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1 Multiple Choice

1.1

A linear equation system Ax = b has a single unique solution if?
A.) A is square
B.) A is symmetric
C.) A is square and full rank
D.) A is square and symmetric
Answer: C - A is square and full rank

1.2

Given a K-dimensional vector X (i.e., containing K elements), which one of the following operations cannot be represented by a linear transformation AX, with regard to some appropriate matrix A?
A.) Keeping the first three elements of X and discarding others
B.) Amplifying the vector norm of X by two times
C.) In the polar coordinate system, rotating the angle of X by 45 degrees
D.) Sorting all elements by magnitude in the descending order
Answer: D - Sorting all elements by magnitude in the descending order

2 Short Answer

2.1

2.1.1

If $v_1, ..., v_m \in V$ are linearly independent, and $T : V \to W$ is a linear operator, is it true that $Tv_1, ..., Tv_m \in W$ must be linearly independent? Explain your reasoning.

Answer: It is not true. Suppose the linear operator T transforms all vectors to the 0 vector. Then, every vector $v_1, ..., v_m \in V$ will get transformed to the 0 vector. Thus, $Tv_1, ..., Tv_m \in W$ will all be 0 vectors. Since any zero vector can be represented as a linear combination of any other zero vectors, $Tv_1, ..., Tv_m \in W$ will not be linearly independent.

2.1.2

If $v_1, ..., v_m \in V$ are dependent, and $T : V \to W$ is a linear operator, is it true that $Tv_1, ..., Tv_m \in W$ must be dependent? Explain your reasoning.

Answer: Yes this is true. If $v_1, ..., v_m \in V$ are dependent, then there exist a subset of vectors $v_x, v_y, ..., v_z \in V$ such that $\alpha v_x + \beta v_y + ... + \lambda v_z = 0$ where at least 1 of the coefficients (out of $\alpha, \beta, ..., \lambda$) are non-zero. Applying the linear transformation T, we get the following:

$$\alpha v_x + \beta v_y + ... + \lambda v_z = 0$$
$$T(\alpha v_x + \beta v_y + ... + \lambda v_z) = T(0)$$
$$T(\alpha v_x) + T(\beta v_y) + ... + T(\lambda v_z) = 0$$
Since T is a linear operator, we can factor out the coefficients
$$\alpha T(v_x) + \beta T(v_y) + ... + \lambda T(v_z) = 0$$

Since at least 1 of the coefficients are non-zero for the equation $\alpha T(v_x) + \beta T(v_y) + ... + \lambda T(v_z) = 0$, we can conclude that $Tv_1, ..., Tv_m \in W$ is dependent.

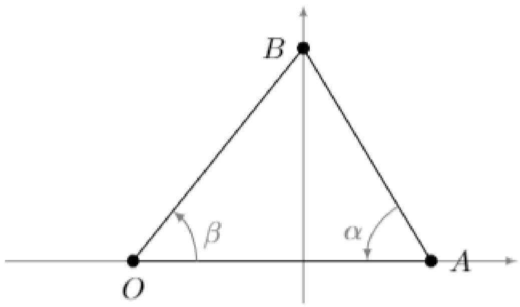
2.2

Hand-compute the euclidean distance of the following two vectors: $\vec{a} = [10, 20, 15, 10, 5]$, $\vec{b} = [12, 24, 18, 8, 7]$. Please write down your computation process.

Answer: Suppose $d(\vec{a}, \vec{b})$ represents the euclidean distance between $\vec{a}$ and $\vec{b}$.

$$\begin{aligned} d(\vec{a}, \vec{b}) &= \sqrt{(12-10)^2 + (24-20)^2 + (18-15)^2 + (8-10)^2 + (7-5)^2} \\ &= \sqrt{2^2 + 4^2 + 3^2 + (-2)^2 + 2^2} \\ &= \sqrt{4 + 16 + 9 + 4 + 4} \\ &= \sqrt{37} \end{aligned}$$

2.3



2.3.1

Express $|OA|$ (the vector norm, or length) using $|OB|$, $|AB|$, α and β. Hint: use projections.

Answer: Consider the vertical line that goes through B and intersects line OA perpendicularily. Consider this point of intersection on line OA as Θ. Thus, $|O\Theta| + |\Theta A| = |OA|$.

Given the above premise, we can calculate $|OA|$ using the following logic:

$$\begin{aligned}
cos(\beta) &= \frac{|O\Theta}{|OB|} \rightarrow |O\Theta| = cos(\beta) \cdot |OB| \\
cos(\alpha) &= \frac{|\Theta A|}{|AB|} \rightarrow |\Theta A| = cos(\alpha) \cdot |AB| \\
|OA| &= |O\Theta| + |\Theta A| \\
&= (cos(\beta) \cdot |OB|) + (cos(\alpha) \cdot |AB|)
\end{aligned}$$

2.3.2

Express $|AB|$ using $|OB|$, $|OA|$, and β.

Answer: Using the Law of Cosines, we get the following:

$$|AB| = \sqrt{|OB|^2 + |OA|^2 - 2|OB||OA|cos(\beta)}$$

2.4

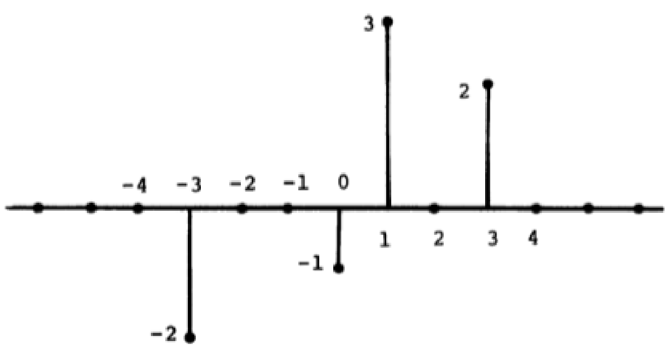
Hand-compute the matrix-vector product: $B = A\vec{x}$, where $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$.

Answer:

$$\begin{aligned} B &= A\vec{x} \\ &= \begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (1)(5) + (-1)(7) + (2)(1) \\ (0)(5) + (-3)(7) + (1)(1) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -20 \end{bmatrix} \end{aligned}$$

2.5

A sample sequence $x(n)$ is shown in Figure 1. The unit impulse signal is denoted by $\delta(n)$ and is defined as $\delta(n) = \{1 \text{ for } n = 0, 0 \text{ for } n \neq 0\}$. Express $x(n)$ as a linear combination of weighted and delayed unit samples.



Answer: $x(n) = -2\delta(n+3) - \delta(n) + 3\delta(n-1) + 2\delta(n-3)$

2.6

Hand-compute eigenvalues of the following matrix: $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. Please write down your computation process.

Answer: Suppose that λ represents the eigenvalues. Then:

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \\
I &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
A - \lambda I &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\
&= \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} \\
|A - \lambda I| &= | \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} | \\
&= (-\lambda)(-3-\lambda) - (1)(-2) \\
&= (3\lambda + \lambda^2) + 2 \\
&= \lambda^2 + 3\lambda + 2 \\
&= (\lambda + 1)(\lambda + 2) \\
\lambda &= -1 \\
\lambda &= -2
\end{aligned}$$

Thus, the eigenvalues of matrix $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ are -1 and -2.