Advanced Risk Management

Week 1

Value at Risk and Expected Shortfall; Backtesting and Stress Testing

Copyright statement

Instructor coordinates

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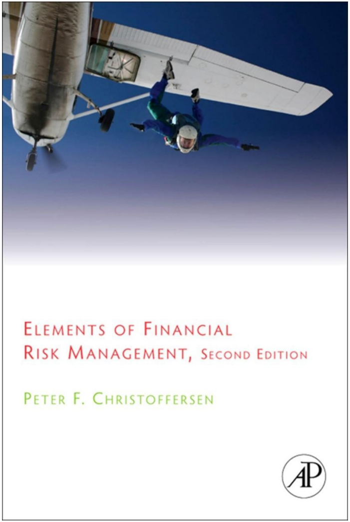
Course setup

- Lectures: Mondays 9.00–11.00 and Tuesdays 9.00–10.00
  - slides to be made available through Canvas
- Exercises: Tuesdays 10.00–11.00
  - made available through Canvas
- Computer lab: Thursdays 13.00–16.00 (1 hour per group)
  - assignments in weeks 2 and 5 (in groups of 3)
  - computer exercises in other weeks
  - based on Stata and Python (but can be combined with Excel, R, ...)
- Exam: written, closed-book final exam (2 hours)
- Final grade: 30% assignments, 70% final exam

Textbook


ELEMENTS OF FINANCIAL
RISK MANAGEMENT, SECOND EDITION
PETER F. CHRISTOFFERSEN

Course outline in weeks

1. Value at Risk and Expected Shortfall; backtesting and stress testing (Ch 1, 2, 13)
2. Univariate volatility models (Ch 4, 5)
3. Non-normal distributions; extreme value theory (Ch 6)
4. Multivariate volatility models; dynamic simulation (Ch 7, 8)
5. Tail risk and copulas (Ch 9)
6. Credit risk (Ch 12)

Outline

- Financial risk management
- Asset returns
- Value at Risk
- Historical Simulation
- Expected Shortfall
- Backtesting
- Stress testing

Risk management

- Why should firms manage risk?
- Classical portfolio theory:
  - Diversifiable risk not priced, investors will hold market porfolio and risk-free asset;
  - hence investors do not care about firm-specific risk.
  - Modigliani-Miller: firm value independent of risk structure $\Rightarrow$ firms only care about expected profit.
- In practice:
  - Bankruptcy costs $\Rightarrow$ reduce probability of default.
  - Reduce volatility of earnings $\Rightarrow$ reduce taxes.
  - Capital structure does affect risk and cost of capital $\Rightarrow$ higher debt-equity ratio requires RM.
  - Compensation packages $\Rightarrow$ employees have unhedged exposure to firm risk.
- Empirical evidence indicates that RM reduces cash flow and share price volatility $\Rightarrow$ lower cost of capital $\Rightarrow$ more investment, outperformance.

Financial risk management

- Why should *banks* and financial institutions manage risk?
- Because banks are highly leveraged, probability of default should be minimal.
- Furthermore, a bank's default or bankruptcy may affect stability of financial system.
- Hence regulators (central banks) require risk measurement and management by banks, as laid down in the Basel Accords I, II and III by the Basel Committee on Banking Supervision (Bank for International Settlements).

Taxonomy of risks

- **Market risk**: portfolio risk from movements in equity prices, exchange rates, interest rates, . . . ;
  - financial derivatives are important tools to manage market risk.
- **Liquidity risk**: risk from transactions in markets with low trading volume, high bid-ask spread;
  - in such markets it will be costlier and take longer to unwind a position.
- **Operational risk**: risk of loss due to physical catastrophe, technical failure, human error, ...
- **Credit risk**: risk of counter-party not (fully) fulfilling its obligations (default, late payment, . . .);
  - core business of banks; many kinds of credit risk not fully hedgeable.
- **Business risk**: external risks (business cycle, competitiveness) affecting business plan.
- **Systemic risk**: risks affecting the financial system's stability;
  - with potentially adverse consequences for the real economy.

Asset returns

- Stock with price S_t at time t: simple rate of return r_{t+1} and log return R_{t+1}:

$$r_{t+1} = \frac{S_{t+1} - S_t}{S_t}, \qquad R_{t+1} = \ln\left(\frac{S_{t+1}}{S_t}\right) = \ln(1 + r_{t+1}).$$

  Taylor approximation $\ln(1+x) \approx x$ implies $R_{t+1} \approx r_{t+1}$.

