<u>Chapter 3 – INDUCTION AND OTHER TRICKS</u>

3.1. <u>GENERALITIES</u>

Statements can be *general*, like the following three examples

- All men are mortal.
- Every citizen of United States over the age of 18 is entitled to vote in elections.
- Every positive integer whose sum of digits is a multiple of 3, is divisible by 3.

Or they can be *particular*, like

- Socrates is a man.
- John Green is a citizen of United States and he is 20 years old.
- The integer 87 is divisible by 3.

Arguing *from general to particular* is often called a **<u>deduction</u>**, e.g. the classic example:

- All men are mortal.
- Socrates is a man.
- <u>Hence</u>: Socrates is mortal.

or

- Every positive integer whose sum of digits is a multiple of 3, is divisible by 3.
- The sum of the digits of the integer 87 is 15.
- 15 is a multiple of 3.
- <u>Hence</u>: 87 is divisible by 3.

Arguing *from particular to general* is called **<u>induction</u>**. Inductive arguments must be made very cautiously so as not to lead to false conclusions. For instance

1. 135 is divisible by 5.
2. <u>Conclusion</u>: every integer whose last digit is 5 is divisible by 5.

In this case the conclusion is correct. But when we draw

3. <u>Conclusion</u>: every 3-digit integer is divisible by 5.

we obtain a false conclusion. So we must be very careful.

Here is another example, used by Euler[1] to illustrate the pitfalls of careless inductive reasoning. Consider the polynomial $x^2 + x + 41$. If we substitute $x = 0$ we obtain the value 41 which is a prime number. Similarly, if we substitute $x = 1$ we obtain 43, again a prime number. Substituting $x = 2$ gives 47, prime number again. Coincidence? It turns out that substituting $x = 0, 1, 2, 3, \ldots, 39$ will always yield a prime number – quite amazing! Unfortunately, substituting $x = 40$ results 41^2, which is a composite number.

In mathematics we formulate a principle, called the ***<u>principle of mathematical induction</u>***, that allows us to draw correct conclusions in precisely and carefully specified circumstances. It is a powerful method that has many applications. We will spend some time on the ideas behind this method and do many examples.

3.2. <u>PRINCIPLE OF MATHEMATICAL INDUCTION</u> (<u>PMI</u>)

The ***Principle of Mathematical Induction***, abbreviated PMI, can be expressed in many forms. Here is one:

> Let $n_0 \in \mathbb{Z}$ and let $S \subseteq \mathbb{Z}$ such that the following conditions hold:
>
> 1. $[Basis]\, n_0 \in S$.
> 2. $[Induction\ Step]\, \forall n \geq n_0 \left[n \in S \ \Rightarrow \ n+1 \in S \right]$.
>
> Then every integer $n \geq n_0$ belongs to *S*.

Comments: (a) Note that n_0 can be positive, negative or zero. In many cases n_0 is 0 or 1. (b) When $n_0 = 0$, the consequence can be expressed as $\mathbb{N} \subseteq S$.

Let *P* be a property of integers, i.e. for any specific integer *n*, the statement *P*(*n*) is either true or false. For example, *P*(*n*) could be the statement "*n* is divisible by 3" which happens to be true for $n = 21$ but is false for $n = 20$. *P*(*n*) could also be a compound statement, like "*n* is divisible by 7 and by 20, but is not divisible by 5."

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[1] **Leonhard Euler** (1707 –1783) was a Swiss mathematician, who spent most of his life in Russia and Germany. Euler is considered to be the preeminent mathematician of the 18th century and one of the greatest of all time.

PMI can also be expressed in another form

> Let $n_0 \in \mathbb{Z}$ and let P be a property of integers such that the following conditions hold:
>
> 1. [*Basis*] $P(n_0)$ is true.
> 2. [*Induction Step*] $\forall n \geq n_0 \left[P(n) \quad \Rightarrow \quad P(n+1) \right]$.
>
> Then for every integer $n \geq n_0$, $P(n)$ is true.

The connection between the two forms of PMI is easy to see: (a) starting from the first PMI with a given set *S*, the property (a.k.a. *predicate*) *P* is defined by $P(n) \equiv "n \in S"$. Conversely, given a predicate *P*, define the set *S* by $S \triangleq \{n \in \mathbb{Z} \mid P(n) \text{ holds}\}$. The *Induction Step* is abbreviated as *IS* in both versions.

Important Comment. In the Induction Step of both versions of PMI <u>you are supposed to prove that the **implication** holds</u>; you should not try to prove the premise $P(n)$ or the consequence $P(n+1)$. The premise $P(n)$ in the implication in the *IS*-part, is called the <u>*induction hypothesis*</u> (sometimes abbreviated as <u>*IH*</u>).

