Chapter 5 - RELATIONS

5.1. RELATIONS AND CARTESIAN PRODUCTS OF SETS

Let $B_1, B_2, ..., B_n$ be sets. We have defined the *Cartesian product* of these sets as follows

$$B_1 \times B_2 \times ... \times B_n \triangleq \left\{(a_1, a_2, ..., a_n) \middle| \ \forall i, a_i \in B_i\right\}.$$

Any subset $A \subseteq B_1 \times B_2 \times ... \times B_n$ is an ***n-ary relation*** on the sets $B_1, B_2, ..., B_n$ (in this order). Obviously, $|B_1 \times B_2 \times ... \times B_n| = |B_1||B_2| ... |B_n|$ and, denoting this product by β, the total number of *n*-ary relations is 2^β.

Example 1. Let $A = \{a, b\}$, $B = \{1, 2, 3\}$, and $C = \{+, -\}$. Then the Cartesian product is

$$A \times B \times C = \left\{ \begin{array}{l} (a,1,+), (a,1,-), (a,2,+), (a,2,-), (a,3,+), (a,3,-), \\ \qquad (b,1,+), (b,1,-), (b,2,+), (b,2,-), (b,3,+), (b,3,-) \end{array} \right\}$$

How many relations we have in this case? Since the set $A \times B \times C$ has 12 elements and any subset of it is a relation of the required type, there are 2^{12} such relations. Two specific examples, in addition to the trivial ones $\varnothing$ and $A \times B \times C$, are

(a) $R_1 \subseteq A \times B \times C$ defined by taking those triples that have 1 in the second coordinate: $R_1 = \{(a,1,+), (a,1,-), (b,1,+), (b,1,-)\}$.
(b) $R_2 \subseteq A \times B \times C$ defined by taking those triples that have + in the third coordinate: $R_2 = \{(a,1,+), (a,2,+), (a,3,+), (b,1,+), (b,2,+), (b,3,+)\}$. ■

Example 2. Define the "attribute" sets as follows

Initials = $\left\{\alpha\beta\gamma \middle| \alpha, \beta, \gamma \in \{A, B, C, D, ..., Z, -\}\right\}$; **Gender** = $\{M, F\}$; **Age** = $[150]$;
EyeColor = $\{$Blue, Green, Brown, Gray$\}$; **Major** = $\{CS, CE, MATH, PHYS, BIO\}$.

We can create the Cartesian product which will comprise all the possible 5-tuples of attributes, representing real or imaginary students:

Initials × **Gender** × **Age** × **EyeColor** × **Major.**

An example of a 5-ary relation can be defined by the students registered in this class

CS330 = $\{s_1, s_2, s_3, ..., s_{20}\}$ where each s_k is a 5-tuple representing one student. For example $s_1 = (ABC, M, 22, \text{Brown}, CS)$ is, perhaps, a student in this class? ■

5.2. BINARY RELATIONS

From now on we will restrict our study to binary relations. This is the most important class of relations and one that enjoys the widest variety of applications. We first note that functions can be viewed as special kind of relations.

Definition 3. Let $f : X \to Y$ be a function. Then the binary relation induced by (or, corresponding to) f is

$$\boxed{G_f \triangleq \{(x, y) \in X \times Y \mid y = f(x)\}}\ .$$

G_f is sometimes referred to as ***the graph of*** f.

Let X and Y be non-empty sets. A subset $R \subseteq X \times Y$ is thus a <u>*binary relation from*</u> X <u>*to*</u> Y. If $\boxed{(a,b) \in R}$, we say that a <u>*is related to*</u> b <u>*via*</u> (<u>*through*</u>, <u>*by*</u>) R. Sometimes we use an alternative way to denote this fact: $\boxed{aRb}$. <u>Always keep in mind that relations are sets</u>. We define the ***domain*** and ***range*** of a relation $R \subseteq X \times Y$ by

$$dom(R) \triangleq \{x \in X \mid \exists y \in Y[(x, y) \in R]\}\ ,$$

and

$$range(R) \triangleq \{y \in Y \mid \exists x \in X[(x, y) \in R]\}.$$

Obviously, $dom(R) \subseteq X$ and $range(R) \subseteq Y$.

Example 4. Let $A = \{1, 2, 7, 9\}$. Define a binary relation from A to A, $R_< \subseteq A \times A$ by

$$(a,b) \in R_< \quad \Leftrightarrow \quad \text{"}a \text{ is smaller than } b\text{"} \quad \Leftrightarrow \quad a < b\ .$$

Then, $R_< = \{(1,2),(1,7),(1,9),(2,7),(2,9),(7,9)\}$. In fact, we might just as well denote this relation by “<”. Next, we can define an extended relation

$$(a,b) \in R_\le \quad \Leftrightarrow \quad \text{"}a \text{ is smaller than or equal to } b\text{"} \quad \Leftrightarrow \quad a \le b.$$

Clearly, $R_{\le} = R_{<} \cup \{(a,a) \mid a \in A\} = R_{<} \cup \{(1,1),(2,2),(7,7),(9,9)\}$. As for the domains and ranges of these relations we have

$$dom(<) = \{1,2,7\} \text{ and } range(<) = \{2,7,9\}, \text{ whereas } dom(\le) = range(\le) = A. \qquad \blacksquare$$

Remarks & Definitions.

(1) The relations of Example 4 have the property that their domains and ranges are subsets of the same "base set" *A*. Thus, $dom(<), range(<), dom(\le), range(\le) \subseteq A.$ Equivalently, we can write $<, \le \;\subseteq\; A \times A$. In general, when $R \subseteq X \times X$ we say that *R* is a relation defined *<u>on the set</u>* *X* (rather than saying that it is from *X* to *X*).

(2) The relation used in Example 4, $\{(a,a) \mid a \in A\} = \{(1,1),(2,2),(7,7),(9,9)\}$, is important enough to deserve a definition and special notation. For any set *X* we define (and denote) the *<u>diagonal relation on</u>* *X* by $\boxed{\Delta_X \triangleq \{(x,x) \mid x \in X\}}$.

Example 5. Define a binary relation **|** on the natural numbers $\mathbb{N}$ by

$$a \mid b \quad \Leftrightarrow \quad "a \text{ divides } b" \quad \Leftrightarrow \quad \exists x \in \mathbb{N}\left[b = ax\right] .$$

For instance, 4 **|** 12, but $4 \nmid 22$ (does not divide). Since every natural number divides and is divisible by itself, $dom(|) = range(|) = \mathbb{N}.$ It is easy to see that as a set of pairs we have

$$| \;=\; \bigcup_{m=1}^{\infty}\{(m,nm) \mid n \in \mathbb{N}\} \;=\; \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\{(m,mn)\} \;. \qquad \blacksquare$$

Example 6. For an integer $m \ge 2$ define a binary relation $\equiv_m$ on the set of all integers $\mathbb{Z}$ by: $\boxed{x \equiv_m y \quad \Leftrightarrow \quad m \mid (x-y)}$. For example, for $m = 5$ we have:

$$2 \equiv_5 7, \quad 2 \equiv_5 12, \quad 15 \equiv_5 5, \quad 121 \equiv_5 11, \text{ etc.}$$

On the other hand, $7 \not\equiv_5 10.$ Because this particular binary relation is very important and has a long history and many applications, it has an associated terminology and a special notation. Whenever $x \equiv_m y$ we say that x *<u>is congruent to</u>* y *<u>modulo</u>* *m* and we write

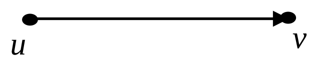
$$\boxed{x \equiv y (\operatorname{mod} m)}. \qquad \blacksquare$$

Let $Q, R \subseteq X \times Y$ be two binary relations. We say that R is ***compatible*** with, or is an ***extension*** of, Q if $Q \subseteq R$. I.e., all the relationships that hold under Q hold under R as well.

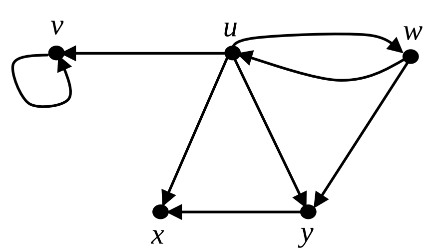
5.3. REPRESENTING BINARY RELATIONS

There are two common ways to represent finite binary relations: as directed graphs and as Boolean matrices.[1] We shall now briefly discuss these two representations.

