General Information on the Final Examination

- The exam will be 2h, closed book, open questions. A pen and calculator are the only permitted tools.
- The relevant exam material is defined by the lecture slides. The reading list in the slides for Week 1 is intended as background material to help you study, but won't be tested *per se*.
- The focus of the exam is on the financial (and statistical) theory from Weeks 3–5. You will not be required to write code. You may, however, be required to *read* code and explain what it does, with the following exceptions:
  - Week 1, *Advanced Material on Functions*.
  - Week 2, anything related to `pandas`, `panda_datareader`, and `statsmodels`; in other words, only the material on `numpy` is relevant.
  - All of Week 3. Note that only the programming part is excluded, not the financial theory. This means that while I don't expect you to be able to, e.g., reproduce the code for plotting a histogram with a fitted density, I *do* expect you to be able to explain and interpret the output.

  The code discussed in Weeks 4–5, the exercises, and the second assignment are just applications of the material in Week 1 (plus `numpy`), so all these are fair game.
- The intent of this document is to give you a flavor of the type of questions that you may expect. In particular, there is no implication that only the material discussed herein is relevant for the final examination; *all* the material in the slides is relevant, with the exception of the list in the previous point.
- I am deliberately not calling this document a mock exam, because as an exam, it would be too long. The material would probably more adequate for an exam of 2.5h or so, not 2h. This is to give you more material to practice on.
- The layout however is identical to the final exam; a list of technical results will be included, as well as statistical distribution tables.

Practice Questions Computational Finance

1. Assume we have done `import numpy as np`.

   (a) For each of the following expressions, state whether the two arrays can be broadcast together, and if so, what the expression will evaluate to:

      i. `np.array([1, 2]) + np.array([7])`

      ii. `np.array([[1, 2], [3, 4]]) + np.array([7, 8])`

      iii. `np.array([[1, 2], [3, 4]]) + np.array([7, 8, 9])`

      iv. `np.array([[1, 2], [3, 4], [5, 6]]) + np.array([[7], [8], [9]])`

   (b) Suppose we define the following function:

```
def f(a, b):
    j = 0
    for e in b:
        a[j] -= e
        j += 1
    return a
```

   What will the following return?

      i. `a=np.array([1, 2]); b=np.array([1, 1]); f(a, b)`

      ii. `a=np.array([1, 2]); b=np.array([1, 1]); f(b, a)`

      iii. `a=np.array([1, 2]); b=np.array([1, 1]); f(b, a); f(b, a)`

2. You are given a set of N returns $\{R_t\}_{t=T-N+1}^{T}$.

   (a) Suppose that $N = 10$ and $\{R_t\} = \{-0.07, 0.87, 1.13, -0.48, 0.22, 0.94, -0.62, -0.01, 2.06, -0.08\}$. What is the historical 20% VaR?

   (b) Name 2 problems associated with the historical VaR and offer a solution for each of them.

   (c) You gather some more returns and produce the graph in Figure 1. Interpret it.

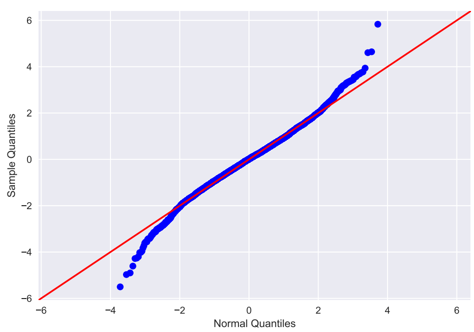

Figure 1

   (d) You decide to use a parametric model for VaR^p_{t+1}, in the form of a Student's t distribution. You fit the distribution to the data and obtain the following estimates: $(m, h, \nu) = (0.01, .3, 4)$, where m, h, and ν are respectively the location and scale parameters and the degrees of freedom. Obtain the (unconditional) 1% Student's t VaR.

3. Consider a two-period binomial stock price tree ($T = 2$, $N = 2$, $\delta t \equiv T/N = 1$), where $S_t = S_{t-1}u$ or $S_t = S_{t-1}d$, with $S_0 = 4$, $u = 2$ and $d = 1/u = \frac{1}{2}$. The risk-free bond satisfies $B_t = e^{rt}$, where $e^{r\delta t} = 5/4$.

   (a) Derive the risk free probability of an up move from the condition that the stock must earn the risk free rate under $\mathbb{Q}$.

   (b) Let $S_t^* \equiv \max_{0\leq j\leq t} S_j$. Obtain C_0, the no-arbitrage at time zero of a European *lookback option* whose payoff at time T is

   $$C_T = S_T^* - S_T.$$

   Hint: the payoff of the option is path-dependent, implying a non-recombining option tree.

