# Chapter 1 - NAIVE SET THEORY

## 1.1. SETS & SUBSETS

A ***set*** is a collection of objects. The objects are called ***elements*** or ***members*** of the set. There are no restrictions on the "nature" of elements; they can be numbers, programs, houses, people, DNA sequences, configurations of some game, functions of some type, and even sets.

The fact that object $a$ ***is an elements of set*** $S$ is denoted by $\boxed{a \in S}$. The opposite fact, that object $a$ is not an element of set $S$, is denoted by $\boxed{a \notin S}$. We also say that an element ***belongs*** ($\in$) or ***does not belong*** ($\notin$) to a set. The identity of a set is completely determined by its elements: "two" sets that have exactly the same elements are considered identical (even if described in different ways). We will say more on this issue later on.

We shall use the following ***logical symbols*** in an informal way:

| | | | | |
|---|---|---|---|---|
| $\wedge$ | & | | and | |
| $\vee$ | | | or | |
| $\Rightarrow$ | | | implies | |
| $\Leftrightarrow$ | *iff* | $\equiv$ | if and only if, is equivalent to | |
| $\sim$ | $\neg$ | | not | |
| $\forall x$ | | | for every $x$, for all $x$ | [***universal quantifier***] |
| $\exists x$ | | | there exists an $x$, for some $x$ | [***existential quantifier***] |

One way that can be used to describe a set is by ***listing*** its elements enclosed within brackets, for example

Days = {Mon, Tue, Wed, Thu, Fri, Sat, Sun}

but the order of listing ***is immaterial***. Moreover, the same element may be listed several times and this does not make any difference.

Another way to define a set is by means of a ***property*** (a.k.a. ***predicate***). If $P$ is a relevant property, then we can define a set as the collection of all objects that have property $P$:

$$\boxed{S = \{s \mid s \text{ has property } P\}}\ .$$

For example, the set of people at least seven feet tall can be defined by

$$S_7 \;=\; \left\{ p \,\middle|\, p \text{ is a person and } p \text{ is at least 7 feet tall} \right\}$$

and it just so happens that (a) Michael Jordan $\notin S_7$, and (b) Shaquille O'Neal $\in S_7$.

The following sets are often used in mathematics and computer science and we shall use them in many examples:

$$\boxed{\begin{array}{l}
\mathbb{Z} = \{\text{integers}\} = \{0,1,-1,2,-2,3,-3,\ldots\} \\
\mathbb{N} = \{\text{natural numbers}\} = \{n \in \mathbb{Z} \mid n \geq 0\} = \{0,1,2,3,\ldots\} \\
\mathbb{Q} = \{\text{rational numbers}\} = \{\tfrac{m}{n} \mid m,n \in \mathbb{Z} \;\&\; n \neq 0\} \\
\mathbb{R} = \{\text{real numbers}\} \\
\mathbb{C} = \{\text{complex numbers}\} = \{a+bi \mid a,b \in \mathbb{R}\} \\
\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n > 0\} = \{1,2,3,\ldots\} \\
\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\} \\
\mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}
\end{array}}$$

In many situations all the objects under consideration are exclusively elements of a given fixed set. Such a set is then called ***the universe*** (*of discourse*) or a ***universal set***. We will often use $U$ to denote a universal set. For example, if $U = \mathbb{R}$ then the description

$$A = \left\{ x^2 + y^2 \,\middle|\, 0 \leq x \leq 1 \;\&\; 1 \leq y \leq 2 \right\}$$

is adequate and we do not need to specify that *x* and *y* are real numbers.

The description of a set may be such that no element fulfills it, i.e. there are no elements that satisfy the defining property of the set. Such a set is called the ***empty set*** (a.k.a. ***null set***, ***void set***). The empty set is unique and it is denoted by $\varnothing$. Thus,

$\varnothing$ = "empty set of onions" = "empty set of people" = "empty set of dragons".

Another, more mathematical, example is: $\{n \in \mathbb{Z} \mid n > 3 \wedge n < 2\} = \varnothing$. A set with a single element is often called a ***singleton***. A set is ***finite*** if it has a finite number of elements. Otherwise it is an ***infinite set***. (Note the circularity in the definition of finite sets. We rely on readers' experience and will have more to say about a more rigorous approach.)

**Important Remark**. The ability to describe a set does not guarantee its existence. For a set to exist it must be possible, in principle, to describe it in a non-paradoxical way: in particular, it must be impossible for any object to both belong and not to belong to the set. Here is an example first brought up by Bertrand Russell[1] which played an important part in the early development of set theory: consider a set whose elements are sets that have the property that they do not belong to themselves: $\mathcal{R} = \{A \mid A \text{ is a set and } A \notin A\}$. Since we have allowed sets to be members of other sets this definition seems reasonable. For instance, $\{1,2,3\} \in \mathcal{R}$ as well as $\mathbb{N}, \mathbb{R}, \mathbb{Z} \in \mathcal{R}$. On the other hand "the set of all infinite sets", denoted by $\mathcal{S}_{\text{inf}}$ say, is itself infinite, so it does belong to itself: $\mathcal{S}_{\text{inf}} \in \mathcal{S}_{\text{inf}}$. Thus, $\mathcal{S}_{\text{inf}} \notin \mathcal{R}$. However, a set being a member of itself is, on the intuitive level, "circular", and seems rather pathological. But without appropriate restrictions on the concept of "set" we are bound to have (in addition to $\mathcal{S}_{\text{inf}}$) sets like "the set of all sets" which, being a set, is a member of itself.

Now, $\mathcal{R}$ being a set, we may ask: $\mathcal{R} \in \mathcal{R}$? If $\mathcal{R} \in \mathcal{R}$, then, by its own definition, $\mathcal{R} \notin \mathcal{R}$; and if $\mathcal{R} \notin \mathcal{R}$, then $\mathcal{R}$ satisfies the defining property of $\mathcal{R}$ which implies $\mathcal{R} \in \mathcal{R}$. This is known as the ***Russell's paradox***. ■

We said before that a set is fully determined by its elements. Now we will expand on the related concepts and go into more details and notation.