A directed graph can be viewed as a pictorial representation of a relation on a finite set. We define a *directed graph* (sometimes abbreviated as *digraph*) as a structure $G = (V, E)$ where V is a finite set of objects called *vertices* (a.k.a. *nodes* or *points*), and E is a binary relation on V. Elements of E are called the *edges* (or *arcs*) of the graph G. The *diagram* representing G (indeed, called *the graph* G) depicts the vertices as points (or little circles) in the plane and the edges as arrows between the points: an edge $(u, v) \in E$ is being represented as an arrow from vertex u to vertex v:



Example 7. The digraph $G = (V, E)$ with vertex set $V = \{u, v, w, x, y\}$ is depicted below



The corresponding binary relation is

$$E = \{(v,v),(u,v),(u,w),(u,x),(u,y),(w,u),(w,y),(y,x)\}. \qquad \blacksquare$$

The *Boolean matrix representation* of a binary relation R from a (finite) set X to a finite set Y is a Boolean array (matrix) of dimension $|X| \times |Y|$, with entries either 1, to represent the membership (of an ordered pair in R), and 0, to represent the non-membership. More precisely,

[1] There are other ways of representing binary relations that are more appropriate for computational tasks, e.g. adjacency lists.

suppose that $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_m\}$ and *R* is a binary relation from *X* to *Y*. The *Boolean matrix representation* of *R* is an $n \times m$ matrix $M_R = (m_{ij})$ where

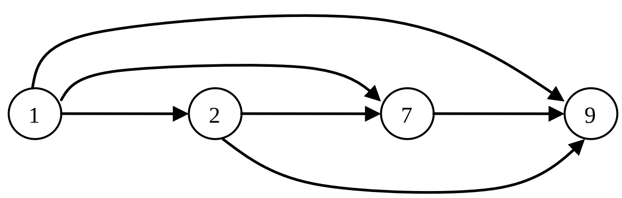
$$m_{ij} = \begin{cases} 1 & (x_i, y_j) \in R \\ 0 & (x_i, y_j) \notin R \end{cases} .$$

Example 8. Take the binary relation (and the digraph) of Example 7, assuming the alphabetic vertex ordering: $u, v, w, x, y.$ Then the Boolean matrix representing the binary relation (and the digraph) of Example 7, is given by the matrix

$$M_E = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

More formally, given any binary relation $R \subseteq X \times X$, with X a finite set, we define the corresponding digraph $G_R = (X, R)$; i.e., the set of vertices is X, the underlying set, and the edges are just the ordered pairs in the relation *R*. ■

Example 9. Take the binary relation $R = \{(1,2),(1,7),(1,9),(2,7),(2,9),(7,9)\}$ over the underlying set $X = \{1, 2, 7, 9\}$. The corresponding directed graph is

$G_R =$ 

■

5.4. <u>OPERATIONS ON BINARY RELATIONS</u>

When we define and study a class objects, one of the first things we do is consider ways to combine simpler objects into more involved ones. I.e., we consider operations on the objects. Thus, when we studied sets we spent a lot of time discussing operations on sets: union, intersection, complement, difference, symmetric difference and various properties of these operations. When we studied functions we have defined composition between them and considered various properties of functions and how they relate to composition.

Since binary relations are sets, we can apply all the operations that are applicable to sets to relations as well. We only need to caution that when considering relations from set X to set Y, i.e. subsets of $X \times Y$, the universal set is $X \times Y$ and the operation of ***complement*** must be performed with respect to it. Thus, if $R \subseteq X \times Y$, then the complement of R is

$$\boxed{\overline{R} \triangleq X \times Y - R}.$$

Another simple operation is ***inverse***: for a binary relation $R \subseteq X \times Y$, the inverse is defined and denoted by

$$\boxed{R^{-1} \triangleq \{(y,x) \mid (x,y) \in R\}}.$$

Note that when R is a relation from X to Y, R^{-1} is in the opposite direction, from Y to X. Arguably, one of the most important operation on relations is ***composition***, precisely as in the case of functions. In order for the composition of two relations R and S to be meaningful they must be compatible in the following sense: for the composition $S \circ R$ make sense it must be the case that $range(R) \subseteq domain(S)$. Suppose $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are two such compatible relations. Then their composition $S \circ R \subseteq X \times Z$ is a relation from X to Z defined by

$$\boxed{S \circ R \triangleq \{(x,z) \mid \exists y \in Y \text{ s.t. } (x,y) \in R \ \& \ (y,z) \in S\}}.$$

Example 10. What happens when we compose R with R^{-1}? Generally, we cannot say much as the following examples show. First take $X = \{a,b\}$ and $R = \{(a,b),(b,a)\}$. Then, as can be easily checked, $R \circ R^{-1} = R^{-1} \circ R = \Delta_X$. As another illustration, take $X = \{a,b,c\}$ and $R = \{(a,c),(b,c)\}$. Then, $R^{-1} = \{(c,a),(c,b)\}$, $R^{-1} \circ R = \{(a,a),(b,b),(a,b),(b,a)\}$ and $R \circ R^{-1} = \{(c,c)\}$. ■

Some basic results regarding these operations are given in the next two theorems.

Theorem 1. [***Composition is associative***] Let $R \subseteq A \times B, S \subseteq B \times C,$ and $T \subseteq C \times D.$ Then

$$\boxed{T \circ (S \circ R) = (T \circ S) \circ R}.$$

Proof. The proof amounts to logical manipulation of the definition of composition:

$$
\begin{aligned}
(a,d) \in T \circ (S \circ R) \quad &\Leftrightarrow \quad \exists c \in C\big[(a,c) \in S \circ R \;\wedge\; (c,d) \in T\big] \\
&\Leftrightarrow \quad \exists c \in C\Big[\exists b \in B\big[(a,b) \in R \;\wedge\; (b,c) \in S\big] \;\wedge\; (c,d) \in T\Big] \\
&\Leftrightarrow \quad \exists c \in C\, \exists b \in B\Big[(a,b) \in R \;\wedge\; \big[(b,c) \in S \;\wedge\; (c,d) \in T\big]\Big] \\
&\Leftrightarrow \quad \exists b \in B\, \exists c \in C\Big[(a,b) \in R \;\wedge\; \big[(b,c) \in S \;\wedge\; (c,d) \in T\big]\Big] \\
&\Leftrightarrow \quad \exists b \in B\Big[(a,b) \in R \;\wedge\; \exists c \in C\big[(b,c) \in S \;\wedge\; (c,d) \in T\big]\Big] \\
&\Leftrightarrow \quad \exists b \in B\big[(a,b) \in R \;\wedge\; (b,d) \in T \circ S\big] \\
&\Leftrightarrow \quad (a,d) \in (T \circ S) \circ R
\end{aligned}
$$

■

Theorem 2. Let $R \subseteq A \times B$ and $S \subseteq B \times C$. Then, $\boxed{(S \circ R)^{-1} = R^{-1} \circ S^{-1}}$.

