Advanced Risk Management

Week 4

Multivariate Volatility Models; Dynamic Simulation

University of Amsterdam

Amsterdam Business School Business

Outline

- Covariance and correlation
- RiskMetrics and GARCH covariance models
- Dynamic conditional correlation
- Realized covariance and correlation
- Monte Carlo simulation of term structure of risk
- Filtered historical simulation
- Simulation with constant or time-varying correlations

Covariance and correlation

- Conditional covariance between returns $R_{i,t+1}$ and $R_{j,t+1}$ on assets i and j:

$$\sigma_{ij,t+1} = E_t\left[(R_{i,t+1} - \mu_{i,t+1})(R_{j,t+1} - \mu_{j,t+1})\right].$$

- Note that $\sigma_{ii,t+1} = \mathrm{Var}_t(R_{i,t+1}) = \sigma^2_{i,t+1}$. Conditional correlation:

$$\rho_{ij,t+1} = \frac{\sigma_{ij,t+1}}{\sigma_{i,t+1}\sigma_{j,t+1}}.$$

- Portfolio conditional variance with $n = 2$ assets and weights $w_t' = (w_{1t}, w_{2t})$:

$$\begin{aligned}
\sigma^2_{PF,t+1} &= w_{1t}^2\sigma^2_{1,t+1} + w_{2t}^2\sigma^2_{2,t+1} + 2w_{1t}w_{2t}\sigma_{12,t+1} \\
&= \begin{bmatrix} w_{1t} & w_{2t} \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t+1} & \sigma_{12,t+1} \\ \sigma_{12,t+1} & \sigma^2_{2,t+1} \end{bmatrix} \begin{bmatrix} w_{1t} \\ w_{2t} \end{bmatrix} \\
&= w_t' \Sigma_{t+1} w_t.
\end{aligned}$$

- $\sigma^2_{PF,t+1} > 0$ for all $w_t \neq 0$ requires $\left|\rho_{12,t+1}\right| < 1$: covariance matrix Σ_{t+1} is **positive definite**.

Covariance and correlation

- Generalization to n assets:

$$\sigma^2_{PF,t+1} = w_t' \Sigma_{t+1} w_t = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{it} w_{jt} \sigma_{ij,t+1},$$

  where w_t is a vector with elements w_{it}, and Σ_{t+1} is a matrix with elements $\sigma_{ij,t+1}$.
- Σ_{t+1} positive definite requires $\left|\rho_{ij,t+1}\right| < 1$, plus additional conditions.
- Model for Σ_{t+1} implies VaR and ES. Under normality:

$$VaR^p_{t+1} = -\sigma_{PF,t+1} \Phi_p^{-1}, \qquad ES^p_{t+1} = \sigma_{PF,t+1} \frac{\phi(\Phi_p^{-1})}{p}.$$

- Such a model for Σ_{t+1} is needed for active risk management (changing w_t to manage risk).

Exposure mappings

- If n is very large, then building a full covariance matrix Σ_{t+1} may not be practical.
- Full portfolio risk may be picked up be a factor model:

$$R_{PF,T+1} = \beta_1 F_{1,t+1} + \ldots + \beta_k F_{k,t+1} + \varepsilon_{t+1},$$

  where $F_{1,t+1}, \ldots, F_{k,t+1}$ are factor returns, and ε_{t+1} is the idiosyncratic risk.
- If ε_{t+1} has (small and) constant variance σ_ε^2, then

$$\begin{aligned} \sigma^2_{PF\ t+1} &= \beta' \Sigma^F_{t+1} \beta + \sigma^2_\varepsilon \\ &\approx \beta' \Sigma^F_{t+1} \beta, \end{aligned}$$

  so then we need to model the covariance matrix Σ^F_{t+1} of the factors.

RiskMetrics and GARCH covariance models

- RiskMetrics EWMA model:

$$\sigma_{ij,t+1} = (1-\lambda)R_{it}R_{jt} + \lambda\sigma_{ij,t}.$$

Same parameter $\lambda = 0.94$ for all variances and covariances $\Rightarrow \Sigma_{t+1}$ is positive definite.

