Advanced Risk Management

Week 5

Tail Dependence and Copulas

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Outline

- Integrated risk management
- Tail dependence: threshold correlation
- Bivariate standard normal and standardized t distributions
- Bivariate skewed t distribution
- Copulas
- Normal and Student's t copulas
- Applied copula modelling
- Systemic risk: ΔCoVaR

Integrated risk management

- So far we looked at risk of individual assets, or portfolios of similar assets (stocks).
- For portfolios, we used multivariate volatility / correlation model (DCC) with normal distribution.
- Under normality and zero mean, we find for portfolio VaR with $R = w_1 R_1 + w_2 R_2$

$$\begin{aligned} VaR^p &= -\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \rho\sigma_1\sigma_2} \times \Phi_p^{-1} \\ &= \sqrt{\left(w_1 VaR_1^p\right)^2 + \left(w_2 VaR_2^p\right)^2 + 2w_1 w_2 \rho VaR_1^p VaR_2^p} \\ &\overset{\rho=1}{=} w_1 VaR_1^p + w_2 VaR_2^p. \end{aligned}$$

  So aggregation of risks is possible using (conditional) correlations.
- Actual distributions may deviate from normal; in particular, dependence in tails may be stronger.
- Also, financial institutions need to aggregate risks over different business units, with very different P&L distributions. Hence need for method to “couple” different distributions: copulas.

Tail dependence

- Consider a pair (z_{1t}, z_{2t}) of standardized returns on two assets, with percentiles $(z_1(p), z_2(p))$. Coefficient of lower tail dependence:

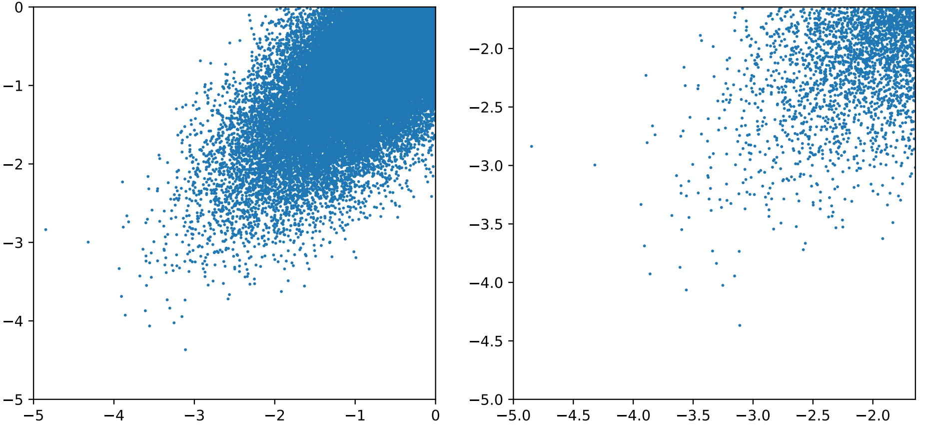
$$\begin{aligned}
\lambda &= \lim_{p\to 0} \Pr(z_{1t} < z_1(p) | z_{2t} < z_2(p)) \\
&= \lim_{p\to 0} \frac{\Pr(z_{1t} < z_1(p), z_{2t} < z_2(p))}{p},
\end{aligned}$$

  with an analogous definition for upper tail dependence.
- Christoffersen focusses on the threshold correlation:

$$\rho(p) = \mathrm{Corr}\left(z_{1t}, z_{2t} | z_{1t} < z_1(p), z_{2t} < z_2(p)\right), \qquad p \leq 0.5.$$

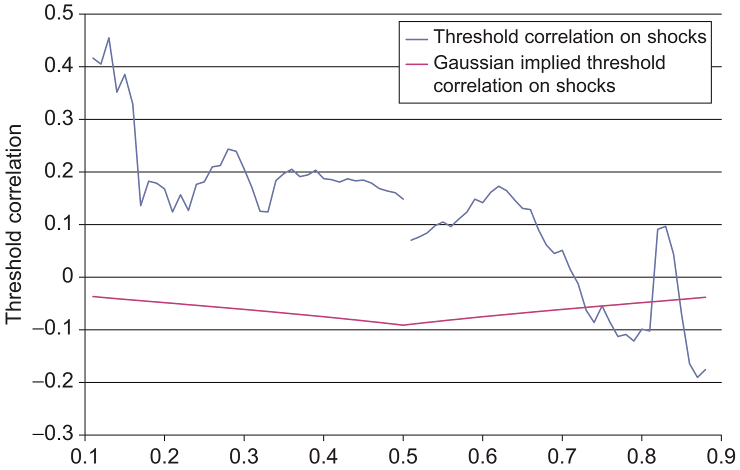
Example – simulations from bivariate normal with $\rho = 0.8$

- Threshold correlations $\rho(p)$ are 0.6 for $p = 0.5$ (left) and 0.4 for $p = 0.05$ (right).