Let $A$ and $B$ be two sets. We say that $A$ ***is a subset of*** $B$, notation $\boxed{A \subseteq B}$, if every element of $A$ is an element of $B$. We also say that $A$ ***is contained in*** $B$, or that $B$ ***contains***, or ***is a superset of***, $A$. The relation $\subseteq$ is called ***inclusion*** or ***containment***. Using logical notation we can express the meaning of $A \subseteq B$ formally

$$\boxed{A \subseteq B \qquad \text{means} \qquad \forall x\left[x \in A \;\Rightarrow\; x \in B\right]}.$$

Two sets $A$ and $B$ ***are equal***, notation $\boxed{A = B}$, if they have exactly the same elements, i.e. if every element of $A$ is an element of $B$ and conversely, every element of $B$ is an element of $A$. More formally, we have the following equivalences:

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[1] Bertrand Arthur William Russell, 3rd Earl Russell (1872 – 1970) was a British philosopher, logician, and mathematician. He co-authored, with Alfred N. Whitehead, *Principia Mathematica*, an attempt to ground mathematics on logic. Read more at [http://en.wikipedia.org/wiki/Russell%27s_paradox](http://en.wikipedia.org/wiki/Russell%27s_paradox) .

$$\begin{aligned} A=B \quad & \textit{iff} \quad \forall x[x\in A \;\Leftrightarrow\; x\in B] \\ & \textit{iff} \quad \forall x[(x\in A \;\Rightarrow\; x\in B) \wedge (x\in B \;\Rightarrow\; x\in A)] \\ & \textit{iff} \quad \forall x(x\in A \;\Rightarrow\; x\in B) \wedge \forall x(x\in B \;\Rightarrow\; x\in A) \\ & \textit{iff} \quad [A\subseteq B \wedge B\subseteq A] \end{aligned}$$

It follows that whenever we want to prove that two sets are equal, we must show ***two inclusions***, as indicated above.

We say that set *A* ***is not a subset*** of set *B*, notation $\boxed{A \not\subseteq B}$, means that not every element of *A* is also an element of *B*, or in logical notation: $\exists x[x \in A \;\wedge\; x \notin B]$. Sets *A* and *B* ***are not equal***, notation $\boxed{A \neq B}$, if either $A \not\subseteq B$ or $B \not\subseteq A$.

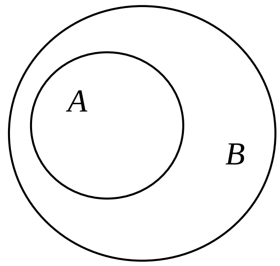
**Claim 1**. For every set *S*, $\varnothing \subseteq S \subseteq S$.
**Proof**. $\varnothing \not\subseteq S$ means $\exists x[x \in \varnothing \;\wedge\; x \notin S]$ which logically implies $\exists x[x \in \varnothing]$. This contradicts the definition of the empty set. The second inclusion is trivial ("*trivial*" means that the claim follows directly from the definitions). Thus, any set is both a subset and a superset if itself. ■

A set *A* ***is a proper subset of*** set *B*, notation $\boxed{A \subsetneq B}$, if *A* is a subset of *B* but not equal to it; i.e., $(A \subseteq B \wedge A \neq B)$. Another notation for that is $A \subset B$. Thus, $A \subsetneq B$ implies that <u>there is an element in the set *B* which is not an element of *A*</u>. Compare this to $A \not\subseteq B$ which means that <u>there is an element of *A* that does not belong to *B*</u>. It follows that, $A \subsetneq B \Rightarrow B \not\subseteq A$. Clearly, the opposite implication does not hold.

**Example 1**. Clearly, $\mathbb{Q} \not\subseteq \mathbb{Z}$, as there are rational numbers (fractions) that are not integers. We also have the proper inclusions: $\mathbb{Z}^+ \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$ as well as $\mathbb{Z}^+ \subsetneq \mathbb{Q}^+ \subsetneq \mathbb{R}^+$. ■

The ***cardinality of a set*** is the "number" of <u>distinct</u> elements in the set. For finite sets we can take the quotes off. The cardinality of infinite sets will be discussed later. We use the notation $|S|$ to denote the cardinality of set *S*. For instance, the cardinality of the empty set is $0$ (i.e., $|\varnothing| = 0$), singletons have cardinality 1, $|\{1, 77, 77, 777\}| = 3$, and $|\{32, 55, 49, 11111\}| = 4$. Note that $|\{\varnothing\}| = 1$. When a finite set has *n* distinct elements, we sometimes say that it is an <u>*n*-set</u>.

It is sometimes useful to give a pictorial representation of inclusion relationships between sets. Such pictures are called ***Venn diagrams*** (after John Venn, 19th century). For example, the inclusion $A \subseteq B$ can be depicted by the Venn diagram:




The set of all subsets of a set *S* is called the ***power set*** of *S*. We use the following notation

$$\boxed{\mathcal{P}(S) \; = \; 2^S \; \triangleq \; \{A \mid A \subseteq S\}}.$$

For example, $\mathcal{P}(\{a\}) = \{\varnothing, \{a\}\}$ and $\mathcal{P}(\{a,b\}) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$. Note that by Claim 1, for any set *S*, $\varnothing \subseteq S \subseteq S$ and so: $\varnothing, S \in \mathcal{P}(S)$. Hence, $\mathcal{P}(\varnothing) = \{\varnothing\}$. Also observe that

$$|\mathcal{P}(\varnothing)| = 1, \quad |\mathcal{P}(\{a\})| = 2, \quad |\mathcal{P}(\{a,b\})| = 4$$

and indeed it is true that for any finite set *S*: $\boxed{|\mathcal{P}(S)| \; = \; 2^{|S|}}$. The proof will be given later.

**Example 2**. As we noted above, $|\varnothing| = 0$ and $\mathcal{P}(\varnothing) = \{\varnothing\}$; hence, $|\mathcal{P}(\varnothing)| = |\{\varnothing\}| = 1$. Let's do it again: $\mathcal{P}(\mathcal{P}(\varnothing)) = \mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$, and so $|\mathcal{P}(\mathcal{P}(\varnothing))| = 2$. And once more:

$$\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))) = \mathcal{P}(\{\varnothing, \{\varnothing\}\}) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$$

and so $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing)))| = 4$. This agrees with the general formula given above. ■

## 1.2. ALGEBRA OF SETS

We first define operations on sets. Each of these operations takes two sets "as input" and defines a new set as the "output" of (the result of performing) the operation. Think, for example, of addition of integers: 3 + 5 = 8. The operation of addition takes two integer inputs and produces an integer output. Here we are interested in operations on sets.

Let $A$ and $B$ be two sets. We define the operations on sets and, at the same time, introduce the corresponding notation.

