Sets

The fundamental discrete structure on which all other discrete structures are built.

Sets

- *Definition*. A *Set* is any well defined collection of "objects."
- *Definition*. The *elements* of a set are the objects in a set.
- *Notation.* Usually we denote sets with upper-case letters, elements with lower-case letters. The following notation is used to show set membership
- $x \in A$ means that *x* is a member of the set A
- $x \notin A$ means that *x* is not a member of the set A.

Ways of Describing Sets

- List the elements

$$A = \{1,2,3,4,5,6\}$$

- Give a verbal description
  - "A is the set of all integers from 1 to 6, inclusive"
- Give a mathematical inclusion rule

$$A = \{\text{Integers } x \mid 1 \le x \le 6\}$$

Some Special Sets

- The Null Set or Empty Set. This is a set with no elements, often symbolized by

$$\varnothing$$

- The Universal Set. This is the set of all elements currently under consideration, and is often symbolized by

U or Ω

The Empty Set

- The empty set is a special set. It contains no elements. It is usually denoted as { } or

  $\varnothing$
- The empty set is always considered a subset of any set.
- Do not be confused by this question:
- Is this set {0} empty?
- It is not empty! It contains the element zero.

Universal Set and Subsets

- The **Universal Set** denoted by *U* is the set of all possible elements used in a problem.
- When every element of one set is also an element of another set, we say the first set is a **subset**.
- Example A={1, 2, 3, 4, 5} and B={2, 3}

  We say that B is a subset of A. The notation we use is B $\subseteq$ A.
- Let S={1,2,3}, list all the subsets of S.
- The subsets of S are $\varnothing$, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}.

Membership Relationships

- *Definition.* Subset.

  $A \subseteq B$ "A is a subset of B"

We say "A is a subset of B" if $x \in A \Rightarrow x \in B$, i.e., all the members of A are also members of B. The notation for subset is very similar to the notation for "less than or equal to," and means, in terms of the sets, "included in or equal to."

Membership Relationships

- *Definition.* Proper Subset.

$A \subset B$ A is a proper subset of B”

We say “A is a proper subset of B” if all the members of A are also members of B, but in addition there exists at least one element *c* such that $c \in B$ but $c \notin A$. The notation for subset is very similar to the notation for “less than,” and means, in terms of the sets, “included in but not equal to.”

Combining Sets – Set Union

- $$A \cup B$$

- "*A* union *B*" is the set of all elements that are in *A*, or *B*, or both.

- This is similar to the logical "or" operator.

Combining Sets – Set Intersection

- $$A \cap B$$
- "*A* intersect *B*" is the set of all elements that are in *both* *A* and *B*.
- This is similar to the logical "and"

Set Complement

- $$\overline{A}$$
- "*A* complement," or "not *A*" is the set of all elements not in *A*.
- The complement operator is similar to the logical not, and is reflexive, that is,

$$\overline{\overline{A}} = A$$

Set Difference

- $$A - B$$

- The set difference "A minus B" is the set of elements that are in A, with those that are in B subtracted out. Another way of putting it is, it is the set of elements that are in A, *and* not in B, so

