Advanced Risk Management

Week 6

Credit Risk

UNIVERSITY OF AMSTERDAM

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Outline

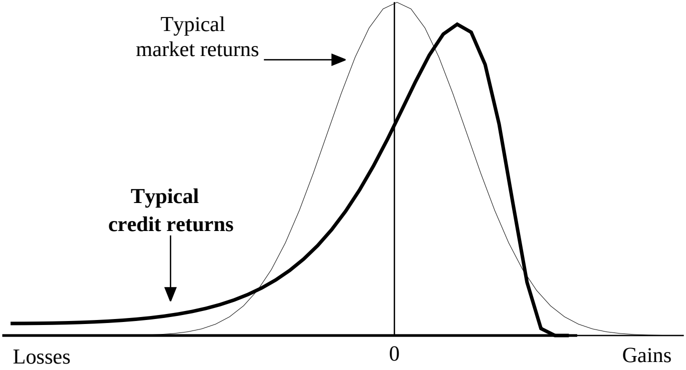
- Elements of credit risk
- Credit ratings and probability of default
- Merton's model
- Portfolio credit risk
- Value at Risk
- Rating migrations

Elements of credit risk

- Credit risk: "risk of loss due to a counterparty's failure to honour an obligation in part or in full".
- Credit risk involves:
  - The risk of default;
  - Uncertainty about recovery rate in case of default;
  - The risk of a credit rating change, which affects the market value of credit portfolio.
- The expansion of the credit derivatives market in the 2000s — in particular the growth of CDO-type products — is considered an important determinant for the 2007–2008 financial crisis.
- The typical shape of the loss distribution of credit portfolios differs from equity portfolios, which motivates the development of separate risk measurement techniques, such as implemented in CreditMetrics and prescribed by Basel III.

Elements of credit risk

Chart 1.1
Comparison of distribution of credit returns and market returns



Source: CreditMetrics Technical Document

Credit ratings

- Credit rating agencies (Moody's, S&P) rate companies based on data on corporate defaults.
- Moody's has the following ratings, with increasing credit risk:
  - **Investment grades**: Aaa, Aa, A and Baa; from minimal to modest credit risk;
  - **Speculative grades**: Ba, B, Caa, Ca, C; from substantial credit risk to in default.
- In this system, default is seen as the final stage of gradual credit risk increase.
- Each grade is associated with a default rate at a particular horizon:

Table 24.1 Average cumulative default rates (%), 1970–2015 (*Source*: Moody's).

Term (years):	*1*	*2*	*3*	*4*	*5*	*7*	*10*	*15*	*20*
Aaa	0.000	0.011	0.011	0.031	0.087	0.198	0.396	0.725	0.849
Aa	0.022	0.061	0.112	0.196	0.305	0.540	0.807	1.394	2.266
A	0.056	0.170	0.357	0.555	0.794	1.345	2.313	4.050	6.087
Baa	0.185	0.480	0.831	1.252	1.668	2.525	4.033	7.273	10.734
Ba	0.959	2.587	4.501	6.538	8.442	11.788	16.455	23.930	30.164
B	3.632	8.529	13.515	17.999	22.071	29.028	36.298	43.368	48.071
Caa–C	10.671	18.857	25.639	31.075	35.638	41.812	47.843	50.601	51.319

Probability of default

- Let T_d denote the remaining time until a company defaults — its survival time.
- T_d is a random variable with a distribution $F(t) = \Pr(T_d \leq t)$.
- Usually such survival times are characterized in terms of intensity or **hazard rate**:

$$\lambda(t) = \lim_{\Delta \to 0} \frac{\Pr(T_d \in (t, t+\Delta] | T_d > t)}{\Delta}.$$

- Relationship between hazard rate and default probability:

$$\Pr(\text{default by time } t) = \Pr(T_d < t) = 1 - e^{-\bar{\lambda}(t)t},$$

  where $\bar{\lambda}(t) = \frac{1}{t}\int_0^t \lambda(s)ds$.
- Reduced form credit risk models decribe $\lambda(t)$ by some random process.