- To allow for mean-reversion in (co)variances, consider

$$\sigma_{ij,t+1} = \omega_{ij} + \alpha R_{it}R_{jt} + \beta\sigma_{ij,t},$$

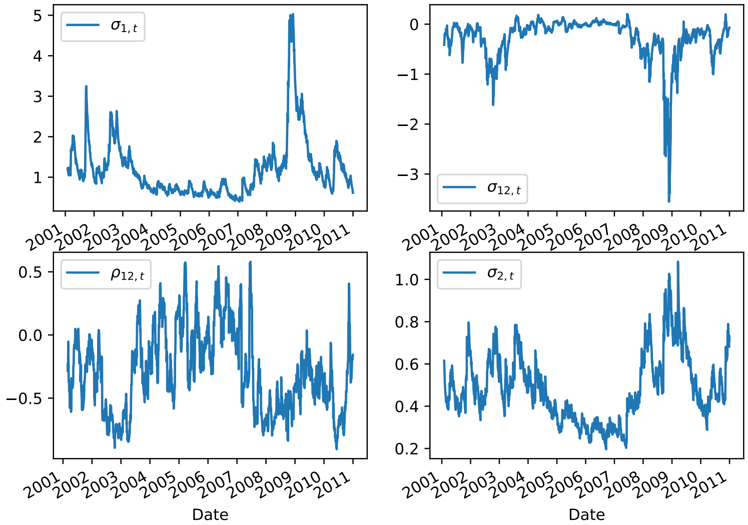
with α and β the same for all i, j, and $\alpha + \beta < 1$

- Using unconditional covariance $\sigma_{ij} = \omega_{ij}/(1-\alpha-\beta)$, we get

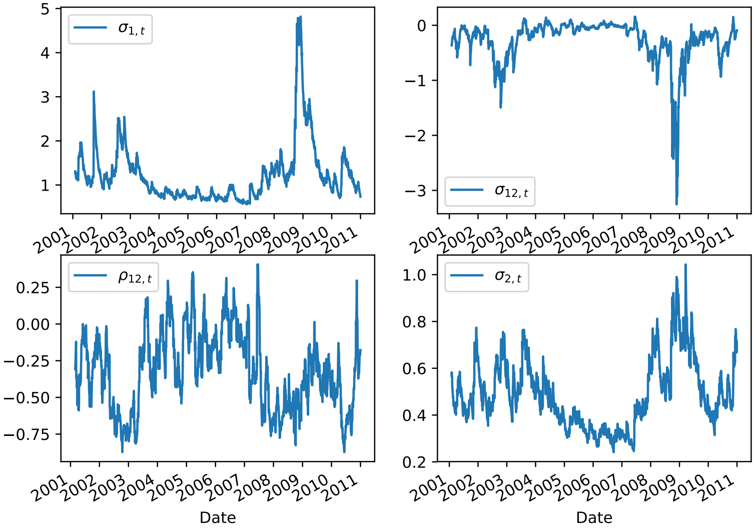
$$\sigma_{ij,t+1} = \sigma_{ij}(1-\alpha-\beta) + \alpha R_{it}R_{jt} + \beta\sigma_{ij,t},$$

so we can estimate $\hat{\sigma}_{ij} = T^{-1}\sum_{t=1}^{T} R_{it}R_{jt}$ separately, and α and β by MLE.

RiskMetrics – S&P500 versus 10-year treasury note returns



GARCH – S&P500 versus 10-year treasury note returns



Dynamic Conditional Correlation

- Model is similar to GARCH, but in terms of correlations instead of covariances.
- Idea is that conditional variances $\sigma^2_{i,t+1}$ can be modelled and estimated for all assets separately.
- Next, construct $z_{it} = R_{it}/\sigma_{it}$ as usual. (Make sure they have mean zero and variance 1.)
- Average correlations are simply estimated by sample covariance

$$\bar{\rho}_{ij} = \frac{1}{T}\sum_{t=1}^{T} z_{it}z_{jt}.$$

- Time-varying correlations are:

$$\rho_{ij,t+1} = \frac{q_{ij,t+1}}{\sqrt{q_{ii,t+1}q_{jj,t+1}}},$$

with

$$q_{ij,t+1} = \bar{\rho}_{ij}(1-\alpha-\beta) + \alpha z_{it}z_{jt} + \beta q_{ij,t}.$$

