Jay Patel Exam-4A

1] $f(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ e^{1} & \text{if } \pi \le t < 2\pi \\ \sin t & \text{if } t > 2\pi \end{cases}$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$L(f(t)) = \int_0^\pi e^{-st}(1)\,dt + \int_\pi^{2\pi} e^{-st}(1)\,dt + \int_{2\pi}^\infty e^{-st}(\sin t)\,dt$$

$$= \int_0^\pi e^{-st}\,dt + \int_{2\pi}^\infty e^{-st}(\sin t)\,dt$$

$$= \left[\left(\frac{e^{-st}}{-s}\right)\right]_0^\pi + \left[\frac{e^{-st}}{1+s^2}(-s\sin t - \cos t)\right]_{2\pi}^\infty$$

$$= \left[\left(\frac{e^{-s\pi}}{-s}\right) - \left(\frac{e^{-0}}{-s}\right)\right] + \left[0 - \frac{e^{-2x}}{1+s^2}(-s\sin 2\pi - \cos 2\pi)\right]$$

FA $\Rightarrow$ $$= \left[\frac{1-e^{-s\pi}}{s}\right] + \left[\frac{e^{-2\pi}}{1+s^2}\right]$$

2] $y'' + 2y' + 5y = 15$

$y(0) = 2$

$y'(0) = 1$

$\rightarrow$ $L(y'') + 2L(y') + 5L(y) = L(15)$

$s^2 y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 5y(s) = 15/s.$

$s^2 y(s) - 2s - 1 + 2[sY(s) - 2] + 5Y(s) = 15/s$

$s^2 y(s) - 2s - 1 + 2sY(s) - 4 + 5Y(s) = 15/s$

$s^2 Y(s) - 2s - 5 + 2sY(s) + 5Y(s) = 15/s$

$s^2 Y(s) + 2sY(s) + 5Y(s) = 15/s + 2s + 5$

$(s^2 + 2s + 5)Y(s) = \dfrac{15 + 2s^2 + 5s}{s}$

$Y(s) = \dfrac{15 + 2s^2 + 5s}{s(s^2 + 2s + 5)}$

$\dfrac{15 + 2s^2 + 5s}{s(s^2 + 2s + 5)} = \dfrac{A}{s} + \dfrac{Bs + C}{s^2 + 2s + 5}$

$15 + 2s^2 + 5s = A(s^2 + 2s + 5) + (Bs + C)s$

$15 + 2s^2 + 5s = As^2 + 2As + 5A + Bs^2 + Cs$

$15 + 2s^2 + 5s = (A + B)s^2 + (2A + C)s + 5A$

Now,

$S = 5A = 15 \Rightarrow A = 3$

$2A + C = 5$

$C = 5 - 2A$

$C = 5 - 2(3)$

$\boxed{C = -1}$

$S^2 = \quad A + B = 2$

$B = 2 - A$

$B = 2 - 3$

$\boxed{B = -1}$

$$\Rightarrow \frac{15 + 2s^2 + 5s}{s(s^2+2s+5)} = \frac{3}{s} + \frac{-s-1}{s^2+2s+5}$$

