Jay Patel Exam 3.

Q1] $2y' - y = \sinh(x)$

we know that $\sinh(x) = \frac{1}{2}[e^x - e^{-x}]$

to expand e^x and e^{-x}

$\therefore\ e^x = \sum_{x=0}^{\infty} \frac{1}{m!} x^n$

$e^{-x} = \sum_{x=0}^{\infty} \frac{1}{n!} (x)^n = \sum_{x=0}^{\infty} \frac{(-1)^n}{n!} x^n$

now, consider the difference

$e^x - e^{-x} = \sum_{n\to 0}^{\infty} \frac{1}{n} (x^n - (-1)^n x^n)$

→ for even $(-1)^n = 1$

So $x^n = (-1)^n x^n = x^n - x^n = 0$.

→ for odd we have,

$(-1)^n = -1$ so we find

$x^n - (-1)^n x^n = x^n + x^n = 2x^n$

So,

$e^x - e^{-x} = \sum_{n\to 0}^{\infty} \frac{2}{(2n+1)!} x^{2n+1}$

since, they both are similar

$\sinh(x) = \sum_{n\to 0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$

Q2] Given: $(1-2x)y'' - y' + xy = 0$, $y(0) = y'(0) = 1$

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the solution of the eq.

Then $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

Sub y, y', y'' .....

$$(1-2x) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 2 \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2+1} - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} [2n(n-1)+n] a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} [2n^2 - n] a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Replace n by $n+1$ in the first summation

$$\sum_{n+1=2}^{\infty} (n+1)(n+1-1) a_{n+1} x^{n+1-2} - \sum_{n=1}^{\infty} [2n^2 - n] a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^{n-1} - \sum_{n=1}^{\infty} [2n^2 - n] a_n x^{n-1} = - \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$\sum_{n=1}^{\infty} \left((n+1)(n)\,a_{n+1} - (2n^2-n)\,a_n\right) x^{n-1} =$$

$$-\sum_{n=0}^{\infty} a_n x^{n+1}$$

Eqaⁿ→1 $(2a_2 - a_1) + (6a_3 - 6a_2)x + (12a_4 - 15a_3)x^2 +$
$(20a_5 - 28a_4)x^3 + ..$
$= -a_0 x - a_1 x^2 - a_2 x^3 \dots$

Given $y(0) = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

put $x=0$ in above target

$y(0) = a_0 \Rightarrow \boxed{a_0 = 0}$

Given $y'(0) = 1$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = 0 + a_1 + a_2 x + \dots$$

put $x=0$ in above

$y'(0) = a_1 \Rightarrow \boxed{a_1 = 1}$

equating on both of tanx

$2a_2 - a_1 = 0 \Rightarrow 2a_2 - 1 = 0 \Rightarrow \boxed{a_2 = \frac{1}{2}}$

coff in eqn → 1

$6a_3 - 6a_2 = -a_0 \Rightarrow 6a_3 - 6\left(\frac{1}{2}\right) = 0 \Rightarrow \boxed{a_3 = \frac{1}{2}}$

equating x^2 in equation 1

$$12a_4 - 15a_3 = -a_1 \Rightarrow 12a_4 - 15\left(\frac{1}{2}\right) = -1 \Rightarrow \boxed{a_4 = \frac{13}{24}}$$

Same for x^3,

$$20a_5 - 28a_4 = -a_2 \Rightarrow 20a_5 - 28\left(\frac{13}{24}\right) = \frac{-1}{2}$$

$$\Rightarrow \boxed{a_5 = \frac{11}{15}}$$

So,

$$\boxed{y(x) = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{13}{24}x^4 + \frac{11}{15}x^5 + \dots}$$

Q3] $x^3(x^2-1)^2(x^2+1)y'' + (x-1)xy' + y = 0$

Singular points:

$x^3(x^2-1)^2(x^2+1) = 0$

$x = 0, 1, -1, i, -i$

By using /writing in standard form

$$y'' + \frac{(x-1)x}{x^3(x^2-1)^2(x^2+1)} y' + \frac{1}{x^3(x^2-1)^2(x^2+1)} y = 0$$

$$y'' + \frac{(x-1)}{x^2(x-1)^2(x+1)^2(x+i)(x-i)} y' + \frac{1}{x^3(x-1)^2(x+1)^2(x+i)(x-i)} y = 0.$$

$$y'' + \frac{1}{x^2(x-1)^2(x+1)^2(x+i)(x-i)} y' + \frac{1}{x^3(x-1)^2(x+1)^2(x+i)(x-i)} y = 0.$$

