Advanced Risk Management

Exercises Week 3

1. The QQ plot based on the theoretical $N(0,1)$ distribution is a scatter plot of $\{x_i\}_{i=1}^T$ against $\Phi_{p_i}^{-1}$, where the x_i's have been sorted from smallest to largest, and where $p_i = (i-0.5)/T$. We will assume that $T = 99$, and that the sample median of $\{x_i\}_{i=1}^T$ is equal to 0.

   (a) Show that the QQ plot passes through the origin $(0,0)$.

   (b) Suppose that $x_i < \Phi_{p_i}^{-1}$ for all $p_i < 0.5$, and $x_i > \Phi_{p_i}^{-1}$ for all $p_i > 0.5$, so that the QQ plot is steeper than the diagonal. Explain that the sample variance of $\{x_i\}_{i=1}^T$ is in that case larger than one (you may assume the sample average $\bar{x}$ to be 0).

   (c) Suppose that $x_i < \Phi_{p_i}^{-1}$ for all p_i except at $p_i = 0.5$, so that the QQ plot lies below the diagonal, touching it only in the origin. Explain that the distribution of $\{x_i\}_{i=1}^T$ is skewed to the left.

2. Suppose that we have $T = 1000$ observations on the standardized shocks z_i of a GARCH model. We sort them from smallest to largest ($z_1 < \ldots < z_T$) and we define $u = -z_{51}$, so that $T_u = 50$ observations of $x_i = -z_i$ are larger than u. In a particular example, we find $u = 2$, and the Hill estimate of the parameter ξ is equal to $\hat{\xi} = 0.5$.

   (a) Calculate the implied VaR_{T+1}^p for $p = 0.02$, $p = 0.01$ and $p = 0.005$ from the formula on Slide 21 of Week 3, assuming $\sigma_{T+1} = 1$.

   (b) Suppose that we would continue this for many other values of $p < 0.05$, and then plot $\ln \mathit{VaR}_{T+1}^p$ against $\ln p$. Explain that this plot will be a straight line, i.e., the relationship between $\ln \mathit{VaR}_{T+1}^p$ and $\ln p$ is linear. What is the slope of this line?

3. A random variable $X \geq 1$ with a Pareto distribution has cumulative distribution function

$$F(x) = \Pr(X \leq x) = 1 - x^{-1/\xi}, \qquad x \geq 1;$$

   for $x < 1$ we have $F(x) = 0$.

   (a) For any $u > 1$, show that the probability density function of X, conditional on $X > u$, is given by

$$f_u(x) = \frac{f(x)}{1-F(u)} = \frac{1}{\xi u}\left(\frac{x}{u}\right)^{-1/\xi-1}, \qquad x \geq u.$$

   (b) Suppose we have observations $x_1, \ldots, x_{T_u}$ larger than u. Assuming that these observations are i.i.d. with density $f_u(x)$, show that the log-likelihood function equals

$$\ln L = \sum_{i=1}^{T_u} \ln f_u(x_i) = -T_u \ln u - T_u \ln \xi - \left(\frac{\xi+1}{\xi}\right) \sum_{i=1}^{T_u} \ln\left(\frac{x_i}{u}\right).$$

   (c) Derive the maximum likelihood estimator

$$\hat{\xi} = \frac{1}{T_u} \sum_{i=1}^{T_u} \ln\left(\frac{x_i}{u}\right)$$

   by setting the derivative of $\ln L$ with respect to ξ equal to 0, and solving for ξ.