\text{avg}}t$$

PROJECTILE MOTION

A **projectile** fired with initial velocity v_0 at angle θ to the ground will trace a parabolic path. If air resistance is negligible, its acceleration is the constant **acceleration due to gravity**, $g = 9.8\,\text{m/s}^2$, directed downward.

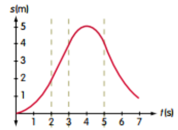
- Horizontal component of velocity is constant:
$$v_x = v_{0x} = v_0\cos\theta.$$
- Vertical component of velocity changes:
$$v_{0y} = v\sin\theta \quad\text{and}\quad v_y = v_{0y} - gt.$$
- After time t, the projectile has traveled
$$\Delta x = v_0t\cos\theta \quad\text{and}\quad \Delta y = v_0t\sin\theta - \tfrac{1}{2}gt^2.$$
- If the projectile is fired from the ground, then the total horizontal distance traveled is $\frac{v_0^2}{g}\sin 2\theta$.



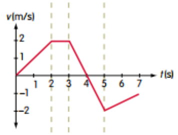
v_x is constant.
$|v_y|$ is the same both times the projectile reaches a particular height.

INTERPRETING GRAPHS

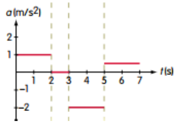
Position vs. time graph
- The **slope** of the graph gives the velocity.



Velociy vs. time graph
- The **slope** of the graph gives the acceleration.
- The (signed) **area** between the graph and the time axis gives the displacement.



Acceleration vs. time graph
- The (signed) **area** between the graph and the time axis gives the change in velocity.



DYNAMICS

Dynamics investigates the cause of an object's motion.

- **Force** is an influence on an object that causes the object to accelerate. Force is measured in Newtons (N), where 1 N of force causes a 1-kg object to accelerate at $1\,\text{m/s}^2$.

NEWTON'S THREE LAWS

1. **First Law:** An object remains in its state of rest or motion with constant velocity unless acted upon by a net external force. (If $\sum F = 0$, then $a = 0$, and v is constant.)
2. **Second Law:** $F_{\text{net}} = ma$.
3. **Third Law:** For every action (i.e., force), there is an equal and opposite reaction ($F_{A\text{ on }B} = -F_{B\text{ on }A}$).

NORMAL FORCE AND FRICTIONAL FORCE

Normal force: The force caused by two bodies in direct contact; perpendicular to the plane of contact.

- The normal force on a mass resting on level ground is its **weight:** $F_N = mg$.
- The normal force on a mass on a plane inclined at θ to the horizonal is $F_N = mg\cos\theta$.

Frictional force: The force between two bodies in direct contact; parallel to the plane of contact and in the opposite direction of the motion of one object relative to the other.

- **Static friction:** The force of friction resisting the relative motion of two bodies at rest in respect to each other.

The maximum force of static friction is given by
$$f_{s,\max} = \mu_sF_N,$$
where μ_s is the **coefficient of static friction,** which depends on the two surfaces.

- **Kinetic friction:** The force of friction resisting the relative motion of two objects in motion with respect to each other. Given by $f_k = \mu_kF_N$, where μ_k is the **coefficient of kinetic friction**.
- For any pair of surfaces, $\mu_k < \mu_s$. (It's harder to push an object from rest than it is to keep it in motion.)

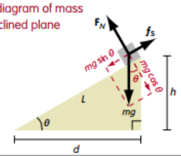
FREE-BODY DIAGRAM ON INCLINED PLANE

A **free-body diagram** shows all the forces acting on an object.

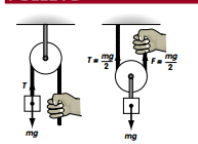
- In the diagram below, the three forces acting on the object at rest on the inclined plane are the force of gravity, the normal force from the plane, and the force of static friction.

Free-body diagram of mass m on an inclined plane

Formulas:
$$\mathbf{F}_N + \mathbf{f}_s + m\mathbf{g} = 0$$
$$F_N = mg\cos\theta$$
$$f_s = mg\sin\theta$$
$$\tan\theta = \tfrac{h}{d}$$
$$\sin\theta = \tfrac{h}{L}$$
$$\cos\theta = \tfrac{d}{L}$$



PULLEYS



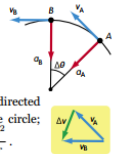
The left pulley is changing the direction of the force (pulling down is easier than up).

The right pulley is halving the amount of force necessary to lift the mass.

UNIFORM CIRCULAR MOTION

An object traveling in a circular path with constant speed experiences **uniform circular motion.**

- Even though the speed v is constant, the velocity **v** changes continually as the direction of motion changes continually. The object experiences **centripetal acceleration,** which is always directed inward toward the center of the circle; its magnitude is given by $a_c = \frac{v^2}{r}$.



- **Centripetal force** produces the centripetal acceleration; it is directed towards the center of the circle with magnitude $F_c = \frac{mv^2}{r}$.

WORK, ENERGY, POWER

WORK

Work is force applied over a distance. It is measured in Joules (J): 1 N of force applied over a distance of 1 m accomplishes 1 J of work. ($1\,\text{J} = 1\,\text{N}{\cdot}\text{m} = 1\,\text{m}^2/\text{s}^2$)

- The work done by force F applied over distance s is $W = Fs$ if $F$and s point in the same direction. In general,
$$W = \mathbf{F}\cdot\mathbf{s} = Fs\cos\theta,$$
where θ is the angle between **F** and **s**.
- If F can vary over the distance, then $W = \int \mathbf{F}\cdot d\mathbf{s}$.

ENERGY

Energy is the ability of a system to do work. Measured in Joules.

- **Kinetic Energy** is the energy of motion, given by
$$KE = \tfrac{1}{2}mv^2.$$
- **Work-Energy Theorem:** Relates kinetic energy and work:
$$W = \Delta KE.$$
- **Potential energy** is the energy "stored" in an object by virtue of its position or circumstance, defined by
$$U_{\text{at }A} - U_{\text{at }B} = -W_{\text{from }A\text{ to }B}.$$
Ex: A rock on a hill has **gravitational potential energy** relative to the ground: it could do work if it rolled down the hill.
Ex: A compressed spring has **elastic potential energy:** it could exert a push if released. See *Oscillations and Simple Harmonic Motion: Springs.*
- **Gravitational potential energy** of mass m at height h:
$$U_g = mgh.$$
- **Mechanical energy:** The total energy is $E = KE + U$.

POWER

Power (P) is the rate of doing work. It is measured in Watts, where 1 Watt = 1 J/s.

- Average power: $P_{\text{avg}} = \frac{\Delta W}{\Delta t}$.
- Instantaneous power: $P = \frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v}$.