- Portfolio value with N_i shares of stock $i \in \{1, \ldots, n\}$

$$V_{PF,t} = \sum_{i=1}^{n} N_i S_{i,t},$$

  hence portfolio return

$$r_{PF,t+1} = \frac{V_{PF,t+1} - V_{PF,t}}{V_{PF,t}} = \sum_{i=1}^{n} w_i r_{i,t+1}, \qquad w_i = \frac{N_i S_{i,t}}{V_{PF,t}}.$$

Asset returns

- Multi-period log return:

$$R_{t+1:t+K} = \ln\left(\frac{S_{t+K}}{S_t}\right) = \sum_{k=1}^{K} \ln\left(\frac{S_{t+k}}{S_{t+k-1}}\right) = \sum_{k=1}^{K} R_{t+k}.$$

- Hence rate of return r_{t+1} most convenient for aggregation over stocks (portfolio), and log return R_{t+1} most convenient for aggregation over time.
- Additional advantage of log return: $R_{t+1} \in \mathbb{R}$ automatically implies

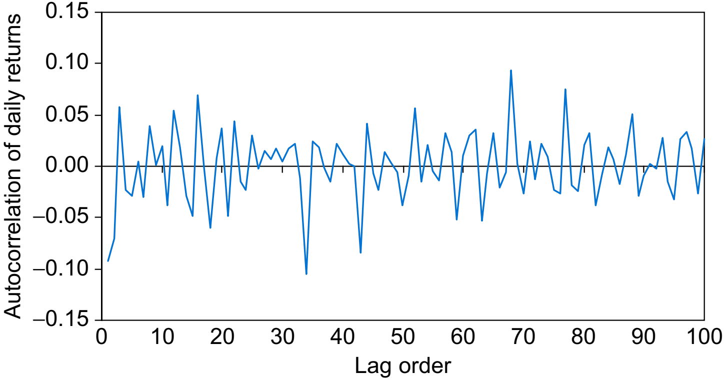
$$r_{t+1} = \exp(R_{t+1}) - 1 \geq -1.$$

Limited liability; no need to impose this on distribution of r_{t+1}.

Stylized facts of asset returns

- **Little autocorrelation**: $\mathrm{Corr}(R_t, R_{t-\tau}) \approx 0$ for all $\tau \geq 1$;
  - no predictability in mean, $E_t(R_{t+1}) \approx \mu$; related to (approximate) market efficiency.

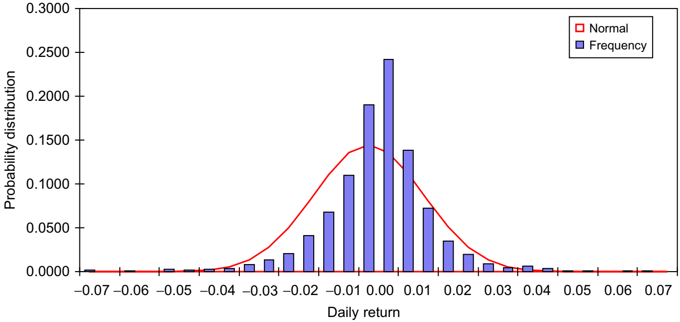
Figure 1.1 Autocorrelation of daily S&P 500 returns January 1, 2001–December 31, 2010.



Stylized facts of asset returns

- **Non-normality**: (daily) returns have higher kurtosis (more peaked, longer tails) than the normal distribution; stock returns are negatively skewed (occasional large drops in value);
  - but longer-horizon returns (e.g. monthly) look more normal.

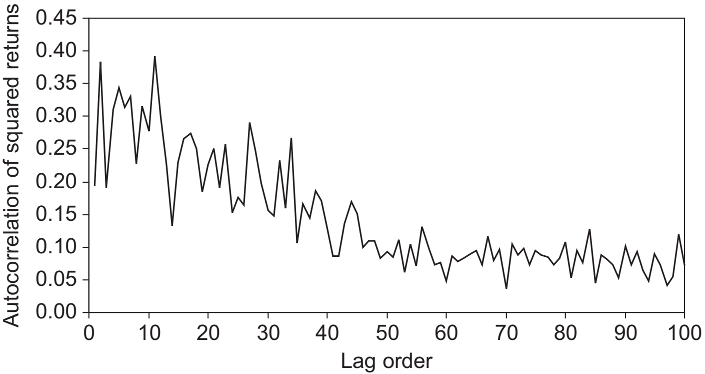
Figure 1.2 Histogram of daily S&P 500 returns and the normal distribution January 1, 2001–December 31, 2010.