So, $y'' + p(x)y' + q(x)y = 0$.

for $x=0$ to be reggular point, factor x can appear on first power in the denominator of $p(x)$

x has the highest power 2 in denominator
so, $x=0$ is irregular singular point

for $x=1$ to be regular singular point, the factor $x-1$ can be appear at most to the first power in the denominator.
so $x=1$ is a regular singular point

for $x=-1$ to be a regular singular point, the factor $x+1$ can appear at most to the second power in denominator of $q(x)$.
so $x=-1$ is irregular singular point

if $x=i$ & $x=-i$ the first condition is $x-i$ and $x+i$ are appearing at most to the first power in denominator of $p(x)$ is satisfied.
so $x=i$ & $x=-i$ are regular singular point

Q4] $xy'' - y' + 2y = 0$ — (1)

$\Rightarrow y'' - \frac{1}{x}y' + \frac{2}{x}y = 0$

comparing this with $y'' + p(x)y' + q(x)y = 0$

$p(x) = \frac{-1}{x}$, $q(x) = 2/x$

$x\,p(x) = -1$, $x\,q(x) = 2$

So, $x = 0$ is a singular point

$p_0 = \lim_{x \to 0} x\,p(x) = -1$

$q_0 = \lim_{x \to 0} x\,q(x) = 2$

Thus, the equation,

$-r^2 + 2r = 0$

$\Rightarrow r_1 = 0$ and $r_2 = 2$

Since, the largest root $= 2$, the series is

$y_1 = x^2 \sum_{n=0}^{\infty} a_n x^n$

$= \sum_{n=0}^{\infty} a_n x^{(n+2)}$

$y' = \sum_{n=0}^{\infty} a_n(x+2)\,x^{n+1}$

$$y'' = \sum_{n=0}^{\infty} a_n (n+2)(n+1)x^n$$

Subsituting in eq 1 we get

$$\sum_{n=0}^{\infty} a_n (x+2)(x+1)x^{(n+1)} - \sum_{n=0}^{\infty} a_n (n+2)x^{(n+1)}$$

$$+2 \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n [(n+2)(n+1) - (n+2)] x^{n+1} + 2 \sum_{n=0}^{\infty}$$

$$a_n x^{+2} = 0. \quad -(2)$$

$$\text{So, } \sum_{n=0}^{\infty} a_{n+1} [(n+3)(n+2) - (n+3)] x^{x+2}$$

$$+ \sum_{n=0}^{\infty} 2a_n x^{n+2} = 0$$

$$\sum_{n=-1}^{\infty} a_{n+1} (x+3)(x+1) x^{n+2} + \sum_{n=0}^{\infty} 2 a_n x^{n+2} = 0$$

comparing the coefficients of x^{n+2} for $n > 0$;

$$a_{n+1} (x+3)(x+1) + 2a_n = 0.$$

$$\Rightarrow a_{n+1} = \frac{-2}{(x+1)(x+3)} a_n$$

Then, $n=0.$ $a_1 = \frac{-2}{3} a_0$

$n=1$ $a_2 = \frac{-2}{8} a_1 = \frac{(-2)(-2)}{3 \cdot 8} a_0 = \frac{1}{24} a_0$

$= \frac{1}{6} a_0$

$n=2$ $a_3 = \frac{-2}{12} a_2 = \frac{(-2)}{12} \times \frac{1}{6} a_0 = \frac{-1}{36} a_0$

$x=3;$ $a_4 = \frac{-2}{24} a_3 = \frac{-2}{24} \times \frac{-1}{36} a_0 = \frac{1}{432} a_0$

So,

$$\boxed{y = y_1 = x^2 - \frac{2}{3} x^3 + \frac{1}{6} x^4 - \frac{1}{36} x^5 + \frac{1}{432} x^6 \ldots}$$