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Q1] we can use standard velocity acceleration as given constant

$V = v_0 + \int a\,dt$

$V = v_0 + at$

as acceleration is constant
in 50's it reaches 3 km = 3000 m

so $s = ut + 0.5 * a * t^2$

$3000 = 15 * 50 + 0.5 * 50^2 * a$

$a = 1.8\ m/s^2$

$V = 15 + 1.8 * 50$

$= 15 + 90$

$= 108\ m/s$

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Q2] $y'' - 3y' - y = e^{4x} \cos 2x$

for $y^c = y'' - 3y' - y = 0$

having roots

$\frac{+3 \pm \sqrt{13}}{2}$ so we can write the left side as,

$$\left(D - \left(\frac{3 - \sqrt{13}}{2}\right)\right)\left(D - \left(\frac{3 + \sqrt{13}}{2}\right)\right) y$$

or

$$(D^2 - 3D - D)y$$

The annihilator of right hand side of the function $e^{4x} \cos 2x$ corresponds to roots $4 \pm 2i = 0$

$$y_c = D^2 - 8D + 20$$

$$y_c = C_1 e^{\frac{+3+\sqrt{13}}{2}} + C_2 e^{\frac{+3-\sqrt{13}}{2}}$$

& $y_p = e^{4x}(A \cos 2x + B \sin 2x)$

$y = y_p + y_c$

$$y = e^{4x}(A \cos 2x + B \cos 2x) + C_1 e^{\frac{3+\sqrt{13}}{2}} + C_2 e^{\frac{3-\sqrt{13}}{2}}$$

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Q3] $y'' + 2y' + 10y = 10 \qquad y(0) = 0,\ y'(0) = 1$

$L(y'') + 2L(y') + 10L(y) = L(10)$

$s^2 y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 10Y(s) = \frac{10}{s}$

$s^2 Y(s) - 1 + 2sY(s) + 10Y(s) = \frac{10}{s}$

$s^2 Y(s) + 2sY(s) + 10Y(s) = \frac{10}{s} + 1$

$s^2 Y(s) + 2sY(s) + 10Y(s) = \frac{10+s}{s}$

$(s^2 + 2s + 10)\, Y(s) = \frac{10+s}{s}$

$Y(s) = \frac{10+s}{s(s^2+2s+10)}$

$\frac{10+s}{s(s^2+2s+10)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+10}$

$10 + s = A(s^2 + 2s + 10) + (Bs + c)s$

$10 + s = As^2 + 2As + 10A + Bs^2 + cs$

$10 + s = (A+B)s^2 + (2A + c)s + (10A)$

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compare coefficient

constant : $10A \Rightarrow 10 \Rightarrow A = 1$

s : $2A + C = 1$

$2(1) + C = 1$

$C = 1-2$

$C = -1$

$s^2 =$ $A + B = 0$

$B = -A$

$B = -1$

$$\Rightarrow \frac{10+s}{s(s^2+2s+10)} = \frac{1}{s} + \frac{-s-1}{s^2+2s+10}$$

$$Y(s) = \frac{1}{s} + \frac{-s-1}{s^2+2s+10}$$

$$Y(s) = \frac{1}{s} + \frac{s+1}{s^2+2s+10}$$

$$Y(s) = \frac{1}{s} - \frac{s+1}{(s+1)^2+(3)^2}$$

take inverse laplace transform, we get

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$$y(t) = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{s+1}{(s+1)^2+(3)^2}\right\}$$

$$y(t) = 1 - L^{-1}\left\{\frac{s+1}{(s+1)^2+(3)^2}\right\}$$

Use shift rule $s+1 \to s$

$$y(t) = 1 - e^{-1} L^{-1}\left\{\frac{s}{s^2+(3)^2}\right\}$$

$$y(t) = 1 - e^{-t} \cos 3t.$$

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Q4] $xy'' - 2y' + y = 0$

5 terms $(a_0 - a_4)$

$y = \sum_{x=0}^{\infty} a_x x^{x+y}$ $\quad y' = \sum_{x=0}^{\infty} (n+r) a_x x^{x+y-1}$

$y'' = \sum_{x=0}^{\infty} (n+r)(n+r-1) a_x x^{n+r-2}$

Substituting these terms in the given differential equation

$$\sum_{x=0}^{\infty} (n+r)(n+r-1) a_x x^{x+r-1} - \sum_{x=0}^{\infty} 2(n+r) a_x x^{x+r-1}$$

$$+ \sum_{x=0}^{\infty} a_x x^{x+r} = 0$$

$$\Rightarrow \sum_{x=0}^{\infty} (n+r)(n+r-3) a_x x^{x+r-1} + \sum_{x=0}^{\infty} a_x x^{n+r} = 0$$

Shifting to $n-1$ in the second summation

$$\sum_{x=0}^{\infty} (n+r)(n+r-3) a_x x^{x+r-1} + \sum_{x=1}^{\infty} a_{x-1} x^{x+r-1} = 0$$

$$\Rightarrow r(r-3) a_0 x^{r-1} + \sum_{x=1}^{\infty} (n+r)(n+r-3) a_x x^{n+r-1}$$

$$+ \sum_{x-1}^{\infty} a_{x-1}\ x+r-1 = 0$$

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the indical equation is $r(r-3)=0$

$\Rightarrow r = 0,3$

further, the identity we get from the summation is,

$$\sum_{x=1}^{\infty} \{(n+r)(n+r-3)\, a_x + a_{x-1}\} x^{x+r-1} = 0$$

$$a_x = \frac{-a_{x-1}}{(n+r)(n+r-3)}$$ this is the recurrence relation.