Proof. Again, the proof amounts to logical manipulation of the definition of inverse:

$$
\begin{aligned}
(c,a) \in (S \circ R)^{-1} \quad &\Leftrightarrow \quad (a,c) \in S \circ R \\
&\Leftrightarrow \quad \exists b \in B\big[(a,b) \in R \;\wedge\; (b,c) \in S\big] \\
&\Leftrightarrow \quad \exists b \in B\big[(b,a) \in R^{-1} \;\wedge\; (c,b) \in S^{-1}\big] \\
&\Leftrightarrow \quad (c,a) \in R^{-1} \circ S^{-1}
\end{aligned}
$$

■

Definition 1. Let X be a set and $R \subseteq X \times X$ a binary relation on X. We define "powers" of the composition operation by induction

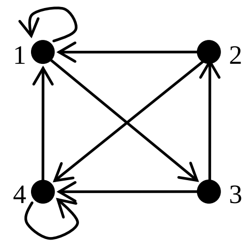
$$
\boxed{\begin{aligned} R^0 &= \Delta_X \\ R^{n+1} &= R^n \circ R \end{aligned}}\,.
$$

We also define the following associated relations

$$
\boxed{\begin{aligned} &R^+ = \bigcup_{n=1}^{\infty} R^n = R \cup R^2 \cup \ldots \cup R^n \cup \ldots \\ &R^* = \Delta_X \cup R^+ = \Delta_X \cup R \cup R^2 \cup \ldots \cup R^n \cup \ldots \end{aligned}}\,.
$$

Example 11. Let $R = \{(1,1),(1,3),(2,4),(2,1),(3,4),(3,2),(4,4),(4,1)\}$ be a binary relation on the set $X = \{1,2,3,4\}$. Here are the matrix and graph representation of R:

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad G_R =$$



We compute $R^2 = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,4),(3,1),(4,4),(4,1),(4,3)\}$. If we compute the product of the above matrix with itself we obtain

$$M_R^2 = M_R \cdot M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

If we now change all the 2's to 1's, it is easy to see that the resulting array is the Boolean matrix representation of R^2. In fact the 2's represent the existence of ***two*** paths! For example, $M_R^2[2,1] = 2$ indicates that there are two paths from vertex 2 to vertex 1. We can verify this in the graph: $2 \to 1 \to 1$ and $2 \to 4 \to 1$. Thus, we can verify that

$$M_{R^2} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad M_{R^3} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad M_{R^4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

It is easy to check that for all $m \geq 4$, $M_{R^m} = M_{R^4} = M_{R^+} = M_{R^*} =$ "4×4 all-1 matrix". ■

5.5. <u>PROPERTIES OF BINARY RELATIONS</u>

The set of all binary relations on the set *X* will be denoted by $Rel_2(X)$. We begin with some basic properties of binary relations (over a fixed set). Each such property may be viewed as a subset of $Rel_2(X)$. We will then define compound properties of relations, which may possess several of the basic properties.

Definition 2. Let X be a set and $R \subseteq X \times X$ a binary relation on X. Then,

1. R is ***reflexive***: $\forall x \in X, (x,x) \in R.$
2. R is ***irreflexive***: $\forall x \in X, (x,x) \notin R.$
3. R is ***symmetric***: $\forall x, y \in X\ [(x,y) \in R \Rightarrow (y,x) \in R].$
4. R is ***asymmetric***: $\forall x, y \in X\ [(x,y) \in R \Rightarrow (y,x) \notin R].$
5. R is ***antisymmetric***: $\forall x, y \in X\ [(x,y) \in R \wedge (y,x) \in R \Rightarrow x = y].$
6. R is ***transitive***: $\forall x, y, z \in X\ [(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R].$
7. R is ***negatively transitive***: $\forall x, y, z \in X\ [(x,y) \notin R \wedge (y,z) \notin R \Rightarrow (x,z) \notin R].$
8. R is ***complete***: $\forall x, y \in X\ [(x,y) \in R \vee (y,x) \in R].$
9. R is ***serial***: $\forall x \in X\ \exists y \in X, (x,y) \in R.$
10. R is ***euclidean***: $\forall x, y, z \in X\ [(x,y) \in R \wedge (x,z) \in R \Rightarrow (y,z) \in R].$
11. R is ***dense***: $\forall x, y \in X\ [(x,y) \in R \Rightarrow \exists z \in X[(x,z) \in R \wedge (z,y) \in R]].$
12. R is a ***partial function***: $\forall x, y, z \in X\ [(x,y) \in R \wedge (x,z) \in R \Rightarrow y = z].$
13. R is ***trivial***: $\forall x, y \in X\ [(x,y) \in R \Rightarrow x = y].$
14. R is ***empty***: $\forall x, y \in X\ [(x,y) \notin R].$

Some obvious relationships between these properties are immediate:

Corollary A. Let $R \subseteq X \times X$ be a binary relation over X.

1. R reflexive $\Rightarrow$ R dense.
2. R asymmetric $\Rightarrow$ R irreflexive.
3. R asymmetric $\Rightarrow$ R antisymmetric.
4. R antisymmetric $\Rightarrow$ $R - \Delta_X$ asymmetric.

Theorem 3. Let X be a set and $R \subseteq X \times X$ a binary relation on X. Then,

1. The relation R^+ is transitive.
2. The relation R^* is reflexive and transitive.

Proof. We prove part (1) and leave (2) as an exercise. Let $(x,y),(y,z) \in R^+$ and recall that $R^+ = \bigcup_{n \geq 1} R^n = R \cup R^2 \cup \ldots \cup R^n \cup \ldots$. Then, clearly, there exist positive integers $r, s \in \mathbb{N}$, such that $(x,y) \in R^r$ and $(y,z) \in R^s$. Thus, by the definition of composition, $(x,z) \in R^s \circ R^r = R^{s+r} \subseteq R^+$, which shows that R^+ is transitive. ■

Remark. Consider now how these properties are reflected in the matrices and the graphs that represent the relations having these properties. For example, if the relation *R* is reflexive then its matrix $M_R = (m_{ij})$ must satisfy $\forall i \in X, m_{ii} = 1$ (and no other condition is required). In other words, the diagonal of M_R must be all 1's. As for the digraph representation, reflexivity of *R* means that for every vertex $x \in X$ the graph has a self-loop on *x*. Similarly, irreflexivity of *R* means that the diagonal entries of M_R are all 0's and that the corresponding graph has no self-loops.

We also define several "compound" binary relations, ones that possess several of the basic properties, and that are of particular importance in various areas.

Definition 3. Let *X* be a set and $R \subseteq X \times X$ a binary relation over *X*. Then,

1. *R* is an ***equivalence relation*** if *R* is reflexive, symmetric and transitive.
2. *R* is a ***preference relation*** if *R* is asymmetric and negatively transitive.
3. *R* is a ***partial order*** if *R* is reflexive, antisymmetric and transitive.
4. *R* is a ***strict partial order*** if *R* is asymmetric and transitive.
5. *R* is a ***quasi order*** if *R* is reflexive and transitive.
6. *R* is a ***total order*** if *R* is a complete partial order.
7.

Examples 12.

1. Let *S* be a set and consider $\mathcal{P}(S)$, the power set of *S*. Define a binary relation on $\mathcal{P}(S)$, by: $A \approx B \Leftrightarrow |A| = |B|$. Obviously, this relation is an equivalence relation.

2. Let $\mathcal{C}$ be the collection of all courses offered at a university and let *P* be the binary relation ***is-prerequisite-of*** defined on $\mathcal{C}$. This relation must be asymmetric and transitive (we might require it to be irreflexive, but this property is implied by asymmetry). Hence *P* is a strict partial order.

3. Let $\mathbb{R}$ be the set of all real numbers and consider the $\leq$ relation on $\mathbb{R}$. It is clearly reflexive, and transitive. It also is antisymmetric: $a \leq b \wedge b \leq a \Rightarrow a = b.$ Finally, $\leq$ is complete. It follows that $\leq$ is a total order on $\mathbb{R}$.

4. Let *S* be a set and consider again $\mathcal{P}(S)$, the power set of *S*. The binary "subset" relation $\subseteq$ on $\mathcal{P}(S)$ is reflexive, antisymmetric and transitive, a partial order. Now consider the "proper subset" relation $\subset$ on $\mathcal{P}(S)$. It is no longer reflexive, in fact it is irreflexive. More generally, $\subset$

is asymmetric (which implies irreflexivity) and it clearly is transitive. Thus, $\subset$ is a strict partial order.

5. Let $\mathcal{L}$ be the set of all lines in the plane and denote by $\|$ the relation of "being parallel". Thus, $l_1 \| l_2$ means that lines l_1 and l_2 are parallel. Assuming that $\|$ is reflexive, it follows that $\|$ is an equivalence relation on $\mathcal{L}$. We can also define the binary relation of "perpendicularity" on $\mathcal{L}$. Thus, $l_1 \perp l_2$ means that lines l_1 and l_2 are perpendicular. Clearly, this relation is symmetric. It is also irreflexive and it is not transitive or negatively transitive. But for every line *l* there is a line perpendicular to it so $\perp$ is serial.