Example 1. Show that $\forall n \geq 1$, $\boxed{\sum_{i=1}^{n} i \;=\; \frac{n(n+1)}{2}}$. The predicate in this example is the equality between the two expressions: $P(n) \quad \equiv \quad \left[\sum_{i=1}^{n} i \;=\; \frac{n(n+1)}{2} \right]$ and the goal is to prove that $P(n)$ is true for all $n \geq 1$.

<u>*Basis*</u>: we need to verify that $P(1)$ is true. The left-hand of $P(1)$ is equal to $\sum_{i=1}^{1} i = 1$ and the right-hand is $\frac{1(1+1)}{2} = 1$ showing that $P(1)$ is true.

<u>*IS*</u>: We assume that $P(n)$ holds, i.e. $\sum_{i=1}^{n} i \;=\; \frac{n(n+1)}{2}$. To show that $P(n+1)$ follows from this assumption we perform the following manipulation:

$$\sum_{i=1}^{n+1} i \;=\; \sum_{i=1}^{n} i + (n+1) \;=\; \frac{n(n+1)}{2} + (n+1) \;=\; (n+1)\left[\frac{n}{2} + 1\right] \;=\; \frac{(n+1)(n+2)}{2}$$

which is precisely the predicate $P(n+1)$. With this the *IS* has been proved. The PMI now implies the initial claim. ■

Example 2. Show that $\forall n \geq ??\left[n^2 \geq 2n+1\right]$. In this case the predicate is an inequality: $P(n) \equiv \left[n^2 \geq 2n+1\right]$. We also want to figure out the start-value for the inductive claim. For $n = 0,1,2$ the inequality does not hold so we try $n_0 = 3$. $P(3)$ obviously holds true. This serves as the *Basis* of the induction. The inductive hypothesis is the inequality $n^2 \geq 2n+1$ and we need to prove that $(n+1)^2 \geq 2(n+1)+1$. This will constitute the *IS*. Now we manipulate

$$(n+1)^2 \;=\; n^2+2n+1 \;\geq\; (2n+1)+2n+1 \;=\; 4n+2 \;\geq\; 2n+3 \;=\; 2(n+1)+1.$$

The first inequality reflects the application of the inductive hypothesis. By the PMI, the claim is proved. ■

Example 3. Show that $\forall n \geq 1$, $\boxed{\sum_{k=1}^{n} 2^k \;=\; 2^{n+1}-1}$. The *Basis* is easy: the left-hand side of $P(1)$ is $\sum_{k=0}^{1} 2^k \;= 2^0+2^1 \;= 3$ and the right-hand side is $2^{1+1}-1 = 3$. For the *IS* we argue as follows:

$$\sum_{k=1}^{n+1} 2^k = \sum_{k=1}^{n} 2^k + 2^{n+1} = (2^{n+1}-1)+2^{n+1} = 2^{(n+1)+1}-1. \qquad ■$$

Example 4. [***Bernoulli***[2] ***Inequality***] Let $\alpha \in \mathbb{R}$ such that $\alpha \neq 0$ and $\alpha > -1$. Show that $\forall n \geq 2$, $\boxed{(1+\alpha)^n \;> 1+n\alpha}$. The *Basis*, $n = 2$, is easy: $(1+\alpha)^2 = 1+2\alpha+\alpha^2 > 1+2\alpha$, since $\alpha \neq 0$ implies $\alpha^2 > 0$. The *IS* is argued as follows:

$$(1+\alpha)^{n+1} \;=\; (1+\alpha)^n(1+\alpha) \;>\; (1+n\alpha)(1+\alpha) \;=\; 1+(n+1)\alpha+n\alpha^2 \;>\; 1+(n+1)\alpha$$

where, in the first inequality, we have used the assumption that $1+\alpha > 0$. ■

[2] **Daniel Bernoulli** (1700 –1782) was a Dutch-Swiss mathematician and was one of the many prominent mathematicians in the Bernoulli family. He is particularly remembered for his applications of mathematics to mechanics, especially fluid mechanics, and for his pioneering work in probability and statistics

Let $a,b\in\mathbb{Z}$. We say that *a* <u>*divides*</u> *b*, notation $a\,|\,b$, if $a\neq 0$ and for some $c\in\mathbb{Z}$, $b=ac$. Alternatively, we say that *b* <u>*is divisible by*</u> *a* or that *b* <u>*is a multiple of*</u> *a*. For instance, $2\,|\,8$, and $7\,|\,-21$, but $7\nmid 24$.