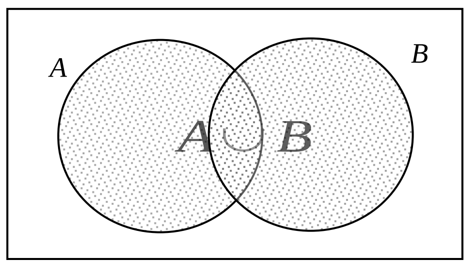
- The ***union*** of $A$ and $B$ is the set

$$\boxed{A \cup B \triangleq \{x \mid x \in A \vee x \in B\}},$$

i.e., the set of elements that belong to either $A$ or $B$. Thus, for every object $x$,

$$x \in A \cup B \quad \Leftrightarrow \quad (x \in A \vee x \in B)$$

and the Venn diagram is



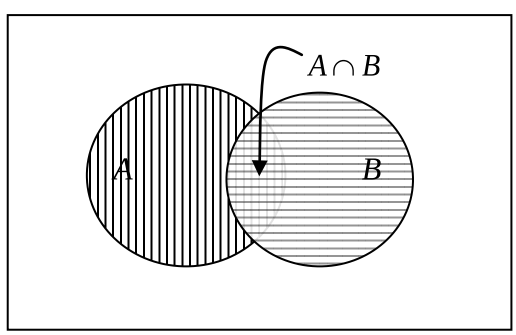

- The ***intersection*** of $A$ and $B$ is the set

$$\boxed{A \cap B \triangleq \{x \mid x \in A \wedge x \in B\}} \ ,$$

i.e., the set of elements that belong to both $A$ and $B$. Thus, for every object $x$,

$$x \in A \cap B \quad \Leftrightarrow \quad (x \in A \wedge x \in B)$$

and the Venn diagram is



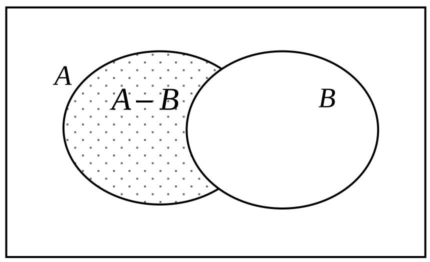
- The ***difference*** of $A$ and $B$, also called the *complement of* $B$ *w.r.t*. (or *relative to*) $A$, is the set

$$\boxed{A - B \triangleq \{x \mid x \in A \wedge x \notin B\}} \, .$$

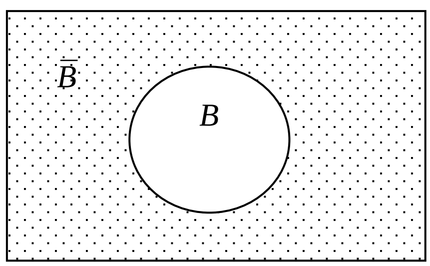
Thus, for every object $x$,

$$x \in A - B \quad \Leftrightarrow \quad (x \in A \wedge x \notin B)$$

and the Venn diagram is




If $U$ is the universal set containing $B$, then the difference $U - B$ is denoted by $\overline{B}$ and called the ***complement*** of $B$. Thus, $x \in \overline{B} \Leftrightarrow x \notin B.$ The Venn diagram is

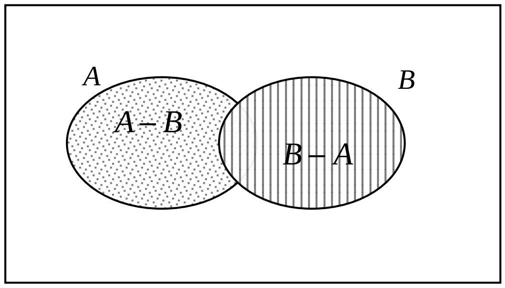



Obviously, in the context of a universal set, we have $\boxed{A - B = A \cap \overline{B}}$ .

- The ***symmetric difference*** of sets $A$ and $B$ (a.k.a. ***exclusive or***, often abbreviated as *XOR*) is defined by

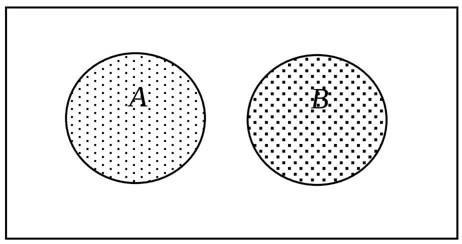
$$\boxed{A \oplus B \triangleq (A - B) \cup (B - A)}$$

and the Venn diagram is




The symmetric difference consists of those elements (of the universe) regarding whose membership the sets $A$ and $B$ "disagree." It is sometimes denoted by $A \Delta B$.

Two sets $A$ and $B$ are said to be ***disjoint*** if $\boxed{A \cap B = \varnothing}$. I.e., $\neg\exists x[x \in A \wedge x \in B]$ or equivalently, $\forall x[x \notin A \vee x \notin B]$. This condition can be depicted in a Venn diagram




We proceed with several simple properties of the concepts defined so far.

**Claim 2**. For all sets $A$ and $B$, $\boxed{A \cap B \subseteq A \subseteq A \cup B}$.

**Proof**. A simple logical manipulation:

$$x \in A \cap B \quad \Leftrightarrow \quad x \in A \wedge x \in B \quad \Rightarrow \quad x \in A\,,$$

implies the first inclusion. And similarly, the logical argument

$$x \in A \quad \Rightarrow \quad x \in A \vee x \in B \quad \Leftrightarrow \quad x \in A \cup B\,,$$

implies the second inclusion. ■

**Claim 3.** For all sets $A$, $B$, and $C$,

(a) $\boxed{A \subseteq B \wedge B \subseteq C \quad \Rightarrow \quad A \subseteq C}$. [ "*transitivity*" of $\subseteq$ ]

(b) $\boxed{A \subseteq C \wedge B \subseteq C \quad \Leftrightarrow \quad A \cup B \subseteq C}$.

(c) $\boxed{A \subseteq B \wedge A \subseteq C \quad \Leftrightarrow \quad A \subseteq B \cap C}$.

**Proof**. (a). For any element $x$

$$\begin{aligned} x \in A \quad &\Rightarrow \quad x \in B \qquad \text{[follows from assumption } A \subseteq B\text{]} \\ x \in B \quad &\Rightarrow \quad x \in C \qquad \text{[follows from assumption } B \subseteq C\text{]} \end{aligned}$$

and combining the two implications we get that for any $x$: $x \in A \Rightarrow x \in C$, which means, by definition, that $A \subseteq C$.