6. Let $\varnothing \neq H \subseteq \mathbb{R}\times\mathbb{R}$ be an arbitrary set of pairs of real numbers. Define the following two binary relations on H:

$$\sigma: \text{ for } (x,y),(u,v)\in H, \quad ((x,y),(u,v))\in\sigma \iff x+y=u+v, \text{ and}$$
$$\tau: \text{ for } (x,y),(u,v)\in H, \quad ((x,y),(u,v))\in\tau \iff x-y=u-v.$$

It is easy to check that both relations are equivalence relations.

7. The binary relation of congruence modulo m, $x \equiv_m y$, was defined (in Example 6) on the set of integers $\mathbb{Z}$ by: $m \mid x-y$. It is easily checked that this is an equivalence relation:

- reflexivity: $\forall x\in\mathbb{Z}, m \mid (x-x) \Rightarrow \forall x\in\mathbb{Z}, x\equiv_m x \Rightarrow \equiv_m$ is reflexive.
- symmetry: $\forall x,y\in\mathbb{Z}: \ x\equiv_m y \Rightarrow m\mid x-y \Rightarrow m\mid y-x \Rightarrow y\equiv_m x$.
- transitivity: $\forall x,y,z\in\mathbb{Z}$ we have:

$$\begin{aligned} x\equiv_m y \wedge y\equiv_m z \quad &\Rightarrow \quad m\mid(x-y)\wedge m\mid(y-z) \\ &\Rightarrow \quad m\mid(x-y)+(y-z) \quad\Rightarrow\quad m\mid(x-z) \quad\Rightarrow\quad x\equiv_m z. \end{aligned}$$

■

5.6. EQUIVALENCE RELATIONS AND PARTITIONS

Recall from our study of set theory that a ***partition of a set*** X is a collection of subsets of X, $\pi = \{A_i \mid i\in I\}$, satisfying two conditions:

1. [***non-emptiness***] $\forall i\in I,\ \varnothing\neq A_i\subseteq S$,
2. [***mutual disjointness***] $\forall i,j\in I: \ i\neq j \Rightarrow A_i\cap A_j=\varnothing$,
3. [***coverage***] $X=\bigcup_{i\in I}A_i$.

The sets A_i sometimes called the ***blocks*** of the partition π.

Example 13. Let $\Sigma = \{a, b, c, ..., z\}$ be the English alphabet and consider the set of all strings over this alphabet, usually denoted by Σ^*. For each string (word) $w \in \Sigma^*$ we let $\alpha(w) \triangleq \{a \in \Sigma \mid a \text{ occurs in } w\}$. For each subset of $A \subseteq \Sigma$ (and there are 2^{26} of those), define $W_A = \{w \in \Sigma^* \mid \alpha(w) = A\}$, i.e. W_A comprises precisely all those strings that use all and only letters from *A*. For instance,

$$W_\varnothing = \{\lambda\},\ W_{\{a\}} = \{a, aa, aaa, ...\},\ W_{\{x,y\}} = \{xy, yx, xxy, xyx, yxx, ...\}, \text{ etc.}$$

The collection of subsets of Σ^*: $\pi_\Sigma = \{W_A \mid A \subseteq \Sigma\}$ is a partition of Σ^*. ■

There is a very important connection between partitions and equivalence relations which we study next. Let *X* be a set, *R* an equivalence relation on *X*, and $a \in X$. The set of all elements of *X* related to *a* via *R* is defined and denoted $\boxed{[a]_R = \{x \in X \mid (a, x) \in R\}}$. It is called the ***equivalence class of** a* ***w.r.t.*** *R*. Sometimes we shall omit the subscript "*R*" from this notation and just write $[a]$, especially when the identity of the equivalence relation is clear from context.

The collection of all equivalence classes of *R* is defined and denoted by

$$\boxed{X/R \triangleq \{[x]_R \mid x \in X\}}.$$

Observation 1. For any set *X*, any equivalence relation *R* on *X*, and any element $x \in X$, the equivalence class of *x* w.r.t. *R* is never empty. This is because

$$x \in X \quad \Rightarrow \quad (x, x) \in R \text{ (because } R \text{ is reflexive)} \quad \Rightarrow \quad x \in [x]_R .$$

Example 14. Using the notation of Example 13, define the following binary relation on Σ^*: for $x, y \in \Sigma^*$, $x \approx y \iff \alpha(x) = \alpha(y)$. It is checked easily that $\approx$ is an equivalence relation. Moreover, we can calculate the equivalence class of any string

$$[x]_\approx = \{y \in \Sigma^* \mid y \approx x\} = \{y \in \Sigma^* \mid \alpha(y) = \alpha(x)\} = W_{\alpha(x)} .$$

It follows that (at least in this example) the equivalence classes define a partition of the underlying set Σ^*. ■

Example 15. Consider the binary congruence relation $\equiv_m$ on the set of all integers $\mathbb{Z}$, as defined in Example 6. We have shown, Example 12 (part 7), that $\equiv_m$ is an equivalence relation. Let $a \in \mathbb{Z}$. What is the equivalence class of *a* w.r.t. $\equiv_m$? We claim that

$$[a] \triangleq \{x \in \mathbb{Z} \mid x \equiv_m a\} = \{a, a \pm m, a \pm 2m, a \pm 3m, ..., a \pm km, ...\}. \qquad (*)$$

(where the first equality $\triangleq$ is just the definition of $[a]$ and not part of the claim). To prove this claim we must prove two sub-claims:

(a) $\forall k \in \mathbb{Z}\left[a \pm km \equiv a(\operatorname{mod} m)\right]$; this shows that the r.h.s. of (*) is a subset of $[a]$. (b) $\forall b \in [a] \exists k \in \mathbb{Z}^+\left[b = a \pm km\right]$ for some *k* (and either + or −); this implies that $[a]$ is a subset of the r.h.s. of (*).

Claim (a) is rather obvious because $a \pm km \equiv a(\operatorname{mod} m)$ is equivalent to $m \mid (a \pm km) - a$ which is true. For claim (b) let $b \in [a]$. This implies, by definition, that $b \equiv a(\operatorname{mod} m)$ which is equivalent to $m \mid b - a$. Hence, $b - a = km$ (for some *k*) which can be rewritten as $b = a + km$. This proves that *b* belongs to the r.h.s. of (*).

Thus, we have the following equivalence classes:

$$\begin{array}{ll}
[0] = \{0, \pm m, \pm 2m, \pm 3m, ...\} & \text{(multiples of } m\text{)} \\
[1] = \{1, 1 \pm m, 1 \pm 2m, 1 \pm 3m, ...\} & \text{(remainder 1)} \\
[2] = \{2, 2 \pm m, 2 \pm 2m, 2 \pm 3m, ...\} & \text{(remainder 2)} \\
\quad \ddots\ddots & \\
\qquad \ddots\ddots & \\
[m-1] = \{m-1, (m-1) \pm m, (m-1) \pm 2m, (m-1) \pm 3m, ...\} & \text{(remainder } m-1\text{)}
\end{array}$$

but the "next" equivalence class is actually the same as a previous one:

$$[m] = \{m, m \pm m, m \pm 2m, m \pm 3m, ...\} = [0] \ .$$

Recall that every integer $a \in \mathbb{Z}$, when divided by *m*, gives a remainder *r*, $0 \le r < m$; e.g. for $m = 5$ we have: $77 = 15 \cdot 5 + 2$ and $-71 = (-15) \cdot 5 + 4$. It follows from this observation that: $\mathbb{Z} = \bigcup_{k=0}^{m-1} [k]$. We now argue the any two distinct classes are disjoint. This will complete the proof that $\{[0],[1],[2],...,[m-1]\}$ is a partition of $\mathbb{Z}$. Let $0 \le a \ne b < m$. We claim that $[a] \cap [b] = \emptyset$. If not, then for some $k, l \in \mathbb{Z}$: $a + km = b + lm$. W.l.o.g. assume that $a > b$. Then, $0 < a - b = (l-k) \cdot m$ and since $a - b < m$, we have $0 < (l-k) \cdot m < m$ which is impossible.

■

We have shown that (at least in Example 15) the equivalence classes form a partition of the underlying domain. We next argue that this is generally true.