Example 5. Show that $\forall n\geq 0,\ 3|n^3-n$. The *Basis* is trivial since any number different from 0 (and $3\neq 0$) divides 0. The *IS* reduces to simple manipulation:

$$\begin{aligned}(n+1)^3-(n+1)&=(n+1)[(n+1)^2-1]=(n+1)[n^2+2n]\\&=n^3+3n^2+2n=(n^3-n)+3n^2+3n\end{aligned}$$

and the last expression is divisible by 3 by *IH*. ■

Example 6. [***Sum of Squares***] Show that $\forall n\geq 1$, $\boxed{\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}}$. The *Basis* is easy: the left-hand side is equal to $\sum_{k=1}^{1}k^2=1^2=1$, and the right-hand side is equal to $\frac{1(1+1)(2\cdot 1+1)}{6}=1$. For the *IS* we proceed as usual, while using the *IH* in the second equality:

$$\begin{aligned}\sum_{k=1}^{n+1}k^2&=\sum_{k=1}^{n}k^2+(n+1)^2=\frac{n(n+1)(2n+1)}{6}+(n+1)^2=(n+1)\left[\frac{n(2n+1)}{6}+\frac{6(n+1)}{6}\right]\\&=\frac{n+1}{6}\left[n(2n+1)+6(n+1)\right]=\frac{n+1}{6}\left[2n^2+7n+6\right]=\frac{(n+1)(n+2)(2n+3)}{6}\\&=\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}\end{aligned}$$

which proves the *IS*. ■

It is important to stress that we must prove the *Basis* and without it "the whole structure may collapse." The next example illustrates this point.

Example 7. Johnny comes home and declares to his dad that "all positive integers are equal." The dad asks Johnny to prove his claim. Johnny explains that the proof is by induction: for arbitrary positive integer *k* it suffices to show $k=k+1\ \Rightarrow\ k+1=k+2$. And indeed, assuming $k=k+1$ add 1 to both sides to obtain $k+1=k+2$. And Johnny smiles triumphantly and the father walks away scratching his head. ■

Now that we have used *PMI* (unquestioningly) in many examples, it is high time that we prove that it is in fact a <u>*correct*</u> principle.

Theorem 8. PMI is correct. That is, if $n_0 \in \mathbb{Z}$ and *P* is a property of integers such that the conditions:

1. $[\textit{Basis}]\, P(n_0)$ is true.
2. $[\textit{Induction Step}]\, \forall n \geq n_0 \left[P(n) \quad \Rightarrow \quad P(n+1) \right]$.

hold, then for every integer $n \geq n_0$, $P(n)$ is true.

Proof. Suppose that the conditions (1) and (2) hold but the conclusion does not, i.e. there are integers $n \geq n_0$ for which $P(n)$ is not true. In that case there must exist a smallest integer $m \geq n_0$ such that $P(m)$ is false. By the *Basis* condition, $m > n_0$ and hence $m-1 \geq n_0$. However, since *m* is a smallest integer $m \geq n_0$ for which $P(m)$ is false, it must be the case that $P(m-1)$ is true! Then however, by condition (2), $P(m)$ is true as well, contradicting the assumption. It follows that for every integer $n \geq n_0$, $P(n)$ must be true. ■

In our correctness proof we have used an important property of sets of integers that can be formulated as follows:

Well-Ordering Principle (**WOP**). Let $n_0 \in \mathbb{Z}$. Then any non-empty set of integers, all of which are $\geq n_0$, has a least (i.e., smallest) element.

Remark. Since we have used WOP to prove PMI, it follows that WOP $\Rightarrow$ PMI. The converse is also true and it is given as a non-trivial exercise.

3.3. <u>INDUCTIVE DEFINITIONS</u>

One can view a sequence of numbers (integer, real, or complex) as a function whose domain is the set of natural numbers $\mathbb{N}$. For example, the sequence $\left\{a_n\right\}_{n=1}^{\infty}$ defined by

$a_n \triangleq \cos 5n$ can be viewed as a function $a:\mathbb{N}\to\mathbb{R}$, given by $a(n)=\cos 5n.$ And the sequence $0,1,0,1,0,1,0,1,\ldots$ which can be defined by the formula $b_n \triangleq \frac{1}{2}[1+(-1)^n]$, can also be viewed as the function $b(n)=\frac{1}{2}[1+(-1)^n].$

In Computer Science we often specify a sequence or a function by means of an ***inductive*** (or *recursive*) *definition*. Using the "sequence terminology," in such cases we explicitly specify a_1, the first element of the sequence, and then we give a definition of a_{n+1} using a_n. More generally, we sometimes specify several initial values, e.g. both a_1 and a_2, and then define a_{n+1} using a_n and possibly other previously defined elements of the sequence.