4. (a) State the definition of a continuous-time martingale, and explain what it means.

   (b) Let

   $$dX_t = \nu dt + \sigma dW_t, \quad \nu = -\frac{1}{2}\sigma^2,$$

   a Brownian motion with drift. For each of the following functions, derive the SDE satisfied by $Y_t \equiv f(t, X_t)$, and state whether Y_t is a martingale.

   i. $f(t, X_t) = \exp(X_t)$;

   ii. $f(t, X_t) = tX_t$.

5. A researcher wants to obtain a Monte Carlo estimate of the price of a European-style option with payoff $C_T(S_T)$. The risk-neutral dynamics of S_t are specified by the SDE

   $$dS_t = \mu_t dt + \sigma_t dW_t.$$

   The market is free from arbitrage, so that $C_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT} C_T(S_T)]$.

   (a) Explain how the researcher could simulate a draw from the risk-neutral distribution of $C_T(S_T)$.

   (b) Let $X \equiv e^{-rT} C_T(S_T)$ and denote by $\{X_i\}_{i=1}^n$ independent draws of X as in the previous question. The researcher uses $\bar{X}_n \equiv \frac{1}{n}\sum_{i=1}^n X_i$ as an estimate of C_0. Derive a (asymptotic) confidence interval for C_0.

Technical results

- **Jarque-Bera Test**. Given a sample $\{R_t\}_{t=1}^T$ of returns, the Jarque-Bera test statistic is

$$JB \equiv T\left(\frac{S^2}{6} + \frac{(K-3)^2}{24}\right),$$

  where S and K are respectively the sample skewness and kurtosis. Under the null hypothesis that R_t is normally distributed, JB is (asymptotically) distributed as χ_2^2.

- **European Call Price in a Binomial Tree**. Let the stock price $\{S_t\}_{t\geq 0}$ be generated by a binomial tree with N steps, so that $S_{i+1} = S_i u$ or $S_{i+1} = S_i d$, $d = 1/u$. The continuously compounded interest rate r is constant. The no-arbitrage price of a European call option with strike price K and exercise date T is

$$C_0 = S_0 \bar{F}(a-1; N, p_*) - \bar{F}(a-1; N, p) e^{-rT} K,$$

  where $\delta t \equiv T/N$, $p \equiv \frac{e^{r\,\delta t} - d}{u-d}$, $p_* \equiv e^{-r\delta t} pu$, a is the smallest integer greater than $N/2 + \log(K/S_0)/(2\log u)$, and $\bar{F}$ is the survivor function of the binomial distribution.

- **Approximate Risk Neutral Probability for large** N. As $N \to \infty$ so that $\delta t \to 0$,

$$p \approx \frac{1}{2}\left(1 + \sqrt{\delta t}\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right) \text{ and } p^* \equiv e^{-r\delta t} pu \approx \frac{1}{2}\left(1 + \sqrt{\delta t}\frac{r + \frac{1}{2}\sigma^2}{\sigma}\right).$$

- **Binomial p.m.f.** If $K \sim \mathrm{Bin}(N, p)$ under $\mathbb{Q}$, then $\mathbb{Q}[K = k] = f(k; N, p) \equiv \binom{N}{k} p^k (1-p)^{N-k}$.

- **Black-Scholes Formula**. Let the index price $\{S_t\}_{t\geq 0}$ be generated by a geometric Brownian motion with volatility σ. The index pays a continuous dividend stream at rate q, and the continuously compounded interest rate r is constant. The no-arbitrage price of a European call option with strike price K and exercise date T is

$$C_t = BS(S_0, K, T, r, \sigma, q) \equiv e^{-q(T-t)} S_t \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2),$$

  where $\Phi(\cdot)$ is the cumulative standard normal distribution function, and

$$d_{1,2} = \frac{\log(S_t/K) + (r - q \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

- **Put-Call Parity**. For vanilla European options,

$$C_t - P_t = S_t e^{-q(T-t)} - K e^{-r(T-t)}.$$

- **Tehranchi Bounds**. For European calls in the Black-Scholes model, the implied volatility σ_I satisfies

$$-\Phi^{-1}\left(\frac{S_0 - C_0^{obs}}{2\min(S_0, e^{-rT}K)}\right) \leq \frac{\sqrt{T}}{2}\sigma_I \leq -\Phi^{-1}\left(\frac{S_0 - C_0^{obs}}{S_0 + e^{-rT}K}\right).$$

- **Mean of Lognormal**. Let $z \sim \mathrm{N}(\mu, \sigma^2)$. Then $y \equiv \exp(z)$ has a lognormal distribution, and $\mathbb{E}[y] = \exp(\mu + \frac{1}{2}\sigma^2)$.