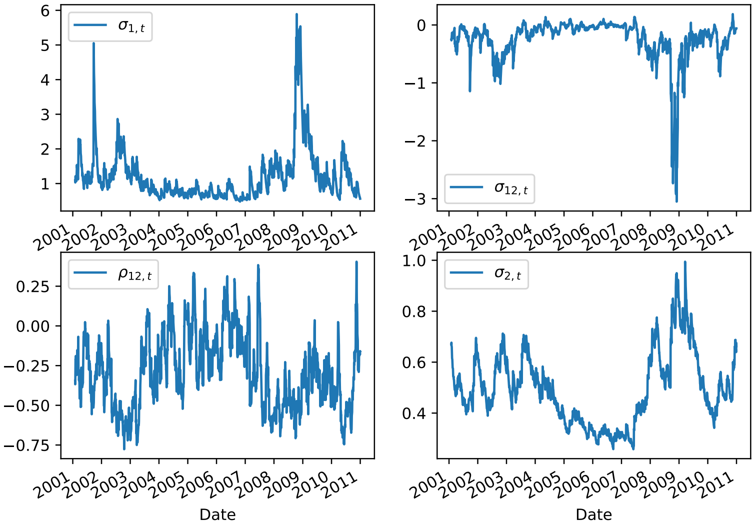
Dynamic Conditional Correlation

- DCC: $\rho_{ij,t+1} = q_{ij,t+1}/\sqrt{q_{ii,t+1}q_{jj,t+1}}$, with

$$q_{ij,t+1} = \bar{\rho}_{ij}(1-\alpha-\beta) + \alpha z_{it}z_{jt} + \beta q_{ij,t}.$$

- EWMA version: $\alpha = 0.06$, $\beta = 0.94$.
- Σ_{t+1} consists of univariate GARCH $\sigma^2_{i,t+1}$, and implied $\sigma_{ij,t+1} = \sigma_{i,t+1}\sigma_{j,t+1}\rho_{ij,t+1}$;
  - common α and β for all pairs of assets (i,j) guarantees that Σ_{t+1} is positive definite.
- Estimation by QMLE with "correlation targeting" (estimating $\bar{\rho}_{ij}$ separately).
- Christoffersen discusses composite likelihood estimation, useful for very large n.
- Correlations go up in times of crisis; hence asymmetric correlation model, stronger effect if $z_{it} < 0$ and $z_{jt} < 0$.

DCC – S&P500 versus 10-year treasury note returns

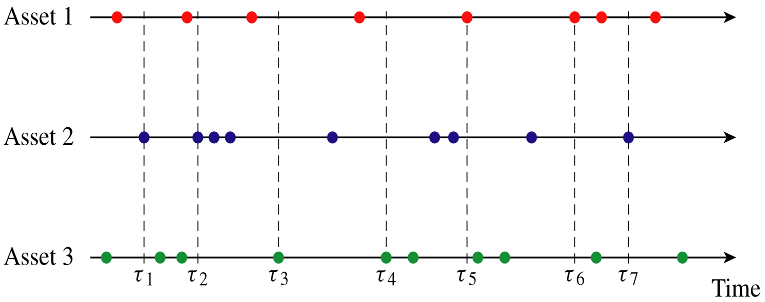


Realized covariance and correlation

- Suppose we have intraday returns for two assets: $(R_{1t,j}, R_{2t,j}), j = 1, \ldots, m$.
- In addition to RV^m_{1t} and RV^m_{1t}, we can define realized covariance and correlation:

$$RCov^m_{12,t} = \sum_{j=1}^{m} R_{1t,j} R_{2t,j}, \qquad RCorr^m_{12,t} = \frac{RCov^m_{12,t}}{\sqrt{RV^m_{1t} RV^m_{2t}}}.$$

- Bias from asynchronous trading; resolved using refresh times instead of 5-minute intervals:



Realized covariance and correlation

- Realized covariances can also be obtained from portfolio and individual realized variances, using e.g.

$$\mathrm{Cov}(R_1, R_2) = 2\,\mathrm{Var}(\tfrac{1}{2}R_1 + \tfrac{1}{2}R_2) - \tfrac{1}{2}\,\mathrm{Var}(R_1) - \tfrac{1}{2}\,\mathrm{Var}(R_2).$$

- $RCov^m_{12,t}$ and $RCorr^m_{12,t}$ can be used similarly as realized variances:
    - to predict tomorrow's correlation, e.g. as extra variables in a DCC model;
    - to evaluate predictive abilities of different covariance/correlation models.