Example – threshold correlations, stock and bond market shocks

Figure 9.2 Threshold correlation for S&P 500 versus 10-year treasury bond GARCH shocks.



Bivariate standard normal distribution

- Probability density function:

$$f(z_1, z_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{z_1^2 + z_2^2 - 2\rho z_1 z_2}{1-\rho^2}\right).$$

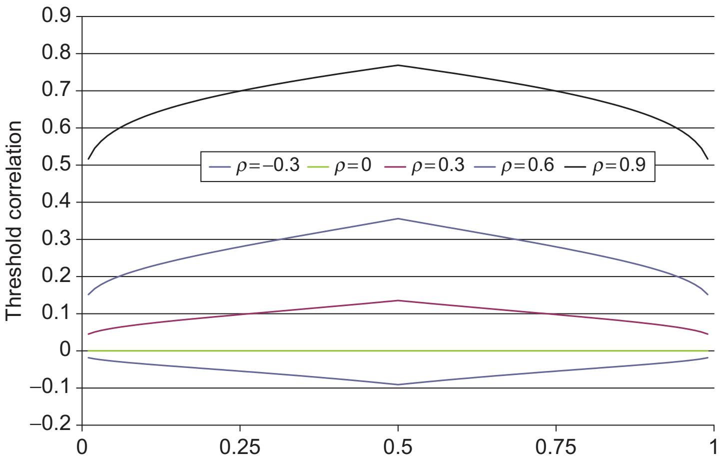
- Multivariate normal density ($n > 2$) involves determinant and inverse of correlation matrix Υ.
- If $z \sim N(0, \Upsilon)$, then for any weight vector w,

$$w'z = \sum_{i=1}^{n} w_i z_i \sim N(0, w'\Upsilon w).$$

- Coefficient of tail dependence $\lambda = 0$, regardless of ρ; threshold correlation $\rho(p)$ converges to 0 as $p \to 0$.

Tail dependence of bivariate normal distribution

Figure 9.3 Simulated threshold correlations from bivariate normal distributions with various linear correlations.



Bivariate standardized t distribution

- Probability density function:

$$f_{\tilde{t}(d,\pi)}(z_1, z_2; d, \rho) = C(d, \rho)\left(1 + \frac{1}{d-2}\frac{z_1^2 + z_2^2 - 2\rho z_1 z_2}{1-\rho^2}\right)^{-(d+2)/2}.$$

- Implies that z_1 and z_2 both have a $\tilde{t}(d)$ distribution, and correlation ρ.
- Generalization to $n > 2$; see Christoffersen, p. 199.
- Simple way to simulate vector z:

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \sqrt{\frac{d-2}{d}}\begin{pmatrix} X_1/S \\ \vdots \\ X_n/S \end{pmatrix},$$

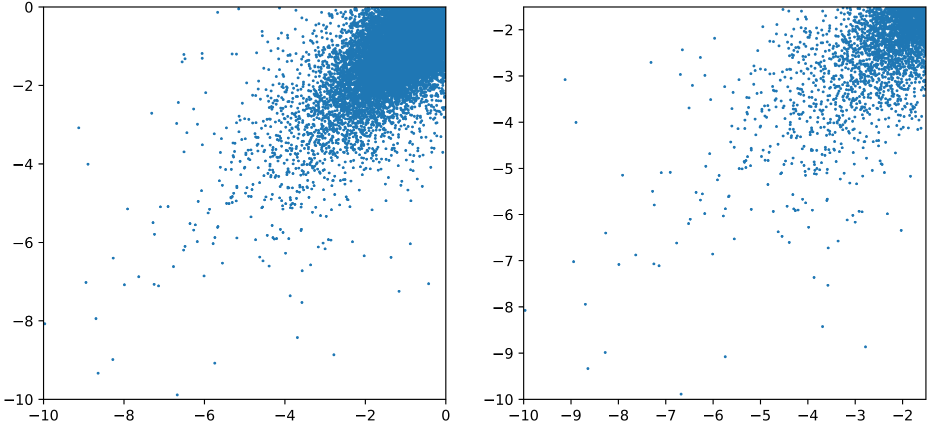
where $(X_1, \ldots, X_n) \sim N(0, \Upsilon)$, and $S^2 \sim \chi^2(d)/d$.
- Also implies that linear combinations $w'z$ have a $t(d)$ distribution with variance $w'\Upsilon w$.