CONSERVATION OF ENERGY

A **conservative force** affects an object in the same way regardless of its path of travel. Most forces encountered in introductory courses (e.g., gravity) are conservative, the major exception being friction, a **non-conservative force.**

- **Conservation of energy**: If the only forces acting on a system are conservative, then the total mechanical energy is conserved: $KE_1 + U_1 = KE_2 + U_2$.

CENTER OF MASS, LINEAR MOMENTUM, IMPULSE

CENTER OF MASS

For any object or system of particles there exists a point, called the **center of mass,** which responds to external forces as if the entire mass of the system were concentrated there.

- **Disrete system:** The position vector $\mathbf{R}_{\text{cm}}$ of the center of mass of a system of particles with masses $m_1, \ldots, m_n$ and position vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$, respectively, satisfies
$$M\mathbf{R}_{\text{cm}} = \sum_i m_i\mathbf{r}_i,$$
where $M = \sum_i m_i$ is the total mass.
- **Continuous system:** If dm is a tiny bit of mass at **r**, then
$$M\mathbf{R}_{\text{cm}} = \int \mathbf{r}\,dm,$$
where $M = \int dm$ is again the total mass.
- **Newton's Second Law** for the center of mass:
$$\mathbf{F}_{\text{net}} = M\mathbf{A}_{\text{cm}}.$$

LINEAR MOMENTUM

Linear momentum accounts for both mass and velocity:
$$\mathbf{p} = m\mathbf{v}.$$

- For a system of particles: $\mathbf{P}_{\text{total}} = \sum_i m_i\mathbf{v}_i = M\mathbf{V}_{\text{cm}}$.
- Newton's Second Law restated: $\mathbf{F}_{\text{avg}} = \frac{\Delta\mathbf{p}}{\Delta t}$ or $\mathbf{F} = \frac{d\mathbf{p}}{dt}$.
- Kinetic energy reexpressed: $KE = \frac{p^2}{2m}$.

Law of Conservation of Momentum
When a system experiences no net external force, there is no change in the momentum of the system.

IMPULSE

Impulse is force applied over time; it is also change in momentum.

- For a constant force, $\mathbf{J} = \mathbf{F}\Delta t = \Delta\mathbf{p}$.
- For a force that varies over time, $\mathbf{J} = \int \mathbf{F}\,dt = \Delta\mathbf{p}$.

COLLISIONS

Mass m_1, moving at $\mathbf{v}_1$, collides with mass m_2, moving at $\mathbf{v}_2$. After the collision, the masses move at $\mathbf{v}_1'$ and $\mathbf{v}_2'$, respectively.

- Conservation of momentum (holds for all collisions) gives
$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'.$$
- **Elastic collisions:** Kinetic energy is also conserved:
$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1(v_1')^2 + \tfrac{1}{2}m_2(v_2')^2.$$
The relative velocity of the masses remains constant:
$$\mathbf{v}_2 - \mathbf{v}_1 = -(\mathbf{v}_2' - \mathbf{v}_1').$$
- **Inelastic collisions:** Kinetic energy is not conserved. In a **perfectly inelastic collision**, the masses stick together and move at $\mathbf{v} = \mathbf{V}_{\text{cm}} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1+m_2}$ after the collision.
- **Coefficient of restitution:** $e = \frac{v_2' - v_1'}{v_1 - v_2}$. For perfectly elastic collisions, $e = 1$; for perfectly inelastic collisions, $e = 0$.

ROTATIONAL DYNAMICS

Rotational motion is the motion of any system whose every particle rotates in a circular path about a common axis.

- Let **r** be the position vector from the axis of rotation to some particle (so **r** is perpendicular to the axis). Then $r = |\mathbf{r}|$ is the radius of rotation.

ROTATIONAL KINEMATICS: DEFINITIONS

Radians: A unit of angle measure. Technically unitless.
$$1\text{ revolution} = 2\pi\text{ radians} = 360°$$

Angular displacement θ: The angle swept out by rotational motion. If s is the linear displacement of the particle along the arc of rotation, then $\theta = \frac{s}{r}$.

Angular velocity ω: The rate of change of angular displacement. If v is the linear velocity of the particle tangent to the arc of rotation, then $\omega = \frac{v}{r}$.

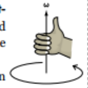
- Average angular velocity: $\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t}$.
- Instantaneous angular velociy: $\omega = \frac{d\theta}{dt}$.

Angular acceleration α: The rate of change of angular velocity. If a_t is the component of the particle's linear acceleration tangent to the arc of rotation, then $\alpha = \frac{a_t}{r}$.

- Average angular velocity: $\alpha_{\text{avg}} = \frac{\Delta\omega}{\Delta t}$.
- Instantaneous angular velociy: $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$.

NOTE: The particle's total linear acceleration **a** can be broken up into components: $\mathbf{a} = \mathbf{a}_c + \mathbf{a}_t$, where $\mathbf{a}_c$ is the **centripetal acceleration,** which does not affect the magnitude of **v**, and $\mathbf{a}_t$ is the **tangential acceleration** related to α.

- **Angular velocity and acceleration as vectors:** It can be convenient to treat ω and α as vector quantities whose directions are perpendicular to the plane of rotation.
- Find the direction of $\vec{\omega}$ using the **right-hand rule:** if the fingers of the right hand curl in the direction of rotation, then the thumb points in the direction of ω.



- Equivalently, $\vec{\omega}$ points in the direction of $\mathbf{r}\times\mathbf{v}$. The equation $\vec{\omega} = \frac{\mathbf{r}\times\mathbf{v}}{r^2}$ gives both the magnitude and the direction of $\vec{\omega}$.

ROTATIONAL KINEMATICS: EQUATIONS

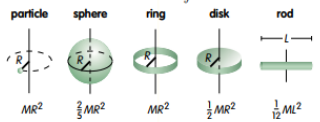
These equations hold if the angular acceleration α is constant.

$$\omega_f = \omega_0 + \alpha t \qquad \theta = \theta_0 + \omega_0t + \tfrac{1}{2}\alpha t^2$$
$$\omega_{\text{avg}} = \tfrac{1}{2}(\omega_0 + \omega_f) \qquad = \theta_0 + \omega_ft - \tfrac{1}{2}\alpha t^2$$
$$\omega_f^2 = \omega_0^2 + 2\alpha(\theta_f - \theta_0) \qquad = \theta_0 + \omega_{\text{avg}}t$$

ROTATIONAL DYNAMICS

Moment of inertia is a measure of an object's resistance to change in rotation; it is the rotational analog of mass.

- For a discrete system of masses m_i at distance r_i from the axis of rotation, the moment of inertia is
$$I = \sum_i m_ir_i^2.$$
- For a continuous system, $I = \int r^2\,dm$.