(b) $(\Leftarrow)$ direction follows from Claim 2 and part (a):

$$\begin{aligned} &A \subseteq A \cup B \quad \text{[by Claim 2]} \quad \& \\ &A \cup B \subseteq C \quad \text{[by assumption]} \end{aligned}$$

together imply, by part (a), that $A \subseteq C$. Similar proof works for showing that $B \subseteq C$.
For the $(\Rightarrow)$ direction we argue as follows:

$$x \in A \cup B \quad \Leftrightarrow \quad x \in A \vee x \in B \quad \Rightarrow \quad x \in C \vee x \in C \quad \Rightarrow \quad x \in C.$$

(c) For the $(\Rightarrow)$ direction we have

$$\begin{aligned} x \in A \quad &\Rightarrow \quad x \in B \ \& \ x \in C \qquad \text{[by assumption]} \\ &\Rightarrow \quad x \in B \cap C \qquad \text{[by the definition of } \cap\text{]} \end{aligned}$$

showing that $x \in A \Rightarrow x \in B \cap C$, which, by the definition of inclusion, means: $A \subseteq B \cap C$.
For the $(\Leftarrow)$ direction, by Claim 2, we have $B \cap C \subseteq B$ and $B \cap C \subseteq C$ which, by assumption and part (a), yields $A \subseteq B$ and $A \subseteq C$. ■

**Claim 4**. For all sets $A$ and $B$ the following conditions are equivalent (***TFAE***):

(1) $A \subseteq B$.
(2) $A \cup B = B$.
(3) $A \cap B = A$.
(4) $A - B = \varnothing$.

**Proof**. A nice way to prove equivalence of several statements is by a "cycle of implications," in this case: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$: $B \subseteq A \cup B$ by Claim 2. By Claim 1, $B \subseteq B$ and by the assumption (1), $A \subseteq B$; hence, by Claim 3 part (b), $A \cup B \subseteq B$ and the equality (2) is proved.

$(2) \Rightarrow (3)$: $A \cap B \subseteq A$ by Claim 2. From Claim 2 and assumption (2), $A \subseteq A \cup B = B$ and so by, Claim 3 part (c), $A \subseteq A \cap B$ and (3) is proved.

$(3) \Rightarrow (4)$: Let $x$ be arbitrary. Then,

$$
\begin{aligned}
x \in A - B \quad &\Leftrightarrow \quad x \in A \wedge x \notin B \quad \Rightarrow \quad x \in A \wedge x \notin A \cap B \\
&\Leftrightarrow \quad x \in A \wedge x \notin A \qquad \text{(by part (3))} \\
&\Leftrightarrow \quad \text{contradiction!}
\end{aligned}
$$

Hence (4) follows.

$(4) \Rightarrow (1)$: Let $x \in A$. If $x \notin B$, then $x \in A - B = \varnothing$, by (3). This is impossible, so $x \in B$. ■

**Claim 5**. For all sets A and B, $\boxed{A = B \quad \Leftrightarrow \quad A \oplus B = \varnothing}$.

**Proof**. We argue using definitions and Claim 4:

$$
\begin{aligned}
\varnothing = A \oplus B \quad &\Leftrightarrow \quad \varnothing = (A - B) \cup (B - A) && \text{(by the definition of } \oplus \text{)} \\
&\Leftrightarrow \quad \varnothing = A - B \wedge \varnothing = B - A \\
&\Leftrightarrow \quad A \subseteq B \wedge B \subseteq A && \text{(by Claim 4)} \\
&\Leftrightarrow \quad A = B \ . \ ■
\end{aligned}
$$

<u>**Theorem 6**</u>. [Summary of properties of set-theoretic operations.] Let *A*, *B*, and *C* be sets.

(1) <u>*Commutative Laws*</u>

(a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$ (c) $A \oplus B = B \oplus A$

(2) <u>*Associative Laws*</u>

(a) $A \cup (B \cup C) = (A \cup B) \cup C$ (b) $A \cap (B \cap C) = (A \cap B) \cap C$
(c) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

(3) <u>*Distributive Laws*</u>

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(c) $A \cap (B - C) = A \cap B - A \cap C$
(d) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

**Proof**.

$$\begin{aligned} A \cap (B \oplus C) &= A \cap [(B - C) \cup (C - B)] = (A \cap (B - C)) \cup (A \cap (C - B)) \\ &= (A \cap B - A \cap C)) \cup (A \cap C - A \cap B)) = (A \cap B) \oplus (A \cap C). \end{aligned}$$

(4) *Idempotent Laws*

(a) $A \cup A = A$ (b) $A \cap A = A$
(c) $A - A = \varnothing$ (d) $A \oplus A = \varnothing$

(5) *Properties of the Empty Set*

(a) $A \cup \varnothing = A$ (b) $A \cap \varnothing = \varnothing$ (c) $A \oplus \varnothing = A$
(d) $A - \varnothing = A$ (e) $\varnothing - A = \varnothing$

(6) *Properties of the Universal Set*

(a) $A \cup U = U$ (b) $A \cap U = A$ (c) $A - U = \varnothing$
(d) $A \oplus U = U - A = \overline{A}$

(7) *Complement Laws*

(a) $\overline{\overline{A}} = A$ or more generally: (a*) $A \subseteq B \Rightarrow B - (B - A) = A$
(b) $A \cup \overline{A} = U$ (c) $A \cap \overline{A} = \varnothing$
(d) $\overline{\varnothing} = U$ (e) $\overline{U} = \varnothing$

(8) *De Morgan Laws*

(a) $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ or more generally: (a*) $A - (B \cup C) = (A - B) \cap (A - C)$
(b) $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ or more generally: (b*) $A - (B \cap C) = (A - B) \cup (A - C)$

## 1.3. FAMILIES OF SETS

The union and intersection (as well as the symmetric difference) operations can be applied simultaneously to more than two sets. Let $I$ be a set such that $A_i$ is a set for each $i \in I$. When used in this manner, the set $I$ is called an ***index set***.