Stylized facts of asset returns

- **Volatility clustering**: squared daily returns have positive autocorrelation;
  - hence large absolute returns tend to be followed by large absolute returns.

Figure 1.3 Autocorrelation of squared daily S&P 500 returns January 1, 2010–December 31, 2010.



Stylized facts of asset returns

- **Mean blur**: for daily or intra-day returns, standard deviation dominates the mean;
  - hence large confidence interval for mean returns, including zero;
  - example: daily S&P 500 returns have a mean of 0.006% and a standard deviation of 1.377%.
- **Leverage effect**: negative correlation between (past) returns and variance;
  - large price drops push up the volatility more than positive returns.
- **Time-varying correlations** between different asset returns.

Value at Risk

- *Definition*: Loss over next K trading days will exceed VaR with probability p.
- If $V_{PF,t}$ is value of portfolio, dollar loss is $\$Loss = V_{PF,t} - V_{PF,t+K}$, hence

$$\Pr(\$Loss > \$VaR) = p.$$

- More convenient to express in terms of returns R_{PF} instead of dollar losses:

$$\Pr(-R_{PF,t+1:t+K} > VaR) = p.$$

- Because $V_{PF,t+K} = V_{PF,t} \exp(R_{PF,t+1:t+K})$, implies relationship

$$\$VaR = V_{PF}(1 - \exp(-VaR)).$$

Value at Risk – generic model

- Daily returns R_{t+1} have conditional mean $\mu_{t+1} = E_t(R_{t+1})$ and conditional variance $\sigma^2_{t+1} = \mathrm{Var}_t(R_{t+1})$; standardized returns

$$z_{t+1} = \frac{R_{t+1} - \mu_{t+1}}{\sigma_{t+1}}$$

  assumed independent and identically distributed (i.i.d.) with mean zero and variance 1. Hence

$$R_{t+1} = \mu_{t+1} + \sigma_{t+1} z_{t+1}, \qquad z_{t+1} \sim \text{i.i.d. } D(0,1).$$

- For many asset classes, $\mu_t \approx 0$ (mean blur).
- Volatility clustering suggests following conditional variance (RiskMetrics):

$$\begin{aligned} \sigma^2_{t+1} &= \sigma^2_t + (1-\lambda)(R^2_t - \sigma^2_t) \\ &= \lambda\sigma^2_t + (1-\lambda)R^2_t, \end{aligned}$$

  with $\lambda = 0.94$.

Value at Risk – generic model

- If we assume $z_{t+1} \sim N(0,1)$, so $\Pr(z_{t+1} \leq z) = \Phi(z)$, then for one-day VaR:

$$\begin{aligned}
\Pr\left(R_{t+1} < -VaR^p_{t+1}\right) &= p \qquad \Leftrightarrow \\
\Pr\left(\frac{R_{t+1} - \mu_{t+1}}{\sigma_{t+1}} < \frac{-VaR^p_{t+1} - \mu_{t+1}}{\sigma_{t+1}}\right) &= p \qquad \Leftrightarrow \\
\Pr\left(z_{t+1} < \frac{-VaR^p_{t+1} - \mu_{t+1}}{\sigma_{t+1}}\right) &= p \qquad \Leftrightarrow \\
\Phi\left(\frac{-VaR^p_{t+1} - \mu_{t+1}}{\sigma_{t+1}}\right) &= p.
\end{aligned}$$

- The final expression implies

$$VaR^p_{t+1} = -\mu_{t+1} - \sigma_{t+1}\Phi^{-1}_p,$$

where $\Phi^{-1}_p = \Phi^{-1}(p)$ is the 100pth percentile of the $N(0,1)$ distribution; e.g., $\Phi^{-1}_{0.01} = -2.33$.