Tail dependence of bivariate standardized *t* distribution

- Coefficient of tail dependence λ:
  - $\lambda > 0$ for all ρ and $d < \infty$;
  - λ increases with ρ and decreases with d.
- Threshold correlation $\rho(p)$ fairly constant, sometimes even increases as $p \to 0$.

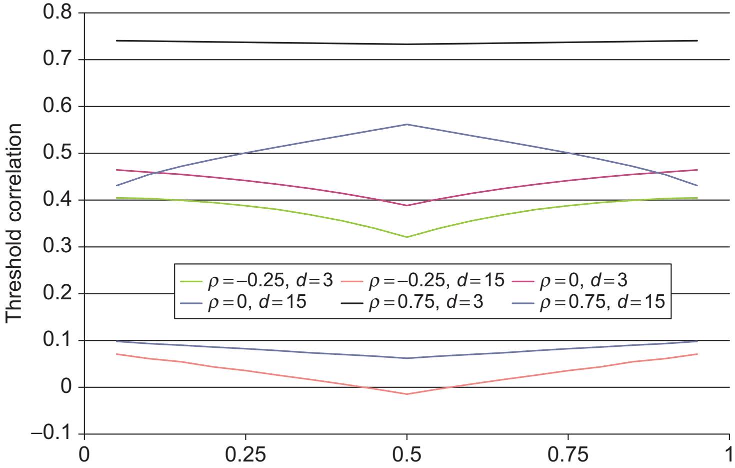
Example – simulations from bivariate standardized $t(4)$, $\rho = 0.8$

- Threshold correlations $\rho(p)$ are 0.7 for $p = 0.5$ (left) and 0.7 for $p = 0.05$ (right).



Tail dependence of bivariate standardized t distribution

Figure 9.4 Simulated threshold correlations from the symmetric t distribution with various parameters.



Bivariate skewed *t* distribution

- General density formula presented in Section 9.3.3 of Christoffersen – hard to interpret.
- For $n = 1$ and $\Upsilon = 1$, this formula does not reduce to asymmetric *t* distribution of Section 7.6: different generalization.
- Vector z can be simulated as

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \sqrt{\frac{d-2}{d}} \begin{pmatrix} X_1/S \\ \vdots \\ X_n/S \end{pmatrix} + \begin{pmatrix} \dot{\mu}_1 + \lambda_1/S^2 \\ \vdots \\ \dot{\mu}_n + \lambda_n/S^2 \end{pmatrix},$$

where:

  - $(X_1, \ldots, X_n) \sim N(0, \dot{\Upsilon})$, and $S^2 \sim \chi^2(d)/d$;
  - $\lambda_i, i = 1, \ldots, n$, are skewness parameters;
  - $\dot{\mu}$ and $\dot{\Upsilon}$ are chosen such that $E(z) = 0$ and $\mathrm{Var}(z) = \Upsilon$.

Copulas

- Idea: to give dependence to risks which may have very different distributions.
- Copula approach: build a bivariate cumulative distribution function (CDF)

$$F(z_1, z_2) = \Pr(z_{1t} \leq z_1, z_{2t} \leq z_2)$$

  from:
  - marginal CDFs $F_1(z_1) = \Pr(z_{1t} \leq z_1)$ and $F_2(z_2) = \Pr(z_{2t} \leq z_2)$;
  - copula CDF $G(u_1, u_2)$, with $u_1 \in [0, 1]$ and $u_2 \in [0, 1]$.
- Sklar's Theorem: For every joint distribution, there exists a copula CDF such that