Theorem 4. Let *R* be an equivalence relation on the set *X*. Then $\forall x, y \in X$

1. $[x] = [y] \quad \Leftrightarrow \quad (x, y) \in R.$
2. $[x] \ne [y] \quad \Leftrightarrow \quad [x] \cap [y] = \emptyset.$

Proof. Recall that $[a] = \{x \in X \mid (a, x) \in R\}$.

[1] One direction follows easily from the definition of equivalence classes and by Observation 1: $[x] = [y] \Rightarrow y \in [x] \Rightarrow (x, y) \in R.$ For the other direction suppose that $(x, y) \in R.$ Then, since *R* is symmetric, $(y, x) \in R$ holds as well and so by the definition of equivalence classes, we get $x \in [y]$ and $y \in [x].$ We show only that $[x] \subseteq [y]$ (the proof of the other inclusion being similar):

$$a \in [x] \Rightarrow (x, a) \in R \Rightarrow (y, a) \in R \Rightarrow a \in [y],$$

where the second implication follows from the transitivity of *R*.

[2] For both directions we argue contrapositively. For the $\Leftarrow$-direction, if $[x] = [y],$ then the intersection is non-empty because equivalence classes are non-empty, see Observation 1. For the other direction suppose that $z \in [x] \cap [y].$ By part (1), it suffices to show that $(x, y) \in R.$

$z \in [x] \Rightarrow (x, z) \in R$
$z \in [y] \Rightarrow (y, z) \in R \Rightarrow (z, y) \in R$ (because *R* is reflexive)

and since *R* is transitive, $(x, y) \in R.$ ■

One of the main results of this section is the following theorem.

Theorem 5. Let R be an equivalence relation on the set X. Then the set X/R of all the equivalence classes of R is a partition of X.

Proof. We have to show two things:

- mutual disjointness: $[x] \neq [y] \Leftrightarrow [x] \cap [y] = \varnothing$, and
- coverage: $\bigcup_{x \in X} [x] = X$.

The second claim is immediate because $x \in [x]$ by Observation 1. The first claim is precisely the content of part (2) of Theorem 3. ■

From Example 15 we have: $\mathbb{Z}/{\equiv_m} \triangleq \mathbb{Z}_m = \{[0],[1],[2],...,[m-1]\}$.

Theorem 4 tells us that every equivalence relation induces a partition whose blocks are precisely the equivalence classes of the given equivalence relation. The opposite claim is also true.

Theorem 6. Given a partition $\pi = \{A_i \mid i \in I\}$ of a set X, there is an equivalence relation R such that $X/R = \pi$.

Proof. Define R as follows: $(x, y) \in R \quad \Leftrightarrow \quad \exists i \in I \left[x \in A_i \wedge y \in A_i \right]$. We first show that R is an equivalence relation.

- R is reflexive:

$$\begin{aligned} \pi \text{ is a partition of } X \quad &\Rightarrow \quad \forall x \in X \, \exists i \in I \left[x \in A_i \right] \\ &\Rightarrow \quad \forall x \in X \, \exists i \in I \left[x \in A_i \wedge x \in A_i \right] \\ &\Rightarrow \quad \forall x \in X \left[(x, x) \in R \right] \\ &\Rightarrow \quad R \text{ is reflexive.} \end{aligned}$$

- R is symmetric:

$$\begin{aligned} (x, y) \in R \quad &\Rightarrow \quad \exists i \in I \left[x \in A_i \wedge y \in A_i \right] \\ &\Rightarrow \quad \exists i \in I \left[y \in A_i \wedge x \in A_i \right] \\ &\Rightarrow \quad (y, x) \in R \end{aligned}$$

- R is transitive:

$$\begin{aligned}(x,y)\in R \wedge (y,z)\in R \quad &\Rightarrow \quad \exists i \in I\left[x\in A_i \wedge y \in A_i\right] \\ &\qquad\qquad \wedge\ \exists j\in I\left[y\in A_j \wedge z\in A_j\right] \\ &\Rightarrow \quad \exists i,j\in I\left[y\in A_i \cap A_j\right] \\ &\Rightarrow \quad \exists i,j\in I\left[A_i = A_j\right] \\ &\Rightarrow \quad \exists i\in I\left[x\in A_i \wedge z\in A_i\right] \\ &\Rightarrow \quad (x,z)\in R\end{aligned}$$

To prove $X/R=\pi$ it suffices to show that $\forall x\in X$, if $x\in A_k$ then $[x]=A_k$. This is because both X/R and π are partitions of *X*. Now, suppose that $x\in A_k$ and consider the equivalence class $[x]$. If $y\in[x]$, then $(x,y)\in R$ and, by the definition of *R*, $y\in A_k$. This shows that $[x]\subseteq A_k$. Conversely, let $y\in A_k$. Then both *x* and *y* belong to A_k so, by the definition of *R*, $(x,y)\in R$ and so $y\in[x]$. Hence, $A_k\subseteq[x]$ and the equality is proved.

■

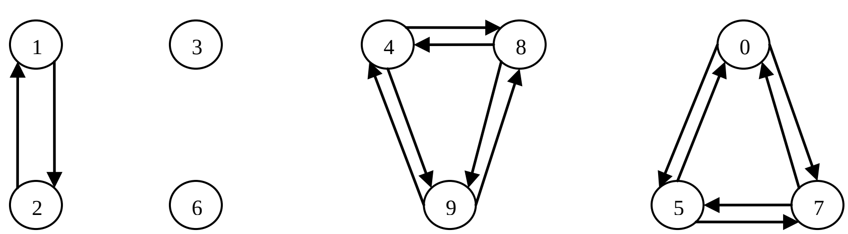
Example 16. Let $X=\{0,1,2,3,4,5,6,7,8,9\}$ and consider the partition

$$\pi=\{\{1,2\},\{3\},\{4,8,9\},\{0,5,7\},\{6\}\}\ .$$

The corresponding equivalence relation is

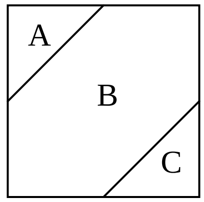
$$\begin{aligned}R_\pi=\{&(1,1),(1,2),(2,1),(2,2),(3,3),(4,4),(4,8),(8,4),(8,8),(4,9),(9,4),(8,9),\\ &(9,8),(9,9),(6,6),(0,0),(0,5),(5,0),(5,5),(0,7),(7,0),(7,7),(5,7),(7,5)\}\end{aligned}$$

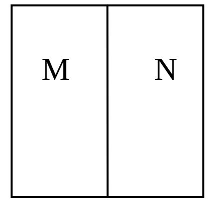
which is better visualized as a graph (all self-loops are omitted):



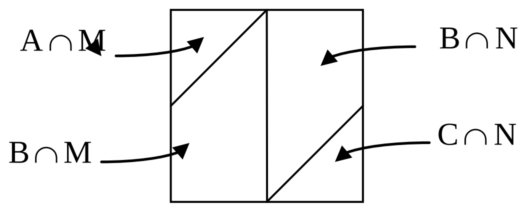
■

Let *R* and *S* be two equivalence relations on a set *X* and let π_R and π_S be the corresponding partitions of *X*. This is illustrated below:

$\pi_R =$ 

$\pi_S =$ 

Now, the intersection $R \cap S$ is also an equivalence relation (see Exercise ??) and we would like to know what the partition $\pi_{R\cap S}$ corresponding to $R \cap S$ "look like". We note that: $(x, y) \in R \cap S \Leftrightarrow (x, y) \in R \wedge (x, y) \in S,$ i.e. two elements in the same equivalence class w.r.t. $R \cap S$ must be in the same equivalence w.r.t. *R* as well as w.r.t. *S*. In the simple illustration given above $\pi_{R\cap S}$ "looks" as follows:



More generally, the blocks in the partition $\pi_{R\cap S}$ are simply all the intersections of the blocks of π_R with the blocks of π_S, omitting the empty intersections (which are not allowed in partitions):

$$\boxed{\pi_{R\cap S} = \{A \cap B \mid A \in \pi_R, B \in \pi_S \text{ s.t. } A \cap B \neq \varnothing\}}\ .$$

5.7. <u>PARTIAL AND RELATED ORDERS</u>

Recall that a binary relation *R* on a set *X* is a *partial order* if *R* is reflexive, antisymmetric and transitive. We have also defined a *strict partial order* by replacing the two properties of reflexivity and antisymmetry with asymmetry. Thus, perhaps strangely, a strict partial order is not a partial order! We have also defined a *quasi order* to be a reflexive and transitive relation. Finally, a *total partial order* was defined as a partial order which is complete. Clearly, every partial order is a quasi order and every total partial order is a partial order.