Value at Risk – example

- Suppose we have an investment of \$2 million in a portfolio with expected return $\mu_{PF,t+1} = 0$ and conditional standard deviation $\sigma_{PF,t+1} = 0.025$.
- Under the normality assumption, the one-day 1% Value at Risk is

$$VaR_{t+1}^{0.01} = -\sigma_{t+1}\Phi_{0.01}^{-1} = -0.025 \times -2.33 = 0.05825.$$

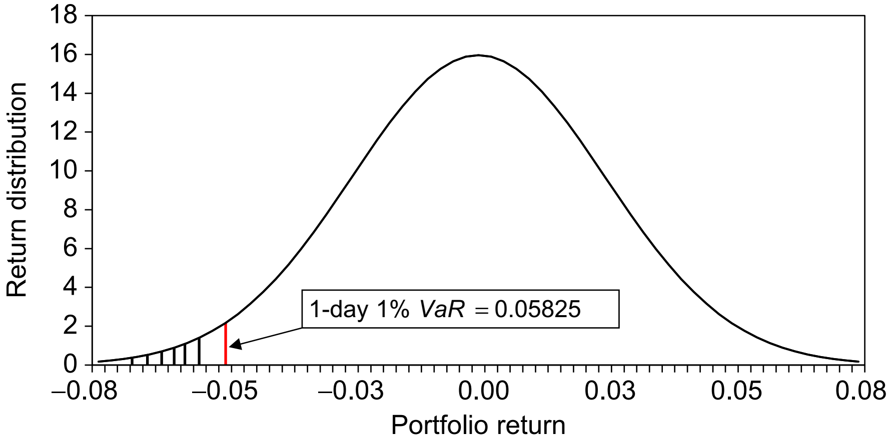
- In dollar terms, the Value at Risk becomes

$$\begin{aligned}
\$VaR &= V_{PF}(1 - \exp(-VaR_{t+1}^{0.01})) \\
&= \$2,000,000 \times (1 - \exp(-0.05825)) \\
&= \$113,172.
\end{aligned}$$

(Note rounding error in $\Phi_{0.01}^{-1} = -2.33$; using more decimals gives $\$VaR = \$113,000.$)

Value at Risk – example

Figure 1.4 Value at Risk (*VaR*) from the normal distribution return probability distribution (top panel) and cumulative return distribution (bottom panel).



Value at Risk – implementation

- To apply the method illustrated above, we need a history of daily portfolio values

$$V_{PF,t-m}, \ldots, V_{PF,t},$$

from which we construct a times series of daily log-returns

$$R_{PF,t+1-\tau} = \ln(V_{PF,t+1-\tau}/V_{PF,t-\tau}), \qquad \tau = 1, \ldots, m.$$

- This leads to a porfolio conditional variance for $t+1$:

$$\sigma^2_{PF,t+1} = \lambda\sigma^2_{PF,t} + (1-\lambda)R^2_{PF,t},$$

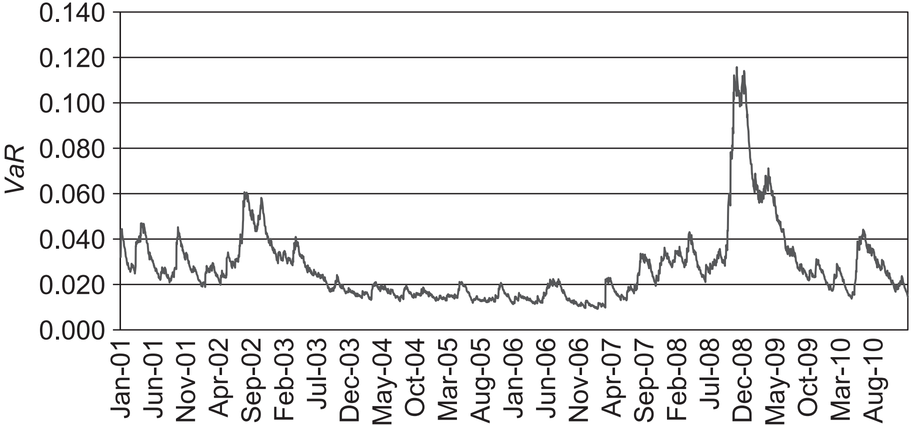
which can be solved backward to

$$\sigma^2_{PF,t+1} = (1-\lambda)\sum_{\tau=1}^{\infty}\lambda^{\tau-1}R^2_{PF,t+1-\tau} \approx (1-\lambda)\sum_{\tau=1}^{m}\lambda^{\tau-1}R^2_{PF,t+1-\tau}.$$

- From this, calculate $VaR^p_{t+1} = -\sigma_{PF,t+1}\Phi^{-1}_p$ and hence $\$VaR^p_{t+1} = V_{PF,t}(1-\exp(-VaR^p_{t+1}))$.