Torque is the rotational analog of force.

- A force **F** applied at a distance **r** from the axis produces torque
$$\tau = rF\sin\theta,$$
where θ is the angle between **F** and **r**.
- Torque may be **clockwise** or **counterclockwise**. Keep track of the direction by using the vector definition of torque:
$$\vec{\tau} = \mathbf{r}\times\mathbf{F}.$$
- Analog of Newton's second law: $\tau_{\text{net}} = I\alpha$.

Angular momentum is the rotational analog of momentum.

- A particle moving with linear momentum **p** at distance **r** away from the pivot has angular momentum
$$L = rmv\sin\theta \quad\text{and}\quad \mathbf{L} = \mathbf{r}\times\mathbf{p},$$
where θ is the angle between **v** and **r**.
- For a rigid body, $\mathbf{L} = I\vec{\omega}$.
- Analog of Newton's Second Law: $\vec{\tau}_{\text{net}} = \frac{d\mathbf{L}}{dt}$.
- **Conservation of angular momentum:** If no net external torque acts on a system, the total angular momentum of the system remains constant.

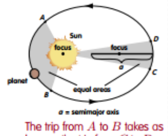
More rotational analogs:

- **Kinetic energy:** $KE_{\text{rot}} = \frac{1}{2}I\omega^2$.
The total kinetic energy of a cylindrical object of radius r rolling (without slipping) with angular velocity ω is
$$KE_{\text{tot}} = \tfrac{1}{2}m\omega^2r^2 + \tfrac{1}{2}I\omega^2.$$
- **Work:** $W = \tau\theta$ or $W = \int \tau\,d\theta$.
- **Power:** $P = \tau\omega$.

GRAVITY

KEPLER'S LAWS

1. **First Law**: Planets revolve around the Sun in elliptical paths with the Sun at one focus.
2. **Second Law**: The segment joining the planet and the Sun sweeps out equal areas in equal time intervals.
3. **Third Law**: The square of the period of revolution (T) is proportional to the cube of the orbit's semimajoir axis a: $T^2 = \frac{4\pi^2a^3}{GM}$.



The trip from A to B takes as long as the trip from C to D.

Here a is the semimajor axis of the ellipse of revolution, M is the mass of the Sun, and $G = 6.67\times10^{-11}\ \text{N}{\cdot}\text{m}^2/\text{kg}^2$ is the **universal gravitational constant.**

NEWTON'S LAW OF UNIVERSAL GRAVITATION

Any two objects of mass m_1 and m_2 attract each other with force
$$F = G\frac{m_1m_2}{r^2},$$
where r is the distance between them (their centers of mass).

- Near the Earth, this reduces to the equation for weight: $F_W = mg$, where $g = \frac{GM_{\text{Earth}}}{R^2_{\text{Earth}}}$ is the acceleration due to gravity.

GRAVITATIONAL POTENTIAL ENERGY

Gravitational potential energy of mass m with respect to mass M measures the work done by gravity to bring mass m from infinitely far away to its present distance r.
$$U(r) = -\int_r^\infty \mathbf{F}\cdot d\mathbf{r} = -G\frac{Mm}{r}$$

- Near the Earth, this reduces to $U(h) = mgh$.

Escape velocity is the minimum surface speed required to completely escape the gravitational field of a planet. For a planet of mass M and radius r, it is given by
$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}.$$

OSCILLATIONS AND SIMPLE HARMONIC MOTION

DEFINITIONS

An **oscillating system** is a system that always experiences a restoring force acting against the displacement of the system.

- **Amplitude** (A): The maximum displacement of an oscillating system from its equilibrium position.
- **Period** (T): The time it takes for a system to complete one cycle.
- **Frequency** (f or ν): The rate of oscillation, measured in Hertz (Hz), or "cycles per second." Technically, $1\,\text{Hz} = 1/\text{s}$.
- **Angular frequency** (ω): Frequency measured in "radians per second," where 2π radians $= 360°$. The unit of angular frequency is still the Hertz (because, technically, radian measure is unitless). For any oscillation, $\omega = 2\pi f$.

Period, frequency, and **angular frequency,** are related as follows:

$$T = \frac{1}{f} = \frac{2\pi}{\omega}.$$

- **Simple harmonic motion** is any motion that experiences a restoring force proportional to the displacement of the system. It is described by the differential equation

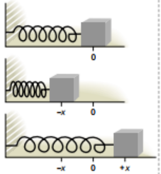
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

SIMPLE HARMONIC MOTION: MASS-SPRING SYSTEM

Each spring has an associated **spring constant** k, which measures how "tight" the spring is.

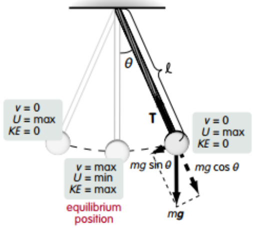
- **Hooke's Law:** The restoring force is given by $F = -kx$, where x is the displacement from equilibrium.
- **Period:** $T = 2\pi\sqrt{\frac{m}{k}}$.
- **Frequency:** $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.
- **Elastic potential energy:**

$$U = \frac{1}{2}kx^2.$$



SIMPLE HARMONIC MOTION: PENDULUM

- **Restoring force:** At angle θ, $F = mg\sin\theta$.
- **Period:** $T = 2\pi\sqrt{\frac{\ell}{g}}$.
- **Frequency:** $f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$.

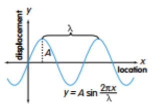


WAVES

A **wave** is a means of transmitting energy through a medium over a distance. The individual particles of the medium do not move very far, but the wave can. The direction in which the energy is transmitted is the **direction of propagation.**

DEFINITIONS

- **Transverse wave:** A type of wave where the medium oscillates in a direction perpendicular to the direction of propagation (**Ex:** pulse on a string; waves on water). A point of maxium displacement in one direction (up) is called a **crest;** in the other direction (down), a **trough.**
  - Transverse waves can either be graphed by plotting displacement versus time in a fixed location, or by plotting displacement versus location at a fixed point in time.



Displacement vs. location graph. Time is fixed.

- **Longitudinal wave:** A type of wave where the medium oscillates in the same direction as the direction of propagation (**Ex:** sound waves).
  - Longitudinal waves are graphed by plotting the density of the medium in place of the displacement. A **compression** is a point of maximum density, and corresponds to a crest. A **rarefraction** is a point of minimum density, and corresponds to a trough.

Also see definitions of **amplitude** (A), **period** (T), **frequency** (f), and **angular frequency** (ω) above.

- **Wavelength** (λ): The distance between any two successive crests or troughs.
- **Wave speed** (v): The speed of energy propagation (not the speed of the individual particles): $v = \frac{\lambda}{T} = \lambda f$.
- **Intensity:** A measure of the energy brought by the wave. Proportional to the square of the amplitude.

