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Exam 5A

$$x'(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & -1 & 1 \\ 1 & 1 & 0 & -2 \end{bmatrix} x(t)$$

1) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & -1 & 1 \\ 1 & 1 & 0 & -2 \end{bmatrix}$

So characteristic ~~of~~ equation is $|A - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 2 & 2-\lambda & 0 & 0 \\ 2 & 2 & -1-\lambda & 1 \\ 1 & 1 & 0 & -2-\lambda \end{bmatrix} = 0$$

$$(1-x) \begin{bmatrix} 2-\lambda & 0 & 0 \\ 2 & -1-\lambda & 1 \\ 1 & 0 & -2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) \begin{vmatrix} -1-\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(-1-\lambda)(-2-\lambda) = 0$$

$$\Rightarrow \lambda = -1, -2, 1, 2$$

The eigen values of A are $\lambda = -1, -2, 1, 2$.

2] eigenvectors

consider the homogenous system $(A - \lambda I)x = 0$

when $\lambda = -1$ then $(A + I)x = 0$.

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from 1st row: $2x_1 = 0 \Rightarrow x_1 = 0$

from 2nd row: $2x_1 + 3x_2 = 0 \Rightarrow x_2 = 0$

from 3rd row: $2x_1 + 2x_2 + x_4 = 0 \Rightarrow x_4 = 0$

choose $x_3 = k_1$ where k_1 is an arbitary constant

$\therefore$ the eigen vector x_1 when $\lambda_1 = -1$ is

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

from $k_1 = 1$

Now we find the eigenvectors of A corresponding to the eigen values

Consider the homogeneous system $(A-\lambda I)_x=0$
When $\lambda=-2$ then $(A+2I)_x=0$.

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from the 1st row $= 3x_1=0 \Rightarrow x_1=0$
from the 2nd $= 2x_1+4x_2=0 \Rightarrow x_2=0$.
from the 3rd $= 2x_1+2x_2+x_3+x_4=0$
$\Rightarrow x_3=-x_4$

choose $x_3=k_1$ where k_1 is an arbitary
$\therefore$ The eigen vector x_1 when $\lambda=-1$ is

$$x_1=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k_1 \\ -k_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ for } k_1=1$$

when $\lambda = 1$, we have $(A - I)x = 0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & -2 & 0 \\ 2 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from 2rd row : $2x_1 + x_2 = 0 \Rightarrow x_2 = -2x_1$

from 3rd row : $2x_1 + 2x_2 - 2x_3 = 0 \Rightarrow$

$x_3 = x_1 + x_2 = -x_1$

from 4th row : $2x_1 + x_2 - 3x_4 = 0$

$\Rightarrow x_4 = 0$.

$\therefore$ The eigen vector when $\lambda = 1$ is

$$x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \\ -x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

when $\lambda = 2$ we have $(A - 2I)x = 0$.

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 2 & -3 & 1 \\ 2 & 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from 1st row: $-x_1 = 0 \Rightarrow x_1 = 0$

from 4th row: $2x_1 + x_2 - 4x_3 = 0 \Rightarrow$

$x_2 = 4x_3$

from 2nd row: $2x_1 + 2x_2 - 3x_3 + x_4 = 0$

$= 2x_2 - 3x_3 + x_4 = 0$

$= 8x_3 - 3x_3 + x_4 = 0$

$= x_4 = -5x_3$

So $x_4 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4x_3 \\ x_3 \\ -5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \\ -5 \end{bmatrix}$

☆ General solution, $x' = Ax$ is

$x(t) = 4e^{-t}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + c_3e^{t}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix}$

$+ c_4e^{2t}\begin{bmatrix} 0 \\ 4 \\ 1 \\ -5 \end{bmatrix}$