The set

$$\boxed{\Im \;=\; \{A_i \mid i \in I\}}$$

is called a ***family of sets indexed by*** *I*. The union and intersection of this family are defined

$$\bigcup_{i\in I} A_i \;\equiv\; \bigcup \Im \;=\; \{x \mid \text{for some } i \in I, x \in A_i\} \;=\; \{x \mid \exists i \in I\left[x \in A_i\right]\}$$

and

$$\bigcap_{i\in I} A_i \;\equiv\; \bigcap \Im \;=\; \{x \mid \text{for all } i \in I, x \in A_i\} \;=\; \{x \mid \forall i \in I\left[x \in A_i\right]\} \;.$$

A useful case is when the index set is some set of positive consecutive integers, in which case we have

$$\bigcup_{i\in I} A_i = \bigcup_{i=1}^{k} A_i = A_1 \cup A_2 \cup \ldots \cup A_k \qquad \text{and} \qquad \bigcap_{i\in I} A_i = \bigcap_{i=1}^{k} A_i = A_1 \cap A_2 \cap \ldots \cap A_k \;.$$

or

$$\bigcup_{i\in I} A_i = \bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \ldots \cup A_k \cup \ldots \qquad \text{and} \qquad \bigcap_{i\in I} A_i = \bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \ldots \cap A_k \cap \ldots$$

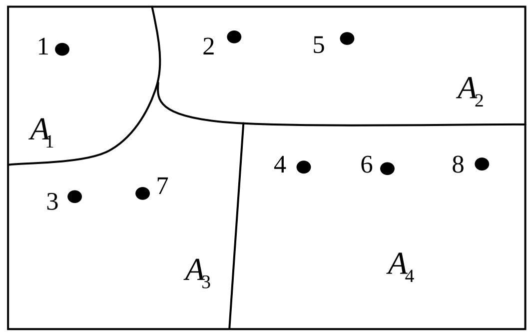
**Example 3**. Let $\Im = \left\{R_{KC}, R_{NYC}, R_{DSM}, R_{LA}, R_{SF},\; R_{MIA}\right\}$ be the family of sets of residents in the respective cities $I = \{KC, NYC, DSM, LA, SF, MIA\}$. Then $I$ is an index set that *helps name* the sets in the family $\Im$. ■

Family of sets $\Im \;=\; \{A_i \mid i \in I\}$ is ***pairwise disjoint*** if every two distinct sets in $\Im$ are disjoint. I.e., $\boxed{\forall i, j \in I\,[i \neq j \Rightarrow A_i \cap A_j = \varnothing]}$. Note that this is more than saying that $\Im$ is ***disjoint*** which just means that there is no element that belongs to every $A_i$, i.e. that the intersection $\bigcap_{i\in I} A_i$ is empty.

Let *S* be a set. We say that $\Im$ ***covers*** set *S* if $\boxed{S \subseteq \bigcup_{i\in I} A_i}$. A ***partition*** of a set *S* is a family of sets $\prod = \{A_i \mid i \in I\}$ such that the following conditions hold

(1) $\forall i \in I,\; \varnothing \neq A_i \subseteq S$,
(2) $\prod$ is pairwise disjoint, and
(3) $\prod$ covers *S*.

**Example 4**. Let $S=\{1,2,3,4,5,6,7,8\}$, $I=\{1,2,3,4\}$, $\Im=\{A_1,A_2,A_3,A_4\}$ where the set are $A_1=\{1\}$, $A_2=\{2,5\}$, $A_3=\{3,7\}$, and $A_4=\{4,6,8\}$. Clearly $\Im$ is a partition of *S*. We can draw a "picture" of the partition




■

**Example 5**. Consider $\mathbb{Z}$, the set of all integers. Define three non-empty subsets of $\mathbb{Z}$:

$X_0 \triangleq \{n \in \mathbb{Z} \mid 3 \text{ divides } n \text{ evenly}\} = \{\ldots,-6,-3,0,3,6,9,\ldots\}$,
$X_1 \triangleq \{n \in \mathbb{Z} \mid \text{when dividing } n \text{ by 3 we get remainder } 1\} = \{\ldots,-5,-2,1,4,7,10,\ldots\}$, and
$X_2 \triangleq \{n \in \mathbb{Z} \mid \text{when dividing } n \text{ by 3 we get remainder } 2\} = \{\ldots,-4,-1,2,5,8,11,\ldots\}$.

The family of sets $\{X_0,X_1,X_2\}$ covers $\mathbb{Z}$ because every integer (positive, negative or zero), when divided by 3 gives a remainder that is 0, 1 or 2. For example, $17=5\cdot 3+2$ and $21=7\cdot 3+0$, implying that $17\in X_2$ and $21\in X_0$. It follows that $\mathbb{Z}\subseteq X_0\cup X_1\cup X_2$. The sets $X_i$ and $X_j$, for $i\neq j$, are disjoint since the remainder when dividing an integer by 3, is unique. Thus, $\{X_0,X_1,X_2\}$ forms a partition of $\mathbb{Z}$. ■

## 1.4. <u>CARTESIAN PRODUCT</u>

Intuitively, an ***<u>ordered pair</u>*** $(a,b)$ is an "ordered" 2-set which has *a* as its *<u>first</u>* element and *b* as its *<u>second</u>* element. Such pairs are useful in Calculus, Analytic Geometry and Physics where they are sometimes called *vectors*. Recall that vectors in the plane (with coordinates) can be thought of as a points in the plane, labeled by ordered pairs of real numbers. They are ordered in that a pair $(3,4)$ is not the same as the pair $(4,3)$ because they denote different points in the plane. In Computer Science, linear arrays are ordered sequences (not necessarily pairs) and they serve as very useful data structures.

The *defining property* of ordered pairs is the requirement

$$(a,b)=(c,d) \Leftrightarrow \left[a=c \wedge b=d\right].$$

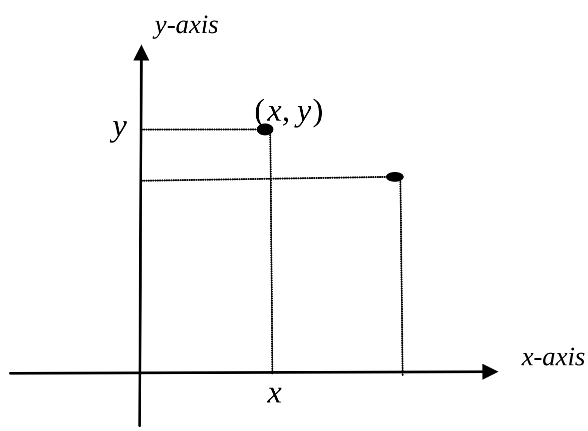
An ordered pair can be defined as a set (not "ordered") and this is left as an exercise.

Let $A$ and $B$ be two sets. The ***Cartesian product***[2] of $A$ and $B$ is a set of all ordered pairs that can be formed with an element of $A$ as a first "coordinate" and an element of $B$ as a second coordinate. The notation and a more formal definition is

$$\boxed{A\times B \triangleq \left\{(a,b)\middle|\ a\in A\wedge b\in B\right\}}.$$

For example, $\{0,1,2\}\times\{a,b\}=\{(0,a),(0,b),(1,a),(1,b),(2,a),(2,b)\}$. Obviously, for all sets $A$, $A\times\varnothing=\varnothing\times A=\varnothing$. It is also easy to see that, generally, $A\times B\neq B\times A$, for example $\{0\}\times\{1\}=\{(0,1)\}\neq\{(1,0)\}=\{1\}\times\{0\}$.