$$= \frac{3}{s} - \frac{s+1}{s^2+2s+5}$$

$$Y(s) = \frac{3}{s} - \frac{s+1}{s^2+2s+5}$$

$$Y(s) = \frac{3}{s} - \frac{s+1}{(s+1)^2+2^2}$$

take integral

$$y(t) = 3 - L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}$$

use shift rule $s+1 \to s$

$$y(t) = 3 - e^{-t} L^{-1}\left\{\frac{s}{s^2+2^2}\right\}$$

FA= $\boxed{y(t) = 3 - e^{-t}\cos 2t}$

3] $y'' + x + 2y = 0$

$x'' + x + 2y = 0$

$x(0) = 0, \; y(0) = 0$

$x'(0) = 1, \; y'(0) = 1$

$\Rightarrow y'' + x + 2y = 0$

$L(y'' + x + 2y = 0)$

$s^2 y(s) - s y(0) - y'(0) + x(s) + 2Y(s) = 0$

put $y(0) = 0, \; y'(0) = 1$

$(s^2 + 2)Y(s) - 1 + X(s) = 0$

$X(s) = 1 - (s^2 + 2)Y(s)$

$x'' + x + 2y = 0$

$L(x'' + x + 2y = 0)$

$s^2 X(s) - s x(0) - x'(0) + x(s) + 2Y(s) = 0$

put $x(0) = 0, \; x'(0) = 1$

$(s^2 + 1)X(s) - 1 + 2Y(s) = 0$

$(s^2+1)[1 - (s^2 + 2)Y(s)] - 1 + 2Y(s) = 0$

$(s^2 + 1) - (s^2 + 1)(s^2 + 2)Y(s) - 1) + 2Y(s) = 0$

$-(s^4 + 3s^2 + 2)Y(s) + 2(Y)(s) = -s^2$

$(s^4 + 3s^2)Y(s) = s^2$

$Y(s) = \dfrac{1}{(s^2+3)}$

$$L^{-1}[Y(s)] = \frac{1}{\sqrt{3}} L^{-1}\left[\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right]$$

$$y(t) = \frac{1}{\sqrt{3}} \sin(\sqrt{3}t)$$

put $Y(s) = \frac{1}{s^2+3}$ in $X(s) = 1-(s^2+2)Y(s)$

$$X(s) = 1-(s^2+2)\frac{1}{s^2+3}$$

$$X(s) = \frac{1}{s^2+3}$$

$$L^{-1}[X(s)] = \frac{1}{\sqrt{3}} L^{-1}\left[\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right]$$

$$x(t) = \frac{1}{\sqrt{3}} \sin(\sqrt{3}t)$$

Solution,

$$x(t) = \frac{1}{\sqrt{3}} \sin(\sqrt{3}t)$$

$$y(t) = \frac{1}{\sqrt{3}} \sin(\sqrt{3}t)$$

4] $f(t) = \cosh(kt)\cos(kt)$

Since $\cosh(kt) = \dfrac{e^{kt}+e^{-kt}}{2}$, therefore

$$f(t) = \frac{e^{kt}+e^{-kt}}{2}\cos(kt)$$

$$f(t) = \frac{1}{2}e^{kt}\cos(kt) + \frac{1}{2}e^{-kt}\cos(kt)$$

$$L(\cos(kt)) = \frac{s}{s^2+k^2} = f(s)$$

$$L(e^{kt}\cos(kt)) = f(s-k) = \frac{s-k}{(s-k)^2+k^2}$$

$$L(e^{-kt}\cos(kt)) = F(s-(-k)) = F(s+k) = \frac{s+k}{(s+k)^2+k^2}$$

So,

$$L\left\{\frac{e^{kt}+e^{-kt}}{2}\cos(kt)\right\}$$

$$= \frac{1}{2}L\{e^{kt}\cos(kt)\} + \frac{1}{2}L\{e^{-kt}\cos(kt)\}$$

FA $\Rightarrow$ $$= \frac{1}{2}\,\frac{s-k}{(s-k)^2+k^2} + \frac{1}{2}\,\frac{s+k}{(s+k)^2+k^2}$$

5] $f(s) = \ln\left(\frac{s+2}{s+4}\right)$

$f(t) = L^{-1}\left\{\ln\left(\frac{s+2}{s+4}\right)\right\}$

$= \frac{-1}{t} L^{-1}\left\{\frac{d}{ds}\ln\left(\frac{s+2}{s+4}\right)\right\}$

$= \frac{-1}{t} L^{-1}\left\{\frac{d}{ds}\left(\ln(s+2)-\ln(s+4)\right)\right\}$

$= \frac{-1}{t} L^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+4}\right\}$

$= \frac{-1}{t}\left(e^{-2t} - e^{-4t}\right)$

FA $\Rightarrow$ $= \frac{e^{-4t} - e^{-2t}}{t}$