WAVE EQUATIONS

- Fixed location x, varying time t: $y(t) = A\sin\omega t = A\sin\left(\frac{2\pi t}{T}\right)$.
- Fixed time t, varying location x: $y(x) = A\sin\left(\frac{2\pi x}{\lambda}\right)$.
- Varying both time t and location x: $y(x,t) = A\sin\left(\omega\left(\frac{x}{v} - t\right)\right) = A\sin\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right)$.

WAVE BEHAVIOR

- **Principle of Superposition:** You can calculate the displacement of a point where two waves meet by adding the displacements of the two individual waves.
- **Interference:** The interaction of two waves according to the principle of superposition.

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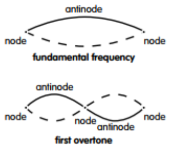
- **Reflection:** When a wave hits a barrier, it will reflect, reversing its direction and orientation (a crest reflects as a trough and vice versa). Some part of a wave will also reflect if the medium through which a wave is traveling changes from less dense to more dense.
- **Refraction:** When a wave encounters a change in medium, part or all of it will continue on in the same general direction as the original wave. The frequency is unchanged in refraction.
- **Diffraction:** The slight bending of a wave around an obstacle.

STANDING WAVES

A **standing wave** is produced by the interference of a wave and its in-synch reflections. Unlike a **traveling wave,** a standing wave does not propagate; at every location along a standing wave, the medium oscillates with a particular amplitude. Standing transverse waves can be produced on a string (**Ex:** any string instrument); standing longitudinal waves can be produced in a hollow tube (**Ex:** any woodwind instrument).

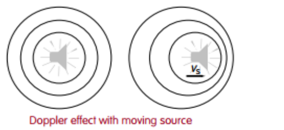
- **Node:** In a standing wave, a point that remains fixed in the equilibrium position. Caused by destructive interference.
- **Antinode:** In a standing wave, a point that oscillates with maximum amplitude. Caused by constructive interference.
- **Fundamental frequency:** The frequency of the standing wave with the longest wavelength that can be produced. Depends on the length of the string or the tube.



DOPPLER EFFECT

When the source of a wave and the observer are not stationary with respect to each other, the frequency and wavelength of the wave as perceived by the observer (f_{eff}, λ_{eff}) are different from those at the source (f, λ). This shift is called the **Doppler effect.**

- For instance, an observer moving toward a source will pass more crests per second than a stationary observer ($f_{\text{eff}} > f$); the distance between successive crests is unchanged ($\lambda_{\text{eff}} = \lambda$); the effective velocity of the wave past the observer is higher ($v_{\text{eff}} > v$).
- **Ex:** Sound: Siren sounds higher-pitched when approaching, lower-pitched when receding. Light: Galaxies moving away from us appear redder than they actually are.



Doppler effect with moving source

WAVES ON A STRING

The behavior of waves on a string depends on the force of tension F_T and the mass density $\mu = \frac{\text{mass}}{\text{length}}$ of the string.

- **Speed:** $v = \sqrt{\frac{F_T}{\mu}}$.
- **Standing waves:** A string of length L fixed can produce standing waves with

  $\lambda_n = \frac{2L}{n}$ and $f_n = nf_1$, where $n = 1, 2, 3, \ldots$.

SOUND WAVES

- **Loudness:** The intensity of a sound wave. Depends on the square of the amplitude of the wave.
- **Pitch:** Determined by the frequency of the wave.
- **Timbre:** The "quality" of a sound; determined by the interference of smaller waves called **overtones** with the main sound wave.
- **Beats:** Two interfering sound waves of different frequencies produce beats—cycles of constructive and destructive intereference between the two waves. The frequency of the beats is given by $f_{\text{beat}} = |f_1 - f_2|$.

DOPPLER EFFECT EQUATIONS

motion of observer	motion of source: stationary	toward observer at velocity v_s	away from observer at velocity v_s
stationary	v λ f	$v_{\text{eff}} = v$ $\lambda_{\text{eff}} = \lambda\left(\frac{v-v_s}{v}\right)$ $f_{\text{eff}} = f\left(\frac{v}{v-v_s}\right)$	$v_{\text{eff}} = v$ $\lambda_{\text{eff}} = \lambda\left(\frac{v+v_s}{v}\right)$ $f_{\text{eff}} = f\left(\frac{v}{v+v_s}\right)$

WAVE EQUATIONS

- Fixed location x, varying time t: $y(t) = A\sin\omega t = A\sin\left(\frac{2\pi t}{T}\right)$.
- Fixed time t, varying location x: $y(x) = A\sin\left(\frac{2\pi x}{\lambda}\right)$.
- Varying both time t and location x: $y(x,t) = A\sin\left(\omega\left(\frac{x}{v} - t\right)\right) = A\sin\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right)$.

WAVE BEHAVIOR

- **Principle of Superposition:** You can calculate the displacement of a point where two waves meet by adding the displacements of the two individual waves.
- **Interference:** The interaction of two waves according to the principle of superposition.

  - **Constructive interference:** Two waves with the same period and amplitude interfere constructively when they meet **in phase** (crest meets crest, trough meets trough) and reinforce each other.
  - **Destructive interference:** Two waves with the same period and amplitude interfere destructively when they meet **out of phase** (crest meets trough) and cancel each other.

DOPPLER EFFECT

When the source of a wave and the observer are not stationary with respect to each other, the frequency and wave-

- **Beats:** Two interfering sound waves of different frequencies produce beats—cycles of constructive and destructive intereference between the two waves. The frequency of the beats is given by $f_{\text{beat}} = |f_1 - f_2|$.