**Example 6**. The Euclidean plane with xy-coordinates:




can be viewed as the Cartesian product $\mathbb{R}\times\mathbb{R}$. ■

We shall use ordered pairs and Cartesian products when we discuss (binary relations). Cartesian products of more than just two sets are used extensively in Relational Data Bases.

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[2] **René Descartes** (1596 – 1650) was a French philosopher. The Cartesian coordinate system—allowing algebraic equations to be expressed in 2D coordinate system—was named after him. He is credited as the father of analytical geometry, the bridge between algebra and geometry, crucial to the discovery of infinitesimal calculus. http://en.wikipedia.org/wiki/Ren%C3%A9_Descartes

## 1.5. CARDINALITY OF FINITE SETS

Recall that the cardinality of a finite set was defined as the number of distinct elements in the set. We also noted (but have not proved) that for the power set of a finite sets we have

$$|\mathcal{P}(S)| \equiv |2^S| = 2^{|S|}.$$

With respect to the cardinality of finite sets obtained by set-theoretic operations we have the following properties.

**Claim 7**. Let *A* and *B* be two finite sets.

1. If $A$ and $B$ are disjoint (i.e., $A \cap B = \varnothing$), then
   - $|A \cup B| = |A| + |B|$.
   - $|A \cap B| = 0$.
   - $|A \oplus B| = |A| + |B|$.
2. More generally (when *A* and *B* are not necessarily disjoint)
   - $|A \cup B| = |A| + |B| - |A \cap B|$.
   - $|A \oplus B| = |A| + |B| - 2|A \cap B|$.
3. $|A \times B| = |A||B|$.

**Proof**. Just talk the reasoning through. ■

Suppose one wants to count how many different subsets of a particular cardinality a finite set has. For example, a set of four elements, say $S = \{1,2,3,4\}$, has six distinct subsets of cardinality 2: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, and $\{3,4\}$. It should be obvious that exactly the same answer is obtained if any other set of four elements is considered instead of *S*. We introduce the following notation. Let $0 \le k \le n.$ Then

$$\boxed{\binom{n}{k} \equiv \text{"}n \text{ choose } k\text{"} \triangleq \begin{cases} \text{number of } k\text{-subsets} \\ \text{of an } n\text{-set} \end{cases}}$$

**Example 7**. $\binom{n}{0}=1$, $\binom{n}{1}=n$, and $\binom{n}{n}=1$. Also note that, $\binom{n}{k} = \binom{n}{n-k}$. Discuss briefly. ■

INTERNET RESOURCES ON SET THEORY

There are thousands of links to set theory on WWW – at various levels. Here are a few good ones at the beginners' level:

History: Highly recommended review of how set theory started.

http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Beginnings_of_set_theory.html

Stanford Encyclopedia of Philosophy http://plato.stanford.edu/entries/set-theory/

Wikipedia http://en.wikipedia.org/wiki/Naive_set_theory

## EXERCISES

(01) Use quantifier notation to write a formula that says that sets $A$ and $B$ are not equal without using the equality symbol.

(02) [Quine's[3] paradoxical set] Recall the Russell's paradox and the Russell set of all sets that do not belong to themselves $\mathcal{R} = \{A \mid A \text{ is a set and } A \notin A\}$. Let's define another set of sets, which you are asked to prove to be "paradoxical",

$$\boldsymbol{Q}_1 = \{A \mid A \text{ is a set } \& \neg\exists \text{set } B[A \in B \wedge B \in A]\}.$$

(a) Prove that $\boldsymbol{Q}_1 \notin \boldsymbol{Q}_1$.

(b) Despite part (a), show that the definition of the set $\boldsymbol{Q}_1$ is paradoxical (self-contradictory) by proving that: $\{\boldsymbol{Q}_1\} \in \boldsymbol{Q}_1 \;\Leftrightarrow\; \{\boldsymbol{Q}_1\} \notin \boldsymbol{Q}_1$.

(03) We have defined $A = B$ if $[A \subseteq B \;\wedge\; B \subseteq A]$ and $A \neq B$ if $[A \not\subseteq B \;\vee\; B \not\subseteq A]$. Now define two sets $A$ and $B$ to be ***incomparable*** if $[A \not\subseteq B \;\wedge\; B \not\subseteq A]$. What does it mean to say that $A$ and $B$ are not incomparable?

(04) [$\subsetneq$ and $\not\subseteq$] Make sure you understand the difference between these two concepts. In particular, prove the following

(a) $A \subsetneq B \;\Rightarrow\; B \not\subseteq A$.
(b) $A \subsetneq B \;\Leftrightarrow\; A \subseteq B \wedge B \not\subseteq A$.

(05) Prove that for any two sets $A$ and $B$: $A - B = B - A \;\Rightarrow\; A = B$.

(06) Prove $A \oplus B = (A \cup B) - (A \cap B)$.

(07) Prove $(A \cap B) \cup C = A \cap (B \cup C) \;\Leftrightarrow\; C \subseteq A$.

(08) Prove or disprove:

(a) $(A - B) - C = (A - C) - (B - C)$.
(b) $A - (B - C) = (A - B) - C$.

(09) Let $A, B \subseteq U$. Prove $A = B \Leftrightarrow (A \cap \overline{B}) \cup (\overline{A} \cap B) = \varnothing$.

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[3] Willard Van Orman Quine (1908 – 2000) was an American philosopher and logician in the analytic tradition. From 1930 until his death, Quine was continuously affiliated with Harvard University. He was born Akron, Ohio. For more details see, http://en.wikipedia.org/wiki/Willard_Van_Orman_Quine .

(10) Let $A, B, C \subseteq U$. Prove that $C = A \cup B$ if, and only if, the following two conditions hold (a) $A \subseteq C \wedge B \subseteq C$, and (b) $\forall D \subseteq U[(A \subseteq D \wedge B \subseteq D) \Rightarrow C \subseteq D]$.

(11) Specify necessary and sufficient conditions on the sets *A*, *B*, and *C* such that the following equality holds: $(A - C) \cup B = (A \cup B) - C$.

(12) Let *A* and *B* be two "general" subsets of the universal set *U*. Using these it is possible to define a total of 16 sets (why?). Draw the Venn diagrams of these sets.