Term structure of risk

- Recall RiskMetrics results for multiple-day VaR and ES:

$$VaR^p_{t+1:t+K} = \sqrt{K}\,VaR^p_{t+1}, \qquad ES^p_{t+1:t+K} = \sqrt{K}\,ES^p_{t+1}.$$

  Based on $\mu_{t+k} = 0$ and persistent variance, $E_t(\sigma^2_{t+k}) = \sigma^2_{t+1}$.
- Both assumptions can be problematic at longer horizons; even if they are valid, the shape of the distribution of $R_{t+1:t+K}$ is unknown.
- Therefore, we use alternative method to determine $VaR^p_{t+1:t+K}$ (and $ES^p_{t+1:t+K}$), based on simulation.
- Result $VaR^p_{t+1:t+K}/\sqrt{K}$, plotted against horizon K, represents the “term structure of risk”.

Monte Carlo simulation

- Starting point for Monte Carlo simulation is GARCH-type model, together with assumed distribution for z_t, e.g. $N(0,1)$ or $\tilde{t}(d)$.
- Current time is t, so we know σ^2_{t+1} and R_t.
- Generate *MC* different scenarios for $R_{t+1}, \dots, R_{t+K}$, as follows:

  1. for $i = 1, \dots, MC$, draw i.i.d. $(\check{z}_{i,1}, \dots, \check{z}_{i,K})$ from assumed distribution;
  2. for $i = 1$, make a loop for $k = 1, \dots, K$:

  $$\begin{aligned} \check{R}_{1,t+k} &= \check{\sigma}_{1,t+k}\check{z}_{1,k}, \\ \check{\sigma}^2_{1,t+k+1} &= \omega + \alpha \check{R}^2_{1,t+k} + \beta \check{\sigma}^2_{1,t+k}, \end{aligned}$$

     where $\check{\sigma}_{1,t+1} = \sigma_{t+1}$;
  3. define $\check{R}_{1,t+1:t+K} = \sum_{k=1}^{K} \check{R}_{1,t+k}$;
  4. repeat steps 2–3 for $i = 2, \dots, MC$, giving in total *MC* replications $\check{R}_{i,t+1:t+K}$, $i = 1, \dots, MC$.

- Can be easily extended to allow for $\mu_{t+k} \neq 0$, GJR-GARCH, etc.

Monte Carlo simulation

- Schematic representation of MC simulation: rows are replications, columns are time steps:

$$
\begin{array}{cccccc}
 & & \check{z}_{1,1} \rightarrow \check{R}_{1,t+1} \rightarrow \check{\sigma}^2_{1,t+2} & \check{z}_{1,2} \rightarrow \check{R}_{1,t+2} \rightarrow \check{\sigma}^2_{1,t+3} & \ldots & \check{z}_{1,K} \rightarrow \check{R}_{1,t+K} \\
 & \nearrow & \check{z}_{2,1} \rightarrow \check{R}_{2,t+1} \rightarrow \check{\sigma}^2_{2,t+2} & \check{z}_{2,2} \rightarrow \check{R}_{2,t+2} \rightarrow \check{\sigma}^2_{2,t+3} & \ldots & \check{z}_{2,K} \rightarrow \check{R}_{2,t+K} \\
\sigma^2_{t+1} & \longrightarrow & \ldots & \ldots & \ldots & \ldots \\
 & \searrow & \ldots & \ldots & \ldots & \ldots \\
 & & \check{z}_{MC,1} \rightarrow \check{R}_{MC,t+1} \rightarrow \check{\sigma}^2_{MC,t+2} & \check{z}_{MC,2} \rightarrow \check{R}_{MC,t+2} \rightarrow \check{\sigma}^2_{MC,t+3} & \ldots & \check{z}_{MC,K} \rightarrow \check{R}_{MC,t+K}
\end{array}
$$