DOPPLER EFFECT EQUATIONS

motion of observer	motion of source: stationary	toward observer at velocity v_s	away from observer at velocity v_s
stationary	v λ f	$v_{\text{eff}} = v$ $\lambda_{\text{eff}} = \lambda\left(\frac{v-v_s}{v}\right)$ $f_{\text{eff}} = f\left(\frac{v}{v-v_s}\right)$	$v_{\text{eff}} = v$ $\lambda_{\text{eff}} = \lambda\left(\frac{v+v_s}{v}\right)$ $f_{\text{eff}} = f\left(\frac{v}{v+v_s}\right)$
toward source at v_o	$v_{\text{eff}} = v + v_o$ $\lambda_{\text{eff}} = \lambda$ $f_{\text{eff}} = f\left(\frac{v+v_o}{v}\right)$	$v_{\text{eff}} = v \pm v_o$ $\lambda_{\text{eff}} = \lambda\left(\frac{v \pm v_s}{v}\right)$ $f_{\text{eff}} = f\left(\frac{v \pm v_o}{v \pm v_s}\right)$	
away from source at v_o	$v_{\text{eff}} = v - v_o$ $\lambda_{\text{eff}} = \lambda$ $f_{\text{eff}} = f\left(\frac{v-v_o}{v}\right)$		

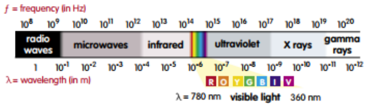
LIGHT WAVES AND OPTICS

ELECTROMAGNETIC WAVES

Light waves are a special case of transverse traveling waves called electromagnetic waves, which are produced by mutually inducing oscillations of electric and magnetic fields. Unlike other waves, they do not need a medium, and can travel in a vacuum at a speed of

$$c = 3.00 \times 10^8\ \text{m/s}.$$

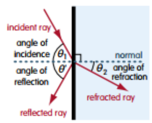
- **Electromagnetic spectrum:** Electromagnetic waves are distinguished by their frequencies (equivalently, their wavelengths). We can list all the different kinds of waves in order.
- The order of colors in the spectrum of visible light can be remembered with the mnemonic **Roy G. Biv.**



REFLECTION AND REFRACTION

At the boundary of one medium with another, part of the incident ray of light will be **reflected,** and part will be **transmitted** but **refracted.**

- All angles (of incidence, reflection, and refraction) are measured from the **normal** (perpendicular) to the boundary surface.
- **Law of reflection:** The angle of reflection equals the angle of incidence.



- **Index of refraction:** Ratio of the speed of light in a vacuum to the speed of light in a medium: $n = \frac{c}{v}$. In general, the denser the substance, the higher the index of refraction.
- **Snell's Law:** If a light ray travels from a medium with index of refraction n_1 at angle of incidence θ_1 into a medium with index of refraction n_2 at angle of refraction θ_2, then

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

- Light passing into a denser medium will bend toward the normal; into a less dense medium, away from the normal.
- **Total internal reflection:** A light ray traveling from a denser into a less dense medium ($n_1 > n_2$) will experience total reflection (no light is transmitted) if the angle of incidence is greater than the **critical angle,** which is given by

$$\theta_c = \arcsin\frac{n_2}{n_1}.$$

DISPERSION

Dispersion is the breaking up of visible light into its component frequencies.

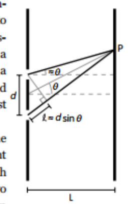
- A **prism** will disperse light because of a slight difference in refraction indices for light of different frequencies:

  $n_{\text{red}} < n_{\text{violet}}$.

DIFFRACTION

Light bends around obstacles slightly; the smaller the aperture, the more noticeable the bending.

- **Young's double-slit experiment** demonstrates the wave-like behavior of light: If light of a single wavelength λ is allowed to pass through two small slits a distance d apart, then the image on a screen a distance L away will be a series of alternating **bright** and **dark fringes,** with the brightest fringe in the middle.



- More precisely, point P on the screen will be the center of a bright fringe if the line connecting P with the point halfway between the two slits and the horizontal make an angle of θ such that $d\sin\theta = n\lambda$, where n is any integer.
- Point P will be the center of a dark fringe if $d\sin\theta = \left(n + \frac{1}{2}\right)\lambda$, where n is again an integer.
- A **single slit** will also produce a bright/dark fringe pattern, though much less pronounced: the central band is larger and brighter; the other bands are less noticeable. The formulas for which points are bright and which are dark are the same; this time, let d be the width of the slit.

OPTICAL INSTRUMENTS: MIRRORS AND LENSES

Lenses and **curved mirrors** are designed to change the direction of light rays in predictable ways because of refraction (lenses) or reflection (mirrors).

- **Convex** mirrors and lenses bulge outward; **concave** ones, like caves, curve inward.
- **Center of curvature** (C): Center of the (approximate) sphere of which the mirror or lens surface is a slice. The radius (r) is called the **radius of curvature.**
- **Principal axis:** Imaginary line running through the center.
- **Vertex:** Intersection of principal axis with mirror or lens.
- **Focal point** (F): Rays of light running parallel to the principal axis will be reflected or refracted through the same focal point. The **focal length** (f) is the distance between the vertex and the focal point. For spherical mirrors, the focal length is half the radius of curvature: $f = \frac{r}{2}$.
- An image is **real** if light rays actually hit its location. Otherwise, the image is **virtual;** it is perceived only.

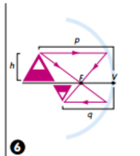
Ray tracing techniques

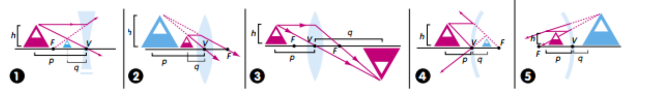
1. Rays running parallel to the principal axis are reflected or refracted toward or away from the focal point (toward F in concave mirrors and convex lenses; away from F in convex mirrors and concave lenses).
2. Conversely, rays running through the focus are reflected or refracted parallel to the principal axis.
3. The normal to the vertex is the principal axis. Rays running through the vertex of a lens do not bend.
4. Concave mirrors and convex lenses use the near focal point; convex mirrors and lenses use the far focal point.
5. Images formed in front of a mirror are real; images formed behind a mirror are virtual. Images formed in front of a lens are virtual; images formed behind are real.

LENSES AND CURVED MIRRORS

Formulas: $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ $\qquad \frac{\text{image size}}{\text{object size}} = -\frac{q}{p}$

Optical instrument	Focal distance f	Image distance q	Type of image
Mirror:			
Concave	positive	$p > f$ positive (same side)	real, inverted 6
		$p < f$ negative (opposite side)	virtual, erect 5
Convex	negative	negative (opposite side)	virtual, erect 4
Lens:			
Convex	positive	$p > f$ positive (opposite side)	real, inverted 3
		$p < f$ negative (same side)	virtual, erect 2
Concave	negative	negative (same side)	virtual, erect 1





THERMODYNAMICS

TERMS AND DEFINITIONS

Temperature measures the average molecular kinetic energy of a system or an object.

Heat is the transfer of thermal energy to a system via thermal contact with a reservoir.

Heat capacity of a substance is the heat energy required to raise the temperature of that substance by 1° Celsius.

- **Heat energy** (Q) is related to the **heat capacity** (C) by the relation $Q = C\Delta T$.

Substances exist in one of three states (**solid**, **liquid**, **gas**). When a substance is undergoing a physical change of state referred to as a **phase change**:

- Solid to liquid: **melting, fusion, liquefaction**
- Liquid to solid: **freezing, solidification**
- Liquid to gas: **vaporization**
- Gas to liquid: **condensation**
- Solid to gas (directly): **sublimation**
- Gas to solid (directly): **deposition**

Entropy (S) is a measure of the disorder of a system.

