Elementary matrix algebra

A **matrix** is a rectangular array of numbers:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

where a_{ij} are numbers. In this example, the matrix is $m \times n$, meaning that it has m **rows** and n **columns**. The **element** a_{ij} can be interpreted as "the entry of the matrix $\mathbf{A}$ in the ith row and jth column".

The transpose of a matrix $\mathbf{A}$, denoted $\mathbf{A}'$, is the $n \times m$ matrix with elements a_{ji} in ith row and j column. For example,

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{A}' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

If $m = n$, the matrix $\mathbf{A}$ is **square**. A square matrix which has $a_{ij} = a_{ji}$ for all i and j, so that $\mathbf{A} = \mathbf{A}'$, is called **symmetric**. An example is

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

For a square $n \times n$ matrix, the elements $a_{ii}, i = 1, \dots, n$ are on the **diagonal**; the other elements a_{ij} with $i \neq j$ are called off-diagonal elements.

A vector is an ordered array of numbers $\mathbf{v} = (v_1, \dots, v_n)$. A **column vector** can be seen as an $n \times 1$ matrix, whereas a **row vector** is an $1 \times n$ matrix. In the following example $\mathbf{v}$ is a column vector and $\mathbf{w}$ is a row vector:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}.$$

The transpose of a column vector is a row vector, and vice versa. Note that an $m \times n$ matrix $\mathbf{A}$ can be seen as a collection of n of column vectors $\mathbf{a}_{\bullet 1}, \dots, \mathbf{a}_{\bullet n}$, each with m components:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{\bullet 1} & \cdots & \mathbf{a}_{\bullet n} \end{bmatrix}, \qquad \mathbf{a}_{\bullet j} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix};$$

and as a collection of m row vectors $\mathbf{a}_{1\bullet}, \ldots, \mathbf{a}_{m\bullet}$, each with n components:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{1\bullet} \\ \vdots \\ \mathbf{a}_{m\bullet} \end{bmatrix}, \qquad \mathbf{a}_{i\bullet} = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix}.$$

For two column vectors $\mathbf{a} = (a_1, \ldots, a_n)'$ and $\mathbf{x} = (x_1, \ldots, x_n)'$, with the same number of components, the **inner product** is defined as

$$\mathbf{a}'\mathbf{x} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + \ldots + a_n x_n = \sum_{i=1}^{n} a_i x_i.$$

Note that $\mathbf{a}'\mathbf{x} = \mathbf{x}'\mathbf{a}$.

The **product** of an $m \times n$ matrix $\mathbf{A}$ and an n-vector $\mathbf{x}$ is defined analogously:

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix}.$$

For example:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 \times 1 + 3 \times 2 + 5 \times 3 \\ 2 \times 1 + 4 \times 2 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 28 \end{bmatrix}.$$

If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{x}$ is an n-vector, then the previous definitions implies the following **quadratic form**:

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'(\mathbf{A}\mathbf{x}) = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

Such quadratic forms typically involve a symmetric matrix $\mathbf{A}$. The matrix is $\mathbf{A}$ is said to be **positive definite** if $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all vectors $\mathbf{x}$ except a vector of zeros; when $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$ for all such vectors $\mathbf{x}$, we say that $\mathbf{A}$ is **positive semi-definite** (or non-negative definite).

An important application is the following. Let $\mathbf{r}$ be a vector of returns on n assets, and let $\mathbf{w}$ be a vector of portfolio weights:

$$\mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}.$$

Then their inner product defines the portfolio return:

$$r_{PF} = \mathbf{w}'\mathbf{r} = w_1 r_1 + \ldots + w_n r_n.$$

Let the variances $\sigma_{ii} = \mathrm{Var}(r_i)$ and covariances $\sigma_{ij} = \mathrm{Cov}(r_i, r_j)$ be collected in the $n \times n$ matrix $\boldsymbol{\Sigma}$:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \ldots & \sigma_{nn} \end{bmatrix}.$$

Such a matrix is always symmetric, because $\mathrm{Cov}(r_i, r_j) = \mathrm{Cov}(r_j, r_i)$. Then the portfolio variance is

$$\sigma_{PF}^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} w_i w_j = \sum_{i=1}^{n} \sigma_{ii} w_i^2 + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} w_i w_j \sigma_{ij}.$$

Because σ_{PF}^2 cannot be negative, $\boldsymbol{\Sigma}$ should be positive semi-definite. In fact, it should be positive definite, unless there exists a portfolio zero variance, i.e., unless $\mathrm{Var}(\mathbf{w}'\mathbf{r}) = 0$ for some $\mathbf{w} \neq \mathbf{0}$.

For example,

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

is positive semi-definite but not positive definite: for $\mathbf{w}' = (\frac{2}{3}, \frac{1}{3})$ we have $\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = 0$. On the other hand,

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$$

is positive definite. For $n = 2$, a necessary and sufficient condition for $\boldsymbol{\Sigma}$ to be positive definite is

$$\sigma_{11} > 0, \qquad \sigma_{22} > 0, \qquad |\rho_{12}| = \left| \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} \right| < 1.$$