Derivation of the Black-Scholes formula from the binomial tree

- This notebook is an extension of the week 5 slides for Computational Finance. It includes additional steps of the derivation of the Black-Scholes formula from the binomial tree for a European call option.
- The derivation is divided in two parts:
  1. We start with deriving the closed form expression for European call options based on a given tree.
  - Next, we let $N \to \infty$ to derive the Black-Scholes formula.

A Closed Form for European Options

- The price of a European option

$$C_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\max(S_T - K, 0)\right]$$

  depends only on S_T, so there is no need to use a tree explicitly to evaluate it.

- Let k denote the number of up moves of the stock , so that $N-k$ is the number of down moves. Then

$$S_T = S_0 u^k d^{N-k} = u^k u^{-(N-k)} = S_0 u^{2k-N},$$

  where we use that $u = 1/d = d^{-1}$.
- Under $\mathbb{Q}$, $k \sim \mathrm{Bin}(N,p)$, with pmf $f(k;N,p) = \binom{N}{k}p^k(1-p)^{N-k}$. Thus

$$C_0 = e^{-rT}\sum_{k=0}^{N} f(k;N,p)\max(S_0u^kd^{N-k} - K, 0).$$

- Note that this is an extension of the one period case: if $N=1$, then $f(0;1,p) = (1-p)$ and $f(1;1,p) = p$ and we get the risk-neutral pricng formulas

- For the cases that $S_T > K$, the maximum in the sum is zero and adds nothing to the value of the option. Therefore, we only have to consider the non-zero pay-offs, i.e. when $S_T > K$. Since the finaly payoff only increases in k, we need to find the values of k for which this holds to truncate the sum.
- Let a denote the minimum number of up moves so that $S_T > K$, i.e., it should hold that

$$\begin{aligned} S_T = S_0u^{2k-N} &> K \\ u^{2k-N} &> K/S_0, \end{aligned}$$

  taking logs on both sides gives

$$\begin{aligned} (2k-N)\log(u) &> \log(K/S_0) \\ 2k-N &> \log(K/S_0)/\log(u) \\ k &> N/2 + \log(K/S_0)/(2\log u). \end{aligned}$$

  This implies that $S_T > K$ if k is any of the integers greater than $N/2 + \log(K/S_0)/(2\log u)$. Note that Hull (2012) uses $\log(K/S_0) = -\log(S_0/K)$.

- Let a denote the minimum number of up moves so that $S_T > K$, i.e., it should hold that a is the smallest integer greater than $N/2 + \log(K/S_0)/(2\log u)$. Then

$$C_0 = e^{-rT}\sum_{k=a}^{N} f(k;N,p)\left[S_0u^kd^{N-k} - K\right].$$

- We can divide this into the sum of two terms:

$$C_0 = e^{-rT}\sum_{k=a}^{N} f(k;N,p)S_0u^kd^{N-k} - e^{-rT}\sum_{k=a}^{N} f(k;N,p)K.$$

- The *second term* is

$$\begin{aligned} \left[\sum_{k=a}^{N} f(k;N,p)\right]e^{-rT}K &= \left[\sum_{k=a}^{N} f(k;N,p)\right]e^{-rT}K \\ &= [1 - F(a-1;N,p)]e^{-rT}K \\ &= \bar{F}(a-1;N,p)e^{-rT}K, \end{aligned}$$

  where F is the binomial cdf and $\bar{F}$ is the survivor function.
- To derive the above expression, we use (i) that the binomial cdf is (by definition) $F(x;N,p) = \sum_{k=1}^{x} f(k;N,p)$, (ii) that $F(x;N,p) = 1$ because the probabilities should sum to one, and (iii) that the survivor function is $\bar{F}(k;N,p) = 1 - F(k;N,p)$.

- Let $p_* = \frac{pu}{pu+(1-p)d} = e^{-r\delta t}pu$, where we use that $pu + (1-p)d = \mathbb{E}^{\mathbb{Q}}\left[R_i\right] = e^{r\delta t}$. Note that $1 - p_* = 1 - \frac{pu}{pu+(1-p)d} = \frac{(1-p)d}{pu+(1-p)d} = e^{-r\delta t}(1-p)d$.