Example 17.

1. Recall the divisibility relation | on $\mathbb{Z}^+$ (or $\mathbb{N}$), e.g. $3|6, 1|22$, and $17|51$. We noted that this is a partial order. However, **|** is not antisymmetric on the larger domain $\mathbb{Z}$; this is because we have $5|-5$ and also $-5|5$ but $5 \neq -5$. Hence, the relation | on $\mathbb{Z}$ is a quasi order.

2. Let $\Sigma = \{0,1\}$ be the binary alphabet and Σ^* the set of all (finite length) strings over Σ. For two strings $u, v \in \Sigma^*$ we say that *u* is a ***prefix*** of *v* if for some string $w \in \Sigma^*, v = uw$. We will denote this relation by $(u,v) \in \boldsymbol{pref}$. We can similarly define the ***suffix*** relation (denoted by ***suff***) and also the ***substring*** relation (denoted by ***subs***):

$$(u,v) \in \boldsymbol{suff} \Leftrightarrow \exists w \in \Sigma^*[v = wu] \quad \& \quad (u,v) \in \boldsymbol{subs} \Leftrightarrow \exists x, y \in \Sigma^*[v = xuy].$$

It is easy to see that all three relations are partial orders.

3. [Intuitive] Let $\mathcal{S}$ be a set of sentences (in some language), each being either true or false. For $\alpha, \beta \in \mathcal{S}$ we write $\alpha \Rightarrow \beta$ if (informally) α implies β. Then, the binary relation $\Rightarrow$ is reflexive and transitive, i.e. a quasi order. However, it is not a partial order because it is not antisymmetric.

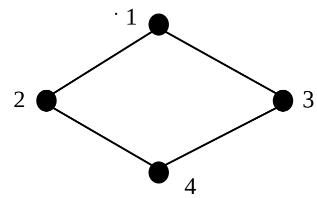
4. The "strictly smaller than" relation $<$ on the set $\mathbb{R}$ (or $\mathbb{Z}$, or $\mathbb{Q}$, or $\mathbb{N}$) is strict partial order as is the "proper subset" relation $\subsetneq$. ■

A convenient way to depict partial orders, at least when the underlying set is not large, is by means of a Hasse diagram[2]. The underlying set is represented by points in the Hasse diagram. We connect a point *x* to a point *y* (with *y* below *x* in the diagram) precisely when the pair (x, y) belongs to the given partial order, say *R*, and there is no other point *z* (in the underlying set) that is "in between" *x* and *y*; i.e., there is no $z \notin \{x, y\}$ such that $(x, z) \in R$ and $(z, y) \in R$. Drawing the point *y* below the point *x* whenever $(x, y) \in R$ allows us to omit the directions from the edges of the diagram. Edges whose existence is implied by the reflexivity or the transitivity of the partial order are omitted.

Example 18.

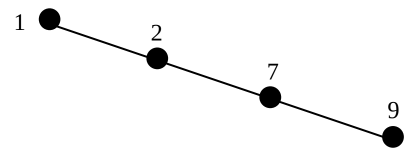
1. Let $R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(2,4),(3,4)\}$ be a binary relation over $X = \{1, 2, 3, 4\}$. It can be easily checked that it is a partial order. The Hasse diagram is:

[2] Hasse diagrams are named after Helmut Hasse (1898–1979). Although Hasse diagrams were originally devised as a technique for making drawings of partially ordered sets by hand, they have more recently been created automatically using "graph drawing" techniques.



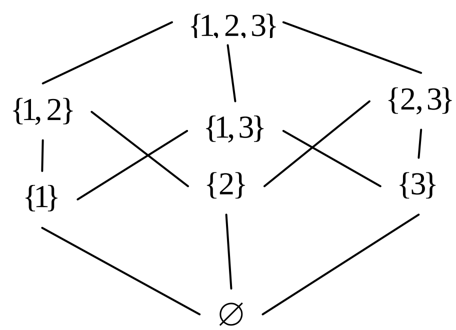
Note that the self-loops are omitted as well as the edge $(1,4)$ whose existence follows from the transitivity of *R*.

2. Consider the binary relation $R = \{(1,2),(1,7),(1,9),(2,7),(2,9),(7,9)\} \cup \Delta_X$ on the set $X = \{1,2,7,9\}$. See Example 9. The Hasse diagram here is

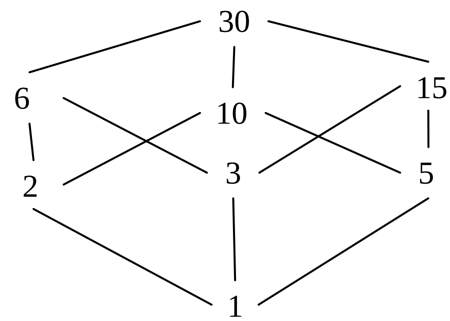


3. We show that "essentially the same" Hasse diagram can represent entirely different partial orders. This illustrates the frequently observed fact that quite different definitions and "circumstances" may lead to the same ordering relation between the corresponding objects.

(a) Consider the powerset $\mathcal{P}(\{1,2,3\})$ with the inclusion operation $\subseteq$. The Hasse diagram corresponding to this partial order is the "3-dimensional cube"



(b) Now consider the set of divisors of the integer 30: $D = \{1,2,3,5,6,10,15,30\}$ with the divisibility relation "|". The Hasse diagram in this case is



This Hasse diagram is “structurally” the same as the one in part (a).

EXERCISES

(01) For each of the relations defined below, (i) write down the Boolean matrix representation of this relation, and (ii) draw its digraph representation.

(a) The relation $<$ on the set $X = \{1, 2, 3, 4\}$.
(b) The relation $x \equiv y (\bmod 3)$ on the set $X = \{0,1, 2, 3, 4, 5, 6, 7\}$.
(c) The relation $y = x + 1$ on $X = \{0,1, 2, 3, 4, 5, 6, 7\}$.
(d) The relation $y \equiv x + 1 (\bmod 6)$ on the set $X = \{0,1, 2, 3, 4, 5\}$.

(02) For each of the conditions defined below check whether or not there is a binary relation on $\mathbb{N}$ that satisfies that condition. Explain.

(a) $R^{-1} \not\subseteq R.$
(b) $R \neq \varnothing$ & $R \circ R = R$ & $R \cap \Delta_{\mathbb{N}} = \varnothing.$
Hint. Since, as we have seen in past lectures that $\mathbb{N}$ and $\mathbb{Q}$ are countable, you may assume (without proof) that there is a bijection $\rho : \mathbb{N} \to \mathbb{Q}.$
(c) $R^{-1} = \overline{R}.$

(03) [(02) cont.] The same problem as (02) except that all the relations are on the set of rational numbers $\mathbb{Q}.$

(04) Let $R, S \subseteq X \times X.$ Prove

(a) $R \subseteq S \quad \Rightarrow \quad R^{-1} \subseteq S^{-1}.$
(b) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}.$
(c) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}.$
(d) $R \subseteq S \quad \Rightarrow \quad \forall n \geq 0, R^n \subseteq S^n.$
(e) $(R \cap S)^2 \subseteq R^2 \cap S^2.$

(05) Consider the relation $\leq$ on the set of real numbers $\mathbb{R}$. Your answer to each of the questions below must be some well known relation.

(a) What is the complement of $\leq$?
(b) What is the inverse $\leq^{-1}$ of $\leq$?
(c) What is the relation $\leq \cap \geq$?
(d) What is the relation $\leq \cup \geq$?

(06) Consider the properties of binary relations given in Definition 2. In the discussion following Corollary A we have considered the properties of the matrix and digraph representations corresponding to reflexive and irreflexive relations. Provide similar discussion for all the other properties listed in Definition 1.

(07) Consider again the properties of binary relation given in Definition 2. Try to find which of these properties imply other properties listed in that definition, similarly to Corollary A.