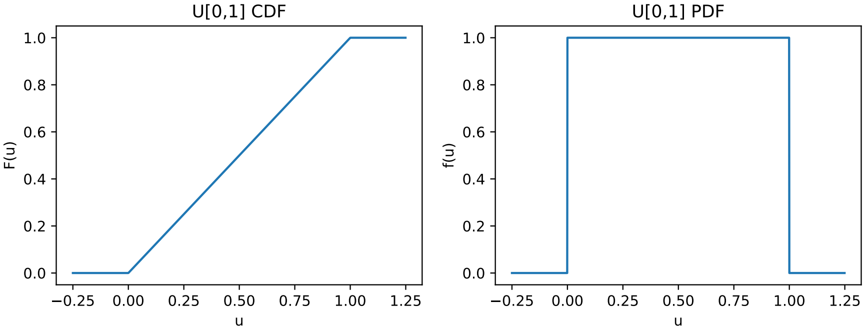
$$F(z_1, z_2) = G(F_1(z_1), F_2(z_2)).$$

- $G(u_1, u_2)$ is joint CDF of $u_{1t} = F_1(z_{1t})$ and $u_{2t} = F_2(z_{2t})$, the **probability integral transforms**.

Probability integral transforms

- If z_t has CDF $F(z)$, then $u_t = F(z_t)$ has a uniform distribution on $[0,1]$:

$$\begin{aligned}\Pr\left(u_t \leq u\right) &= \Pr\left(F(z_t) \leq u\right) \\ &= \Pr\left(z_t \leq F^{-1}(u)\right) \\ &= F(F^{-1}(u)) = u.\end{aligned}$$



Copulas

- Copula is joint distribution of two random variables (u_{1t}, u_{2t}) with uniform marginal distribution.
- Examples:
  - Independence copula: $G(u_1, u_2) = u_1 u_2$.
  - Clayton copula

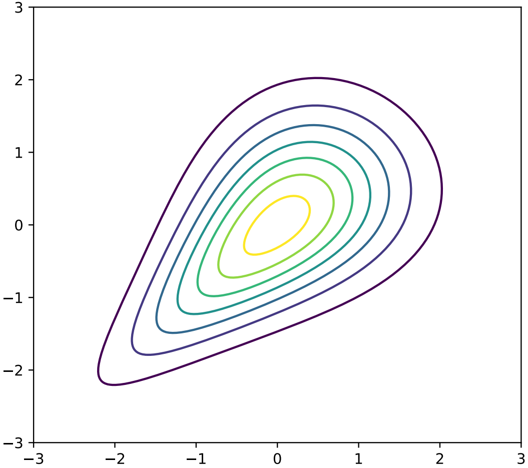
    $$G(u_1, u_2) = (u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1/\alpha}, \qquad \alpha > 0;$$

    lower tail dependence increasing with α; approaches independence as $\alpha \to 0$.
  - Normal / Gaussian and (skewed) Student's t copula – more below.
- Copula density is the PDF $g(u_1, u_2)$ corresponding to CDF $G(u_1, u_2)$. Differentiation gives:

$$f(z_1, z_2) = g(F_1(z_1), F(z_2)) \times f_1(z_1) \times f_2(z_2).$$

Example – Clayton copula with normal marginals

- Contour plot of $f(z_1, z_2)$, with standard normal f_1 & f_2 and Clayton copula, $\alpha = 1$.