- Taking MC sufficiently large (e.g. $MC = 10,000$), use histogram of $\left\{\check{R}_{i,t+1:t+K}\right\}_{i=1}^{MC}$ for VaR:

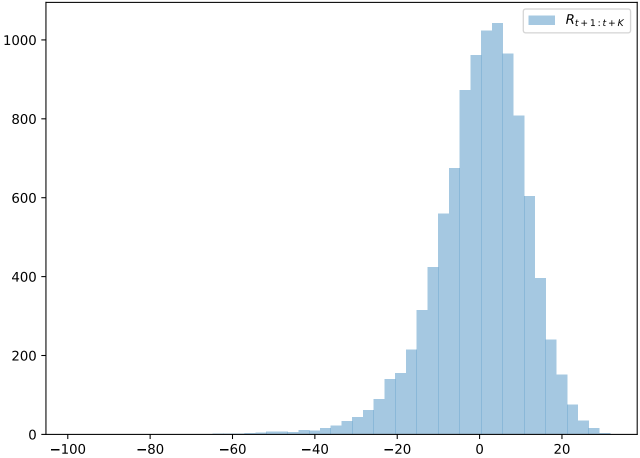
$$VaR^p_{t+1:t+K} = -Percentile\left\{\left\{\check{R}_{i,t+1:t+K}\right\}_{i=1}^{MC}, 100p\right\}.$$

Monte Carlo simulation - application to S&P500

- Consider 1-month, 1% Value at Risk ($K = 22$ days), on two dates: December 31, 2004 and December 31, 2008.
- Results for December 31, 2004:
  - RiskMetrics VaR: 6.21;
  - MC GJR-GARCH-N VaR: 8.98.
- Results for December 31, 2008:
  - RiskMetrics VaR: 27.14;
  - MC GJR-GARCH-N VaR: 32.64.
- Because of asymmetric GARCH, distribution of $R_{t+1:t+K}$ is skewed.

Monte Carlo simulation - application to S&P500

- Histogram of $R_{t+1:t+K}$ on December 31, 2008 ($SK = -0.92$, $EK = 2.67$):



Filtered historical simulation

- FHS is entirely analogous to MCS, except that $(\check{z}_{i,1}, \ldots, \check{z}_{i,K})$ are replaced by $(\hat{z}_{i,1}, \ldots, \hat{z}_{i,K})$, drawn (with replacement) from m most recent standardized shocks:

$$\hat{z}_{t-m+1}, \ldots, \hat{z}_t.$$

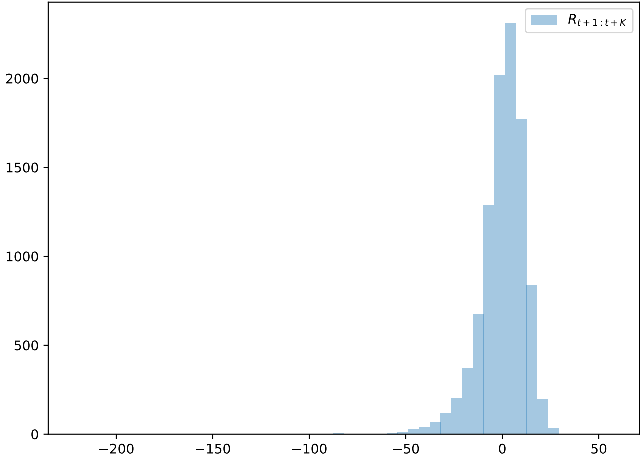
- This avoids using (possibly wrong) parametric model.
- As usual with HS, assumption is that recent history is representative for the near future.
- To obtain enough variation in the different scenarios, m should be taken as large as possible.

FHS - application to S&P500

- Consider again 1-month, 1% Value at Risk.
- Results for December 31, 2004:
  - RiskMetrics VaR: 6.21;
  - MC GJR-GARCH-N VaR: 8.98;
  - FHS GJR-GARCH VaR: 9.12.
- Results for December 31, 2008:
  - RiskMetrics VaR: 27.14;
  - MC GJR-GARCH-N VaR: 32.64;
  - FHS GJR-GARCH VaR: 37.52.

