Advanced Risk Management

Solutions to Exercises Week 3

1. It is convenient to define $z_i = \Phi_{p_i}^{-1}$, the $100p_i$th percentile of the standard normal distribution.

   (a) With $T = 99$, the sample median is the middle observation, i.e., $x_{50} = 0$. The corresponding probability level is $p_i = (50 - 0.5)/99 = 0.5$, and the normal quantile corresponding to this is $z_{50} = \Phi_{0.5}^{-1} = 0$. So the scatter plot has a point $(z_i, x_i) = (0, 0)$ for $i = 50$, i.e., it passes through the origin.

   (b) If $\bar{x} = 0$, then the sample variance becomes

   $$s_x^2 = \frac{1}{T-1}\sum_{i=1}^{T} x_i^2 = \frac{1}{98}\left(\sum_{i=1}^{49} x_i^2 + \sum_{i=51}^{99} x_i^2\right).$$

   Note that $x_i < z_i < 0$ for $i = 1, \ldots, 49$, so that $\sum_{i=1}^{49} x_i^2 > \sum_{i=1}^{49} z_i^2$. Similarly, $x_i > z_i > 0$ for $i = 51, \ldots, 99$, so that $\sum_{i=51}^{99} x_i^2 > \sum_{i=51}^{99} z_i^2$. In summary, the sample variance s_x^2 satisfies

   $$s_x^2 > \frac{1}{98}\left(\sum_{i=1}^{49} z_i^2 + \sum_{i=51}^{99} z_i^2\right) = \boxed{\frac{1}{98}\sum_{i=1}^{99} z_i^2 \approx 1.}$$

   [The final approximation is not exact; direct computation based on the percentiles $z_i = \Phi_{p_i}^{-1}$ gives $\sum_{i=1}^{99} z_i^2/98 = 0.997$. For larger sample sizes, the sample variance of the percentiles will converge to the population variance of the $N(0,1)$ distribution, i.e., 1.]

   (c) The assumption $x_i < z_i$ for all $i \neq 50$ and $x_{50} = z_{50} = 0$, implies for the third (uncentered) sample moment of $\{x_i\}_{i=1}^{99}$:

   $$\frac{1}{99}\sum_{i=1}^{99} x_i^3 < \frac{1}{99}\sum_{i=1}^{99} z_i^3 = 0,$$

   where the final equality follows from symmetry of the normal distribution. This is an indication that the distribution of $\{x_i\}_{i=1}^{99}$ is skewed to the left (it is harder to show the same result for the third *centered* moment). Note also that $x_i < z_i$ for all $i \neq 50$ implies $\bar{x} < \bar{z} = 0$, so that the mean is less than the median; this is another indication of skewness.

2. (a) The formula, with $\sigma_{T+1} = 1$, $u = 2\,T_u/T = 0.05$ and $\xi = \frac{1}{2}$, implies

   $$VaR_{T+1}^p = 2\left(\frac{p}{0.05}\right)^{-1/2} = 2\sqrt{\frac{0.05}{p}}.$$

   So for $p = 0.02$, 0.01, and 0.005 we find $VaR_{T+1}^p = 3.162$, 4.472 and 6.325, respectively.

   (b) The formula $VaR_{T+1}^p = 2\,(p/0.05)^{-1/2}$ directly implies

   $$\begin{aligned}\ln VaR_{T+1}^p &= \ln 2 - \frac{1}{2}\ln p + \frac{1}{2}\ln 0.05\\ &= a - \frac{1}{2}\ln p,\end{aligned}$$

   with $a = \ln 2 + \frac{1}{2}\ln 0.05 = -0.805$. So the slope is $-\frac{1}{2}$.

   (a) From $F(x) = 1 - x^{-1/\xi}$, we have

   $$f(x) = \frac{dF(x)}{dx} = \frac{1}{\xi}x^{-1/\xi-1},$$

and hence

$$f_u(x) = \frac{f(x)}{1-F(u)} = \frac{x^{-1/\xi-1}}{\xi u^{-1/\xi}} = \frac{1}{\xi u}\left(\frac{x}{u}\right)^{-1/\xi-1}.$$

(b) The previous result implies

$$\ln f_u(x_i) = -\ln\xi - \ln u - (1/\xi+1)\ln(x_i/u),$$

so summing this over the observations gives

$$\begin{aligned}\ln L &= \sum_{i=1}^{T_u}\left(-\ln\xi - \ln u - \frac{\xi+1}{\xi}\ln\left(\frac{x_i}{u}\right)\right)\\ &= -T_u\ln u - T_u\ln\xi - \frac{\xi+1}{\xi}\sum_{i=1}^{T_u}\ln\left(\frac{x_i}{u}\right).\end{aligned}$$

(c) We find

$$\frac{d\ln L}{d\xi} = -\frac{T_u}{\xi} + \frac{1}{\xi^2}\sum_{i=1}^{T_u}\ln\left(\frac{x_i}{u}\right) = 0,$$

and solving this for ξ gives

$$\hat{\xi} = \frac{1}{T_u}\sum_{i=1}^{T_u}\ln\left(\frac{x_i}{u}\right).$$