$$A - B = A \cap \overline{B}$$

Examples

$$\Omega = \{1,2,3,4,5,6\}$$

$$A = \{1,2,3\} \qquad B = \{3,4,5,6\}$$

$$A \cap B = \{3\} \qquad A \cup B = \{1,2,3,4,5,6\}$$

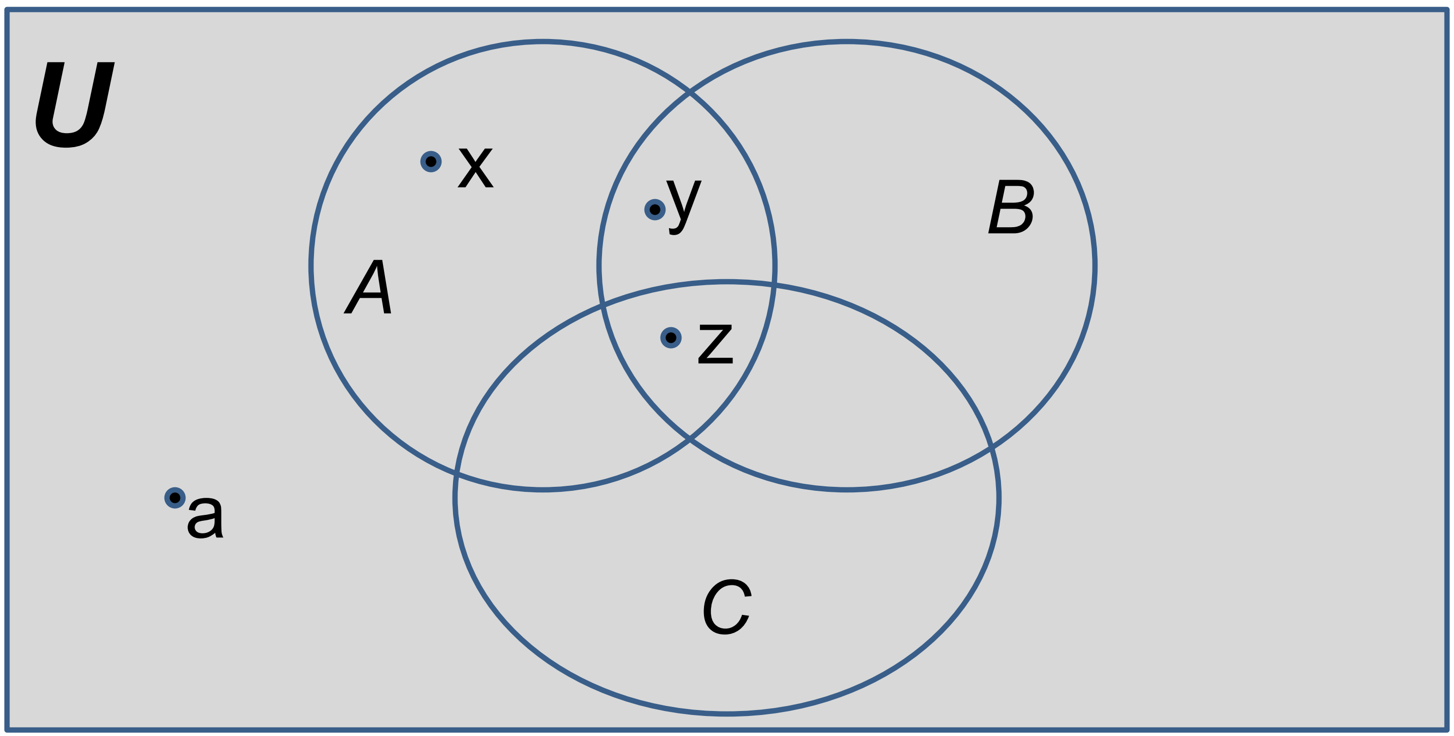
$$B - A = \{4,5,6\} \qquad \overline{B} = \{1,2\}$$

Special Sets of Numbers

- **N** = The set of natural numbers.
  = {1, 2, 3, …}.
- **W** = The set of whole numbers.
  ={0, 1, 2, 3, …}
- **Z** = The set of integers.
  = { …, -3, -2, -1, 0, 1, 2, 3, …}
- **Q** = The set of rational numbers.
  ={x| x=p/q, where p and q are elements of **Z and**
  q ≠ 0 }
- **H** = The set of irrational numbers.
- **R** = The set of real numbers.
- **C** = The set of complex numbers.