- **Brownian Motion**. A standard Brownian Motion is a stochastic process $\{W_t\}_{t\geq 0}$ satisfying

  1. $W_0 = 0$;
  2. The increments $W_t - W_s$ are independent for all $0 \leq s < t$;
  3. $W_t - W_s \sim N(0, t-s)$ for all $0 \leq s \leq t$;
  4. Continuous sample paths.

- **Ito's Lemma**. Let $\{X_t\}$ be an Ito process given by $dX_t = \mu_t dt + \sigma_t dW_t$, and let $f(t,x)$ be a function with continuous first- and second-order partial derivatives

  $$\dot{f}(t,x) \equiv \frac{\partial f(t,x)}{\partial t}, \qquad f'(t,x) \equiv \frac{\partial f(t,x)}{\partial x}, \qquad f''(t,x) \equiv \frac{\partial^2 f(t,x)}{\partial x^2}.$$

  Consider a process Y_t defined as $Y_t \equiv f(t, X_t)$. Clearly Y_t is a stochastic process; specifically, Ito's lemma says that

  $$dY_t \equiv df(t,X_t) = \dot{f}(t,X_t)dt + f'(t,X_t)dX_t + \frac{1}{2}f''(t,X_t)(dX_t)^2,$$

  where $(dX_t)^2 = \sigma_t^2 dt$. This implies that Y_t is an Ito process as well, with drift $\dot{f}(t,X_t) + f'(t,X_t)\mu_t + \frac{1}{2}f''(t,X_t)\sigma_t^2$ and volatility $f'(t,X_t)\sigma_t$.

- **CLT and Weak LLN**. Let $\theta \equiv \mathbb{E}[X]$, where X is a random variable with finite variance σ^2. Suppose we have a sample $\{X_i\}_{i\in\{1,\ldots,n\}}$ of independent draws for X, and let $\bar{X}_n \equiv \frac{1}{n}\sum_{i=1}^n X_i$. Then

  $$\bar{X}_n \xrightarrow{p} \theta \quad \text{(Weak Law of Large Numbers);}$$
  $$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N\left(0, \sigma^2\right) \quad \text{(Central Limit Theorem).}$$

