Advanced Risk Management

Solution Exercises Week 4

1. Consider the portfolio weights

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{R_{2t} - R_{1t}} \begin{bmatrix} R_{2t} \\ -R_{1t} \end{bmatrix}.$$

   We see that for this choice of w, we have

$$w' R_t = \frac{1}{R_{2t} - R_{1t}} \begin{bmatrix} R_{2t} & -R_{1t} \end{bmatrix} \begin{bmatrix} R_{1t} \\ R_{2t} \end{bmatrix} = 0,$$

   so that

$$w' \Sigma_{t+1} w = w' R_t R_t' w = 0.$$

   For any other w, we have $R_{PF,t} = w' R_t \neq 0$, and hence

$$w' \Sigma_{t+1} w = w' R_t R_t' w = (R_{PF,t})^2 > 0.$$

2. The EWMA covariance matrix satisfies

$$\begin{aligned} \Sigma_{t+1} &= (1-\lambda) R_t R_t' + \lambda \Sigma_t \\ &= (1-\lambda) R_t R_t' + \lambda(1-\lambda) R_{t-1} R_{t-1}' + \lambda^2 \Sigma_{t-1} \\ &\vdots \\ &= (1-\lambda) \sum_{\tau=1}^{\infty} \lambda^{\tau-1} R_{t+1-\tau} R_{t+1-\tau}'. \end{aligned}$$

   For any w, define again $R_{PF,t+1-\tau} = w' R_{t+1-\tau}$. We may choose w that some of these $R_{PF,t+1-\tau}$ are zero, but they cannot all be zero for the same w. Therefore

$$w' \Sigma_{t+1} w = (1-\lambda) \sum_{\tau=1}^{\infty} \lambda^{\tau-1} R_{PF,t+1-\tau}^2 > 0.$$

3. (a) If $\sigma_{t+1} = \sigma_{t+2} = 1$, then R_{t+1} and R_{t+2} are both ± 1, with probability $\frac{1}{2}$. This means that

$$R_{t+1:t+2} = \begin{cases} 2, & \text{with probability } \frac{1}{4}, \\ 0, & \text{with probability } \frac{1}{2}, \\ -2 & \text{with probability } \frac{1}{4}. \end{cases}$$

   This implies

$$\begin{aligned} E(R_{t+1:t+2}) &= 2 \times \tfrac{1}{4} + 0 \times \tfrac{1}{2} + (-2) \times \tfrac{1}{4} = 0, \\ \mathrm{Var}(R_{t+1:t+2}) &= 2^2 \times \tfrac{1}{4} + 0^2 \times \tfrac{1}{2} + (-2)^2 \times \tfrac{1}{4} = 2, \\ E(R_{t+1:t+2}^3) &= 2^3 \times \tfrac{1}{4} + 0^3 \times \tfrac{1}{2} + (-2)^3 \times \tfrac{1}{4} = 0, \\ E(R_{t+1:t+2}^4) &= 2^4 \times \tfrac{1}{4} + 0^4 \times \tfrac{1}{2} + (-2)^4 \times \tfrac{1}{4} = 8, \end{aligned}$$

   so that the skewness is 0 and the kurtosis is $E(R_{t+1:t+2}^4)/\,\mathrm{Var}(R_{t+1:t+2})^2 = 8/4 = 2.$

(b) In this case, $\sigma_{t+2}^2 = 0.4 + 0.21R_{t+1}^2 + 0.6\sigma_{t+1}^2 = 1.21$, so that $\sigma_{t+2} = 1.1$. So now R_{t+2} is ± 1.1, with probability $\frac{1}{2}$, and

$$R_{t+1:t+2} = \begin{cases} 2.1, & \text{with probability } \frac{1}{4}, \\ 0.1, & \text{with probability } \frac{1}{4}, \\ -0.1, & \text{with probability } \frac{1}{4}, \\ -2.1, & \text{with probability } \frac{1}{4}. \end{cases}$$