Example 9. We inductively define several sequences $\{a_n\}_{n=1}^{\infty}$:

(a) $\begin{cases} a_1 = 6 \\ a_{n+1} = a_n + 3 \end{cases}$ this gives the sequence $6,9,12,15,18,\ldots$

(b) $\begin{cases} a_1 = 3 \\ a_{n+1} = 3a_n \end{cases}$ this gives the sequence $3,3^2,3^3,3^4,\ldots$

(c) $\begin{cases} a_1 = 5 \\ a_{n+1} = a_n^5 \end{cases}$ this gives the sequence $5^{5^0},5^{5^1},5^{5^2},5^{5^3},\ldots$

(d) $\begin{cases} a_1 = 2 \\ a_{n+1} = 2^{a_n} \end{cases}$ this gives the sequence $2,2^2,2^{2^2},2^{2^{2^2}},\ldots$

■

Example 10. We define the function $\boxed{\begin{cases} f(1)=1 \\ f(n+1)=(n+1)f(n) \end{cases}}$ and some of its values for small values of *n* are: $f(2)=2,\ f(3)=6,\ f(4)=24,\ f(5)=120,\ f(6)=720,\ldots$ Clearly, this is the familiar "factorial" function and we shall write, as is the custom, $n!$ for $f(n)$. Thus, $\boxed{n! \triangleq n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1}$. It is also customary to extend the factorial to 0 with the convention that $\boxed{0!=1}$. ■

It may be useful to mention that $n!$ has a *combinatorial meaning*: it is the number of ways in which one can arrange *n* distinct objects in a sequence. This is because one may pick the first element of the sequence in n ways, and once that was chosen, the second element of the sequence can be chosen in $(n-1)$ ways, the third in $(n-2)$ and so on.

Important Remark. The number of bijections between two n-sets is $n!$. Why?

Example 11. Define the Fibonacci sequence $\{f_n\}_{n=1}^{\infty}$:

$$\boxed{\begin{cases} f_0 = 0 \\ f_1 = 1 \\ f_{n+1} = f_n + f_{n-1} \qquad n \geq 2 \end{cases}}$$

The sequence starts like: $0,1,1,2,3,5,8,13,21,34,...$ each "next" element is the sum of the two preceding ones. The Fibonacci sequence has many wonderful and surprising properties. I will leave several of these for exercises and here prove an explicit formula for the general Fibonacci number. The claim is that $\forall n \geq 0$

$$\boxed{f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}.$$

This formula is sometimes called the Binet's[3] formula although it was known to Euler. The proof is by induction. For the *Basis* we need to verify two cases: $n=0$ and $n=1$.

$\boxed{n=0}$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right] = \frac{1}{\sqrt{5}}[1-1] = 0 = f_0.$$

$\boxed{n=1}$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\left[\sqrt{5}\right] = 1 = f_1.$$

In this example <u>the simple form of the *IS* does not work</u>: this is because the Fibonacci sequence has been inductively defined so that f_{n+1} is expressed using both f_n and f_{n-1}.

[3] **Jacques Philippe Marie Binet** (1786 - 1856) was a French mathematician, physicist and astronomer born in Rennes.

We therefore extend the PIM without being too formal about it. The *IS* will have this form: $\boxed{\forall n \geq n_0 [P(n) \wedge P(n-1) \;\Rightarrow\; P(n+1)]}$. We now continue with our example.

$$
\begin{aligned}
f_{n+1} = f_n + f_{n-1} &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] \\
&= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\right\} - \frac{1}{\sqrt{5}}\left\{\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right\} \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}+1\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]
\end{aligned}
$$

which is the desired claim. We only need to verify two calculations:

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}+1 \qquad \text{and, similarly}$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2}+1\ . \qquad \blacksquare$$

3.4. BINOMIAL COEFFICIENTS AND NEWTON'S[4] BINOMIAL FORMULA

Let $\mathcal{P}_k(S)$ denote the set of all *k*-subsets of the *n*-set *S*. Recall that, when talking about sets, we have defined the symbol $\binom{n}{k}$, read "*n* choose *k*," as the number of *k*-subsets of an *n*-set, with $0 \leq k \leq n.$ These integers are also known as the ***binomial coefficients***.

We can easily extend the definition for $k > n$ by letting the value to be 0 (there are no subsets of cardinality larger than *n*, in an *n*-set). Thus, by definition, the cardinality of

[4] **Sir Isaac Newton**, (1643 –1727) was an English physicist, mathematician, astronomer, natural philosopher, and alchemist, regarded by many as the greatest figure in the history of science.