- The *first term* of the sum is

$$\begin{aligned} e^{-rT}S_0\sum_{k=a}^{N}\binom{N}{k}p^k(1-p)^{N-k}u^kd^{N-k} &= S_0\sum_{k=a}^{N}\binom{N}{k}[e^{-r\delta t}pu]^k[e^{-r\delta t}(1-p)d]^{N-k} \\ &= S_0\sum_{k=a}^{N}\binom{N}{k}p_*^k(1-p_*)^{N-k}, \end{aligned}$$

  where we use that $e^{rT} = e^{r\delta tN} = e^{r\delta tk}e^{r\delta t(N-k)}$.

- Then, in the same as how we got the expression for the second term, we have that the first term can be written as

$$S_0\sum_{k=a}^{N}\binom{N}{k}p_*^k(1-p_*)^{N-k} = S_0\bar{F}(a-1;N,p_*).$$

- Putting things together,

$$\begin{aligned} C_0 &= S_0\bar{F}(a-1;N,p_*) - \bar{F}(a-1;N,p)e^{-rT}K \\ &= S_0\mathbb{Q}^*(S_T > K) - \mathbb{Q}(S_T > K)e^{-rT}K. \end{aligned}$$

The Black-Scholes Formula as Continuous Time Limit

- Let's consider what happens if we let $N \to \infty$
- A first-order Taylor expansion, together with l'Hopital's rule, can be used to show that, for small δt,

$$p \approx \frac{1}{2}\left(1 + \sqrt{\delta t}\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right).$$

- Similarly,

$$p^* \approx \frac{1}{2}\left(1 + \sqrt{\delta t}\frac{r + \frac{1}{2}\sigma^2}{\sigma}\right).$$

- Next, Let $X_T \equiv \log S_T$. Then, because R_i is either $\log u$ or $\log d = -\log u$, and since $\log u = \sigma\sqrt{\delta t}$ from the tree calibration, we have

$$\begin{aligned} X_T = \log S_0 + \sum_{i=1}^{N} R_i &= \log S_0 + k\log u + (N-k)\log d \\ &= \log S_0 + (2k-N)\log u \\ &= \log S_0 + \sigma\sqrt{\delta t}(2k-N). \end{aligned}$$

- As $k \sim \mathrm{Bin}(N,p)$, we have $\mathbb{E}^{\mathbb{Q}}[k] = Np$ and $\mathrm{var}^{\mathbb{Q}}[k] = Np(1-p)$.
- Thus,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[X_T] &= \log S_0 + \sigma\sqrt{\delta t}N(2p-1) \to \log S_0 + (r - \frac{1}{2}\sigma^2)T \\ \mathrm{Var}^{\mathbb{Q}}[X_T] &= \sigma^2\delta t4Np(1-p) \to \sigma^2 T. \end{aligned}$$

  The derivation of both the expected value and the variance are below.

- The expected value of X_T is given by

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[X_T] &= \log S_0 + \sigma\sqrt{\delta t}(2\mathbb{E}^{\mathbb{Q}}[k] - N) \\ &= \log S_0 + \sigma\sqrt{\delta t}(Np - N) \\ &= \log S_0 + \sigma\sqrt{\delta t}N(2p-1) \end{aligned}$$

- As we let $N \to \infty$, this converges to ($\to$ means converges to)

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[X_T] &= \log S_0 + \sigma\sqrt{\delta t}N(2p-1) \\ &\to \log S_0 + \sigma\sqrt{\delta t}N\left(2\left[\frac{1}{2}\left(1 + \sqrt{\delta t}\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right)\right] - 1\right) \\ &= \log S_0 + \sigma\sqrt{\delta t}N\sqrt{\delta t}\frac{r - \frac{1}{2}\sigma^2}{\sigma} \\ &= \log S_0 + (r - \frac{1}{2}\sigma^2)T, \end{aligned}$$

  where we fill in the approximation for p (if $N \to \infty$) that we derived before and use that $\delta t = T/N$ such that $T = N\delta t$.

