Advanced Risk Management

Exercises Week 4

1. Suppose that $R_t = (R_{1t}, R_{2t})'$ is a vector of returns, and we model its conditional covariance matrix Σ_{t+1} by:
$$\Sigma_{t+1} = R_t R_t' = \begin{bmatrix} R_{1t}^2 & R_{1t}R_{2t} \\ R_{1t}R_{2t} & R_{2t}^2 \end{bmatrix},$$
or in other words
$$\sigma_{ij,t+1} = \mathrm{Cov}_t(R_{i,t+1}, R_{j,t+1}) = R_{it}R_{jt}, \qquad i,j = 1,2.$$
This is an extreme example ($m = 1$) of an equally-weighted moving average:
$$\sigma_{ij,t+1} = \frac{1}{m}\sum_{\tau=1}^{m} R_{i,t+1-\tau}R_{j,t+1-\tau}.$$
Show that Σ_{t+1} is positive semi-definite, but not positive definite.
Note: the same result will appear if we use an m-period moving average for n assets, with $m < n$: in that case one can find portfolios w_t with $w_t'\Sigma_{t+1}w_t = 0$.

2. Show that the EWMA covariance matrix can be expressed as
$$\Sigma_{t+1} = (1-\lambda)\sum_{\tau=1}^{\infty} \lambda^{\tau-1} R_{t+1-\tau}R_{t+1-\tau}',$$
where $R_t = (R_{1t}, \ldots, R_{nt})'$, an n-vector of returns. Show also that Σ_{t+1} is positive definite.

3. Suppose that z_t is i.i.d. with the following discrete distribution:
$$z_t = \begin{cases} +1, & \text{with probability } \frac{1}{2}, \\ -1, & \text{with probability } \frac{1}{2}. \end{cases}$$
This implies $E(z_t) = 0$, $\mathrm{Var}(z_t) = 1$, $E(z_t^3) = 0$, $E(z_t^4) = 1$. This means that if $R_{t+1} = \sigma_{t+1}z_{t+1}$, then the conditional distribution of R_{t+1} given information at time t is symmetric around 0, with variance σ_{t+1}^2. We wish to investigate the (discrete) distribution of the two-day return $R_{t+1:t+2}$, which can be derived from a (possibly non-recombining) two-step binomial tree.

   (a) Suppose that $\sigma_{t+1} = \sigma_{t+2} = 1$, i.e., the variance is constant (no GARCH). Derive the distribution of $R_{t+1:t+2}$, and its first four moments (mean, variance, skewness, kurtosis).

   (b) Next, let $\sigma_{t+1} = 1$ but let σ_{t+2}^2 be generated by the GARCH model
   $$\sigma_{t+2}^2 = 0.4 + 0.21R_{t+1}^2 + 0.6\sigma_{t+1}^2.$$
   Derive again the distribution of $R_{t+1:t+2}$, and its first four moments. Explain why, unlike the case of Question (a), we now have $\mathrm{Var}_t(R_{t+1:t+2}) > 2\sigma_{t+1}^2$. Also, compare the kurtosis in the two cases.

(c) Finally, $\sigma_{t+1} = 1$ but σ^2_{t+2} is generated by the GJR-GARCH model

$$\sigma^2_{t+2} = 0.4 + 0.44 I_{t+1} R^2_{t+1} + 0.6\sigma^2_{t+1},$$

where $I_{t+1} = 1$ if $R_{t+1} < 0$, and zero otherwise. Derive again the distribution of $R_{t+1:t+2}$, and its first four moments; compare these moments with cases (a) and (b).

4. Suppose that $z = (z_1, z_2)'$ is a vector with mean zero, variances 1, and covariance/correlation matrix

$$\Upsilon = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

(a) The Cholesky decomposition of a positive definite matrix Υ is $\Upsilon = \mathbf{L}\mathbf{L}'$, where $\mathbf{L}$ is a so-called lower triangular matrix, which means that it has elements 0 above the diagonal. Check that

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}$$

satisfies the defining property $\Upsilon = \mathbf{L}\mathbf{L}'$.

(b) Let z_1^u and z_2^u be two independent $N(0,1)$ random variables, and define

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mathbf{L} \begin{pmatrix} z_1^u \\ z_2^u \end{pmatrix} = \begin{pmatrix} z_1^u \\ \rho z_1^u + \sqrt{1-\rho^2} z_2^u \end{pmatrix}.$$

Check that $(z_1, z_2)'$ has covariance matrix Υ, so that in particular, $\mathrm{Var}(z_1) = \mathrm{Var}(z_2) = 1$ and $\mathrm{Cov}(z_1, z_2) = \rho$.

5. Discuss the possible advantages and disadvantages of Monte Carlo simulation versus filtered historical simulation.