(13) Prove $X \oplus (Y \oplus Z) = \left(X \cap Y \cap Z\right) \bigcup \left(X \cap \overline{Y} \cap \overline{Z}\right) \bigcup \left(\overline{X} \cap Y \cap \overline{Z}\right) \bigcup \left(\overline{X} \cap \overline{Y} \cap Z\right)$ .

(14) Prove the associative law for the *XOR* operation: $X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$ .

(15) Prove $A \cup B = (A \oplus B) \cup (A \cap B) = A \oplus B \oplus (A \cap B)$ .

(16) Prove or disprove: if $A \oplus B = C$, then all the following equalities are true:

$$B \oplus A = C, \quad A \oplus C = B, \quad C \oplus A = B, \quad B \oplus C = A, \quad C \oplus B = A.$$

(17) Prove in detail: $\left(\bigcup_{n=1}^{\infty} A_n\right) \cap \left(\bigcup_{m=1}^{\infty} B_m\right) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left(A_n \cap B_m\right)$ .

(18) Prove in detail: $\left(\bigcap_{n=1}^{\infty} A_n\right) \cup \left(\bigcap_{m=1}^{\infty} B_m\right) = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \left(A_n \cup B_m\right)$ .

(19) Prove in detail: $X - \left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} \left(X - A_n\right)$.

(20) Prove in detail: $X - \left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \left(X - A_n\right)$ .

(21) Let $a, b \in \mathbb{R}$ with $a < b$. We define four kinds of intervals:

$(a,b) \triangleq \left\{x \in \mathbb{R} \middle| \ a < x < b\right\}$ [this is an *open* interval]

$[a,b] \triangleq \left\{x \in \mathbb{R} \middle| \ a \le x \le b\right\}$ [this is a *closed* interval]

$(a,b] \triangleq \left\{x \in \mathbb{R} \middle| \ a < x \le b\right\}$ [this is a "*half open-half closed*" interval]

$[a,b) \triangleq \left\{x \in \mathbb{R} \middle| \ a \le x < b\right\}$ [this also is a "*half open-half closed*" interval]

Calculate the following unions and intersections:

(a) $\bigcup_{n=1}^{\infty}[\frac{1}{n},1]$ (b) $\bigcup_{n=1}^{\infty}(\frac{1}{n},1]$ (c) $\bigcup_{n=1}^{\infty}[\frac{1}{n},1)$ (d) $\bigcup_{n=1}^{\infty}(\frac{1}{n},1)$

(e) $\bigcap_{n=1}^{\infty}[0,\frac{1}{n}]$ (f) $\bigcap_{n=1}^{\infty}[0,\frac{1}{n})$ (g) $\bigcap_{n=1}^{\infty}(0,\frac{1}{n}]$ (h) $\bigcap_{n=2}^{\infty}(-\frac{1}{n},1-\frac{1}{n})$ .

(22) Clearly, if two sets are equal, their power sets are equal: $A=B \Rightarrow \mathbb{P}(A)=\mathbb{P}(B)$. What about the converse: $\mathbb{P}(A)=\mathbb{P}(B) \Rightarrow A=B$? Prove your claims.

(23) (a) Prove that $A \subseteq B \Rightarrow \mathbb{P}(A) \subseteq \mathbb{P}(B)$.
Since $X,Y \subseteq X \cup Y$, we can immediately conclude from (a) that $P(X), P(Y) \subseteq P(X \cup Y)$ and therefore $P(X) \cup P(Y) \subseteq P(X \cup Y)$.
(b) Give precise conditions under which $P(X) \cup P(Y) = P(X \cup Y)$.

(24) Prove or disprove: for any two sets $A, B$: $P(A \cap B) = P(A) \cap P(B)$.

(25) Let $A$, $B$, and $C$ be sets such that $A \cup C = B \cup C$ and $A \cap C = B \cap C$. Does it follow that $A = B$? Prove your claim.

(26) Define an ordered pair $(a,b)$ as the set $\{\{a\},\{a,b\}\}$ and prove, using this definition, that $(a,b)=(c,d) \Leftrightarrow \left[a=c \wedge b=d\right]$.

(27) Let $\{A_n \mid n=1,2,3,\ldots\}$ be an infinite family (sequence) of sets. Define the following two sets

$$\limsup_{n\to\infty} A_n \triangleq \bigcap_{n=1}^{\infty}\bigcup_{m=0}^{\infty} A_{n+m} \qquad \liminf_{n\to\infty} A_n \triangleq \bigcup_{n=1}^{\infty}\bigcap_{m=0}^{\infty} A_{n+m} .$$

Prove the following properties of these sets.

(a) $\limsup_{n\to\infty} A_n = \left\{x \,\middle|\, x \in A_n \text{ for infinitely many } n\right\}$.

(b) $\liminf_{n\to\infty} A_n = \left\{x \,\middle|\, x \in A_n \text{ for all but finitely many } n\right\}$.

(c) $\forall n \geq 1, A_n \subseteq B_n \quad \Rightarrow \quad \limsup A_n \subseteq \limsup B_n$ & $\liminf A_n \subseteq \liminf B_n$.

(d) $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$ .

(e) $\overline{\limsup_{n\to\infty} A_n} = \liminf_{n\to\infty} \overline{A_n}$ .

(f) $\limsup(A_n \cup B_n) = \limsup A_n \cup \limsup B_n$ .

(g) $\liminf(A_n \cap B_n) = \liminf A_n \cap \liminf B_n$ .

(h) $\limsup(A_n \cap B_n) \subseteq \limsup A_n \cap \limsup B_n$ .

(i) $\liminf A_n \cup \liminf B_n \subseteq \liminf(A_n \cup B_n)$ .

(28) Let $\{A_i \mid i \in I\}$ be a family of sets, where $I$ is some index set. For $J \subseteq I$ define the following two sets:

$$\bigcup J \triangleq \bigcup_{n \in J} A_n \qquad \text{and} \qquad \bigcap J \triangleq \bigcap_{n \in J} A_n \ .$$

(a) Suppose $J \subseteq J'$. What is the relation between $\bigcup J$ and $\bigcup J'$? Prove your claim.
(b) Suppose $J \subseteq J'$. What is the relation between $\bigcap J$ and $\bigcap J'$? Prove your claim.
(c) Express $\bigcup(J \cup J')$ in terms of $\bigcup J$ and $\bigcup J'$ and prove correctness.
(d) Prove that $\bigcup(J \cap J') \subseteq \bigcup J \cap \bigcup J'$.
(e) Express $\bigcap(J \cup J')$ in terms of $\bigcap J$ and $\bigcap J'$ and prove correctness.
(f) Find a relationship between the sets $\bigcap(J \cap J')$, $\bigcap J$, and $\bigcap J'$ and prove it.