Venn Diagram: Example

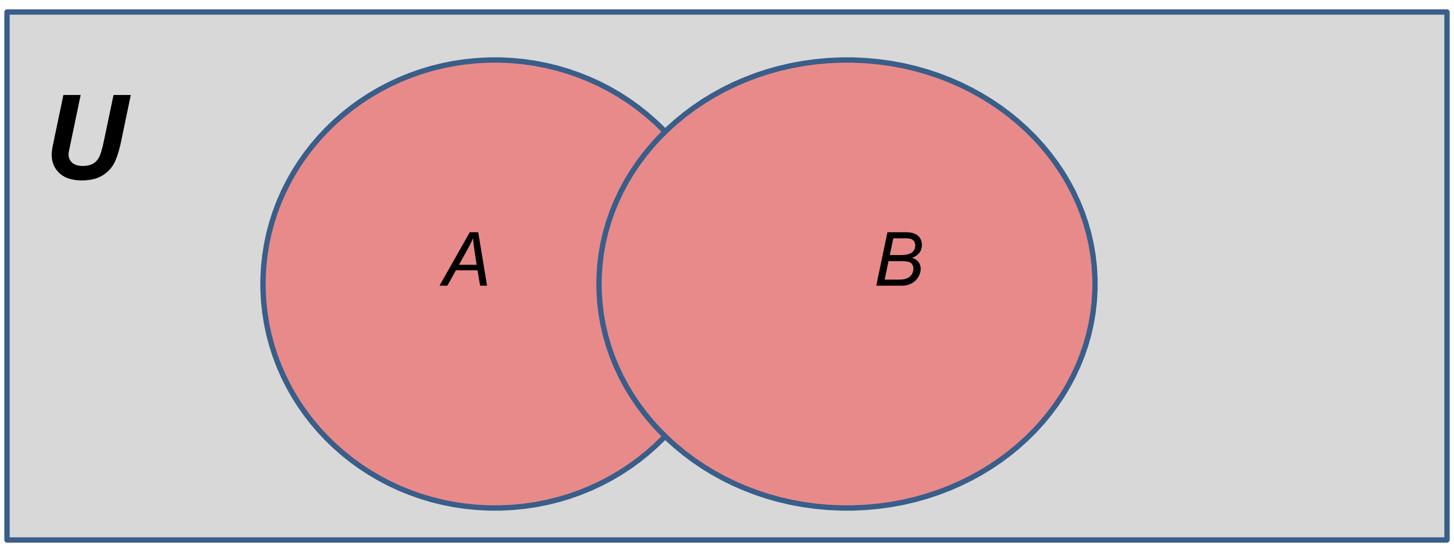
- A set can be represented graphically using a Venn Diagram



Set Operators: Union

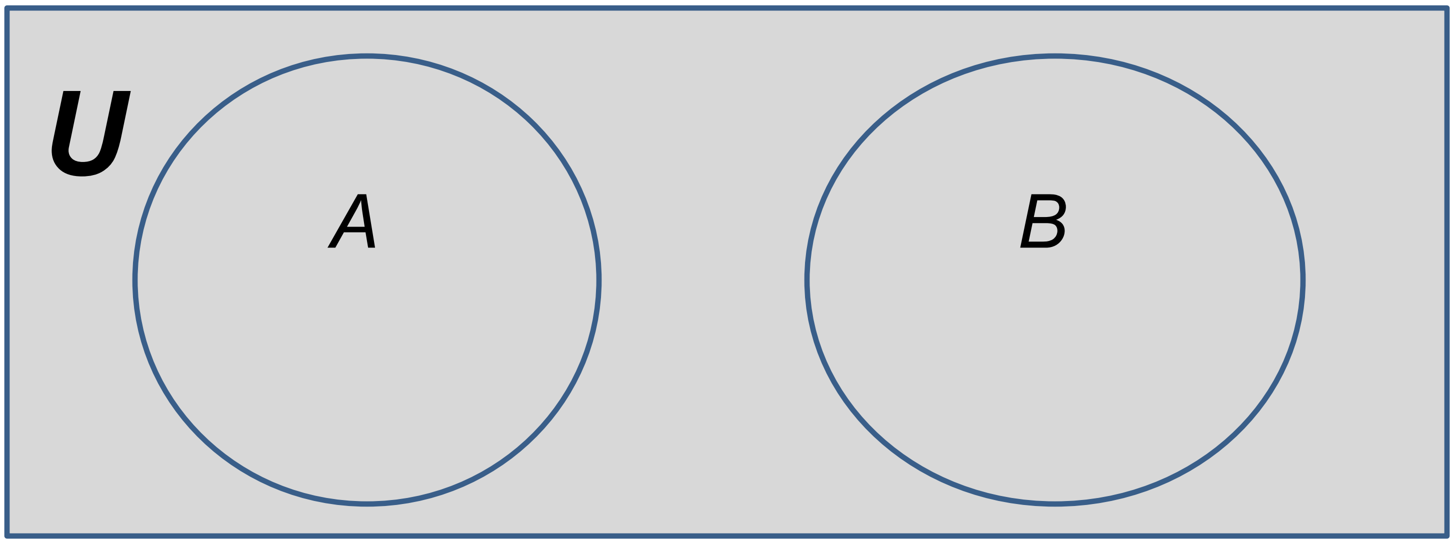
- **Definition**: The union of two sets A and B is the set that contains all elements in A, B, or both. We write:

$$A \cup B = \{ x \mid (a \in A) \vee (b \in B) \}$$



Disjoint Sets

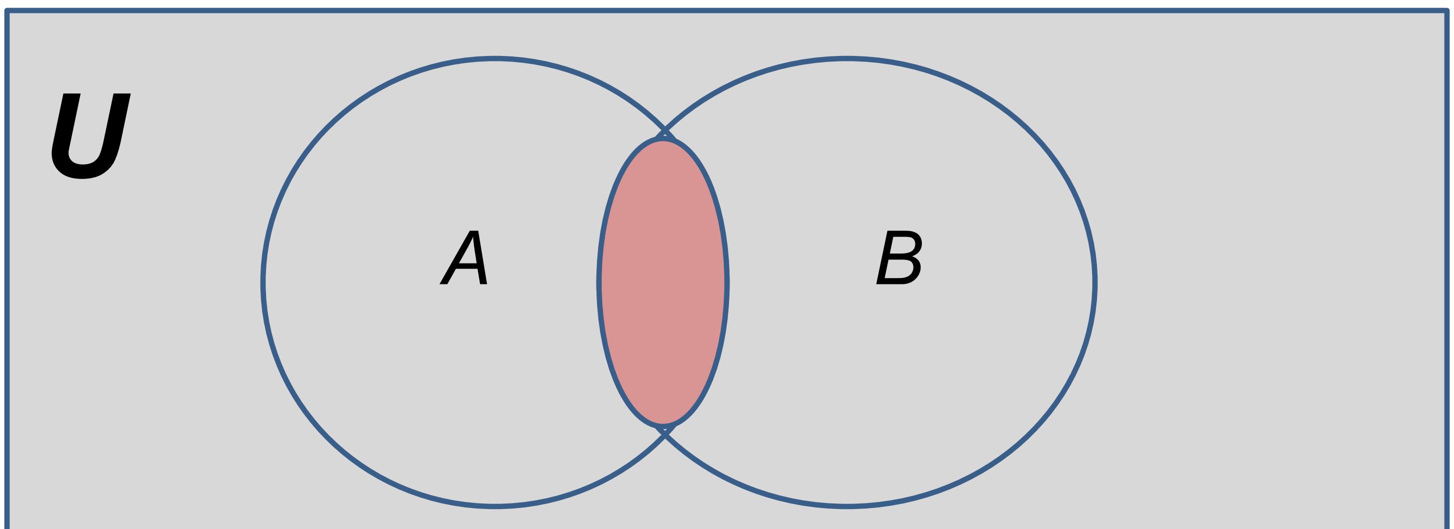
- **Definition**: Two sets are said to be disjoint if their intersection is the empty set: $A \cap B = \varnothing$