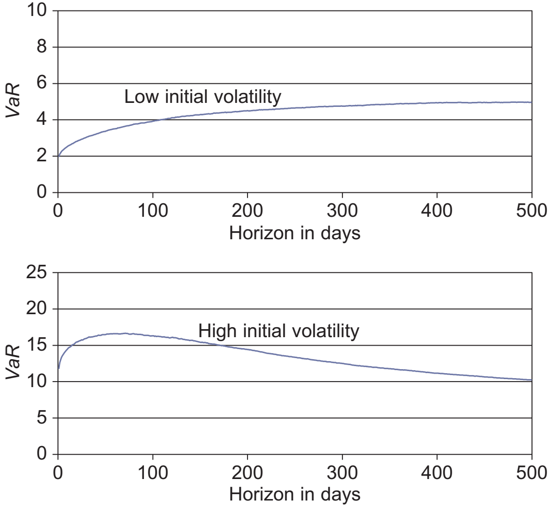
FHS - application to S&P500

- Histogram of $R_{t+1:t+K}$ on December 31, 2008 ($SK = -2.02$, $EK = 17.54$):



Shape of VaR term structure, MC simulation

Figure 8.1 *VaR* term structures using NGARCH and Monte Carlo simulation.



Simulation with constant correlations

- Monte Carlo simulation with constant correlation ρ between z_{1t} and z_{2t}:

$$\begin{pmatrix} \check{z}_{1,i,k} \\ \check{z}_{2,i,k} \end{pmatrix} = \begin{pmatrix} z_1^u \\ \rho z_1^u + \sqrt{1-\rho^2} z_2^u \end{pmatrix},$$

  where (z_1^u, z_2^u) are two independent $N(0,1)$ random variables;
  - based on Cholesky decomposition of correlation matrix.
- FHS with constant correlation: draw pairs $(\hat{z}_1, \hat{z}_2)$ from

$$\begin{pmatrix} \hat{z}_{1,t-m+1} \\ \hat{z}_{2,t-m+1} \end{pmatrix}, \ldots, \begin{pmatrix} \hat{z}_{1,t} \\ \hat{z}_{2,t} \end{pmatrix}.$$

  Preserves dependence between z_{1t} and z_{2t} in the sample.

Simulation with time-varying correlations

- Monte Carlo simulation with DCC correlation ρ_{t+1} between $z_{1,t+1}$ and $z_{2,t+1}$:

$$\begin{pmatrix} \check{z}_{1,i,k} \\ \check{z}_{2,i,k} \end{pmatrix} = \begin{pmatrix} z_1^u \\ \rho_{t+1} z_1^u + \sqrt{1-\rho_{t+1}^2}\, z_2^u \end{pmatrix}.$$

  After creating returns, update $\sigma_{1,t+k}$, $\sigma_{2,t+k}$ and ρ_{t+1} according to GARCH and DCC equations.

- FHS with DCC: after creating standarded shocks $\hat{z}_{1,t-j}$ and $\hat{z}_{2,t-j}$, make them uncorrelated by

$$\begin{pmatrix} \hat{z}^u_{1,t-j} \\ \hat{z}^u_{2,t-j} \end{pmatrix} = \begin{pmatrix} \hat{z}_{1,t-j} \\ (\hat{z}_{2,t-j} - \rho_{t-j}\hat{z}_{1,t-j})/\sqrt{1-\rho_{t-j}^2} \end{pmatrix}$$

  Then draw pairs $(\hat{z}_1^u, \hat{z}_2^u)$ from $(\hat{z}^u_{1,t-m+1}, \hat{z}^u_{1,t-m+1}), \ldots, (\hat{z}^u_{1,t}, \hat{z}^u_{1,t})$, and proceed as with MCS.

Copyright statement

The figure on Slide 12 has been extracted from:

Barndorff-Nielsen, O. E., P. R. Hansen, A. Lunde and N. Shephard (2011), "Multivariate realised kernels: Consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading," *Journal of Econometrics*, 162, 149–169.

Figure 8.1 on Slide 22 has been extracted from:

Christoffersen, P. F. (2012), *Elements of Financial Risk Management*, Second Edition. Waltham (MA): Academic Press. ISBN 978-0-12-374448-7.

Back to first slide