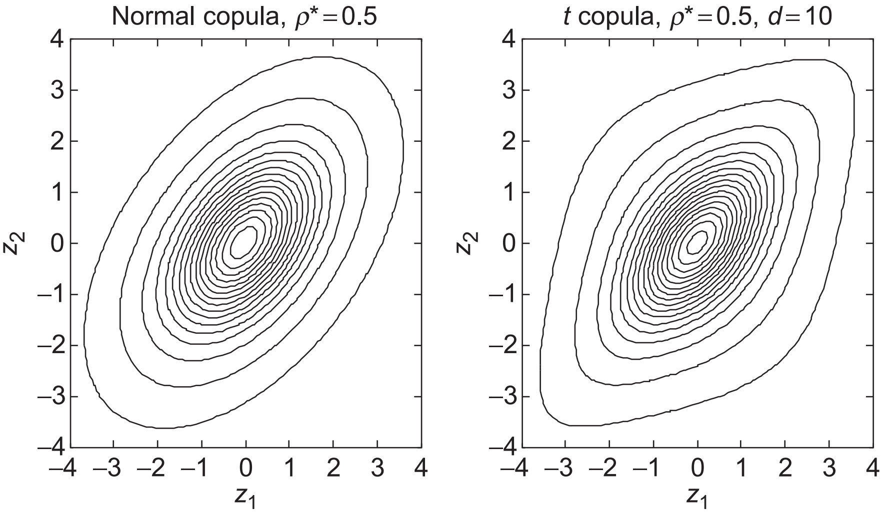
Normal and Student's *t* copula

- Normal / Gaussian copula:
  - Let $\Phi_{\rho^*}(z_1, z_2)$ be bivariate normal CDF, and $\Phi(z)$ univariate standard normal CDF; then
  $$G(u_1, u_2; \rho^*) = \Phi_{\rho^*}(\Phi^{-1}(u_1), \Phi^{-1}(u_2)),$$
  - Letting $\phi_{\rho^*}(z_1, z_2)$ and $\phi(z)$ be the bivariate / univariate normal PDFs, copula PDF is:
  $$g(u_1, u_2; \rho^*) = \frac{\phi_{\rho^*}\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right)}{\phi(\Phi^{-1}(u_1)) \times \phi(\Phi^{-1}(u_2))},$$
- *Note*: using this copula does **not** require that marginal distribution of z_1 and z_2 is normal!
  - If actual CDFs are F_1 and F_2, then implied joint distribution is
  $$F(z_1, z_2) = \Phi_{\rho^*}(\Phi^{-1}(F_1(z_1)), \Phi^{-1}(F_2(z_2)));$$
  - copula correlation ρ^* is correlation between $\Phi^{-1}(F_1(z_{1t}))$ and $\Phi^{-1}(F_2(z_{2t}))$.

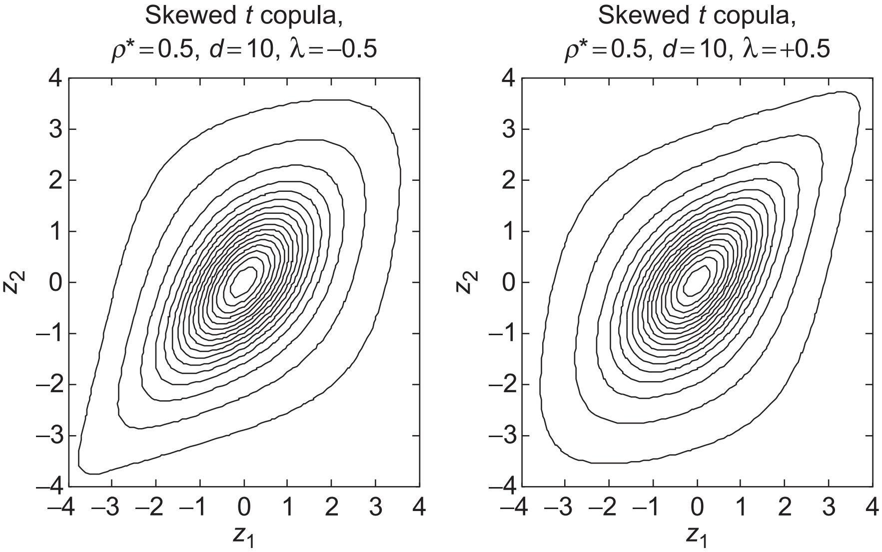
Normal and Student's t copula

- Student's t copula: replace Φ_{ρ^*} and Φ^{-1} by joint CDF $t_{(d,\rho^*)}$ and inverse CDF t_d^{-1}, respectively.
- Can be generalized to skewed t distribution.
- All of these can be easily generalized to multivariate normal and t distributions ($n > 2$).
- Next slides give contour plots of implied PDFs, in all cases with normal marginal distributions.

Normal and Student's *t* copula, normal marginals



Skewed t copula, normal marginals



Applied copula modelling

- Stepwise estimation procedure, applied to time series of returns $\{(R_{1t}, R_{2t})\}_{t=1}^{T}$:
  1. Select and fit GARCH models to R_{1t} and R_{1t} separately; keep σ_{it} and $z_{it} = R_{it}/\sigma_{it}$;
  2. Fit distribution $F_1(z)$ to $\{z_{1t}\}_{t=1}^{T}$, and similarly fit $F_2(z)$ to $\{z_{2t}\}_{t=1}^{T}$ (Student's t, FHS);
  3. Construct $(u_{1t}, u_{2t}) = (F_1(z_{1t}), F_2(z_{2t}))$, $t = 1, \ldots, T$;
  4. Fit parameters θ of copula density $g(u_1, u_2; \theta)$ by maximum likelihood applied to $\{(u_{1t}, u_{2t})\}_{t=1}^{T}$;
     - ★ for normal copula, this is done by sample correlation ρ^* of $\Phi^{-1}(u_{1t}), \Phi^{-1}(u_{2t})$.