$\nu \setminus p$	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
1	-63.657	-31.821	-12.706	-6.314	-3.078	3.078	6.314	12.706	31.821	63.657
2	-9.925	-6.965	-4.303	-2.920	-1.886	1.886	2.920	4.303	6.965	9.925
3	-5.841	-4.541	-3.182	-2.353	-1.638	1.638	2.353	3.182	4.541	5.841
4	-4.604	-3.747	-2.776	-2.132	-1.533	1.533	2.132	2.776	3.747	4.604
5	-4.032	-3.365	-2.571	-2.015	-1.476	1.476	2.015	2.571	3.365	4.032
6	-3.707	-3.143	-2.447	-1.943	-1.440	1.440	1.943	2.447	3.143	3.707
7	-3.499	-2.998	-2.365	-1.895	-1.415	1.415	1.895	2.365	2.998	3.499
8	-3.355	-2.896	-2.306	-1.860	-1.397	1.397	1.860	2.306	2.896	3.355
9	-3.250	-2.821	-2.262	-1.833	-1.383	1.383	1.833	2.262	2.821	3.250
10	-3.169	-2.764	-2.228	-1.812	-1.372	1.372	1.812	2.228	2.764	3.169
11	-3.106	-2.718	-2.201	-1.796	-1.363	1.363	1.796	2.201	2.718	3.106
12	-3.055	-2.681	-2.179	-1.782	-1.356	1.356	1.782	2.179	2.681	3.055
13	-3.012	-2.650	-2.160	-1.771	-1.350	1.350	1.771	2.160	2.650	3.012
14	-2.977	-2.624	-2.145	-1.761	-1.345	1.345	1.761	2.145	2.624	2.977
15	-2.947	-2.602	-2.131	-1.753	-1.341	1.341	1.753	2.131	2.602	2.947
16	-2.921	-2.583	-2.120	-1.746	-1.337	1.337	1.746	2.120	2.583	2.921
17	-2.898	-2.567	-2.110	-1.740	-1.333	1.333	1.740	2.110	2.567	2.898
18	-2.878	-2.552	-2.101	-1.734	-1.330	1.330	1.734	2.101	2.552	2.878
19	-2.861	-2.539	-2.093	-1.729	-1.328	1.328	1.729	2.093	2.539	2.861
20	-2.845	-2.528	-2.086	-1.725	-1.325	1.325	1.725	2.086	2.528	2.845
21	-2.831	-2.518	-2.080	-1.721	-1.323	1.323	1.721	2.080	2.518	2.831
22	-2.819	-2.508	-2.074	-1.717	-1.321	1.321	1.717	2.074	2.508	2.819
23	-2.807	-2.500	-2.069	-1.714	-1.319	1.319	1.714	2.069	2.500	2.807
24	-2.797	-2.492	-2.064	-1.711	-1.318	1.318	1.711	2.064	2.492	2.797
25	-2.787	-2.485	-2.060	-1.708	-1.316	1.316	1.708	2.060	2.485	2.787
26	-2.779	-2.479	-2.056	-1.706	-1.315	1.315	1.706	2.056	2.479	2.779
27	-2.771	-2.473	-2.052	-1.703	-1.314	1.314	1.703	2.052	2.473	2.771
28	-2.763	-2.467	-2.048	-1.701	-1.313	1.313	1.701	2.048	2.467	2.763
29	-2.756	-2.462	-2.045	-1.699	-1.311	1.311	1.699	2.045	2.462	2.756
30	-2.750	-2.457	-2.042	-1.697	-1.310	1.310	1.697	2.042	2.457	2.750
40	-2.704	-2.423	-2.021	-1.684	-1.303	1.303	1.684	2.021	2.423	2.704
50	-2.678	-2.403	-2.009	-1.676	-1.299	1.299	1.676	2.009	2.403	2.678
60	-2.660	-2.390	-2.000	-1.671	-1.296	1.296	1.671	2.000	2.390	2.660
70	-2.648	-2.381	-1.994	-1.667	-1.294	1.294	1.667	1.994	2.381	2.648
80	-2.639	-2.374	-1.990	-1.664	-1.292	1.292	1.664	1.990	2.374	2.639
90	-2.632	-2.368	-1.987	-1.662	-1.291	1.291	1.662	1.987	2.368	2.632
100	-2.626	-2.364	-1.984	-1.660	-1.290	1.290	1.660	1.984	2.364	2.626
1000	-2.581	-2.330	-1.962	-1.646	-1.282	1.282	1.646	1.962	2.330	2.581

Table 1: Percentage point function $F_\nu^{-1}(p)$ of the Student's t distribution.

$\nu \setminus p$	0.900	0.950	0.975	0.990	0.995
1	2.706	3.841	5.024	6.635	7.879
2	4.605	5.991	7.378	9.210	10.597
3	6.251	7.815	9.348	11.345	12.838
4	7.779	9.488	11.143	13.277	14.860
5	9.236	11.070	12.833	15.086	16.750
6	10.645	12.592	14.449	16.812	18.548
7	12.017	14.067	16.013	18.475	20.278
8	13.362	15.507	17.535	20.090	21.955
9	14.684	16.919	19.023	21.666	23.589
10	15.987	18.307	20.483	23.209	25.188
11	17.275	19.675	21.920	24.725	26.757
12	18.549	21.026	23.337	26.217	28.300
13	19.812	22.362	24.736	27.688	29.819
14	21.064	23.685	26.119	29.141	31.319
15	22.307	24.996	27.488	30.578	32.801
16	23.542	26.296	28.845	32.000	34.267
17	24.769	27.587	30.191	33.409	35.718
18	25.989	28.869	31.526	34.805	37.156
19	27.204	30.144	32.852	36.191	38.582
20	28.412	31.410	34.170	37.566	39.997
21	29.615	32.671	35.479	38.932	41.401
22	30.813	33.924	36.781	40.289	42.796
23	32.007	35.172	38.076	41.638	44.181
24	33.196	36.415	39.364	42.980	45.559
25	34.382	37.652	40.646	44.314	46.928
26	35.563	38.885	41.923	45.642	48.290
27	36.741	40.113	43.195	46.963	49.645
28	37.916	41.337	44.461	48.278	50.993
29	39.087	42.557	45.722	49.588	52.336
30	40.256	43.773	46.979	50.892	53.672
40	51.805	55.758	59.342	63.691	66.766
50	63.167	67.505	71.420	76.154	79.490
60	74.397	79.082	83.298	88.379	91.952
70	85.527	90.531	95.023	100.425	104.215
80	96.578	101.879	106.629	112.329	116.321
90	107.565	113.145	118.136	124.116	128.299
100	118.498	124.342	129.561	135.807	140.169

Table 2: Percentage point function $F_\nu^{-1}(p)$ of the χ^2_ν distribution.