$\mathcal{P}_k(S)$ is $\binom{n}{k}$. We define $f : \mathcal{P}_k(S) \to \mathcal{P}_{n-k}(S)$ by $f(A) \triangleq S - A$ for each $A \in \mathcal{P}_k(S)$. It is easy to show that f is a bijection (or that f has an obvious inverse - the complement function with opposite domain and co-domain). We thus obtain the

Corollary 12. $\binom{n}{k} = \binom{n}{n-k}$.

Another useful property of the binomial coefficients is

Lemma 13. $\forall n, k \geq 1, \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Proof. First note that if $k > n$ the claim is trivially true because all the expressions are 0. So suppose $1 \leq k \leq n$ and let *S* be an *n*-set. Pick an arbitrary element $s_0 \in S$. Now split the set $\mathcal{P}_k(S)$ into two parts:

$$\Im = \left\{ A \in \mathcal{P}_k(S) \,\middle|\, s_0 \in A \right\} \qquad \text{and} \qquad \Im' = \left\{ B \in \mathcal{P}_k(S) \,\middle|\, s_0 \notin B \right\}.$$

It is easy to see that these two sets (of sets) form a partition of $\mathcal{P}_k(S)$. Consider now the $(n-1)$-set $T \triangleq S - \{s_0\}$. The two sets (of sets) $\Im'$ and $\mathcal{P}_k(T)$ are actually the same (and hence have the same cardinality) which, by the definition of the binomial coefficients, is simply $\binom{n-1}{k}$. The set $\Im$ and the set $\mathcal{P}_{k-1}(T)$ have the same cardinality as established by the bijection $f : \Im \to \mathcal{P}_{k-1}(T)$ defined by $f(A) \triangleq A - \{s_0\}$. Hence, the cardinality of $\Im$ is $\binom{n-1}{k-1}$. Since (clearly) the sets $\Im$ and $\Im'$ are disjoint, the claim follows. □

Theorem 14. Let $n, k \in \mathbb{N}$ be positive integers such that $1 \leq k \leq n.$ Then

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}}.$$

Proof. We use PMI for the variable *n*. For the *Basis*, $n = 1$, the claim is true because $\binom{1}{1} = 1$, and the right-hand side is $\frac{1!}{1!0!} = 1$. For the *IS* we shall use Lemma 13.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$
$$= \frac{n!}{(k-1)!(n-k)!}\left[\frac{1}{n-k+1} + \frac{1}{k}\right] = \frac{n!}{(k-1)!(n-k)!}\left[\frac{k+(n-k+1)}{k(n-k+1)}\right]$$
$$= \frac{n!}{(k-1)!(n-k)!}\left[\frac{n+1}{k(n-k+1)}\right] = \frac{(n+1)!}{k!(n-k+1)!} \;. \qquad \square$$

Remark: Theorem 14 can be extended to allow $k=0$ and $n=k=0$ by letting

$\binom{n}{0} = \frac{n!}{0!n!} = 1$, which works fine even in the case that $n=0$.

Theorem 15. [***Newton's Binomial Formula***] $\forall n \geq 0$,

$$\boxed{(x+y)^n \;=\; \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \;=\; \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n}$$

Proof. The _Basis_ is easily verified:

$\boxed{n=0}$ The left-hand side: $(x+y)^0 = 1$ and right-hand side: $\sum_{k=0}^{0}\binom{0}{k}x^{0-k}y^k = \binom{0}{0}x^0y^0 = 1$.

$\boxed{n=1}$ [We don't actually have to check this case, but …]

The left-hand side: $(x+y)^1 = x+y$. And the right-hand side

$$\sum_{k=0}^{1}\binom{1}{k}x^{1-k}y^k = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 = x+y.$$

Induction Step.

$$(x+y)^{n+1} \;=\; (x+y)^n(x+y) \;=\; \left\{\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right\}(x+y)$$
$$=\; \sum_{k=0}^{n}\binom{n}{k}x^{n+1-k}y^k \;+\; \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1}$$

$$= \binom{n}{0}x^{n+1} + \binom{n}{1}x^n y + \binom{n}{2}x^{n-1}y^2 + \ldots + \binom{n}{n}xy^n$$

$$+ \binom{n}{0}x^n y + \binom{n}{1}x^{n-1}y^2 + \ldots + \binom{n}{n-1}xy^n + \binom{n}{n}y^{n+1}$$

$$= \binom{n+1}{0}x^{n+1} + \left[\binom{n}{1} + \binom{n}{0}\right]x^n y + \left[\binom{n}{2} + \binom{n}{1}\right]x^{n-1}y^2 + \ldots$$

$$+ \left[\binom{n}{n} + \binom{n}{n-1}\right]xy^n + \binom{n+1}{n+1}y^{n+1}$$

and using Lemma 13 we can rewrite the expressions in the square brackets and continue the equality

$$= \binom{n+1}{0}x^{n+1} + \binom{n+1}{1}x^{(n+1)-1}y + \binom{n+1}{2}x^{(n+1)-2}y^2 + \ldots$$

$$+ \binom{n+1}{n}x^{(n+1)-n}y^n + \binom{n+1}{n+1}y^{n+1}$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}x^{(n+1)-k}y^k \qquad \square$$