Case 1:

When $r=0$, $a_x = \frac{-a_{x-1}}{(n)(n-3)}$ $(n \neq 3)$

putting $n=1,2$ we get

$$a_1 = \frac{-a_0}{1-2}, \quad a_2 = \frac{-a_1}{2-1} = \frac{a_0}{(-1)^2 \cdot 2^2}$$

$$a_5 = \frac{-a_4}{5 \cdot 2}, \quad a_6 = \frac{-a_5}{6 \cdot 3} = \frac{a_4}{2 \cdot 3 \cdot 5 \cdot 6}$$

Similarily, $a_7 = \frac{-a_4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$

$$a_8 = \frac{-a_4}{2 \cdot 3 \cdot 4 \cdot 5^2 \cdot 6 \cdot 7 \cdot 8}$$

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$$y = a_0 \left\{ 1 + \frac{1}{2}x + \frac{1}{4}x^2 \right\} + a_4 \left\{ x^4 + \frac{1}{2 \cdot 5} x^5 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} x^7 + \cdots \right\}$$

when $r = 3$, $a_x = \frac{-a_{n-1}}{n(n+3)}$

putting $n = 1, 2, 3$

$$a_1 = \frac{-a_0}{1 \cdot 4}$$

$$a_2 = \frac{a_0}{1 \cdot 2 \cdot 4 \cdot 5}$$

$$a_3 = \frac{-a_0}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

$$a_k = \frac{(-1)^k a_0}{1 \cdot 2 \cdot 3 \cdots k \cdot 4 \cdot 5 \cdot 6 \cdots (k+3)} = \frac{(-1)^k a_0}{k!\,(k+3)!}$$

$$y = a_0 \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+3)!} x^k \right\}$$

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Q5] $x'(t) = \begin{bmatrix} -7 & 0 & 0 & 0 \\ 8 & -3 & 4 & 0 \\ 1 & 0 & -5 & 0 \\ 2 & 1 & 4 & -1 \end{bmatrix} x(t)$

Eigenvalues

$$\det(A-\lambda I) = \begin{bmatrix} -7-\lambda & 0 & 0 & 0 \\ 8 & -3-\lambda & 4 & 0 \\ 1 & 0 & -5-\lambda & 0 \\ 2 & 1 & 4 & -1-\lambda \end{bmatrix}$$

$= (-7-\lambda)(-3-\lambda)(-5-\lambda)(-1-\lambda) = 0.$

Solving we have the eigenvalues

$\lambda_1 = -7,$
$\lambda_2 = -3$
$\lambda_3 = -5$
$\lambda_4 = -1$

-Eigenvectors v_1:
for $v_1 = -7$ $(A - \lambda_1 I)v_1 = 0$, ie.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 8 & 4 & 4 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 4 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

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Let $x_4 = 1$

$$\begin{bmatrix} 8 & 4 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6 \end{Bmatrix}$$

Solving we have

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -6 \\ -2 \end{Bmatrix}$$

eigenvectors are,

$$v_1 = \begin{Bmatrix} 4 \\ -6 \\ -2 \\ 1 \end{Bmatrix} \quad \text{or} \quad v_1 = \begin{Bmatrix} 4/\sqrt{57} \\ -6/\sqrt{57} \\ -2/\sqrt{57} \\ 1/\sqrt{57} \end{Bmatrix}$$

Similar, $\lambda_2 = -3$ $\quad (A - \lambda I)v_2 = 0$

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ 8 & 0 & 4 & 0 \\ 1 & 0 & -2 & 0 \\ 2 & 1 & 4 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Let $x_4 = 1$, and solving we have

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2 \\ 0 \end{Bmatrix}$$

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So v_2 eigenvector is

$$v_2 = \begin{Bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{Bmatrix} \text{ or normalized } v_2 = \alpha \begin{Bmatrix} 0 \\ -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{Bmatrix}$$

similar for $\lambda_3 = -5$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 8 & 2 & 4 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 4 & 4 \end{bmatrix} \alpha \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \alpha \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Let $x_4 = 1$

$$v_3 = \alpha \begin{Bmatrix} 0 \\ 4 \\ -2 \\ 1 \end{Bmatrix} \text{ or normalized } v_3$$

$$= \alpha \begin{Bmatrix} 0 \\ 4/\sqrt{21} \\ -2\sqrt{21} \\ 1/\sqrt{21} \end{Bmatrix}$$

Similar for $\lambda_4 = -1$ $(A - \lambda_4 I)v_4 = 0$ ie

$$\begin{bmatrix} -6 & 0 & 0 & 0 \\ 8 & -2 & 4 & 0 \\ 1 & 0 & -4 & 0 \\ 2 & 1 & 4 & \end{bmatrix} \alpha \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \alpha \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

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Let $x_4 = 1$ & solving we have the eigenvector V3

$$V4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

Checking for each λ_1 $(A - \lambda_1 I) V0 = 0$ is true so answer is correct.

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Q6] $\frac{dy}{dx} = \sinh(x) + 1$ $[0, 1)$ $y(0) = 0$

$h = 0.25$

$y_1 = y_0 + k$ & $x_1 = x_0 + h$

$K_1 = hf(x_0 y_0)$ & $K_2 = h(x_0 + h, y_0 + K_2)$

$K = \frac{K_1 + K_2}{2}$

So, $K_1 = 0.25f(1) = 0.25$

$K_2 = 0.25f(0 + 0.25) - (0 + 0.25) = 0$

$K = \frac{K_1}{2} = \frac{0.25}{2} = 0.125$

$y_1 = y_0 + k$ $x_1 = x_0 + h$

$\boxed{y_1 = 0 + 0.125}$ $\boxed{x_1 = 0.25}$

only one itreation is required.