Advanced Risk Management

Exercises Week 5

1. Let (X_1, X_2) have a bivariate standard normal distribution with correlation coefficient ρ, and let S^2 be a $\chi^2(4)$ random variable, divided by 4; S^2 is independent of (X_1, X_2). Finally, define

$$z_i = \frac{1}{\sqrt{2}} \frac{X_i}{S}, \qquad i = 1, 2,$$

so that (z_1, z_2) has a bivariate standardized $t(4)$ distribution.

   (a) For any w_1 and w_2, show that

$$\frac{w_1 z_1 + w_2 z_2}{\sqrt{w_1^2 + w_2^2 + 2\rho w_1 w_2}}$$

   has a standardized $t(4)$ distribution.

   (b) If $\rho = 0$, then X_1 and X_2 are independent. Would you also expect z_1 and z_2 to be independent in this case? Why / why not?

2. Let u_t be a random variable with a uniform distribution on the interval $[0, 1]$, and define $z_t = \Phi^{-1}(u_t)$, where $\Phi^{-1}(p)$ is the standard normal percentile function. Show that z_t has a standard normal distribution.

3. The empirical distribution function of a sample $\{z_i\}_{i=1}^T$ is defined as

$$\hat{F}(z) = \frac{1}{T} \sum_{i=1}^{T} \mathbb{I}(z_i \leq z),$$

where $\mathbb{I}(A) = 1$ if A is true, and zero otherwise.

   (a) Let $T = 5$ and suppose that $(z_1, \ldots, z_5) = (-2, -1, 0, 1, 2)$. Make a plot of $\hat{F}(z)$, and use this to determine the values of $u_i = \hat{F}(z_i), i = 1, \ldots, 5$.

The figure mentioned in (a) will show that the function $\hat{F}$ is not one-to-one, so that its inverse $\hat{F}^{-1}$ does not exist; (such an inverse would have to satisfy $\hat{F}(\hat{F}^{-1}(u)) = u$ for all $u \in [0, 1]$). However, the empirical quantile function

$$\hat{Q}(u) = \min\{z : \hat{F}(z) \geq u\}$$

is considered a generalized inverse of $\hat{F}(z)$, in the sense that $\hat{Q}(\hat{F}(z_i)) = z_i$ for $i = 1, \ldots, T$.

   (b) Make a plot of $\hat{Q}(u)$ for the same example as mentioned in (a).

   (c) Let $U \sim U[0, 1]$, and define $Z = \hat{Q}(U)$, where $\hat{Q}$ is the empirical quantile function of a sample $\{z_i\}_{i=1}^T$. Show that Z has a discrete probability distribution with $\Pr(Z = z_i) = 1/T$, for $i = 1, \ldots, T$.

The result in (c) implies that (filtered) historical simulation can be carried out as follows:

- draw independent random numbers $u_i, i = 1, \ldots, MC$, from the $U[0, 1]$ distribution;
- define $\check{z}_i = \hat{Q}(u_i)$, where $\hat{Q}(u)$ is the empirical quantile function of the last m standardized residuals $\{\hat{z}_{t-m+1}, \ldots, \hat{z}_t\}$;
- define $\check{R}_{i,t+1} = \sigma_{t+1}\check{z}_i$ for $i = 1, \ldots, MC$;
- determine the Value at Risk at time t as minus a percentile of $\{\check{R}_{1,t+1}, \ldots, \check{R}_{1,t+1}\}$.

The next question tries to generalize this to the bivariate situation.

(d) Compare the filtered historical simulation approach, where (z_{1i}, z_{2i}) are drawn with replacement from the observed vectors $(\hat{z}_{1t}, \hat{z}_{2t})$, with a normal copula approach, in which we take the empirical distribution functions of $\hat{z}_{1t}$ and $\hat{z}_{2t}$ for $F_1(z)$ and $F_2(z)$, respectively, and we use:

  i. an independence copula, i.e., $G(u_1, u_2) = u_1 u_2$;
  ii. a perfect dependence copula, i.e., $G(u_1, u_2) = u_1 \mathbb{I}(u_2 = u_1)$;
  iii. a Gaussian copula with correlation $\rho^* = \mathrm{corr}(\hat{z}_{1t}, \hat{z}_{2t})$.