Value at Risk – implementation

Figure 1.5 1-day, 1% *VaR* using RiskMetrics in S&P 500 portfolio January 1, 2001–December 31, 2010.



Historical Simulation

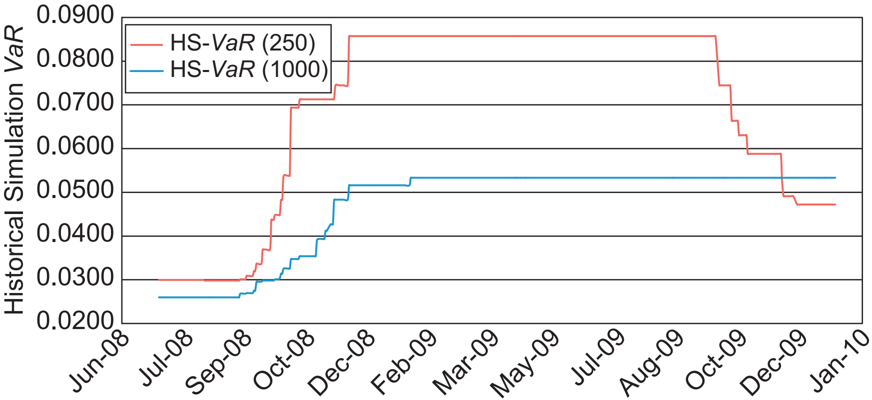
- Alternative to the generic (RiskMetrics) model approach discussed above.
- Assumption: conditional distribution of $R_{PF,t+1}$ captured by histogram of m most recent returns.
- Implies

$$VaR_{t+1}^{p} = -Percentile\left(\{R_{PF,t+1-\tau}\}_{\tau=1}^{m}, 100p\right).$$

  In practice: sort m most recent returns in ascending order, and take minus the pm^{th} of these.
- Advantages:
  - simple to implement;
  - no distributional assumption (such as normality) needed.
- Drawbacks:
  - choice of m is arbitrary, but determines how smooth (large m) or responsive (small m) VaR is;
  - "ghost" feature: a very large negative return stays in the sample for m days, after which VaR suddenly decreases;
  - multiday VaR requires very large number of returns (mK).

Historical Simulation

Figure 2.1 *VaR*s from Historical Simulation using 250 and 1,000 return days: July 1, 2008–December 31, 2009.



Weighted Historical Simulation

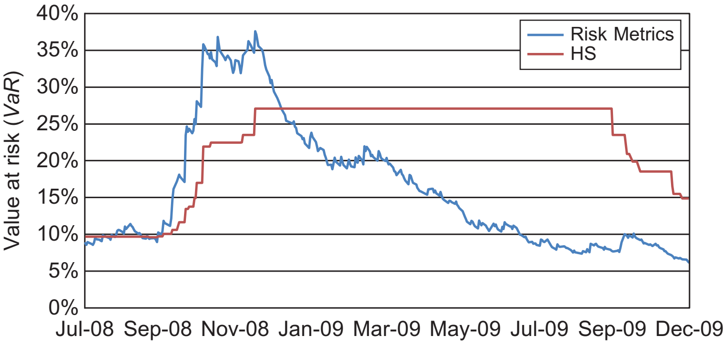
- Choose a number $\eta \in (0,1)$ close to 1, and define weight η_τ for $R_{PF,t+1-\tau}$ as

$$\eta_\tau = \left(\frac{1-\eta}{1-\eta^m}\right)\eta^{\tau-1}, \qquad \tau = 1,\ldots,m.$$

  - weights sum to 1, and decrease geometrically / exponentially as we go further into the past;
  - similar to RiskMetrics approach, which uses weights $(1-\lambda)\lambda^{\tau-1}$ to estimate the variance.
- In practice: sort returns along with their weights, and define $-VaR$ so that smaller returns have weights adding up to p.
- Advantages:
  - reduces dependence on m (but leads to sentivity to η!);
  - eliminates "ghost" feature.
- Disadvantage of (W)HS: VaR responds to negative returns, not to large positive returns;
  - this is actually reasonable for equity returns, because of leverage effect.