The Binomial Theorem has a huge number of interesting consequences whose proofs without using it would be “longish,” to say the least. Below we will give examples of several such consequences.

Example 16. Substituting $x = y = 1$ in the binomial formula we obtain:

$$\boxed{2^n = \sum_{k=0}^{n}\binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{0} + \ldots \binom{n}{n}} \; .$$

Example 17. Substituting $x = 1$ and $y = -1$ in the binomial formula we obtain:

$$\boxed{0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^k = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n}} \; .$$

Example 18. We want to prove the following identity

$$\boxed{\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}}.$$

We can prove the claim in (at least) two different ways. One is by PMI (and this is left as an exercise) and the other by using the Binomial Theorem with a tiny bit of calculus, specifically, the concept of a derivative. Here is how it goes. In the Binomial Theorem substitute $x = 1$ and $y = x.$ Then

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k .$$

Since this is an identity, both sides represent the same function of *x*, and hence their derivatives must also be the same

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1} .$$

Now, substituting $x = 1$ we obtain the claimed formula.

3.5. PASCAL'S TRIANGLE[5]

Based on Lemma 13 we can actually set up a method for calculating the binomial coefficients. The method can be illustrated in the form of a triangular grid called the Pascal's triangle.

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
\dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots
\end{array}$$

Each row represents the values of the binomial coefficients in Newton's Binomial Formula expansion. E.g., the fourth row (the first being the 0^{th}) represents the expansion

[5] **Blaise Pascal**, (June 19, 1623–August 19, 1662) was a French mathematician, physicist, and religious philosopher. He was a child prodigy who was educated by his father.

$$(x+y)^4 \;=\; x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\,.$$

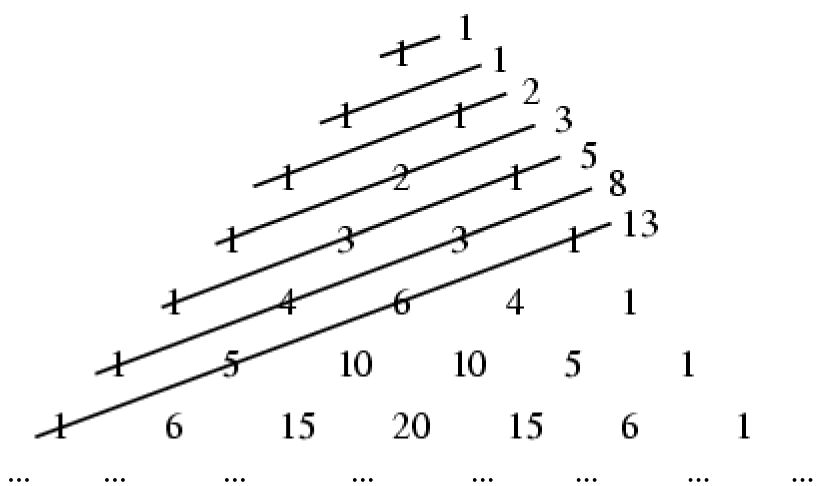
Also, every entry in the Pascal's triangle is the sum of the two entries to the left and to the right of that entry, in the preceding row.

There is a curious connection between binomial coefficients and Fibonacci numbers as shown in the next example.

Example 19. Let $\{f_k\}_{k=0}^{\infty}$ denote the Fibonacci numbers. The claim is that $\forall n \geq 0$,

$$\boxed{\sum_{k=0}^{n}\binom{n-k}{k} \;=\; \sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n-k}{k} \;=\; f_{n+1}}\,.$$

The first equality follows because the terms for $k > \lfloor n/2 \rfloor$ are all 0. This relationship can be visualized in the Pascal triangle as "shallow diagonals"



The proof can be easily done by induction, by first verifying the *Basis* cases $n = 0,1$. The *IS* is based on the observation that the sum along the current line (in the figure above) is equal to the sum along the two previous lines using the identity of Lemma 13. The actual manipulation is not difficult.

EXERCISES

(01) Find the smallest n_0 such that for all $n \geq n_0$, $2^n \geq n^2$ and then prove the inequality.