- We can similarly derive the expression for the variance.

$$\begin{aligned} \mathrm{Var}^{\mathbb{Q}}[X_T] &= \mathrm{Var}^{\mathbb{Q}}[\sigma\sqrt{\delta t}(2k-N)] \\ &= \sigma^2\delta t\mathrm{Var}^{\mathbb{Q}}[(2k-N)] \\ &= \sigma^2\delta t4\mathrm{Var}^{\mathbb{Q}}[k] \\ &= \sigma^2\delta t4Np(1-p), \end{aligned}$$

  using that the variance of a constant (e.g. $\log S_0$ or N) is zero.

- Before we take the limit of the variance, note that if $N \to \infty$, then

$$1 - p \approx 1 - \frac{1}{2}\left(1 + \sqrt{\delta t}\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right) = 1 - \frac{1}{2} - \frac{1}{2}\left(\sqrt{\delta t}\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right) = \frac{1}{2} - \frac{1}{2}\left(\sqrt{\delta t}\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right) = \frac{1}{2}$$

- Next, let's work out the limit of $p(1-p)$ if $N \to \infty$. Define $A \equiv \sqrt{\delta t}\frac{r - \frac{1}{2}\sigma^2}{\sigma}$. We have that if $N \to \infty$, then

$$p(1-p) \approx \frac{1}{2}(1+A)\frac{1}{2}(1-A) = \frac{1}{4}(1 - A + A - A^2) = \frac{1}{4}(1 - A^2) \to \frac{1}{4},$$

  because

$$A^2 = \delta t\left(\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right)^2 \to 0,$$

  since $\delta t \to 0$ and $\left(\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right)^2$ is a constant.

- As we let $N \to \infty$, the variance converges to

$$\begin{aligned} \mathrm{Var}^{\mathbb{Q}}[X_T] &= \sigma^2\delta t4Np(1-p) \\ &\to \sigma^2\delta t4N\frac{1}{4} \\ &= \sigma^2 T, \end{aligned}$$

  where we use that $\delta t = T/N$ such that $T = N\delta t$.

- Finally, as $N \to \infty$, the distribution of X_T tends to a normal. This follows from the *central limit theorem* and the fact that X_T is the sum of N i.i.d. terms.

- Thus, as $N \to \infty$,

$$\begin{aligned} \mathbb{Q}(S_T > K) = \mathbb{Q}(X_T > \log K) &= \mathbb{Q}\left(\frac{X_T - \mathbb{E}^{\mathbb{Q}}[X_T]}{\sqrt{\mathrm{Var}^{\mathbb{Q}}[X_T]}} > \frac{\log K - \mathbb{E}^{\mathbb{Q}}[X_T]}{\sqrt{\mathrm{Var}^{\mathbb{Q}}[X_T]}}\right) \\ &= 1 - \Phi\left(\frac{\log K - \mathbb{E}^{\mathbb{Q}}[X_T]}{\sqrt{\mathrm{Var}^{\mathbb{Q}}[X_T]}}\right) =: 1 - \Phi(-d_2) = \Phi(d_2), \text{where} \\ d_2 \equiv \frac{\mathbb{E}^{\mathbb{Q}}[X_T] - \log K}{\sqrt{\mathrm{Var}^{\mathbb{Q}}[X_T]}} &= \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \end{aligned}$$

- In the derivation, we use that X_T is normally distributed, such that $Z_T = \frac{X_T - \mathbb{E}^{\mathbb{Q}}[X_T]}{\sqrt{\mathrm{Var}^{\mathbb{Q}}[X_T]}} \sim N(0,1)$. Further, $1 - \Phi(-d_2) = 1 - (1 - \Phi(d_2)) = \Phi(d_2)$ holds due to symmetry of the standard normal distribution around zero.

- The same argument can be used to show that as $N \to \infty$, $\mathbb{Q}^*(S_T > K) = \Phi(d_1)$, where

$$d_1 \equiv d_2 + \sigma\sqrt{T} = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

- In summary, we have derived the *Black-Scholes formula*

$$\begin{aligned} C_0 &= S_0\Phi(d_1) - e^{-rT}K\Phi(d_2) \\ &=: BS(S_0, K, T, r, \sigma). \end{aligned}$$