(08) Consider the properties of binary relation as given in Definition 1. Below are listed fourteen properties of relation expressed in terms of composition, complement, inverse, set-theoretic operations of union and intersection and, of course, set inclusion of binary relations, all over a fixed set *X*. These properties precisely match the (14) definitions given in Definition 1. What is this matching?

$(a)\ R \subseteq R \circ R$ $\quad (b)\ \Delta_X \subseteq R^{-1} \circ R$ $\quad (c)\ R \circ R^{-1} \subseteq R$ $\quad (d)\ \Delta_X \subseteq R$

$(e)\ R^{-1} \circ R \subseteq \Delta_X$ $\quad (f)\ R \subseteq \Delta_X$ $\quad (g)\ R \subseteq \varnothing$ $\quad (h)\ X \times X \subseteq R \cup R^{-1}$

$(i)\ R^{-1} \subseteq R$ $\quad (j)\ R \cap \Delta_X \subseteq \varnothing$ $\quad (k)\ R \circ R \subseteq R$ $\quad (l)\ R \circ R^{-1} \subseteq \Delta_X$

$(m)\ \overline{R} \circ \overline{R} \subseteq \overline{R}$ $\quad (n)\ R \cap R^{-1} \subseteq \varnothing$

(09) Let *X* be a set and let $Rel_2(X)$ be the set of all binary relations on *X*. We have shown that the composition operation on $Rel_2(X)$ is associative. **Prove or disprove** the following claims regarding composition on the set $Rel_2(X)$:

(a) it is ***commutative***: $\forall R, S \in Rel_2(X),\ R \circ S = S \circ R.$

(b) it is ***distributive w.r.t. union***: $\forall R, S, T \in Rel_2(X)$

(i) $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$,

(ii) $(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$.

(c) it is ***distributive w.r.t. intersection***: $\forall R, S, T \in Rel_2(X)$

(i) $R \circ (S \cap T) = (R \circ S) \cap (R \circ T)$,

(ii) $(R \cap S) \circ T = (R \circ T) \cap (S \circ T)$.

(10) Since for any $R \in Rel_2(X)$, we have $R^{-1} \in Rel_2(X)$, Theorem 2 gives another interesting property of $Rel_2(X)$: $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. We also have defined iterated composition R^n by induction on $n \geq 0$. Here we extend this definition to negative powers by letting for any $n \geq 1$: $R^{-n} \triangleq (R^{-1})^n$. Prove that for every $n \geq 0$: $R^{-n} = (R^n)^{-1}$.

(11) Let $R, S \subseteq X \times X$. Prove

(a) R reflexive $\Leftrightarrow$ R^{-1} reflexive.

(b) R reflexive $\Leftrightarrow$ $\overline{R}$ irreflexive.

(c) R, S reflexive $\Leftrightarrow$ $R \cap S$ reflexive.

(d) R or S reflexive $\Rightarrow$ $R \cup S$ reflexive.

(e) R, S reflexive $\Rightarrow$ $R \circ S$ reflexive.

(12) Let $R, S \subseteq X \times X$. Prove

(a) R antisymmetric $\Leftrightarrow$ $R \cap R^{-1} \subseteq \Delta_X$.

(b) R symmetric $\Leftrightarrow$ $\overline{R}$ symmetric.

(c) R, S symmetric $\Rightarrow$ $R \cap S$ symmetric.

(d) R, S symmetric $\Rightarrow$ $R \cup S$ symmetric.

(e) R symmetric $\Rightarrow$ $R \circ R$ symmetric.

(f) $R \subseteq S \ \wedge \ S$ symmetric $\Rightarrow$ $R \cup R^{-1} \subseteq S$.

(13) Let $R, S \subseteq X \times X$. Prove

(a) R, S transitive $\Rightarrow$ $R \cap S$ transitive.

(b) R transitive $\Leftrightarrow$ R^{-1} transitive.

(c) R, S equivalence relations $\Rightarrow$ $R \cap S$ equivalence relation.

(d) Give a counterexample to: R, S transitive $\Rightarrow$ $R \circ S$ transitive.

(14) Consider the implication claims in problems (11, parts d and e), (12, parts c, d, e and f), and (13, parts a and c). In each case consider the opposite implication. If you think it holds, prove it. Otherwise, give a counterexample.

(15) Let $X = \{1,2,3\} \times \{1,2,3,4\}$ be a set of pairs. In each case prove that the defined relation is an equivalence relation. Explain your reasoning

(a) Define a binary relation E on X as follows: $(x, y)E(u, v) \Leftrightarrow x + y = u + v$.

(b) Define a binary relation D on X as follows: $(x, y)D(u, v) \Leftrightarrow |x - y| = |u - v|$.

(c) Define a binary relation M on X as follows: $(x, y)M(u, v) \Leftrightarrow x \cdot y = u \cdot v$.

(d) Define a binary relation P on X as follows: $(x, y)P(u, v) \Leftrightarrow x^y = u^v$.

(16) Let $Rel_2(X)$ denote the set of all binary relations on a set *X*. Consider any of the properties, Ψ say, listed in Definition 2. We say that Ψ is *preserved under composition* if for arbitrary $R, S \in Rel_2(X)$: if *R* and *S* have property Ψ, then so does $R \circ S$. Which of the properties listed in Definition 2 are closed under composition and which are not? Give counterexamples or proofs, as appropriate.

(17) Consider the following properties of binary relations: being *reflexive*, *irreflexive*, *symmetric*, *asymmetric*, *antisymmetric*, *transitive*, *negatively transitive*, *serial*, *euclidean*, and the compound properties of being an *equivalence relation*, *partial order* (reflexive, antisymmetric and transitive), or a *preference relation* (asymmetric and negatively transitive). For each of the binary relations defined below specify which of the properties listed above the given relation must satisfy. Present you results in a table.

(a) Relation $|$ on $\mathbb{Z}^+$ defined by: $(x, y) \in | \Leftrightarrow x$ divides *y*.

(b) Relation $\equiv_5$ on $\mathbb{Z}^+$ defined by: $x \equiv_5 y \Leftrightarrow 5$ divides $(x - y)$.

(c) Let $G = (V, E)$ be a directed graph. Define a relation ρ_G on *V* by:

$(x, y) \in \rho_G \Leftrightarrow$ there is a directed path (in *G*) from *x* to *y*.

(d) Let $G = (V, E)$ be a directed graph. Define a relation σ_G on *V* by:

$(x, y) \in \sigma_G \Leftrightarrow$ there is a directed path from *x* to *y* and from *y* to *x*.

(e) Let *S* be a set and $\boldsymbol{P}(S)$ the power set of *S*. Define a relation $\subset$ on $\boldsymbol{P}(S)$, by:

$A \subset B \Leftrightarrow A$ is a <u>proper</u> subset of *B*.

(18) Let $Y = \{1, 2, ..., n\}$ be a set with *n* elements.

(01) How many (binary) relations on *Y* are there?
(02) How many of the relations in (a) are reflexive?
(03) How many of the relations in (a) are irreflexive?
(04) How many of the relations in (a) are symmetric?
(05) How many of the relations in (a) are asymmetric?
(06) How many of the relations in (a) are antisymmetric?
(07) How many of the relations in (a) are both reflexive and symmetric?
(08) How many of the relations in (a) are both neither reflexive nor symmetric?
(09) How many of the relations in (a) are complete?
(10) How many of the relations in (a) are serial?
(11) How many of the relations in (a) are partial functions?
(12) How many of the relations in (a) are trivial?
(13) How many of the relations in (a) are empty?
(14) How many of the relations in (a) are reflexive, symmetric and antisymmetric?

(19) Let $R \subseteq X \times X$ be a binary relation. Define two "projection" functions based on *R*, $\pi_R^i : X \to \mathbb{P}(S),\ \ i = 1, 2,$ by

$$\pi_R^1(y) \triangleq \{x \in X \mid (x, y) \in R\} \quad \text{and} \quad \pi_R^2(x) \triangleq \{y \in X \mid (x, y) \in R\}.$$

(a) Prove that *R* is reflexive $\Leftrightarrow \forall x \in X \left[x \in \pi_R^i(x) \right]$.