Historical Simulation versus RiskMetrics during the crisis

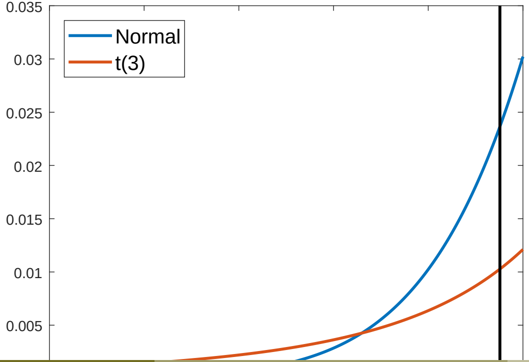
Figure 2.5 10-day, 1% *VaR* from Historical Simulation and RiskMetrics during the 2008–2009 crisis period.



- 10-day VaR has been calculated as $\sqrt{10}$ times 1-day VaR;
  - $\text{Var}(R_{t+1:t+K}) = K\text{Var}(R_{t+1})$ when daily returns have constant variance and zero autocorrelation.

Large exceedances

- Limitation of VaR: when loss exceeds VaR, we would like to know by how much.
- Two distributions may have same VaR, but different probabilities of losses $\geq 1.5VaR$.
  - Example: left tails of rescaled normal and Student's $t(3)$ distribution with same VaR (vertical bar):



Expected Shortfall

- One solution is to report VaR for $p \in \{0.025, 0.01, 0.005, \ldots\}$.
- Other solution: replace VaR by **Expected Shortfall**: expected loss, given that it exceeds VaR:

$$ES^p_{t+1} = -E_t[R_{PF,t+1} | R_{PF,t+1} < -VaR^p_{t+1}].$$

- In "generic model" ($\mu_{t+1} = 0$, $z_{t+1} \sim N(0,1)$):

$$ES^p_{t+1} = -\sigma_{PF,t+1} E[z_{t+1} | z_{t+1} < \Phi^{-1}_p] = \sigma_{PF,t+1} \frac{\phi(\Phi^{-1}_p)}{p},$$

  where $\phi(\cdot)$ is the standard normal density function.
- Example on previous slide: both distributions have $VaR = 2.6$; normal distribution has $ES = 3.0$, whereas $t(3)$ distribution has $ES = 4.0$.

Expected Shortfall

- Theoretical advantage of ES over VaR: ES is *subadditive*, i.e., for two returns R_1 and R_2,

$$ES(R_1 + R_2) \leq ES(R_1) + ES(R_2).$$

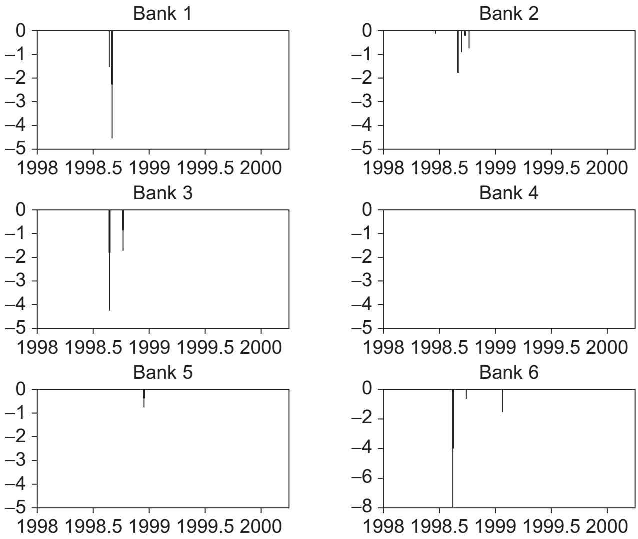
  This property is not always satisfied by VaR.
- Also for this reason, Basel III prescribes ES instead of VaR as market risk measure.

Backtesting

- Any VaR method is based on assumptions:
  - about the functional form of $\mu_{t+1} = E_t(R_{t+1})$ and $\sigma^2_{t+1} = \text{Var}_t(R_{t+1})$;
  - about the distribution of standardized returns $z_{t+1} = (R_{t+1} - \mu_{t+1})/\sigma_{t+1}$;
  - about the recent history being representative for the future.
- To check the validity of such assumptions, we can give a VaR method a "test run" on historical data and test:
  - if losses exceed VaR approximately $100p\%$ of the time (*unconditional coverage*);
  - if exceedances occur randomly over time, not clustered (*independence*);
  - a combination of these two is called *conditional coverage*.
- Figure 13.1 shows that during the financial crisis, exceedances *were* clustered, not only in time but also across banks.