THREE METHODS OF HEAT TRANSFER

1. **Conduction:** Method of heat transfer through physical contact.
2. **Convection:** Method of heat transfer in a gas or liquid in which hot fluid rises through cooler fluid.
3. **Radiation:** Method of heat transfer that does not need a medium; the heat energy is carried in an electromagnetic wave.

LAWS OF THERMODYNAMICS

0. **Zeroth Law of Thermodynamics:** If two systems are in thermal equilibrium with a third, then they are in thermal equilibrium with each other.
1. **First Law of Thermodynamics:** The change in the internal energy of a system U plus the work done by the system W equals the net heat Q added to the system:
$$Q = \Delta U + W.$$
2. **Second Law of Thermodynamics** (three formulations):
   1. Heat flows spontaneously from a hotter object to a cooler one, but not in the opposite direction.
   2. No machine can work with 100% efficiency: all machines generate heat, some of which is lost to the surroundings.
   3. Any system tends spontaneously towards maximum entropy.

The change in **entropy** is a reversible process defined by
$$\Delta S = \int \frac{dQ_{\text{rev}}}{T}.$$

Carnot theorem: No engine working between two heat reservoirs is more efficient than a reversible engine. The efficiency of a **Carnot engine** is given by $\varepsilon_C = 1 - \frac{T_c}{T_h}$.

GASES

Ideal gas law: $PV = nRT$, where n is the number of moles of the gas, T is the absolute temperature (in Kelvin), and $R = 8.314\ \mathrm{J/(mol \cdot K)}$ is the universal gas constant.

The ideal gas law incorporates the following gas laws (the amount of gas is constant for each one):

- **Charles' Law:** $\frac{P_1}{T_1} = \frac{P_2}{T_2}$ if the volume is constant.
- **Boyle's Law:** $P_1V_1 = P_2V_2$ if the temperature is constant.

Translational kinetic energy for ideal gas:
$$N(KE) = N\left(\tfrac{1}{2}mv^2\right)_{\text{avg}} = \tfrac{3}{2}NkT = \tfrac{3}{2}nRT,$$
where N is the number of molecules and $k = 1.381 \times 10^{-23}$ J/K is Boltzmann's constant.

van der Waals equation for real gases:
$$\left(P + \frac{an^2}{V^2}\right)(V - bn) = nRT$$

Here, b accounts for the correction due the volume of the molecules and a accounts for the attraction of the gas molecules to each other.

ELECTRICITY

ELECTRIC CHARGE

Electric charge is **quantized**—it only comes in whole number multiples of the **fundamental unit of charge**, e, so called because it is the absolute value of the charge of one electron.

Because the fundamental unit charge (e) is extremely small, electric charge is often measured in **Coulombs** (C). 1 C is the amount of charge that passes through a cross section of a wire in 1 s when 1 **ampere** (A) of current is flowing in the wire. (An ampere is a measure of **current**; it is a fundamental unit.)
$$e = 1.602210^{-19}\ \mathrm{C}$$

Law of conservation of charge: Charge cannot be created or destroyed in a system: the sum of all the charges is constant.

Electric charge must be **positive** or **negative**. The charge on an electron is negative.

- Two positive or two negative charges are **like** charges.
- A positive and a negative charge are **unlike** charges.

Coulomb's law: Like charges repel each other, unlike charges attract each other, and this repulsion or attraction varies inversely with the square of the distance.

- The electrical force exerted by charge q_1 on charge q_2 a distance r away is
$$F_{1\text{ on }2} = k\frac{q_1q_2}{r^2},$$
where $k = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$ is **Couloumb's constant.**
- Similarly, q_2 exerts a force on q_1; the two forces are equal in magnitude and opposite in direcion:
$$\mathbf{F}_{1\text{ on }2} = -\mathbf{F}_{2\text{ on }1}.$$
- Sometimes, Coulomb's constant is expressed as $k = \frac{1}{4\pi\varepsilon_0}$, where ε_0 is a "more fundamental" constant called the **permittivity of free space.**

ELECTRIC FIELDS

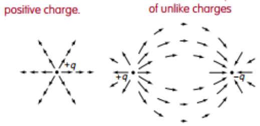
The concept of an **electric field** allows you to keep track of the strength of the electric force on a particle of any charge. If $\mathbf{F}$ is the electric force that a particle with charge q feels at a particular point, the strength of the electric field at that point is given by $\mathbf{E} = \frac{\mathbf{F}}{q}$.

- The electric field is given in units of N/C.
- The direction of the field is always the same as the direction of the electric force experienced by a positive charge.
- Conversely, a particle of charge q at a point where the electric field has strength $\mathbf{E}$ will feel an electric force of $\mathbf{F} = \mathbf{E}q$ at that point.

Electric field due to a point charge: A charge q creates a field of strength $E = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{r^2}$ at distance r away. The field points towards a negative charge and away from a positive charge.

Field lines for a positive charge.

Field lines for a pair of unlike charges



The electric field is stronger when the field lines are closer together.

FLUX AND GAUSS'S LAW

Flux (Φ) measures the number and strength of field lines that go through (flow through) a particular area. The flux through an area A is the product of the area and the magnetic field perpendicular to it:
$$\Phi_E = \mathbf{E} \cdot \mathbf{A} = EA\cos\theta.$$

- The vector $\mathbf{A}$ is perpendicular to the area's surface and has magnitude equal to the area in question; θ is the angle that the field lines make with the area's surface.

Gauss's Law: The relation between the charge Q enclosed in some surface, and the corresponding electric field is given by
$$\Phi_E = \oint_s \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0},$$
where Φ_E is the flux of field lines though the surface.

ELECTRIC POTENTIAL

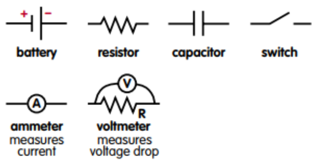
Just as there is a mechanical potential energy, there is an analogous **electrostatic potential energy,** which corresponds to the work required to bring a system of charges from infinity to their final positions. The potential difference and energy are related to the electric field by
$$dV = \frac{dU}{q} = -\mathbf{E} \cdot d\boldsymbol{\ell}.$$

The unit of potential energy is the **Volt** (V).

- This can also be expressed as
$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right).$$

ELECTRIC CURRENT AND CIRCUITS

Symbols used in circuit diagrams



Current

Current (I) is the rate of flow of electric charge through a cross-sectional area. The current is computed as $I = \frac{\Delta Q}{\Delta t}$. Current is measured in amperes, where $1\ \mathrm{A} = 1\mathrm{C/s}$.