This implies

$$\begin{aligned} E(R_{t+1:t+2}) &= (2.1 + 0.1 - 0.1 - 2.1) \times \tfrac{1}{4} = 0, \\ \mathrm{Var}(R_{t+1:t+2}) &= (2.1^2 + 0.1^2 + (-0.1)^2 + (-2.1)^2) \times \tfrac{1}{4} = 2.21, \\ E(R_{t+1:t+2}^3) &= (2.1^3 + 0.1^3 + (-0.1)^3 + (-2.1)^3) \times \tfrac{1}{4} = 0, \\ E(R_{t+1:t+2}^4) &= (2.1^4 + 0.1^4 + (-0.1)^4 + (-2.1)^4) \times \tfrac{1}{4} = 9.7241, \end{aligned}$$

so that the skewness is 0 and the kurtosis is

$$E(R_{t+1:t+2}^4) / \mathrm{Var}(R_{t+1:t+2})^2 = 9.7241/(2.21)^2 = 1.991.$$

The kurtosis in both cases is approximately the same. We now have $\mathrm{Var}_t(R_{t+1:t+2}) > 2\sigma_{t+1}^2$, because this is a stationary GARCH process with ~~unconditional variance~~ $0.4/(1 - 0.21 - 0.6) = 2.1053$, which is larger than σ_{t+1}^2. So the future conditional variance will gradually increase from $\sigma_{t+1}^2 = 1$ to $\sigma^2 = 2.1053$, and this explains why $\mathrm{Var}_t(R_{t+1:t+2}) = \sigma_{t+1}^2 + E_t(\sigma_{t+2}^2) > 2\sigma_{t+1}^2$.

(c) With the GJR-GARCH model, we have that $\sigma_{t+2}^2 = 1$ if $R_{t+1} = 1$, and $\cancel{\sigma_{t+2}^2 = 1.44}$ if $R_{t+1} = -1$. In the former case, $R_{t+2} = \pm 1$ with probability $\frac{1}{2}$; in the latter case, $R_{t+2} = \pm 1.2$ with probability $\frac{1}{2}$. Therefore,

$$R_{t+1:t+2} = \begin{cases} 2, & \text{with probability } \frac{1}{4}, \\ 0.2, & \text{with probability } \frac{1}{4}, \\ 0, & \text{with probability } \frac{1}{4}, \\ -2.2, & \text{with probability } \frac{1}{4}. \end{cases}$$

This implies

$$\begin{aligned} E(R_{t+1:t+2}) &= (2 + 0.2 + 0 - 2.2) \times \tfrac{1}{4} = 0, \\ \mathrm{Var}(R_{t+1:t+2}) &= (2^2 + 0.2^2 + (-2.2)^2) \times \tfrac{1}{4} = 2.22, \\ E(R_{t+1:t+2}^3) &= (2^3 + 0.2^3 + (-2.2)^3) \times \tfrac{1}{4} = -0.66, \\ E(R_{t+1:t+2}^4) &= (2^4 + 0.2^4 + (-2.2)^4) \times \tfrac{1}{4} = 9.8568. \end{aligned}$$

So the skewness is $-0.66/(2.22)^{3/2} = -0.200$, and the kurtosis is $9.8568/(2.22)^2 = 2$. The variance is compa~~rable to (b), a~~nd the kurtosis is comparable to (a) and (b). But we now see that the GJR-GARCH model leads to skewness in the distribution of $R_{t+1:t+2}$.

4. (a) Using the rules of matrix multiplication, we directly find

$$
\begin{aligned}
\mathbf{LL}' &= \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \rho \\ \rho & \rho^2 + 1 - \rho^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.
\end{aligned}
$$

(b) The definitions imply

$$
\begin{aligned}
\operatorname{Var}(z_1) &= \operatorname{Var}(z_1^u) = 1 \\
\operatorname{Var}(z_2) &= \rho^2 \operatorname{Var}(z_1^u) + (1-\rho^2)\operatorname{Var}(z_2^u) = 1, \\
\operatorname{Cov}(z_1, z_2) &= \operatorname{Cov}\left(z_1^u, \rho z_1^u + \sqrt{1-\rho^2} z_2^u\right) \\
&= \rho \operatorname{Cov}\left(z_1^u, z_1^u\right) + 0 \\
&= \rho.
\end{aligned}
$$

5. The main disadvantage of filtered historical simulation is that a specific large negative z in the past m observations will increase the Value at Risk, and this will then stay high for m periods, and not gradually decrease. This is the advantage of Monte Carlo simulation: by choosing a suitable distribution, the large and persistent effect of a single observation in the past is avoided.
The main disadvantage of Monte Carlo simulation is that the assumed distribution may be wrong, and in particular we might underestimate the risk if we choose a distribution with no heavy tails (such as $N(0, 1)$). This is the advantage of filtered historical simulation, which allows the data to speak for itself, without imposing an incorrect distribution.