Backtesting

Figure 13.1 Value-at-Risk exceedences from six major commercial banks.



Backtesting

- In a 1998 article, Peter Christoffersen proposed likelihood ratio (LR) tests for backtesting. These are reviewed and extended in Section 13.2 of the book.
- We focus on simpler regression-based t-tests.
- Suppose we have, for days $t = 1, \ldots, T$, the ex-ante VaR^p_{t+1}, and the realized return $R_{PF,t+1}$.
- From these, define the "hit sequence"

$$I_{t+1} = \begin{cases} 1, & \text{if } R_{PF,t+1} < -VaR^p_{t+1}, \\ 0, & \text{if } R_{PF,t+1} \geq -VaR^p_{t+1}, \end{cases} \qquad t = 1, \ldots, T.$$

- Then we should find:
  - unconditional coverage: $E(I_{t+1}) \quad (= \Pr\{I_{t+1} = 1\}) \quad = p$;
  - independence: $E(I_{t+1} | I_t, I_{t-1}, \ldots) = E(I_{t+1})$;
  - conditional coverage: $E(I_{t+1} | I_t, I_{t-1}, \ldots) = p$.

Unconditional coverage testing

- If $\{I_{t+1}\}_{t=1}^{T}$ is i.i.d., the number of exceedances $T_1 = \sum_{t=1}^{T} I_{t+1}$ follows binomial (T, π) distribution for some $\pi \in [0, 1]$.
- We wish to test $H_0 : \pi = p$, where p is the assumed probability, e.g., $p = 0.01$.
- Standard test statistic based on normal approximation of binomial distribution:

$$t_\pi = \frac{\hat{\pi} - p}{SE(\hat{\pi})} = \frac{\hat{\pi} - p}{\sqrt{p(1-p)/T}}, \qquad \hat{\pi} = \frac{T_1}{T}.$$

- Test that rejects H_0 against $H_a : \pi \neq p$ if $|t_\pi| > 1.96$ has 5% significance level (for large T);
  - other significance levels, and one-sided tests (against $H_a : \pi > p$) can also be considered.
- Christoffersen's LR test statistic:

$$LR_{uc} = -2 \ln \left(\frac{p^{T_1}(1-p)^{T-T_1}}{\hat{\pi}^{T_1}(1-\hat{\pi})^{T-T_1}} \right) \overset{H_0}{\sim} \chi^2_1.$$

Unconditional coverage testing – example

- Suppose that we consider a backtesting period of four years, hence $T = 1000$ trading days.
- After comparing the realized returns with the 1% one-day VaRs, we find $T_1 = 14$ exceedances.
- Hence $\hat{\pi} = 14/1000 = 0.014$; is this significantly different from $p = 0.01$?

$$t_\pi = \frac{\hat{\pi} - p}{\sqrt{p(1-p)/T}} = \frac{0.014 - 0.01}{\sqrt{0.01 \times 0.99/1000}} = 1.271,$$

$$LR_{uc} = -2 \ln \left(\frac{p^{T_1}(1-p)^{T-T_1}}{\hat{\pi}^{T_1}(1-\hat{\pi})^{T-T_1}} \right) = -2 \ln \left(\frac{0.01^{14} 0.99^{986}}{0.014^{14} 0.986^{986}} \right) = 1.437.$$

- The 5% critical values for t_π and LR_{uc} are 1.96 and 3.84, respectively.
- So unconditional coverage cannot be rejected.

Independence testing

- To test if I_{t+1} depends on $I_t, I_{t-1}, \ldots$, we need an alternative to the i.i.d. null hypothesis.
- Simple alternative: AR(1) model

$$I_{t+1} = b_0 + b_1 I_t + e_{t+1}, \qquad t = 1, \ldots, T,$$

  which can be estimated by OLS.
- The independence hypothesis requires $b_1 = 0$; hence reject independence if $t_{b_1} = \hat{b}_1 / SE(\hat{b}_1)$ is larger than 1.96 in absolute value;
  - using standard t-test; heteroskedasticity-robust standard errors not useful here.
- Can be extended to more lags, or other variables observed at time t, collected in vector X_t:

$$I_{t+1} = b_0 + b_1' X_t + e_{t+1}, \qquad t = 1, \ldots, T,$$

  and test $H_0 : b_1 = 0$ using a standard F-test.