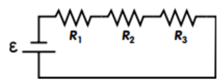
In this chart, the direction of the current corresponds to the direction of positive charge flow, opposite the flow of electrons.

Ohm's Law: The potential difference is proportional to the current:
$$V = IR,$$
where R is the **resistance**, measured in **Ohms** (Ω). $1\,\Omega = 1\,\mathrm{V/A}$.

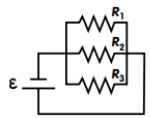
- The resistance of a wire is related to the length L and cross-sectional area A of the current carrying material by
$$R = \rho\frac{L}{A},$$
where ρ is **resistivity,** which depends on the material and is measured in ohm-meters ($\Omega \cdot \mathrm{m}$).

Resistors

- **Combinations of resistors:** Multiple resistors in a circuit may be replaced by a single equivalent resistors R_{eq}.
- **Resistors in series:** $R_{\text{eq}} = R_1 + R_2 + R_3 + \cdots$
- **Resistors in parallel:** $\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$



Resistors in series



Resistors in parallel

The **power** dissipated in a current-carrying segment is given by
$$P = IV = I^2R = \frac{V^2}{R}.$$

The unit for power is the **Watt** (W). $1\ \mathrm{W} = 1\ \mathrm{J/s}$.

Kirchhoff's rules

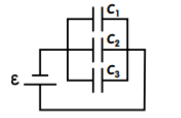
Kirchhoff's rules for circuits in steady state:

- **Loop Rule:** The total change of potential in a closed circuit is zero.
- **Junction Rule:** The total current going into a junction point in a circuit equals the total current coming out of the junction.

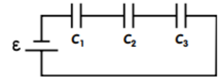
Capacitors

A **capacitor** is a pair of oppositely charged **conductors** separated by an insulator. **Capacitance** is defined as $C = \frac{Q}{V}$, where Q is the magnitude of the total charge on one conductor and V is the potential difference between the conductors. The SI unit of capacitance is the **Farad** (F), where $1\ \mathrm{F} = 1\ \mathrm{C/V}$.

- The **parallel-plate capacitor** consists of two conducting plates, each with area A, separated by a distance d. The capacitance for such a capacitor is $C = \frac{\varepsilon_0 A}{d}$.
- A capacitor stores electrical potential energy given by
$$U = \tfrac{1}{2}CV^2.$$
- Multiple capacitors in a circuit may be replaced by a single equivalent capacitor C_{eq}.
  - **Capacitors in parallel:** $C_{\text{eq}} = C_1 + C_2 + C_3 + \cdots$
  - **Capacitors in series:** $\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$



Capacitors in parallel



Capacitors in series

MAGNETISM AND ELECTROMAGNETIC INDUCTION

MAGNETIC FIELDS

A magnetic field $\mathbf{B}$ is created by a moving charge, and affects moving charges. Magnetic field strength is measured in Tesla (T), where $1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}$.

Magnetic force on a moving charge: A magnetic field $\mathbf{B}$ will exert a force
$$\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B}), \text{ of magnitude}$$
$$F = qvB\sin\theta$$
on a charge q moving with velocity $\mathbf{v}$ at an angle of θ to the field lines.

- Determine the direction of $\mathbf{F}$ using the **right-hand rule** (align fingers along $\mathbf{v}$, curl towards $\mathbf{B}$; the thumb points towards $\mathbf{F}$). If the charge q is negative, then $\mathbf{F}$ will point in the direction opposite to the one indicated by the right-hand rule.

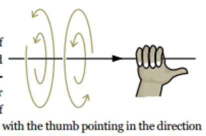
Because this force is always perpendicular to the motion of the particle, it cannot change the magnitude of $\mathbf{v}$; it only affects the direction. (Much like centripetal force affects only the direction of velocity in uniform circular motion.)

- A charged particle moving in a direction parallel to the field lines experiences no magnetic force.
- A charged particle moving in a direction perpendicular to the field lines experiences a force of magnitude $F = qvB$. A uniform magnetic field will cause this particle (of mass m) to move with speed v in a circle of radius $r = \frac{mv}{qB}$.

Magnetic force on a current-carrying wire: A magnetic field $\mathbf{B}$ will exert a force
$$\mathbf{F} = I\,(\boldsymbol{\ell} \times \mathbf{B}), \text{ of magnitude}$$
$$F = I\ell B\sin\theta$$
on a wire of length ℓ carrying current I and crossed by field lines at angle θ. The direction of ℓ corresponds to the direction of the current (which in this SparkChart means the flow of positive charge).

Magnetic field due to a moving charge:
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\,(\mathbf{v} \times \hat{\mathbf{r}})}{r^2},$$
where μ_0 is a constant called the **permeability of free space.**

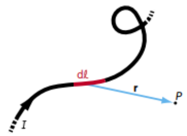
Magnetic field due to a current-carrying wire: The strength of the magnetic field created by a long wire carrying a current I depends on the distance r from the wire:
$$B = \frac{\mu_0}{2\pi}\frac{I}{r}.$$



- The direction of the magnetic field lines are determined by another **right-hand rule:** if you grasp the wire with the thumb pointing in the direction of the (positive) current, then the magnetic field lines form circles in the same direction as the curl of your fingers.

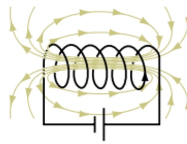
MAGNETISM AND ELECTROMAGNETIC INDUCTION (continued)

Biot-Savart Law: The formula for the magnetic field due to a current-carrying wire is a simplification of a more general statement about the magnetic field contribution of a **current element** $d\vec{\ell}$. Let $d\vec{\ell}$ be a vector representing a tiny section of wire of length $d\ell$ in the direction of the (positive) current I. If P is any point in space, $\mathbf{r}$ is the vector that points from the the current element to P, and $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$ is the unit vector, then the magnetic field contribution from the current element is given by
$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\left(d\vec{\ell} \times \hat{\mathbf{r}}\right)}{r^2}.$$

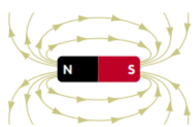


To find the total magnetic field at point P, integrate the magnetic field contributions over the length of the whole wire.

Magnetic field due to a solenoid:
$$B = \mu_0 nI,$$
where n is the number of loops in the solenoid.



BAR MAGNETS



A **bar magnet** has a **north pole** and a **south pole.**

The magnetic field lines run from the north pole to the south pole.