(29) Let $X$ be a set and let $\Im$ be a family of subsets of $X$ (not necessarily all subsets), i.e. $\Im \subseteq \mathcal{P}(X)$, satisfying the following conditions

(i) $X \in \Im$, and
(ii) $\Im$ is closed under arbitrary intersections: $\prod \subseteq \Im \Rightarrow \bigcap \prod \in \Im$.

Prove that for every $A \subseteq X$ <u>there is a unique</u> $A^* \subseteq X$ such that

(a) $A \subseteq A^* \in \Im$, and
(b) $\forall B \subseteq X [A \subseteq B \in \Im \Rightarrow A^* \subseteq B]$.

Also show that

(c) $\forall A \subseteq X, (A^*)^* = A^*$.

(30) Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets such that $\forall A \in \mathcal{A} \exists B \in \mathcal{B} \left[A \subseteq B\right]$ and conversely, $\forall B \in \mathcal{B} \exists A \in \mathcal{A} \left[B \subseteq A\right]$. Prove $\bigcup \mathcal{A} = \bigcup \mathcal{B}$.

(31) Under precisely what condition is $\{A - B, A \cap B\}$ a partition of $A$? Explain in detail.

(32) Give precise conditions under which $X \times Y = Y \times X$.

(33) Let $X$ and $Y$ be sets and let $A$ and $B$ (with or without subscripts) be subsets of $X$ and $Y$, respectively. Prove the following identities.

(a) $(A_1 \times B) \cup (A_2 \times B) = (A_1 \cup A_2) \times B$.
(b) $(A_1 \times B) - (A_2 \times B) = (A_1 - A_2) \times B$.

(c) $(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$.
(d) $X \times Y - A \times B = (X - A) \times Y \cup X \times (Y - B)$ .

(34) Let $X$ be a non-empty set and let $\mathfrak{B}$ be a *covering* of $X$. I.e., $\mathfrak{B}$ is a collection of non- empty subsets of $X$ such that $X = \bigcup_{B \in \mathfrak{B}} B \equiv \bigcup \mathfrak{B}$. We will refer to $\mathfrak{B}$ as a *basis* and to the sets in $\mathfrak{B}$ as *basis sets* if, in addition, it satisfies the following

$$\forall x \in X \ \forall E, F \in \mathfrak{B} \left[ x \in E \cap F \Rightarrow \exists G \in \mathfrak{B} \left[ x \in G \subseteq E \cap F \right] \right]. \qquad (*)$$

(a) Consider the condition: (†) $\forall A, B \in \mathfrak{B} \left[ A \cap B \neq \varnothing \Rightarrow \exists C \in \mathfrak{B} \left( C \subseteq A \cap B \right) \right]$. Does this condition imply (*)? Does (*) imply this condition? Prove your answers.

For a subset $A \subseteq X$ define its *closure* by $A' \triangleq \left\{ x \in X \,\middle|\, \forall B \in \mathfrak{B} \left[ x \in B \Rightarrow A \cap B \neq \varnothing \right] \right\}$. Put in words, the elements of $A'$ are "close" to the set $A$ in the sense that no basis set can "separate" that element from $A$.

(b) Show $\forall A \subseteq X \left( A \subseteq A' \right)$.
(c) $\forall A, C \subseteq X \left( A \subseteq C \Rightarrow A' \subseteq C' \right)$.
(d) Prove that $\forall A, C \subseteq X : (A \cup C)' = A' \cup C'$.

(35) [Continued] Define $A^{\dagger} \triangleq \overline{A'} = X - A'$, i.e. the complement of the closure of the set $A$.

(a) Calculate $\varnothing^{\dagger}$ and $X^{\dagger}$.
(b) Show that $\forall A, C \subseteq X \left( A \subseteq C \ \Rightarrow \ C^{\dagger} \subseteq A^{\dagger} \right)$.
(c) Prove that $\forall A \subseteq X \left( A^{\dagger\dagger} = A^{\dagger\dagger\dagger\dagger} \right)$.

We will say that $A \subseteq X$ is ***regular*** if $A^{\dagger\dagger} = A$. Denote by $\mathfrak{R}_X$ the collection of all regular subsets of $X$.

(d) Show that $\varnothing, \ X \in \mathfrak{R}_X$.
(e) Show that $\forall A \subseteq X \left( A^{\dagger\dagger} \in \mathfrak{R}_X \right)$.
(f) Prove $A, C \in \mathfrak{R}_X \ \Rightarrow \ A \cap C \in \mathfrak{R}_X$.
(g) Lets use the following abbreviation: $(A \vee C) \triangleq (A \cup C)^{\dagger\dagger}$. Prove the following properties

1. $A, C \in \mathfrak{R}_X \ \Rightarrow \ A \vee C \in \mathfrak{R}_X$.
2. $A, C \in \mathfrak{R}_X \ \Rightarrow \ (A \cap C)^{\dagger} = A^{\dagger} \vee C^{\dagger}$.
3. $A, C \in \mathfrak{R}_X \ \Rightarrow \ (A \vee C)^{\dagger} = A^{\dagger} \cap C^{\dagger}$.

4. $A \cap (E \vee C) = (A \cap E) \vee (A \cap C).$
5. $A \vee (E \cap C) = (A \vee E) \cap (A \vee C).$

NEW EXTRA PROBLEMS

1) Prove or disprove:

- $A \cup B = B \cap C \Leftrightarrow A \subseteq B \subseteq C.$
- $A \in B \wedge B \in C \Rightarrow A \in C.$
- $A \subseteq B \wedge B \in C \Rightarrow A \in C.$

2) [Transitive sets] A set $A$ is said to be ***transitive*** if for every $x$ and $B$, $x \in B \in A \Rightarrow x \in A.$

- Prove that a set $A$ is transitive iff for every set $B$: $B \in A \Rightarrow B \subseteq A.$
- Give an example of a transitive set and a non-transitive set. [Can use set $\{\varnothing\}$.]
- Prove that $A$ is transitive iff $A \subseteq \mathbb{P}(A)$.
- Prove that $A$ is transitive $\Rightarrow \mathbb{P}(A)$ is transitive.
- Prove that $A$ is transitive $\Rightarrow A \cup \{A\}$ is transitive.