(02) Find the smallest n_0 such that for all $n \geq n_0$, $3n - 6 < \dfrac{n(n-1)}{4}$.

(03) Find the smallest n_0 such that for all $n \geq n_0$, $n! \geq 3^n$ and then prove the inequality.

(05) Prove that for all $n \geq 1$, $\boxed{\sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}}$.

(06) Prove that for all $n \geq 1$, $n^n \geq n!$.

(07) Prove that for all $n \geq 1$, $\sum_{k=1}^{n} \dfrac{2}{3^k} = 1 - \dfrac{1}{3^n}$.

(08) Prove the following divisibility results,

(a) for all $n \geq 0$, $2^n 3^n \mid (3n)!$.
(b) for all $n \geq 0$, $24 \mid 5^{2n} - 1$.
(c) for all $n \geq 0$, $15 \mid 4^{2n} - 1$.
(d) for all $n \geq 0$, $2^n \mid 3^{2^n} - 1$.

(09) [***Geometric Sum***] Let $q \in \mathbb{R}$, $q \neq 1$. Show that for all $n \geq 0$, $\boxed{\sum_{k=0}^{n} q^k = \dfrac{1 - q^{n+1}}{1 - q}}$.

(10) [***Sum of Cubes***] Prove that for all $n \geq 1$, $\boxed{\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2}$.

(11) [***Sum of 4^th^ powers***] Prove that for all $n \geq 1$, $\boxed{\sum_{k=1}^{n} k^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}}$.

(12) For $n \geq 1$ define $U_n \triangleq \sum_{k=1}^{n} \dfrac{1}{k(k+1)}$. Find a "closed" formula for U_n (i.e., a formula that does not involve summation symbols) and then use mathematical induction to prove its correctness.

(13) For $n \geq 1$ define $T_n \triangleq \sum_{k=1}^{n} \dfrac{2k+1}{k^2(k+1)^2}$. Find a "closed" formula for T_n (i.e., a formula that does not involve summation symbols) and then use mathematical induction (MI) to prove its correctness.

(14) Show that for $n \geq 1$, $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$.

(15) Prove that PMI $\Rightarrow$ WOP. (**Hint 1**. Start by assuming that S is a set of integers (all $\geq n_0$) that does not have a least element. You should use PMI to get a contradiction. For that you need to formulate a predicate $P(n)$.)

(16) Use PMI to prove the *Odometer Principle* with base b: every positive integer can be represented *in base* b as a b-*ary* string $x_{n-1}x_{n-2}\ldots x_1x_0$ where all $0 \leq x_i < b$ are integers. The string $x_{n-1}x_{n-2}\ldots x_1x_0$ represents the integer $N = x_{n-1}b^{n-1} + x_{n-2}b^{n-2} + \ldots + x_1b + x_0$ (similarly to the representation of integers using the decimal notation).

(17) In this problem we deal with recursively (inductively) defined sequences.

(a) Define $u_1 = 2; u_n = u_{n-1} + 2n\ (n \geq 2).$ Prove that $u_n = n(n+1).$

(b) Define $u_1 = 3; u_2 = 5; u_n = 3u_{n-1} - 2u_{n-2}\ (n \geq 3).$ Prove that $u_n = 2^n + 1.$

(c) Let $1 < a < b$ be real numbers. Define two sequences in a simultaneous induction:

$$\begin{cases} u_1 = b \\ u_{n+1} = \frac{1}{2}(u_n + v_n) \end{cases} \qquad \begin{cases} v_1 = a \\ v_{n+1} = \dfrac{2u_nv_n}{u_n + v_n} \end{cases}.$$

Show that $0 < u_{n+1} - v_{n+1} < \dfrac{(b-a)^{2^n}}{2^n}$.

(18) Prove the following properties of the binomial coefficients:

(a) $\forall n \geq 1,\ \sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$ (b) $\forall n \geq 0,\ \sum_{i=0}^{n} \binom{n-i}{k} = \binom{n+1}{k+1}.$

(c) $\forall n \geq 0,\ \sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1}.$ (d) $\forall n \geq 0,\ \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$

(e) Prove (d) by considering the identity: $(1+x)^n(1+x)^n = (1+x)^{2n}$.