Set Operators: Intersection

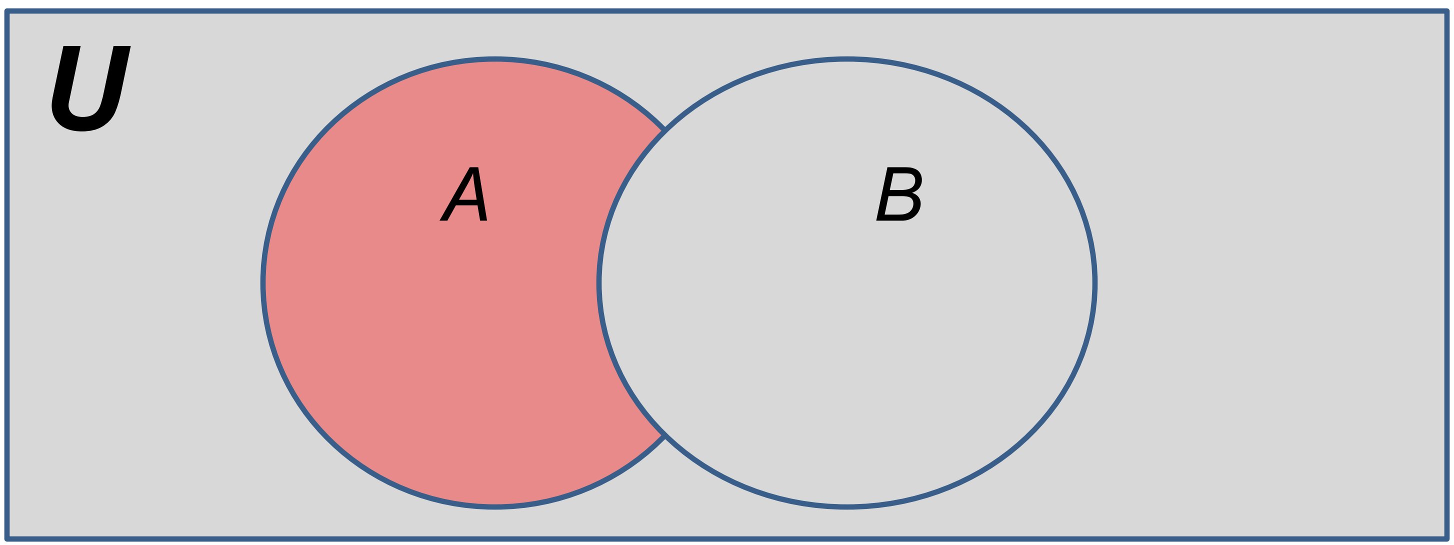
- **Definition**: The intersection of two sets A and B is the set that contains all elements that are element of both A and B. We write:

$$A \cap B = \{ x \mid (a \in A) \wedge (b \in B) \}$$



Set Difference

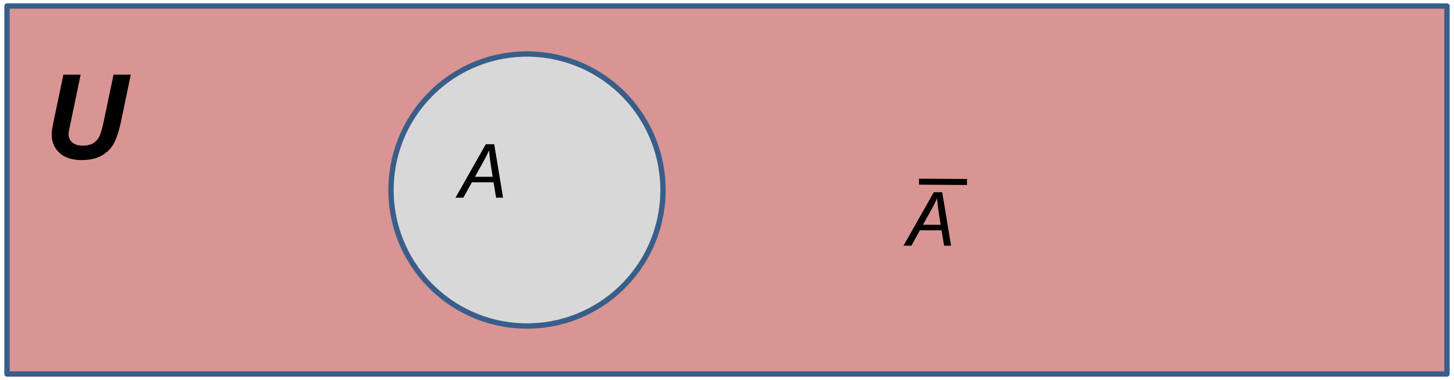
- **Definition**: The difference of two sets A and B, denoted A\B ($\setminus$) or A−B, is the set containing those elements that are in A but not in B



Set Complement

- **Definition**: The complement of a set A, denoted $\bar{A}$ ($\bar$), consists of all elements not in A. That is the difference of the universal set and A: U\A

$$\bar{A} = A^C = \{x \mid x \notin A\}$$



Mutually Exclusive and Exhaustive Sets

- *Definition.* We say that a group of sets is *exhaustive* of another set if their union is equal to that set. For example, if $A \cup B = C$ we say that A and B are exhaustive with respect to C.
- *Definition.* We say that two sets A and B are *mutually exclusive* if $A \cap B = \varnothing$, that is, the sets have no elements in common.