ELECTROMAGNETIC INDUCTION

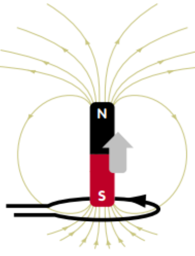
- Just as a changing electric field (e.g., a moving charge) creates a magnetic field, so a changing magnetic field can induce an electric current (by producing an electric field). This is **electromagnetic induction.**
- **Magnetic flux** (Φ_B) measures the flow of magnetic field, and is a concept analogous to Φ_E. See *Electricity: Flux and Gauss's Law* above. The magnetic flux through area A is
$$\Phi_B = \mathbf{B} \cdot \mathbf{A} = BA\cos\theta.$$
Magnetic flux is measured in **Webers** (Wb), where $1\ \mathrm{Wb} = 1\ \mathrm{T \cdot m^2}$.

Faraday's Law: Induced emf is a measure of the change in magnetic flux over time:
$$|\varepsilon_{\text{avg}}| = \frac{\Delta\Phi_B}{\Delta t} \qquad \text{or} \qquad |\varepsilon| = \frac{d\Phi_B}{dt}.$$

- A metal bar rolling in a constant magnetic field $\mathbf{B}$ with velocity $\mathbf{v}$ will induce emf according to $\varepsilon = vB\ell$. The change in flux is due to a change in the area through which the magentic field lines pass.

Lenz's Law: The direction of the induced current is such that the magnetic field created by the induced current opposes the change in the magnetic field that produced it.

- Lenz's Law is a special case of conservation of energy: if the induced current flowed in a different direction, the magnetic field it would create would reinforce the existing flux, which would then feed back to increase the current, which, in turn would increase the flux, and so on.



As the bar magnet moves up throught the loop, the upward magnetic flux decreases.

By Lenz's law, the current induced in the loop must create more upward flux counteracting the changing magnetic field.

The induced current runs counterclockwise (looking down from the top).

An **inductor** allows magnetic energy to be stored just as electric energy is stored in a capacitor. The energy stored in an inductor is given by $U = \frac{1}{2}LI^2$. The SI unit of inductance is the **Henry** (H).

MAXWELL'S EQUATIONS

1. **Gauss's Law:** $\oint_s \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$
2. **Gauss's Law for magnetic fields:** $\oint_s \mathbf{B} \cdot d\mathbf{A} = 0$
3. **Faraday's Law:** $\oint_c \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial\Phi_B}{\partial t} = -\frac{\partial}{\partial t}\oint_s \mathbf{B} \cdot d\mathbf{A}$

AMPERE'S LAW

Ampere's Law is the magnetic analog to Gauss's Law in electrostatics:

$$\oint_s \mathbf{B} \cdot d\ell = \mu_0 I_{\text{enclosed}}.$$

- Lenz's Law and Faraday's Law together make the formula
$$\varepsilon = -\frac{\Delta \Phi_B}{\Delta t} \quad \text{or} \quad \varepsilon = -\frac{d\Phi_B}{dt}.$$
- **Right-hand rule:** Point your thumb opposite the direction of the change in flux; the curl of the fingers indicated the direction of the (positive) current.

4. **Ampere's Law:** $\oint_c \mathbf{B} \cdot ds = \mu_0 I_{\text{enclosed}}$

5. **Ampere-Maxwell Law:**
$$\oint_c \mathbf{B} \cdot ds = \mu_0 I_{\text{enclosed}} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \oint_s \mathbf{E} \cdot d\mathbf{A}$$

MODERN PHYSICS

THE ATOM

Thompson's "Raisin Pudding" model (1897): Electrons are negatively charged particles that are distributed in a positively charged medium like raisins in pudding.

Rutherford's nuclear model (1911): Mass of an atom is concentrated in the central nucleus made up of positively charged protons and neutral neutrons; the electrons orbit this nucleus in definite orbits.

- Developed after **Rutherford's gold foil experiment,** in which a thin foil of gold was bombarded with small particles. Most passed through undeflected; a small number were deflected through 180°.

Bohr's model (1913): Electrons orbit the nucleus at certain distinct radii only. Larger radii correspond to electrons with more energy. Electrons can absorb or emit certain discrete amounts of energy and move to different orbits. An electron moving to a smaller-energy orbit will emit the difference in energy ΔE in the form of photons of light of frequency
$$f = \frac{\Delta E}{h},$$
where $h = 6.63 \times 10^{-34}$ J·s is Planck's constant.

Quantum mechanics model: Rather than orbiting the nucleus at a specific distance, an electron is "more likely" to be found in some regions than elsewhere. It may be that the electron does not assume a specific position until it is observed. Alternatively, the electron may be viewed as a wave whose amplitude at a specific location corresponds to the probability of finding the electron there upon making an observation.

SPECIAL RELATIVITY

Postulates

1. The laws of physics are the same in all inertial reference frames. (An inertial reference frame is one that is either standing still or moving with a constant velocity.)
2. The speed of light in a vacuum is the same in all inertial reference frames:
$$c = 3.0 \times 10^8 \text{ m/s}.$$

Lorentz Transformations

If (x, y, z, t) and (x', y', z', t') are the coordinates in two inertial frames such that the the second frame is moving along the x-axis with velocity v with respect to the first frame, then

- $x = \gamma(x' + vt')$
- $y = y'$
- $z = z'$
- $t = \gamma\left(t' + \frac{x'v}{c^2}\right)$

Here, $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$.

Relativistic momentum and energy

- **Momentum:**
$$\mathrm{p} = \frac{m_0 \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$
- **Energy:**
$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

PHYSICAL CONSTANTS

Acceleration due to gravity	g	$9.8\ \mathrm{m/s^2}$
Avogadro's number	N_A	6.022×10^{23} molecules/mol
Coulomb's constant	k	$9 \times 10^9\ \mathrm{N{\cdot}m^2/C^2}$
Gravitational constant	G	$6.67 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$
Planck's constant	h	6.63×10^{-34} J·s
Ideal gas constant	R	8.314 J/(mol·K) $= 0.082$ atm·L/(mol·K)
Permittivity of free space	ε_0	8.8541×10^{-12} C/(V·m)
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ Wb/(A·m)
Speed of sound at STP		331 m/s
Speed of light in a vacuum	c	3.00×10^8 m/s
Electron charge	e	1.60×10^{-19} C
Electron volt	eV	1.6022×10^{-19} J
Atomic mass unit	u	1.6606×10^{-27} kg $= 931.5\ \mathrm{MeV}/c^2$
Rest mass of electron	m_e	9.11×10^{-31} kg $= 0.000549$ u $= 0.511\ \mathrm{MeV}/c^2$
...of proton	m_p	1.6726×10^{-27} kg $= 1.00728$ u $= 938.3\ \mathrm{MeV}/c^2$
...of neutron		1.6750×10^{-27} kg $= 1.008665$ u $= 939.6\ \mathrm{MeV}/c^2$
Mass of Earth		5.976×10^{24} kg
Radius of Earth		6.378×10^6 m

Thank You and God Bless You