Stationarity conditions for GARCH and GJR-GARCH models

In the GARCH(1,1) model

$$\sigma_{t+1}^2 = \omega + \alpha R_t^2 + \beta \sigma_t^2, \qquad \omega, \alpha, \beta > 0,$$

we know that the unconditional variance $\sigma^2 = \omega/(1-\alpha-\beta)$ exists under the stationarity condition $\alpha + \beta < 1$. Under this condition, σ_t^2 mean-reverts to σ^2. This result may be generalized as follows:

- GARCH(p, q):

$$\sigma_{t+1}^2 = \omega + \sum_{i=1}^{p} \alpha_i R_{t-i+1}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j+1}^2, \qquad \omega, \alpha_i, \beta_j > 0.$$

  Now the stationarity condition and resulting unconditional variance are

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1 \qquad \Rightarrow \qquad \sigma^2 = \frac{\omega}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j}.$$

- GJR-GARCH(p, o, q) (with $I_t = 1$ if $R_t < 0$ and zero otherwise):

$$\sigma_{t+1}^2 = \omega + \sum_{i=1}^{p} \alpha_i R_{t-i+1}^2 + \sum_{k=1}^{o} \gamma_k I_{t-k+1} R_{t-k+1}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j+1}^2, \qquad \omega, \alpha_i, \gamma_k, \beta_j > 0.$$

  Now the stationarity condition and resulting unconditional variance are

$$\sum_{i=1}^{p} \alpha_i + \frac{1}{2} \sum_{k=1}^{o} \gamma_k + \sum_{j=1}^{q} \beta_j < 1 \qquad \Rightarrow \qquad \sigma^2 = \frac{\omega}{1 - \sum_{i=1}^{p} \alpha_i - \frac{1}{2} \sum_{k=1}^{o} \gamma_k - \sum_{j=1}^{q} \beta_j}.$$