Conditional coverage testing

- Engle-Manganelli test: F-test for $b_0^* = 0, b_1 = 0$ in reformulated regression

$$I_{t+1} - p = b_0^* + b_1' X_t + e_{t+1}.$$

  with $b_0^* = b_0 - p$.
- Christoffersen's LR tests are based on the model

$$\Pr(I_{t+1} = j | I_t = i) = \pi_{ij}, \qquad i, j = 0, 1.$$

  - Maximum likelihood estimators are

$$\hat{\pi}_{01} = \frac{T_{01}}{T_{00} + T_{01}}, \qquad \hat{\pi}_{11} = \frac{T_{11}}{T_{10} + T_{11}}, \qquad \hat{\pi}_{00} = 1 - \hat{\pi}_{01}, \qquad \hat{\pi}_{10} = 1 - \hat{\pi}_{11},$$

    with T_{ij} the number of i obervations followed by a j.
  - Independence test statistic LR_{ind} for $H_0 : \pi_{01} = \pi_{11}$, satisfies $LR_{ind} \overset{H_0}{\sim} \chi_1^2$.
  - Conditional coverage test for $H_0 : \pi_{01} = \pi_{11} = p$ is $LR_{cc} = LR_{uc} + LR_{ind} \overset{H_0}{\sim} \chi_2^2$.

Backtesting Expected Shortfall

- Recall that $I_{t+1} = 1$ if $R_{PF,t+1} < -VaR^p_{t+1}$. Expected Shortfall can be written as
$$ES^p_{t+1} = E_t[-R_{PF,t+1} | I_{t+1} = 1].$$
- Let $\mathbb{T}_1 = \{t \in \{1, \ldots, T\} : I_{t+1} = 1\}$, the T_1 exceedances. Then we should find $b_0 = 0$ and $b_1 = 0$ in the regression
$$-R_{PF,t+1} - ES^p_{t+1} = b_0 + b_1' X_t + e_{t+1}, \qquad t \in \mathbb{T}_1.$$
Leads to F-tests for $b_0 = 0, b_1 = 0$ ("conditional coverage") or $b_1 = 0$ ("independence").
- Note that this regression is based on a small number of observations, $T_1 \approx pT$. So need to take p large and/or T large to get reliable results.

Stress testing

- The results of backtesting are by definition determined by the historical period used for the test.
- If there was no serious stock market crash, or other extreme event in financial markets, then the results will be too optimistic about the effectiveness of a VaR / ES method.
- Similarly, it may be that particular financial products have only become available recently, so their impact on risk cannot be determined yet.
- Thus, in the period leading up to the 2008 financial crisis, even advanced risk measures might not have indicated any serious increase in risk, or negative backtesting results.
- To deal with this, regulators have increasingly made use of **stress tests**.
- This involves defining a scenario (generator) of future risk drivers (stock market indices, term structure of interest rates, credit events), and calculating their effect on the P&L of a bank, given the current portfolio weights / trading strategies.

Choosing scenarios

- Because of the limitations of backtesting, ideally the scenarios should contain events that are currently unforeseen, but possible.
- Christoffersen lists the following desirable elements that scenarios should contain:
  - shocks that are more likely to occur than the historical data suggest;
  - shocks that have never occurred but could;
  - shocks reflecting the possibility of a break in statistical patterns;
  - shocks reflecting a structural change in the financial system.
- To end up with a coherent method to measure risk, we should be able to give a probability to extreme scenarios; then we can test risk measures using a weighted average of the benchmark (historical) distribution $f(\cdot)$ and the stress scenario generator $f_{stress}(\cdot)$.

Copyright statement

Figures 1.1–1.5, 2.1, 2.5 and 13.1 on Slides 12–14, 20, 22, 24, 26 and 31 have been extracted from:

> Christoffersen, P. F. (2012), *Elements of Financial Risk Management*, Second Edition. Waltham (MA): Academic Press. ISBN 978-0-12-374448-7.

Back to first slide