- Monte Carlo simulation of portfolio VaR:
  1. Simulate (u_{1i}, u_{2i}), $i = 1, \ldots, MC$ from estimated copula density $g(u_1, u_2; \rho^*)$;
  2. For each replication i, transform (u_{1i}, u_{2i}) to shocks $(z_{1i}, z_{2i}) = \left(F_1^{-1}(u_{1i}), F_2^{-1}(u_{2i})\right)$;
  3. Create simulated returns $R_{1i,T+1} = \sigma_{1,T+1} z_{1i}$ and $R_{2i,T+1} = \sigma_{2,T+1} z_{2i}$;
  4. Create simulated portfolio returns $R_{PF,i,T+1} = w_{1t} R_{1i,T+1} + w_{2t} R_{2i,T+1}$, and hence

$$VaR^p_{T+1} = -Percentile\left\{\{R_{PF,i,T+1}\}_{i=1}^{MC}, 100p\right\}.$$

Systemic risk

- Copula approach can also be applied to dependence between losses of different banks.
- The 2008 financial crisis illustrated the fragility of a strongly interconnected banking system.
- It has shown that banks may have a systemic importance: they are “so interconnected and large that they can generate negative risk spillover effects on others.”
- There may be various causes to the interconnectedness, e.g.:
  - interbank investment and lending;
  - common exposure to risk in particular financial products.
- Adrian and Brunnermeier (2016) propose a measure of an institution’s contribution to the financial system’s risk, based on conditional VaR or CoVaR.
- (Other research on systemic risk has taken the opposite point of view: the institution’s exposure to the system’s risk.)

ΔCoVaR

- Let R_i be the return on equity of institution i, and R_s be the return on the system (market index, or financial sector index).
- The CoVaR is definied from

$$\Pr(R_s < -CoVaR_p^{s|i} | R_i) = p,$$

  and ΔCoVaR is

$$\Delta CoVaR_p^{s|i} = CoVaR_p^{s|R_i = -VaR_p} - CoVaR_p^{s|R_i = -VaR_{0.5}}.$$

- Interpretation: the additional systemic risk when institution i moves from a normal situation ($R_i = -VaR_{0.5}$) to a crisis situation ($R_i = -VaR_p$).
- Adrian and Brunnermeier propose to implement this using quantile regression.
- Alternatively, it can be implemented from a DCC or copula model.

ΔCoVaR based on DCC

- Assuming

$$\begin{pmatrix} R_{i,t+1} \\ R_{s,t+1} \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_{i,t+1} & \rho_{is,t+1}\sigma_{i,t+1}\sigma_{s,t+1} \\ \rho_{is,t+1}\sigma_{i,t+1}\sigma_{s,t+1} & \sigma^2_{s,t+1} \end{bmatrix} \right),$$

then

$$\begin{aligned} R_{s,t+1}|R_{i,t+1} &\sim N\left(\mu_{s|i,t+1}, \sigma^2_{s|i,t+1}\right), \\ \mu_{s|i,t+1} &= \rho_{is,t+1}\frac{\sigma_{s,t+1}}{\sigma_{i,t+1}}R_{i,t+1}, \qquad \sigma^2_{s|i,t+1} = \sigma^2_{s,t+1}(1-\rho^2_{is,t+1}). \end{aligned}$$

- Because assumptions imply $VaR^p_{t+1} = -\sigma_{i,t+1}\Phi^{-1}_p$ and $VaR^{0.5}_{t+1} = 0$, we have

$$\Delta CoVaR^{s|i}_p = -\sigma_{s,t+1}\rho_{is,t+1}\Phi^{-1}_p.$$

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Figure 9.2–9.4 on Slides 6, 8 and 12, and the figures on Slides 20 and 21 have been extracted from:

Christoffersen, P. F. (2012), *Elements of Financial Risk Management*, Second Edition. Waltham (MA): Academic Press. ISBN 978-0-12-374448-7.

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