(b) Prove that *R* is symmetric $\Leftrightarrow \forall x, y \in X \left[x \in \pi_R^i(y) \Rightarrow y \in \pi_R^i(x) \right]$.

(c) Prove that *R* is transitive $\Leftrightarrow \forall i \forall x, y \in X \left[x \in \pi_R^i(y) \Rightarrow \pi_R^i(x) \subseteq \pi_R^i(y) \right]$. ????

(d) Prove that if *R* is an equivalence relation, then $range(\pi_R^i)$ is a partition of *X*.

(e) Let $f : X \to \mathbb{P}(X)$ be a function. Define the relation $\rho(f)$ on X by $\rho(f) \triangleq \{(x, y) \in X \times X \mid x \in f(y)\}$. Prove that $\pi_{\rho(f)}^1 = f$.

(20) Let R be an equivalence relation on a set *X*, let $\mathcal{C}_R$ be the set of corresponding equivalence classes and let $f : X \to Y$ be a function. Show that the following two conditions are equivalent

(A) $\exists g : \mathcal{C}_R \to Y$ such that $\forall a \in X,\ g([a]_R) = f(a)$.

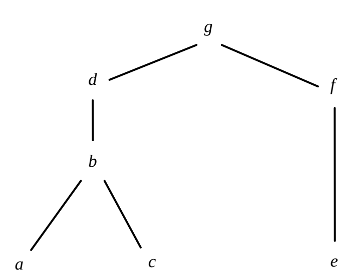
(B) $\forall (a, b) \in R,\ f(a) = f(b)$.

Also show that when these conditions are satisfied, the function *g* referred to in (A) is unique.

(21)[Project Scheduling] Here is a possible scenario. A development project of a construction company requires the completion of seven tasks $\{a, b, c, d, e, f, g\}$. Some of these tasks cannot be started before some other tasks are finished (e.g., cannot put on socks after putting your shoes on). A partial order on the tasks is set up:

task $x \ll$ task y ***iff*** task *y* cannot be started before task *x* is completed.

The Hasse diagram for the seven tasks with respect to this partial order is shown below:



Find a total order in which these tasks can be carried out to complete the project.

(22) Count and exhibit all the possible partial orders of the set $\{1,2,3\}$.

Hint. First find all the Hasse diagrams on 3 vertices and then count the number of partial orderings for each diagram. **Remark**. Using a similar approach we can count the number of partial orders on sets with 4 elements. It turns out that in this case there are 219 partial orders.

(23) Let ρ be a ***partial order*** (reflexive, antisymmetric, and transitive) on a set *S*. Two elements $a,b \in S$ are said to be ***comparable via*** ρ if either $(a,b) \in \rho$ or $(b,a) \in \rho$. Which of the relations defined below is a total order, a partial order, or neither. You need indicate only the most restrictive of the correct answers.

(a) The set $\mathbb{Z}^+$ with the binary relation $\leq$.
(b) The set $\mathbb{Z}^+$ with the binary relation $<$.
(c) The set $\mathbb{Z}^+$ with the binary relation $|$ ("divides").
(d) The set $(0,1]$ with the binary relation $\leq$.
(e) The set $\mathbb{Z}^+ \times \mathbb{Z}^+$ with the binary relation β defined as follows:

$$(a,a')\beta(b,b') \quad \Leftrightarrow \quad \left[a<b \,\vee\, (a=b \wedge a'<b')\right].$$

(This is sometimes called the ***lexicographic ordering***.)

(f) The set $\boldsymbol{P}(\mathbb{Z})$, the ***power set*** of $\mathbb{Z}$, with the binary relation $\subseteq$.

(24) Recall Example 17, part 2. Prove

(a) Prove or disprove: $\boldsymbol{pref} \circ \boldsymbol{suff} = \boldsymbol{suff} \circ \boldsymbol{pref}$.
(b) Prove or disprove: $\boldsymbol{pref} \circ \boldsymbol{suff} = \boldsymbol{subs}$.

(25) Let $f: X \to Y$ be a function and μ_Y be any partial order on *Y*. Define a binary relation μ_X on X by: $(x_1,x_2) \in \mu_X \Leftrightarrow (f(x_1), f(x_2)) \in \mu_Y$.

(a) Prove that μ_X is a quasi order that need not be a partial order.
(b) Prove that if f is injective, then μ_X is a partial order.

(26) Let $\preccurlyeq$ be an arbitrary quasi order on the set *X* and define $x \approx y \Leftrightarrow [x \preccurlyeq y \;\&\; y \preccurlyeq x]$.
Prove that $\approx$ is an equivalence relation.

(27) Let X be a finite set and η a partial order on X. Define the collection of all the total orders that are compatible with η: $\mathcal{T}_\eta = \{\tau \mid \tau \text{ is a total order and } \eta \subseteq \tau\}$. Based on $\mathcal{T}_\eta$ define a probability-like function on $X \times X$:

$$p_\eta(x,y) \triangleq \frac{\left|\{\tau \in \mathcal{T}_\eta \mid (x,y) \in \tau\}\right|}{\left|\mathcal{T}_\eta\right|}.$$

In other words, $p_\eta(x,y)$ is the fraction of total orders in $\mathcal{T}_\eta$ that "rank" y after x.

(a) Consider the 3-element partial order $\eta = \{(a,b)\} \cup \Delta_{\{a,b,c\}}$ on $X = \{a,b,c\}$. Compute the set $\mathcal{T}_\eta$ and calculate $p_\eta(x,y)$ for all $x, y \in X$.

(b) For $n \geq 3$ let $X = \{1,2,...,n\}$. Define the binary relation η on X by:

$$(x,y) \in \eta \quad \Leftrightarrow \quad x = y \vee y - x \geq 2.$$

Clearly, η is a partial order on X. Compute the set $\mathcal{T}_\eta$ and calculate $p_\eta(x,y)$ for all $x, y \in X$.

(c) Show that $x, y \in X$ are incomparable w.r.t. η (i.e., $(x,y) \notin \eta$ and $(y,x) \notin \eta$) if, and only if, $0 < p_\eta(x,y) < 1$.

(d)[Open?] Let η be a partial, but **non-total**, order on a finite set X. Show that there exist elements $x, y \in X$, such that $1/3 < p_\eta(x,y) < 2/3$.

(28) Let X be a set and Π_X be the set of all partitions of X. For $x \in X$ and $\sigma \in \Pi_X$, we denote by $\sigma(x)$ the block of the partition σ to which x belongs. For partitions $\sigma, \tau \in \Pi_X$ we say that σ is a ***refinement*** of τ, denoted by $\sigma \sqsubseteq \tau$, if $\forall x \in X, \sigma(x) \subseteq \tau(x)$.

(a) Prove that $\sqsubseteq$ is a partial order on the set Π_X.

(b) Draw the Hasse diagram for Π_X where $X = \{1,2,3,4\}$.

(29) Let X be a set. Prove or Disprove: R partial order on $X \Rightarrow R^{-1}$ partial order on X.

(30) Let σ be a preference relation (asymmetric and negatively transitive), on a set X. Prove that σ is irreflexive and transitive.

(31) Let σ be a binary relation on a set *X*. Based on σ define a new relation $\sim_{\sigma}$ as follows: $(x, y) \in \simeq_{\sigma} \iff (x, y) \notin \sigma \wedge (y, x) \notin \sigma.$ Prove that if σ is a preference relation, then $\simeq_{\sigma}$ is an equivalence relation.

Remark. A preference relation may represent consumer preferences over some “menu” of items. The definition of $\simeq_{\sigma}$ is essentially expressing “indifference” in the sense that $(x, y) \in \simeq_{\sigma}$ whenever “the consumer” does not prefer *x* over *y* (i.e., $(x, y) \notin \sigma$) nor *y* over *x* (i.e., $(y, x) \notin \sigma$).

(32) Prove that if σ is a preference relation on a set *X*, then for every $x, y \in X$ exactly one of the conditions $(x, y) \in \sigma, (y, x) \in \sigma,$ or $(x, y) \in \simeq_{\sigma}$ holds.

Remark. The claim here is that if σ is a consumer preferences, then for any $x, y \in X$, the consumer either prefers *x* over *y*, or *y* over *x*, or is indifferent between them.