Credit spread

- Let r_t be the risk-free short rate, so price of zero-coupon bond (payoff 1) maturing at T:
$$P(0,T) = E^Q\left[e^{-\bar{r}(T)T} \times 1\right] = e^{-y(T)T},$$
with E^Q risk-neutral expectation, $\bar{r}(T) = \frac{1}{T}\int_0^T r_t dt$, and $y(T)$ yield to maturity.
- With zero recovery, payoff of defaultable bond is $\mathbb{I}(T_d > T)$, with price
$$P_d(0,T) = E^Q\left[e^{-\bar{r}(T)T} \times \mathbb{I}(T_d > T)\right].$$
- If T_d is independent of the risk-free rate, then
$$P_d(0,T) = E^Q\left[e^{-\bar{r}(T)T}\right]\Pr^Q(T_d > T) = e^{-(y(T)+\bar{\lambda}(T))T}.$$
- Hence $\bar{\lambda}(T)$ is obtained from credit spread $s(T) = -(\ln P_d(0,T))/T - y(T)$.
  - In case of recovery rate $R > 0$, this becomes $\bar{\lambda}(T) = s(T)/(1-R)$.

Merton's model

- Merton relates probability of default to evolution of a firm's assets.
- Let A_t, E_t and D denote a firm's asset value, equity value and debt, so

$$A_t = E_t + D_t.$$

- Assume that firm defaults at time $t + T$ if A_{t+T} drops below face value D. Then:
  - Equity value is European call option price on A_t with payoff $E_{t+T} = \max\{A_{t+T} - D, 0\}$;
  - Debt value is short put option price with payoff $D_{t+T} = D - \max\{D - A_{t+T}, 0\}$.
- Under Black-Scholes assumptions on A_t (log-normal distribution):

$$E_t = A_t\Phi(d) - e^{-r_f T} D\Phi\left(d - \sigma_A\sqrt{T}\right), \quad D_t = e^{-r_f T} D\Phi\left(d - \sigma_A\sqrt{T}\right) + A_t\Phi(-d),$$

where

$$d = \frac{\ln(A_t/D) + (r + \sigma_A^2/2)T}{\sigma_A\sqrt{T}}.$$

Merton's model

- Implementation requires knowledge of A_t and σ_A, whereas we observe $E_t = N_t S_t$ and σ_E.
- Ito calculus implies

$$\begin{aligned} E_t &= A_t\Phi(d) - e^{-r_f T} D\Phi\left(d - \sigma_A\sqrt{T}\right) \\ \Rightarrow \qquad \sigma_E E_t &= \Phi(d)\sigma_A A_t. \end{aligned}$$

  Two equations in two unknowns, can be solved for A_t and σ_A.
- Risk-neutral probability of default:

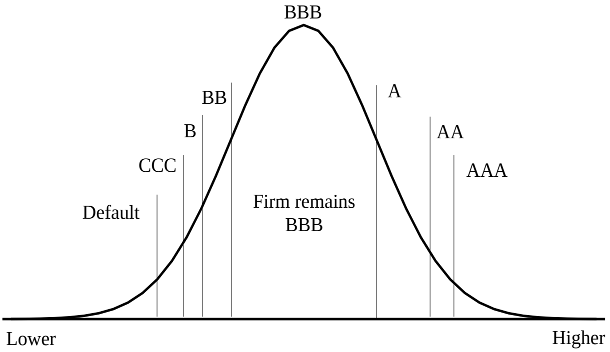
$$\Pr(A_{t+T} < D) = \Phi(\sigma_A\sqrt{T} - d) = \Phi(-dd),$$

  where $dd = d - \sigma_A\sqrt{T}$ represents "distance to default".
- Can be generalized to give probabilities of ending up in particular credit rating.