Some Test Questions

$$\text{If } A \subset B \text{ then}$$

$$A \cap B = ?$$

Some Test Questions

If $A \subset B$ then

$$A \cup B = ?$$

Cardinal Number

- The **Cardinal Number** of a set is the number of elements in the set and is denoted by $n(A)$.
- Let $A=\{2,4,6,8,10\}$, then $n(A)=5$.
- The Cardinal Number formula for the union of two sets is

$$n(A \cup B)=n(A) + n(B) - n(A \cap B).$$

- The Cardinal number formula for the complement of a set is $n(A) + n(A')=n(U)$.

Power Set (1)

- **Definition**: The power set of a set S, denoted P(S), is the set of all subsets of S.
- Examples
  - Let A={a,b,c}, P(A)={∅,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}
  - Let A={{a,b},c}, P(A)={∅,{{a,b}},{c},{{a,b},c}}
- Note: the empty set ∅ and the set itself are always elements of the power set.

Power Set (2)

- The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set
- **Fact**: Let S be a set such that $|S|=n$, then

$$|P(S)| = 2^n$$

Computer Representation of Sets (1)

- There really aren't ways to represent <u>infinite</u> sets by a computer since a computer has a finite amount of memory
- If we assume that the universal set U is finite, then we can easily and effectively represent sets by <u>bit vectors</u>
- Specifically, we <u>force</u> an ordering on the objects, say:

$$U=\{a_1, a_2,\ldots,a_n\}$$

- For a set $A \subseteq U$, a bit vector can be defined as, for $i=1,2,\ldots,n$
  - $b_i=0$ if $a_i \notin A$
  - $b_i=1$ if $a_i \in A$

Computer Representation of Sets

- There are *many* ways to represent sets.
- Which is best *depends* on the particular sets & operations.
- Bit string: Let | U | = *n*, where *n* is not "too" large: U = { a_1, …, a_n }.

  Represent set A as an *n*-bit string.

  ```
  If ( a_i ∈ A )
      bit i = 1;
  else
      bit i = 0.
  ```

  Operations ∪ , ∩ , _ are performed bitwise.
- In Java, [*Set* is the name of an interface]().
- Consider a Java *set* class (e.g., BitStringSet), where | U | is a constructor parameter.
  - What data structures might be useful to implement the interface?
  - What public methods might you want?
  - How would you implement them?

Computer Representation of Sets (2)

- Examples
  - Let U={0,1,2,3,4,5,6,7} and A={0,1,6,7}
  - The bit vector representing A is: 1100 0011
  - How is the empty set represented?
  - How is U represented?
- Set operations become trivial when sets are represented by bit vectors
  - Union is obtained by making the bit-wise OR
  - Intersection is obtained by making the bit-wise AND

Computer Representation of Sets (3)

- Let U={0,1,2,3,4,5,6,7}, A={0,1,6,7}, B={0,4,5}
- What is the bit-vector representation of B?
- Compute, bit-wise, the bit-vector representation of $A \cap B$
- Compute, bit-wise, the bit-vector representation of $A \cup B$
- What sets do these bit vectors represent?

Prove $\underline{A \cup B} = \underline{A} \cap \underline{B}$

Venn diagrams

1. Draw the Venn diagram of the LHS.

2. Draw the Venn diagram of the RHS.

3. Explain that the regions match.

Prove the following using Venn diagrams

- Prove the following using Venn diagrams:
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $(A \cap B)' = A' \cup B'$
- $(A \cap B) \cup (A \cap B') = A$.

- Find the power set of each of these sets where a and b are distinct elements.
- $\{a\}$
- $\{a, b\}$
- $\{\emptyset, \{\emptyset\}\}$

- Let **E** denote the set of even integers and **O** denote the set of odd integers. As usual, let **Z** denote the set of all integers. Determine each of these sets.
- **E** $\cup$ **O**
- **E** $\cap$ **O**
- **Z** – **E**
- **Z** – **O**