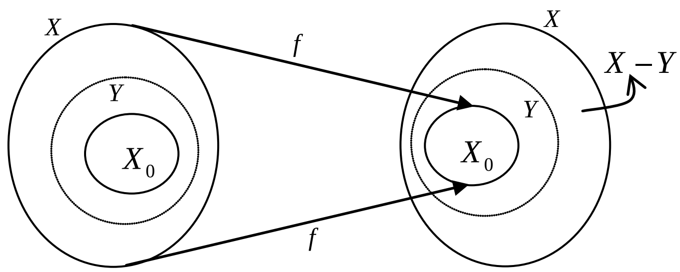
(19) Find the sum and prove correctness: $\sum_{k=1}^{n} k(k+1).$

(20) Prove $\forall n \geq 1,\ a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k.$

(21) Prove $\forall n \geq 3,\ (n+1)^n < n^{n+1}$.

(22) Prove $\frac{1}{3}n^3 < \sum_{k=1}^{n} k^2 < \frac{1}{3}(n+1)^3$.

(23) Let X be a set, $X_0 \subseteq X$ a subset, $f : X \to X_0$ a bijection, and let Y be an "in between" subset: $X_0 \subseteq Y \subseteq X$. Picture of the situation is:



Now, lets define a sequence of sets inductively:

$$\begin{cases} A_0 = X - Y \\ A_{n+1} = f(A_n) \qquad n \geq 0 \end{cases}$$

Prove that the family $\{A_n\}_{n=0}^{\infty}$ is pairwise disjoint. **Hint**: Proving the following Claim by induction on *m*, will imply the required result. **Claim**. $\forall m \geq 0 \left[\forall k \geq 1 \left[A_m \cap A_{m+k} = \varnothing\right]\right]$.

(24) Prove the following properties of Fibonacci numbers $\{f_n\}_{n=0}^{\infty}$.

(a) $\boxed{\forall n \geq 0,\ \sum_{k=1}^{n} f_k^2 = f_n f_{n+1}}$.

(b) $\boxed{\forall n \geq 0,\ \sum_{k=1}^{n} f_k = f_{n+2} - 1}$.

(c) $\boxed{\forall n \geq 1,\ f_{n+1} f_{n-1} - f_n^2 = (-1)^n}$.

(d) $\boxed{\forall n \geq 0, \forall k \geq 1,\ f_{n+k} = f_k f_{n+1} + f_{k-1} f_n}$.

(e) $\boxed{\forall n, m \geq 1,\ \gcd(f_n, f_m) = f_{\gcd(n,m)}}$.

(f) $\boxed{\sum_{k=1}^{n} f_{2k-1} = f_{2n}}$.

(25) You have *n*\$ to spend. Every day you can spend either 1\$ for a candy or 2\$ for an ice cream. In how many ways you can spend your money?

(26) How many subsets of $A \subseteq \{1, 2, ..., n\}$ are there in which *A* does not contain two consecutive integers?

(27) Given an unrestricted supply of 3¢ and 5¢ stamps, prove that any postage $n \geq 8$ can be produced.

(28) Show that *n* lines "in general position" in the plane divide it into $\frac{1}{2}(n^2+n+2)$ regions. A collection of lines is in *general position* if no two lines in the collection are parallel and no three intersect in a single point.

SOME MORE EXERCISES

(01) Suppose $0 < y < x.$ Prove that $\forall n \geq 2,\ n(x-y)y^{n-1} < x^n - y^n < n(x-y)x^{n-1}.$

(02) [Improvement of Exercise (06)] Prove $\forall n \geq 6,\ \left(\frac{n}{2}\right)^n > n!.$

(03) Prove $\forall n \geq 1,\ \dfrac{2^{2n-1}}{\sqrt{n}} \leq \dbinom{2n}{n} \leq \dfrac{2^{2n}}{\sqrt{3n+1}}.$

(04) [*Weird Induction*] Let $n \geq 2$ and let $x_1, x_2, ..., x_n$ be positive reals such that not all are equal and so that $x_1 x_2 \ldots x_n = 1.$ Prove that $x_1 + x_2 + ... + x_n > n.$

(05) [Arithmetic & Geometric Means] Prove that if $a_1, a_2, ..., a_n$ are positive reals, not all equal, then

$$\boxed{\frac{a_1 + a_2 + ... + a_n}{n} > \sqrt[n]{a_1 a_2 \ldots a_n}}.$$

(06) Prove $\forall n \geq 2,\ \sqrt[2]{\sum_{k=1}^{n} k^2} > \sqrt[3]{\sum_{k=1}^{n} k^3}.$ [For $n = 1$ this is an equality.]

(07) Prove $\forall n \geq 3,\ 2\sqrt{n} - 1 \geq 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > \sqrt{n}.$

(08) [See Exercise (07).] Prove $\forall n \geq 1,\ 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2.$

(09) Prove $\forall n \geq 1,\ \sum_{k=1}^{n} a_k = 1 \Rightarrow \sum_{k=1}^{n} a_k^2 \geq 1/n.$