Merton's model

Chart 3.3

Model of firm value and generalized credit quality thresholds



Value of BBB firm at horizon date

Source: CreditMetrics Technical Document

Portfolio credit risk

- Suppose we have a portfolio of corporate debts; how to obtain aggregate loss distribution?
- Vasicek: assume a common factor in asset values, implying a default correlation structure.
- Risk-neutral model for log-asset values:

$$\ln A_{i,t+T} = \ln A_{it} + \left(r_f - \tfrac{1}{2}\sigma_{A,i}^2\right) T + \sigma_{A,i}\sqrt{T} z_i, \qquad i = 1, \dots, n,$$

  where

$$z_i = \sqrt{\rho} F + \sqrt{1-\rho}\,\tilde{z}_i,$$

  and $(\tilde{z}_1, \dots, \tilde{z}_n, F)$ are independent standard normal.
- Implies that $(z_1, \dots, z_n)$ are multivariate standard normal, with correlations

$$\operatorname{corr}(z_i, z_j) = \rho \geq 0, \qquad i \neq j.$$

Portfolio credit risk

- Unconditional probability of default: $PD = \Pr(A_{i,t+T} < D_i) = \Phi(-dd_i)$, so $dd_i = -\Phi^{-1}(PD)$.
- Probability that firm i defaults conditional on F is

$$\Pr\left(z_i < -dd_i \middle| F\right) = \Pr\left(\tilde{z}_i < \frac{-dd_i - \sqrt{\rho}F}{\sqrt{1-\rho}}\right) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}F}{\sqrt{1-\rho}}\right).$$

- Let $L_i = \mathbb{I}(\text{firm } i \text{ defaults})$. Then portfolio loss rate $L = n^{-1}\sum_{i=1}^{n} L_i$ satisfies

$$L \approx \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}F}{\sqrt{1-\rho}}\right).$$

Portfolio credit risk

- Then portfolio loss rate:

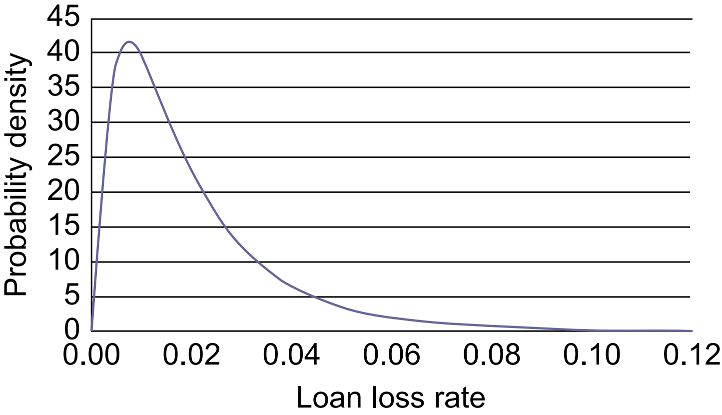
$$L \approx \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}F}{\sqrt{1-\rho}}\right).$$

- Implies

$$\begin{aligned} \Pr(L < x) &= \Pr\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}F}{\sqrt{1-\rho}} < \Phi^{-1}(x)\right) \\ &= \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right). \end{aligned}$$

Portfolio credit risk

Figure 12.6 Portfolio loss rate distribution for $PD = 0.02$ and $\rho = 0.1$.



Portfolio Value at Risk

- Vasicek model implies that VaR^{1-p} should solve

$$\Pr(L < VaR^{1-p}) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(VaR^{1-p}) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right) = 1-p,$$

so

$$VaR^{1-p} = \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}_{1-p} + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right).$$

- To turn this into $\$VaR$, we need to augment this with exposure DV_{PF} and recovery rate RR:

$$\$VaR^{1-p} = DV_{PF} \times (1-RR) \times VaR^{1-p}.$$

- Empirically, recovery rates are time-varying and negatively correlated with default probabilities.

Copula model

- Vasicek's model is closely related to the Gaussian copula model of Li.
- His purpose was to model correlation between survival times T_1 and T_2 of firm 1 and 2.
- Suppose we have distribution functions $F_1(t)$ and $F_2(t)$ for T_1 and T_2;
  - then $u_1 = F_1(T_1)$ and $u_2 = F_2(T_2)$ have a uniform distribution on $[0, 1]$;
  - Li assumed a Gaussian copula, so that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \Phi^{-1}(u_1) \\ \Phi^{-1}(u_2) \end{pmatrix} = \begin{pmatrix} \Phi^{-1}(F_1(T_1)) \\ \Phi^{-1}(F_2(T_2)) \end{pmatrix}$$

    has a bivariate standard normal distribution with correlation ρ, with CDF $\Phi(z_1, z_2; \rho)$;
  - hence probability of joint default within the next year is

$$\begin{aligned} \Pr(T_1 < 1, T_2 < 1) &= \Pr(u_1 < F_1(1), u_2 < F_2(1)) \\ &= \Pr\left(z_1 < \Phi^{-1}(F_1(1)), z_2 < \Phi^{-1}(F_2(1))\right) \\ &= \Phi(\Phi^{-1}(F_1(1)), \Phi^{-1}(F_2(1)); \rho). \end{aligned}$$

- This model was used to price CDOs; hence "the formula that killed Wall street".

Copula model – numerical example

- Suppose firm 1 and 2 both have one-year default probability $F_i(1) = 0.2\%$.
- If $\rho = 0$, then
$$\Pr\left(T_1 < 1, T_2 < 1\right) = (0.2\%)^2 = 0.0004\%.$$
- On the other hand, if $\rho = 0.5$, then:
  - $\Phi^{-1}(F_i(1)) = \Phi^{-1}\left(0.002\right) = -2.878$;
  - hence
$$\Pr\left(T_1 < 1, T_2 < 1\right) = \Phi(-2.878, -2.878; 0.5) = 0.014\%.$$
- Python code:

```
import scipy.stats as stats
rho = 0.5
var = stats.multivariate_normal(mean=[0,0],cov=[[1,rho],[rho,1]])
Prob = var.cdf([stats.norm.ppf(0.002),stats.norm.ppf(0.002)])
```

Rating migrations

- CreditMetrics and related models use Monte Carlo simulation of credit portfolio losses to obtain Value at Risk.
- Part of the simulation design is the possibility that firms' credit rating changes over time.
- The rating transition matrix has entries

$$\Pr(\text{rating}_{t+1} = i | \text{rating}_t = j),$$

  which are estimated from historical data (next slide).
- When simulating from future losses, transitions can be correlated across firms using a copula.

Rating migrations

Table 12.2 Average one-year rating transition rates, 1970–2010

From/To	**Aaa**	**Aa**	**A**	**Baa**	**Ba**	**B**	**Caa**	**Ca_C**	**WR**	**Default**
Aaa	87.395%	8.626%	0.602%	0.010%	0.027%	0.002%	0.002%	0.000%	3.336%	0.000%
Aa	0.971%	85.616%	7.966%	0.359%	0.045%	0.018%	0.008%	0.001%	4.996%	0.020%
A	0.062%	2.689%	86.763%	5.271%	0.488%	0.109%	0.032%	0.004%	4.528%	0.054%
Baa	0.043%	0.184%	4.525%	84.517%	4.112%	0.775%	0.173%	0.019%	5.475%	0.176%
Ba	0.008%	0.056%	0.370%	5.644%	75.759%	7.239%	0.533%	0.080%	9.208%	1.104%
B	0.010%	0.034%	0.126%	0.338%	4.762%	73.524%	5.767%	0.665%	10.544%	4.230%
Caa	0.000%	0.021%	0.021%	0.142%	0.463%	8.263%	60.088%	4.104%	12.176%	14.721%
Ca_C	0.000%	0.000%	0.000%	0.000%	0.324%	2.374%	8.880%	36.270%	16.701%	35.451%

Notes: The table shows Moody's credit rating transition rates estimated on annual data from 1970 through 2010. Each row represents last year's rating and each column represents this year's rating. Ca_C combines two rating categories. WR refers to withdrawn rating. Data is from Moody's (2011).

Copyright statement

Charts 1.1 and 3.3 on Slides 4 and 10 have been extracted from:

Gupton, G. M., C. C. Finger and M. Bhatia (2007), *CreditMetrics^{TM} Technical Document*. RiskMetrics Group, Inc. `https://www.msci.com/documents/10199/93396227-d449-4229-9143-24a94dab122f`

Table 24.1 on Slide 5 has been extracted from:

Hull, J. C. (2018) *Options, Futures and Other Derivatives*, Tenth Edition. New York: Pearson Education. ISBN 978-0-13-447208-9.

Figure 12.6 and Table 12.2 on Slides 14 and 19 have been extracted from:

Christoffersen, P. F. (2012), *Elements of Financial Risk Management*, Second Edition. Waltham (MA): Academic Press. ISBN 978-0-12-374448-7.

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